The Annals of Probability
2002, Vol. 30, No. 3, 1369–1396

LAWS OF THE ITERATED LOGARITHM FOR THE RANGE OF RANDOM WALKS IN TWO AND THREE DIMENSIONS

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Let S_n be a random walk in $\mathbf{Z}^d$ and let R_n be the range of S_n. We prove an almost sure invariance principle for R_n when $d = 3$ and a law of the iterated logarithm for R_n when $d = 2$.

1. Introduction. Let S_n be a random walk taking values in $\mathbf{Z}^d$ and let R_n be the range of S_n. That means that R_n is the number of points visited at least once by S_k, $k \le n$. The subject of the asymptotics of R_n has a long history in probability. Despite this, the problem of proving a law of the iterated logarithm for dimensions $d = 2, 3$ has remained open, even for the case of simple symmetric random walk. Our purpose in this paper is to provide such LILs.

The strong law of large numbers for R_n was proved in Dvoretzky and Erdős [5]. The central limit theorem for $d \ge 3$ can be found in Jain and Pruitt [13, 16], for example, while the case $d = 2$ was proved by Le Gall [18]. See Le Gall and Rosen [21] for a central limit theorem when the random walk is in the domain of attraction of a stable law. The LIL for $d \ge 4$ can be found in Jain and Pruitt [14]. An almost sure invariance principle for R_n in the case $d \ge 4$ was recently proved by Hamana [8]. For information on large deviations, see Donsker and Varadhan [4] and Hamana and Kesten [11, 10]. Questions about the range have as analogues questions about the volume of the Wiener sausage. See, for example, Le Gall [19].

In this paper we first consider the case of dimension 3. We show that under some moment assumptions on S_n an almost sure invariance principle holds. Changing the probability space if necessary, we show there exists a Brownian motion B_t, an explicit constant σ, and another constant $q < 1/2$ such that

$$\frac{R_n - ER_n}{\sigma} - B_{n \log n} = O\big(\sqrt{n}(\log n)^q\big) \qquad \text{a.s.}$$

Our rate is quite poor and can probably be improved. However, our results are strong enough to yield the analogues of the usual LILs for Brownian motion. For

Received December 2000; revised September 2001.

[1]Supported in part by NSF 99-88496 and Grant-in-Aid for Scientific Research (A)(1) 11304003 of Japan.

[2]Supported in part by Grant-in-Aid for Scientific Research (C)(2) 11640713 of Japan.

AMS 2000 subject classifications. Primary 60J10; secondary 60F15, 60G17.

Key words and phrases. Range of random walk, law of the iterated logarithm, almost sure invariance principle, intersection local time.

example, we show

$$\limsup_{n\to\infty} \frac{R_n - ER_n}{\sqrt{n \log n \log\log n}} = c_{1.1} \qquad \text{a.s.}$$

where $c_{1.1}$ is an explicitly determined constant. The extra $\log n$ term in the almost sure invariance principle and in the LIL is a consequence of the fact that $\operatorname{Var} R_n \asymp n \log n$, where $f_n \asymp g_n$ means the ratio f_n/g_n is bounded above and below by positive constants not depending on n.

The case $d = 2$ is considerably harder. Under somewhat stronger assumptions on the random walk, we show there exists a constant $c_{1.2}$ such that

$$\limsup_{n\to\infty} \frac{R_n - ER_n}{n \log\log\log n/(\log n)^2} = c_{1.2} \qquad \text{a.s.}$$

In the case $d = 2$ it is known (see [15]) that $\operatorname{Var} R_n \asymp n^2/(\log n)^4$, which explains part of the rate. The presence of a $\log\log\log n$ term instead of the expected $\log\log n$ term is perhaps surprising.

In Section 2 we give a precise statement of our results. We prove the three-dimensional case in Section 3 and the two-dimensional case in Section 4. Overviews of the proofs of Theorems 2.1 and 2.5 are given near the beginning of Section 3 and after the statements of Propositions 4.1 and 4.4. Throughout the paper $c_{n.i}$ will denote the ith fixed constant in Section n; other positive finite constants c_i will be also be used, but will be fixed within a given proof.

2. Main theorems and known results. In this section, we will recall several known results and state our main theorems. We first explain the setting. Let $\{X_j\}$ be an i.i.d. sequence of random variables taking values in $\mathbf{Z}^d$ ($d = 3$ in Section 2.1 and $d = 2$ in Section 2.2) such that $EX_1 = 0$ and $E[|X_1|^{2+\delta}] < \infty$ for some $\delta > 0$ and set $S_n = \sum_{j=1}^n X_j$. Let R_n be the range of $S_0, \dots, S_n$, that is, R_n is the cardinality of the set $\{S_0, S_1, \dots, S_n\}$.

Define

$$p = P(S_k \neq 0 \text{ for all } k \in \mathbf{N}).$$

Throughout this paper, we assume $p < 1$ as otherwise $R_n = n + 1$ a.s. and there is no interest in this case. We also assume that the random walk $\{S_n\}$ is genuinely d-dimensional; that is, if

$$R^+ = \{x \in \mathbf{Z}^d : P^0(S_n = x) > 0 \text{ for some } n \geq 0\},$$
$$\hat{R} = \{x \in \mathbf{Z}^d : x = y - z \text{ for some } y \in R^+ \text{ and } z \in R^+\},$$

then $\hat{R}$ is d-dimensional. When $\hat{R}$ is a proper subgroup of $\mathbf{Z}^d$, it is isomorphic to $\mathbf{Z}^d$, so by a suitable transformation we can suppose $\hat{R} = \mathbf{Z}^d$; that is, the transformed random walk is aperiodic. As the transformation does not change R_n and p, there is no loss of generality in considering the case $\hat{R} = \mathbf{Z}^d$.

For sequences $\{f_n\}$ and $\{g_n\}$, we write $f_n \sim g_n$ when $\lim_{n\to\infty} f_n/g_n = 1$. Define $\log_2 a = \log\log a$ and $\log_3 a = \log\log\log a$.

2.1. *Main theorem*: *three-dimensional case.* When $d = 3$, our main theorem is an almost sure invariance principle for R_n.

THEOREM 2.1. *Suppose $d = 3$. Let $q = \frac{15}{32}$. Changing the probability space if necessary*, *there exist a one-dimensional Brownian motion and a constant $\sigma > 0$ such that*

$$\frac{R_n - ER_n}{\sigma} - B_{n \log n} = O\big(\sqrt{n}(\log n)^q\big) \qquad a.s. \tag{2.1}$$

As we will see in the proof, $\sigma^2 = 2p^4(2\pi)^{-2}|Q|^{-1}$ where Q is the covariance matrix for X_1.

Using the laws of the iterated logarithm for Brownian motion, we have the following LILs for R_n as an immediate corollary of the theorem.

COROLLARY 2.2. *Suppose $d = 3$. The following hold P-a.s.*:

$$\limsup_{n\to\infty} \frac{R_n - ER_n}{\sqrt{n \log n \log_2 n}} = \sqrt{2}\sigma,$$

$$\liminf_{n\to\infty} \frac{R_n - ER_n}{\sqrt{n \log n \log_2 n}} = -\sqrt{2}\sigma,$$

$$\liminf_{n\to\infty} \frac{\sup_{m\le n} |R_m - ER_m|}{\sqrt{n \log n / \log_2 n}} = \frac{\pi\sigma}{\sqrt{8}}.$$

An analogue of Strassen's LIL also holds.

REMARK 2.3. Let $Q_n^{(p)}$ be the number of distinct sites that $\{S_i : 0 < i \le n\}$ has visited exactly p times. Hamana [9] has informed us that by using our arguments and some estimates for $Q_n^{(p)}$, one can prove the analogue of Theorem 2.1 for $Q_n^{(p)}$ (with a different constant for σ). We will briefly sketch the argument in Remark 3.4.

2.2. *Main theorem*: *two-dimensional case.* When $d = 2$, our main theorem is a law of the iterated logarithm for R_n. In this case, we need the following further assumptions for X_1.

ASSUMPTION 2.4. (a) *X_1 is mean 0 and has covariance matrix equal to σI for some $\sigma > 0$.*

(b) *X_1 is bounded*: *there exists $\Lambda > 0$ such that $P(|X_1| > \Lambda) = 0$.*

We note that (a) is equivalent to $(H3)$ in [18]. Under these conditions, we have the following.

THEOREM 2.5. *Suppose $d = 2$. There exists $c_{2.1} > 0$ such that the following holds P-a.s.*:

$$\limsup_{n\to\infty} \frac{\sup_{j\le n}(R_j - ER_j)}{n\log_3 n/(\log n)^2} = c_{2.1}. \tag{2.2}$$

REMARK 2.6. (i) As we will see from the proof, the same result holds with $R_n - ER_n$ instead of $\sup_{j\le n}(R_j - ER_j)$.

(ii) We do not know the exact value of $c_{2.1}$. Also, we have not obtained the LIL for the lim inf of $R_n - ER_n$.

2.3. *Known results.* Before giving the proofs, we recall some known results. The results in this subsection hold for aperiodic random walks with $EX_1 = 0$ and $E[|X_1|^2] < \infty$. Further estimates will be introduced in the next section.

For the three-dimensional case, the following are known:

$$ER_n = pn + O(\sqrt{n}), \tag{2.3}$$

$$E[(R_n - ER_n)^4] = O(n^2(\log n)^2), \tag{2.4}$$

$$\frac{R_n - ER_n}{\sqrt{n\log n}} \to c_{2.2}\mathcal{N}, \tag{2.5}$$

where $\mathcal{N}$ is the standard normal distribution. The convergence in (2.5) is in the sense of distribution. Equation (2.3) was proved by Dvoretzky and Erdős [5], (2.4) is from Jain and Pruitt [13], Theorem 4, and (2.5) is from Jain and Pruitt [13].

For the two-dimensional case, the following are known:

$$ER_n = \kappa\frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right), \tag{2.6}$$

$$\mathrm{Var}\,(R_n) = O\left(\frac{n^2}{(\log n)^4}\right), \tag{2.7}$$

$$\frac{(\log n)^2}{n}(R_n - ER_n) \to -c_{2.3}\gamma, \tag{2.8}$$

where γ is renormalized self-intersection local time of planar Brownian motion and κ is a constant. The convergence in (2.8) is again in distribution. Equation (2.6) is from Jain and Pruitt [12], Lemma 2.6 with the estimates (2.2) and (2.3) in [7], (2.7) is from Jain and Pruitt [15], Theorem 4.2, and (2.8) is from Le Gall [18].

3. Proof: three-dimensional case. In this section, we will prove Theorem 2.1. We set $\langle x\rangle = n$ if $x \in (n-\frac{1}{2}, n+\frac{1}{2}]$ throughout the paper. Let α be a positive constant that we will choose later. We form a sequence $\{n_j\}$ of positive integers by taking all positive integers in each interval $[2^k, 2^{k+1})$ which are of the form $2^k + \langle i2^k/k^\alpha\rangle$, $k = 1, 2, \dots, i = 0, 1, \dots, k^\alpha$. This choice of the sequence will be important in the proof. Let $n_0 = 0$. For $2^k \le n_i < 2^{k+1}$, we have $2^k/k^\alpha - 1 \le n_{i+1} - n_i \le 2^k/k^\alpha + 1$, so that the following hold:

$$\lim_{n\to\infty} n_{i+1}/n_i = 1, \qquad n_{i+1} - n_i = O\big(n_i/(\log n_i)^\alpha\big).$$

We write $\#A$ for the cardinality of the set A. For any random variable Y we write $\overline{Y}$ for $Y - EY$. Let

$$U_j = \#\big\{S_k : k \in [n_{j-1}, n_j)\big\}.$$

Fix $i < j$ and let

$$V_j = V_j^{(i)} = \#\big(\big\{S_k : k \in [n_{j-1}, n_j)\big\} \cap \big\{S_k : k \in [n_j, n_i]\big\}\big).$$

Then $R_{n_i} = \sum_{j=1}^{i} U_j - \sum_{j=1}^{i-1} V_j$, so that

$$\overline{R}_{n_i} = \sum_{j=1}^{i} \overline{U}_j - \sum_{j=1}^{i-1} \overline{V}_j. \tag{3.1}$$

Let us now give a overview of the proof of Theorem 2.1. We will need three lemmas (Lemmas 3.1, 3.2, 3.3) for the proof. Using Lemma 3.1, we show

$$\sum_{j=1}^{i-1} \overline{V}_j = o\big(\sqrt{n_i}(\log n_i)^q\big) \qquad \text{a.s.}$$

As the $\{\overline{U}_j\}_{j=1}^{i}$ are independent, by Skorohod embedding [22] there exist a Brownian motion B_t and a sequence of nonnegative independent random variables $\{T_j\}_{j=1}^{\infty}$ such that

$$\frac{1}{\sigma}\sum_{j=1}^{i} \overline{U}_j \overset{\mathcal{L}}{\sim} B\left(\sum_{k=1}^{i} T_k\right).$$

We then use Lemma 3.2 and after some computations derive

$$B\left(\sum_{k=1}^{i} T_k\right) = B(n_i \log n_i) + O\big(\sqrt{n_i}(\log n_i)^q\big) \qquad \text{a.s.}$$

Thus, by (3.1), we have (2.1) for the subsequence $\{n_i\}$. Lemma 3.3 will then be used to show the result for all n.

Before stating the lemmas, we give some notation. For $x, y \in \mathbf{Z}^3$, $n \geq 0$ and $A \subset \mathbf{Z}^3$, define

$$\begin{aligned}
P^{(n)}(x, y) &= P^x(S_n = y),\\
P_A^{(n)}(x, y) &= P^x(S_1, \dots, S_{n-1} \notin A,\ S_n = y),\\
F(x, y) &= \sum_{n=1}^{\infty} P_y^{(n)}(x, y) = P^x(T_y < \infty),\\
G_n(x, y) &= \sum_{k=0}^{n} P^{(k)}(x, y),\\
G(x, y) &= \sum_{k=0}^{\infty} P^{(k)}(x, y),
\end{aligned}$$

where $T_A = \inf\{n > 0 : S_n \in A\}$.

Let

$$\begin{aligned}
Z_i^n &= \mathbb{1}_{\{S_i \neq S_{i+1}, \dots, S_i \neq S_n\}} && \text{for } 0 \leq i < n,\ Z_n^n = 1,\\
Z_i &= \mathbb{1}_{\{S_i \neq S_{i+1}, S_i \neq S_{i+2}, \dots\}} && \text{for } i \geq 0,\\
W_i^n &= Z_i^n - Z_i && \text{for } 0 \leq i < n,\\
Y_n &= \sum_{i=0}^{n-1} Z_i,\\
W_n &= \sum_{i=0}^{n-1} W_i^n.
\end{aligned}$$

Note that $R_n = \sum_{i=0}^{n} Z_i^n = Y_n + W_n + 1$. We now state the lemmas. The proofs will be given at the end of this section.

LEMMA 3.1. *For nonnegative integers $a < b$, let $V_{a,b} = \#(\{S_j : j \in [a, b)\} \cap \{S_k : k \in [b, \infty)\})$. There exists $c_{3.1} > 0$ such that*

$$E[V_{a,b}^4] \leq c_{3.1}(b - a)^2. \tag{3.2}$$

Further, for each $l \geq 3$, there exists $c_{3.2} = c_{3.2}(l) > 0$ such that

$$E\big[(W_n)^{2l}\big] \leq c_{3.2} n^l (\log n)^l. \tag{3.3}$$

LEMMA 3.2. *There exists $\sigma > 0$ such that for all $n \in \mathbf{N}$,*

$$\operatorname{Var}(R_n) = \sigma^2 n \log n + O\big(n\sqrt{\log n}\big). \tag{3.4}$$

Further, for each $l \in \mathbf{N}$,

$$E\big[|R_n - E R_n|^l\big] = O\big((n \log n)^{l/2}\big). \tag{3.5}$$

LEMMA 3.3. (a) *For nonnegative integers $a < b$ and l, there exists $c_{3.3} = c_{3.3}(l)$ such that*

$$E\big[|(R_b - ER_b) - (R_a - ER_a)|^l\big] \le c_{3.3}\big((b-a)\log(b-a)\big)^{l/2}.$$

(b) *For nonnegative integers $a < b$ and $l > 2$, there exists $c_{3.4} = c_{3.4}(l)$ such that*

$$P\left(\max_{a \le n \le b} |(R_n - ER_n) - (R_a - ER_a)| > \lambda\right) \le c_{3.4} \frac{((b-a)\log(b-a))^{l/2}}{\lambda^l}.$$

We now give a proof of Theorem 2.1, assuming the above lemmas.

PROOF OF THEOREM 2.1. Let

$$\alpha = \frac{9}{32}, \quad \beta = \frac{15}{32} - \varepsilon, \quad \gamma = \frac{7}{8}, \qquad \varepsilon = 10^{-6}.$$

Recall that for i fixed and for $j \le i$, $V_j = V_j^{(i)}$ is the cardinality of $\{S_k : k \in [n_{j-1}, n_j)\} \cap \{S_k : k \in [n_j, n_i]\}$. We have

$$P\left(\sum_{j=1}^{i} V_j \ge c_1 \sqrt{n_i}(\log n_i)^\beta\right) \le \frac{E[(\sum_{j=1}^{i} V_j)^4]}{c_1^4 n_i^2 (\log n_i)^{4\beta}}. \tag{3.6}$$

By Hölder's inequality and (3.2),

$$E[V_{j_1} V_{j_2} V_{j_3} V_{j_4}] \le \left\{\prod_{m=1}^{4} E[V_{j_m}^4]\right\}^{1/4} \le c_2 \prod_{m=1}^{4} \sqrt{n_{j_m} - n_{j_m - 1}}.$$

Thus, when $2^{k_0} \le n_i < 2^{k_0+1}$,

$$\begin{aligned} E\left[\left(\sum_{j=1}^{i} V_j\right)^4\right] &= \sum_{j_1, j_2, j_3, j_4 = 1}^{i} E[V_{j_1} V_{j_2} V_{j_3} V_{j_4}] \le c_3 \sum_{j_1, j_2, j_3, j_4 = 1}^{i} \prod_{m=1}^{4} \sqrt{n_{j_m} - n_{j_m - 1}} \\ &\le c_4 \left(\sum_{k=1}^{k_0} k^\alpha \sqrt{2^k / k^\alpha}\right)^4 \le c_5 k_0^{2\alpha} 2^{2k_0}, \end{aligned}$$

where in the last inequality we use the elementary fact that

$$\sum_{k=1}^{n} k^p q^k \sim n^p q^n \tag{3.7}$$

as $n \to \infty$ for each $p > 0$, $q > 1$. Thus the right-hand side of (3.6) is bounded from above by $c_5 k_0^{2\alpha - 4\beta}$. The number of n_i in $[2^{k_0}, 2^{k_0+1})$ is less than $c_6 k_0^\alpha$. Since $3\alpha - 4\beta < -1$, then $\sum_{k_0=1}^{\infty} k_0^{3\alpha - 4\beta} < \infty$, and by Borel–Cantelli we see that

$$\limsup_{i \to \infty} \frac{\sum_{j=1}^{i} V_j^{(i)}}{\sqrt{n_i}(\log n_i)^\beta} \le c_1 \qquad \text{a.s.} \tag{3.8}$$

Since $\alpha/2 < \beta$ and $EV_j \le c_7\sqrt{n_j - n_{j-1}}$, we have by similar calculations that

$$\sum_{j=1}^{i} EV_j \le c_8 \sum_{k=1}^{k_0} k^\alpha \sqrt{2^k/k^\alpha} \le k_0^{\alpha/2} 2^{k_0/2} = o\big(\sqrt{n_i}(\log n_i)^\beta\big).$$

Thus we obtain

$$\limsup_i \frac{|\sum_{j=1}^{i-1}(V_j - EV_j)|}{\sqrt{n_i}(\log n_i)^\beta} \le c_9 \qquad \text{a.s.} \tag{3.9}$$

Set $H_j = (U_j - EU_j)/\sigma$. As the $\{H_j\}_{j=1}^\infty$ are independent there exist [22] a Brownian motion B_t and a sequence of nonnegative independent random variables $\{T_j\}_{j=1}^\infty$ such that

$$\left\{B\left(\sum_{k=1}^{j} T_k\right) - B\left(\sum_{k=1}^{j-1} T_k\right)\right\}_j \overset{\mathcal{L}}{\sim} \{H_j\}_j, \tag{3.10}$$

$$E[T_j] = E\big[|H_j|^2\big], \tag{3.11}$$

$$E[T_j^l] \le c_{10} E\big[|H_j|^{2l}\big] \qquad \text{for all } l \ge 2. \tag{3.12}$$

From (3.10), we see that $\sum_{j=1}^{i} H_j$ is equal in law to $B(\sum_{k=1}^{i} T_k)$.

We now prove

$$\limsup_{i\to\infty} \frac{\sum_{j=1}^{i}(T_j - ET_j)}{n_i(\log n_i)^\gamma} < \infty \qquad \text{a.s.} \tag{3.13}$$

It is clear that $\sum_{j=1}^{i} \overline{T}_j$ is a martingale. So by Doob's inequality, for each $l \in \mathbf{N}$,

$$P\left(\sup_{r\le i}\left|\sum_{j=1}^{r} \overline{T}_j\right| \ge n_i(\log n_i)^\gamma\right) \le c_{11} \frac{E[(\sum_{j=1}^{i} \overline{T}_j)^{2l}]}{n_i^{2l}(\log n_i)^{2l\gamma}}. \tag{3.14}$$

Note that

$$\begin{aligned} E\left[\left(\sum_{j=1}^{i} \overline{T}_j\right)^{2l}\right] &= \sum_{j_1, j_2, \ldots, j_{2l}=1}^{i} E[\overline{T}_{j_1} \cdots \overline{T}_{j_{2l}}] \\ &= \sum_{(*)} \frac{(2l)!}{\zeta_1! \cdots \zeta_p!} \sum_{j_1,\ldots,j_p=1}^{i} E[\overline{T}_{j_1}^{\zeta_1}] \cdots E[\overline{T}_{j_p}^{\zeta_p}], \end{aligned} \tag{3.15}$$

where $(*)$ ranges over all $(\zeta_1, \ldots, \zeta_p)$, $1 \le p \le 2l$, such that $\zeta_i \ge 2$ for all $1 \le i \le p$ and $\sum_{t=1}^{p} \zeta_t = 2l$. The second equality holds because $E[\overline{T}_{j_1} \cdots \overline{T}_{j_{2l}}] = 0$ when one of $j_1, j_2, \ldots, j_{2l}$ is different from all the others, as the $\{\overline{T}_j\}_j$ are independent and mean zero.

Observe also that

$$|E[\overline{T}_j^m]| = |ET_j^m - mET_j^{m-1}ET_j + \cdots + (-1)^m(ET_j)^m| \\ \leq c_{12}\{(n_j - n_{j-1})\log(n_j - n_{j-1})\}^m$$

by (3.11), (3.12) and (3.5).

Then when $2^{k_0} \leq n_i < 2^{k_0+1}$,

$$\sum_{j=1}^{i} |[E\overline{T}_j^m]| \leq c_{12}\sum_{j=1}^{i}\{(n_j - n_{j-1})\log(n_j - n_{j-1})\}^m \\ \leq c_{13}\sum_{k=1}^{k_0} k^\alpha \frac{2^{km}}{k^{\alpha m}} k^m \leq c_{14}k_0^{m(1-\alpha)+\alpha_2 mk_0} \tag{3.16}$$

where we used (3.7) for the last inequality. [Note that $c_{12} = c_{12}(m), c_{13} = c_{13}(m), c_{14} = c_{14}(m)$ depend on m.] Using this, (3.15) is estimated from above by $c_{15}\sum_{(*)} k_0^{2l(1-\alpha)+\alpha p}2^{2lk_0}$ for some $c_{15} = c_{15}(l) > 0$. As the term is the biggest when $p = l$, combining with (3.14),

$$\sum_{i=1}^{\infty} P\left(\sup_{r\leq i}\left|\sum_{j=1}^{r}\overline{T}_j\right| \geq n_i(\log n_i)^\gamma\right) \\ \leq c_{16}\sum_{k_0=1}^{\infty} k_0^\alpha \frac{k_0^{l(2-\alpha)}2^{2lk_0}}{2^{2lk_0}k_0^{2l\gamma}} = c_{16}\sum_{k_0=1}^{\infty} k_0^{2l(1-\alpha/2-\gamma)+\alpha},$$

for some $c_{16} = c_{16}(l) > 0$. The last term is finite if we choose l large enough so that $2l(1 - \frac{\alpha}{2} - \gamma) + \alpha < -1$. This proves (3.13).

Let $J_i = \sum_{k=1}^{i} T_k$. Let $\xi_i = n_i(\log n_i)^\gamma$. Then,

$$P(|B(J_i) - B(EJ_i)| > \xi_i^{1/2}(\log n_i)^\varepsilon; |J_i - EJ_i| \leq 2\xi_i) \\ \leq P\left(\sup_{EJ_i - 2\xi_i \leq s,t \leq EJ_i + 2\xi_i} |B_t - B_s| \geq \xi_i^{1/2}(\log n_i)^\varepsilon\right) \\ \leq c_{17}e^{-(\log n_i)^{2\varepsilon}/2}.$$

There are at most $c_{18}k_0{}^\alpha$ values of n_i such that $2^{k_0-1} \leq n_i \leq 2^{k_0}$, so the above is summable in k_0. Combining with (3.13), we deduce

$$B(J_i) - B(EJ_i) = O(n_i^{1/2}(\log n_i)^{(\gamma/2)+\varepsilon}).$$

By (3.11), $\sum_{j=1}^{i} ET_j = \sum_{j=1}^{i}(n_j - n_{j-1})\log(n_j - n_{j-1}) + O(n_i\sqrt{\log n_i})$ and we have by elementary computations that

$$\sum_{j=1}^{i} ET_j = n_i \log n_i + o(n_i(\log n_i)^{2\beta-\varepsilon}).$$

We thus have

$$B\left(\sum_{k=1}^{i} T_k\right) - B(n_i \log n_i) = O\big(\sqrt{n}_i (\log n_i)^{\beta}\big) \qquad \text{a.s.} \tag{3.17}$$

Putting together what we have so far, we have

$$(R_{n_i} - ER_{n_i}) - B(n_i \log n_i) = O\big(\sqrt{n}_i (\log n_i)^{\beta}\big) \qquad \text{a.s.}$$

It remains to take care of values of n that are not one of the n_i. By Lemma 3.3(b),

$$\begin{aligned} P\left(\max_{\substack{n_i \le n \le n_{i+1} \\ n \in \mathbf{N}}} |\overline{R}_n - \overline{R}_{n_i}| > \sqrt{n}_i (\log n_i)^{\beta}\right) &\le c_{19} \frac{((n_{i+1} - n_i) \log(n_{i+1} - n_i))^{l/2}}{(n_i)^{l/2} (\log n_i)^{\beta l}} \\ &\le c_{20} k^{l(1-\alpha-2\beta)/2} \end{aligned}$$

if $2^k \le n_i \le 2^{k+1}$. There are at most $c_{21} k^{\alpha}$ values of n_i such that $2^k \le n_i \le 2^{k+1}$, so taking l large enough, this will be summable and we obtain

$$\max_{n_i \le n \le n_{i+1}} |\overline{R}_n - \overline{R}_{n_i}| = O\big(\sqrt{n}_i (\log n_i)^{\beta}\big) \qquad \text{a.s.}$$

Finally, standard estimates on Brownian motion show that

$$P\left(\sup_{n_i \log n_i \le t \le n_{i+1} \log n_{i+1}} |B_t - B_{n_i \log n_i}| > \sqrt{n}_i (\log n_i)^{\beta}\right)$$

is summable in i so that

$$\sup_{n_i \log n_i \le t \le n_{i+1} \log n_{i+1}} |B_t - B_{n_i \log n_i}| = O\big(\sqrt{n}_i (\log n_i)^{\beta}\big) \qquad \text{a.s.}$$

The proof of Theorem 2.1 is complete. □

REMARK 3.4. As we pointed out in Remark 2.3, similar arguments allow one to deduce Theorem 2.1 for $Q_n^{(p)}$, which is the number of distinct sites that $\{S_i : 0 < i \le n\}$ has visited exactly p times. We sketch how to prove this. For $0 \le a < b$, let $S(a, b] = \{S_k : a < k \le b\}$ and $S^p(a, b]$ be the set of distinct sites where $S(a, b]$ visited exactly p times. (For simplicity we do not count S_0.) Clearly, $Q_n^{(p)} = \#S^p(0, n]$. Now take a sequence $\{n_j\}$ as in the proof of this section, fix i, and define

$$\begin{aligned} U_j^p &= \#S^p(n_{j-1}, n_j], \\ L_j^{(i)} &= \#\{S^p(n_{j-1}, n_j] \cap S(n_j, n_i]\}, \\ M_j^{(i)} &= \#\{S(n_{j-1}, n_j] \cap S^p(n_j, n_i]\}, \end{aligned}$$

$$N_j^{(i)} = \sum_{l=1}^{p-1} \#\{S^l(n_{j-1}, n_j] \cap S^{p-l}(n_j, n_i]\}.$$

Then

$$0 \le L_j^{(i)}, M_j^{(i)}, N_j^{(i)} \le V_j, \tag{3.18}$$

where V_j is the same as above. By a simple calculation similar to [7], (3.1), we have

$$Q_t^{(p)} = \sum_{j=1}^{i} U_j^p - \sum_{j=1}^{i-1} (L_j^{(i)} + M_j^{(i)} - N_j^{(i)}). \tag{3.19}$$

Thanks to (3.18), we can apply (3.3) to derive moment bounds for $L_j^{(i)}$, $M_j^{(i)}$ and $N_j^{(i)}$. Also, an estimate of the variance for $Q_n^{(p)}$ is obtained in [6], Theorem 3.1, so that (3.4) still holds (with a different constant for σ) for $Q_n^{(p)}$. Thus, our proof can be applied to $Q_n^{(p)}$.

In the rest of this section, we will give proofs of Lemmas 3.1, 3.2 and 3.3.

PROOF OF LEMMA 3.1. First, note that because $V_{0,n+1} = 1 - Z_n + W_n$, then $V_{0,n}$ and W_n have the same asymptotics. Also by the Markov property, $V_{a,b}$ and $V_{0,b-a}$ have the same distribution. As $E[(W_n)^4] = O(n^2)$ by Lemma 6.1 of [16], (3.2) follows.

We next prove (3.3) by induction. When $l = 2$ this is from (3.2). Assume that (3.3) holds up to $l - 1$. By the same argument as in the proof of Lemma 6.1 of [16], we have

$$\sum E[W_{i_1}^n \cdots W_{i_{2l}}^n] \le \sum c_1 (n - i_{2l})^{-1/2} G_n(0, x_1) G_n(x_1, x_2) \cdots G_n(x_{2l-2}, x_{2l-1}) \\ \times G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_3) \cdots G(\alpha_{2l-1}, \alpha_{2l}),$$

where i_{2l} is fixed and the first sum is over all $0 \le i_1 < i_2 < \cdots < i_{2l-1} \le n-1$. The second sum is over all $x_1, x_2, \ldots, x_{2l-1} \in \mathbf{Z}^3 \setminus \{0\}$ such that they are all distinct and over all permutations $(\alpha_1, \ldots, \alpha_{2l})$ of $(0, x_1, \ldots, x_{2l-1})$. We will sum over i_{2l}, so what we need to show is the following:

$$\sum G_n(0, x_1) G_n(x_1, x_2) \cdots G_n(x_{2l-2}, x_{2l-1}) \\ \times G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_3) \cdots G(\alpha_{2l-1}, \alpha_{2l}) = O(n^{l-1/2} (\log n)^l). \tag{3.20}$$

By Lemma 3 of [13] we have

$$\sum_x G_n(0, x) \{G(u, x) + G(x, u)\} = O(n^{1/2}), \tag{3.21}$$

$$\sum_x G_n(0, x) G(u, x) G(x, v) = O(\log n), \tag{3.22}$$

uniformly over $u, v \in \mathbf{Z}^3$. First we sum over x_{2l-1} in the left-hand side of (3.20). Depending whether either of α_1 or α_{2l} is x_{2l-1} or not, we use either (3.21) or (3.22). Then we sum over $x_{2l-2}, x_{2l-3}, \dots$. [When (3.22) is used, there is the possibility that for some j, no x_j term will be left as we proceed with our summation. In that case, we use the estimate $\sum_{x_j} G_n(x_{j-1}, x_j) = \sum_{k=0}^{n} \sum_{x_j} P^{(k)}(x_{j-1}, x_j) \le n$.] As a result, we obtain (3.20).

We must also consider the case where at least two of $i_1, \dots, i_{2l-1}$ are equal, say $i_j = i_{j+1}$. In this case, as $W^n_{i_j} \le 1$,

$$\sum_{\substack{i_1,\dots,i_{2l-1}\\ i_j=i_{j+1}}} E[W^n_{i_1} \cdots W^n_{i_{2l}}] = n \sum_{i_3,\dots,i_{2l-1}} E[W^n_{i_3} \cdots W^n_{i_{2l}}],$$

so that by the induction hypothesis, we again obtain the desired estimate. Combining these facts, the proof of (3.3) is complete. □

REMARK 3.5. We believe that the right-hand side of (3.3) can be replaced by $c_2 n^l$. As (3.3) is enough for our use, we did not try to prove this.

The next lemma will be used in the proof of Lemma 3.2. The proof is due to D. Khoshnevisan.

LEMMA 3.6. *Let $EX_1 = 0$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta \in (0, 1)$. Let Q be the covariance matrix of X_1 and let $\varepsilon = \delta/(4 + \delta)$. Then*

$$G(0, x) = \frac{1}{2\pi |Q|^{1/2} (xQ^{-1}x)^{1/2}} \big(1 + O(|x|^{-\varepsilon})\big).$$

PROOF. Let B_t be a standard three-dimensional Brownian motion and let $p_s(x)$ be the transition density for $Q^{1/2} B_s$, where $Q^{1/2}$ is the nonnegative definite symmetric square root of Q.

Considering the cases $|y| \le 1$ and $|y| > 1$ separately for $y \in \mathbf{R}$, note

$$\left| e^{iy} - \left(1 + iy - \frac{y^2}{2}\right)\right| \le c_1(|y|^2 \wedge |y|^3) \le c_2 |y|^{2+\delta}.$$

If φ is the characteristic function of X_1, then

$$\big|\varphi(\alpha) - \big(1 - \tfrac{1}{2}\alpha Q \alpha\big)\big| \le c_2 E|X_1|^{2+\delta} |\alpha|^{2+\delta}.$$

Let $B > 0$. Since $E|X_1|^{2+\delta} < \infty$ and $|a^n - b^n| \le n|a - b|(|a| \vee |b|)^{n-1}$, for $|\alpha| \le B\sqrt{\log n}$, we can deduce

$$\big|\varphi^n(\alpha/\sqrt{n}) - e^{-\alpha Q \alpha/2}\big| \le c_3 n^{-\delta/2} |\alpha|^{2+\delta}.$$

Using this estimate, we now proceed as in the proof of Proposition 3.1 of [1] to obtain

$$|P(S_n = x) - p_n(x)| \le c_4 n^{-(3+\delta)/2} (\log n)^{(5+\delta)/2} \le c_5 n^{-(3/2)-(\delta/4)}. \tag{3.23}$$

It is well known (see [23]) that

$$P(S_n = x) \le c_6 n^{-3/2}. \tag{3.24}$$

When $|x| > n^{1/2}$ we can get a better estimate on $P(S_n = x)$. Let $A = \{z \in \mathbf{Z}^3 : |z| \le |x - z|\}$. Write

$$P(S_n = x) = P(S_n = x,\ S_{\langle n/2\rangle} \in A) + P(S_n = x,\ S_{\langle n/2\rangle} \in A^c). \tag{3.25}$$

By the Markov property, (3.24) and Chebyshev's inequality,

$$\begin{aligned} P(S_n = x,\ S_{\langle n/2\rangle} \in A^c) &= \sum_{z \in A^c} P(S_{\langle n/2\rangle} = z) P(S_{n-\langle n/2\rangle} = x - z) \\ &\le c_6 n^{-3/2} \sum_{z \in A^c} P(S_{\langle n/2\rangle} = z) \\ &\le c_6 n^{-3/2} P(|S_{\langle n/2\rangle}| \ge |x|/2) \\ &\le c_7 n^{-3/2} \frac{\langle n/2\rangle}{|x|^2} \le \frac{c_8}{n^{1/2}|x|^2}. \end{aligned}$$

If $\widetilde{S}_k = S_{n-k}$, then

$$P(S_n = x,\ S_{\langle n/2\rangle} \in A) = P(\widetilde{S}_n = 0,\ \widetilde{S}_{n-\langle n/2\rangle} \in A \mid \widetilde{S}_0 = x).$$

Since $\widetilde{S}_k$ satisfies the same hypotheses as S_k, then by the same argument the first term on the right-hand side of (3.25) is also bounded by $c_8/(n^{1/2}|x|^2)$. We thus have

$$P(S_n = x) \le \frac{2c_8}{n^{1/2}|x|^2}. \tag{3.26}$$

That $p_n(x)$ satisfies the same bound is easy, using Gaussian tail estimates.

Let $r = 2/(1 + \delta/4)$ and $\varepsilon = \delta/(4 + \delta)$. By (3.23), (3.26) and the bound on $p_n(x)$,

$$\begin{aligned} \sum_{n=1}^{\infty} |P(S_n = x) - p_n(x)| &\le \sum_{n=1}^{|x|^r} P(S_n = x) + \sum_{n=1}^{|x|^r} p_n(x) \\ &\quad + \sum_{n=|x|^r+1}^{\infty} |P(S_n = x) - p_n(x)| \\ &\le \frac{4c_8}{|x|^2} \sum_{n=1}^{|x|^r} n^{-1/2} + c_5 \sum_{n=|x|^r}^{\infty} n^{-(3/2)-(\delta/4)} \\ &\le c_9 |x|^{(r/2)-2} + c_9 |x|^{-r((1/2)+(\delta/4))} \\ &\le c_{10} |x|^{-1-\varepsilon}. \end{aligned}$$

It is easy to see that

$$\left|\sum_{n=1}^{\infty} p_n(x) - \int_0^{\infty} p_s(x)\,ds\right| \le \sum_{n=1}^{\infty}\left|p_n(x) - \int_{n-1}^{n} p_s(x)\,ds\right|$$

is $o(|x|^{-1-\varepsilon})$. A direct calculation of $\int_0^\infty p_s(x)\,ds$, now proves the lemma. □

PROOF OF LEMMA 3.2. We first prove (3.4) which is a refinement of Theorem 2 in [13]. Note that in the proof of Theorem 2 in [13], the following fact is obtained:

$$\operatorname{Var}(R_n) = 2\sum_{j=1}^{n-1} a_j + O(n\sqrt{\log n}), \tag{3.27}$$

where

$$a_j = p \sum_{x\in B_j\cap \mathbf{Z}^3} G(0,x)b(x) + O(1), \tag{3.28}$$

$$B_j = \{z \in \mathbf{R}^3 : 1 \le zQ^{-1}z \le j\}, \tag{3.29}$$

$$b(x) = \frac{1 - F(x,0)}{1 - F(x,0)F(0,x)} pF(x,0)F(0,x), \tag{3.30}$$

$$F(x,y) = pG(x,y), \tag{3.31}$$

and Q is the covariance matrix for X_1 (equation (3.30) is proved in Lemma 5 of [13]). We have by (3.28) and (3.30) that

$$a_j = p^4 \sum_{x\in B_j\cap \mathbf{Z}^3} G(0,x)^2 G(x,0) + O\left(\left\{\sum_{x\in B_j\cap \mathbf{Z}^3} G(0,x)^3 G(x,0)\right\} \vee 1\right). \tag{3.32}$$

By Lemma 3.6 with $\varepsilon = \delta/(4+\delta)$,

$$G(0,x) = \frac{1}{2\pi|Q|^{1/2}(xQ^{-1}x)^{1/2}}(1 + O(x^{-\varepsilon})). \tag{3.33}$$

Note that by translation invariance, $G(-x,0) = G(0,x)$, so that $G(x,0)$ has the same asymptotics as $G(0,x)$. Substituting (3.33) in (3.32), we have

$$a_j = p^4(2\pi)^{-3}|Q|^{-3/2} \sum_{x\in B_j\cap \mathbf{Z}^3} (xQ^{-1}x)^{-3/2}$$
$$+ O\left(\left\{\sum_{x\in B_j\cap \mathbf{Z}^3} (xQ^{-1}x)^{-(3/2)-\varepsilon}\right\} \vee 1\right).$$

Since

$$\sum_{x\in B_j\cap\mathbf{Z}^3}(xQ^{-1}x)^{-3/2}=\int_{B_j}(zQ^{-1}z)^{-3/2}\,dz+O(1)$$
$$=\int_{\{1\le|y|^2\le j\}}|y|^{-3}|Q|^{1/2}\,dy+O(1)$$
$$=2\pi|Q|^{1/2}\log j+O(1),$$

and $\sum_{x\in B_j\cap\mathbf{Z}^3}(xQ^{-1}x)^{-(3/2)-\varepsilon}=O(1)$ by a similar computation, we have

$$a_j=\sigma^2(\log j)/2+O(1),$$

where $\sigma^2=2p^4(2\pi)^{-2}|Q|^{-1}$. Substituting this into (3.27), we obtain (3.4).

We next prove (3.5). The basic idea is the same as the proof of Lemma 4.1 in [16]. Set $\rho_n=\sqrt{n\log n}$ and define for each $m,n\in\mathbf{N}$,

$$L_{n,m}=\frac{1}{\rho_n}\{E[(R_n-ER_n)^{2m}]\}^{1/2m}.$$

Our goal is to prove

$$L_{n,m}\le M_m\qquad\text{for all } m,n\in\mathbf{N},\tag{3.34}$$

where $\{M_m\}$ is a sequence of positive bounded numbers independent of n. Once (3.34) is proved, it leads to (3.5) for l odd by using Hölder's inequality.

As seen in (2.4) and (3.4), (3.34) holds for $m=1,2$. Now we assume that (3.34) holds for all $m\le m_0$ [thus, by Hölder's inequality, (3.5) holds for all $l\le 2m_0$], and we will show $\{L_{n,m_0+1}\}$ is bounded for all n. Note that

$$R_{2n}=\sum_{i=0}^{n}Z_i^n+\sum_{i=n}^{2n}Z_i^{2n}-\sum_{i=0}^{n-1}(Z_i^n-Z_i^{2n})-1\tag{3.35}$$

and by (3.3),

$$E\left[\left(\sum_{i=0}^{n-1}(Z_i^n-Z_i^{2n})\right)^{2(m_0+1)}\right]\le E\big[W_n^{2(m_0+1)}\big]=O\big(\rho_n^{2(m_0+1)}\big).\tag{3.36}$$

Recall $\overline{Y}=Y-EY$ for any random variable Y. Noting that $\sum_{i=0}^{n}\overline{Z}_i^n$ and

$\sum_{i=n}^{2n} \overline{Z}_i^{2n}$ are independent and have the same distribution as $\overline{R}_n$, we have

$$\begin{aligned} & E\left[\left(\sum_{i=0}^{n} \overline{Z}_i^n + \sum_{i=n}^{2n} \overline{Z}_i^{2n}\right)^{2(m_0+1)}\right] \\ & \quad = 2E\left[\overline{R}_n^{2(m_0+1)}\right] + 2\binom{2(m_0+1)}{2} E\left[\overline{R}_n^{2m_0}\right] E\left[\overline{R}_n^2\right] + \cdots \\ & \qquad + \binom{2(m_0+1)}{m_0+1} E\left[\overline{R}_n^{m_0+1}\right]^2 \\ & \quad \le \rho_n^{2(m_0+1)}\left(2L_{n,m_0+1}^{2(m_0+1)} + c_1\right), \end{aligned} \tag{3.37}$$

for some $c_1 = c_1(m_0) > 0$, where the last inequality is due to the induction hypothesis. By (3.35), (3.36) and (3.37), we have

$$\left\{E\left[\overline{R}_{2n}^{2(m_0+1)}\right]\right\}^{1/2(m_0+1)} \le \rho_n\left(2L_{n,m_0+1}^{2(m_0+1)} + c_1\right)^{1/2(m_0+1)} + O(\rho_n).$$

Dividing both sides by $\rho_{2n} \sim \sqrt{2}\rho_n$, we have

$$L_{2n,m_0+1} \le \left(\frac{1}{2^{m_0}} L_{n,m_0+1}^{2(m_0+1)} + c_2\right)^{1/2(m_0+1)} + c_3. \tag{3.38}$$

Now choose N large so that

$$\left(\frac{1}{2^{m_0}} + \frac{c_2}{N^{2(m_0+1)}}\right)^{1/2(m_0+1)} + \frac{c_3}{N} \le 1.$$

Either $L_{m,m_0+1} \le N$ for every m that is a power of 2 or for some $m \in \mathbf{N}$ that is a power of 2, we have $L_{m,m_0+1} \ge N$. In the latter case, for $n \ge m$, we have by (3.38) that

$$\begin{aligned} & \frac{L_{2n,m_0+1}}{L_{m,m_0+1}} \\ & \quad \le \left(\frac{1}{2^{m_0}}\left(\frac{L_{n,m_0+1}}{L_{m,m_0+1}}\right)^{2(m_0+1)} + \frac{c_2}{(L_{m,m_0+1})^{2(m_0+1)}}\right)^{1/2(m_0+1)} + \frac{c_3}{L_{m,m_0+1}}. \end{aligned}$$

Thus $L_{2m,m_0+1} \le L_{m,m_0+1}$ and by induction it follows that $L_{n,m_0+1} \le L_{m,m_0+1}$ for all $n > m$ which are powers of 2. Thus $\{L_{2^n,m_0+1}\}$ is bounded.

Next consider $n/2 < m < n$ where n is a power of 2. We can write

$$R_n = \sum_{i=0}^{m} Z_i^m + \sum_{i=m}^{n} Z_i^n - \sum_{i=0}^{m-1} (Z_i^m - Z_i^n) - 1.$$

By a similar argument to the above, we obtain

$$L_{m,m_0+1} \le c_4 L_{n,m_0+1} + c_5.$$

The boundedness of $\{L_{n,m_0+1}\}$ follows. $\square$

REMARK 3.7. Hamana [9] has informed us that (3.27) holds with $O(n)$ (instead of $O(n\sqrt{\log n})$). Using this, the extra term in (3.4) can be sharpened to $O(n)$.

PROOF OF LEMMA 3.3. Let

$$A = \#\{S_k : a < k \le b\},$$
$$B = \#\big(\{S_k : a < k \le b\} \cap \{S_k : 0 \le k \le a\}\big).$$

Then

$$\overline{R}_b - \overline{R}_a = \overline{A} - \overline{B}.$$

The law of A is equal to the law of R_{b-a}, so by Lemma 3.2 we have

$$E[(\overline{A})^l] \le c_1((b-a)\log(b-a))^{l/2}.$$

Consider the sequence $\{\widetilde{S}_k\} = \{S_b, S_{b-1}, \dots, S_0\}$. Then $\widetilde{S}_k$ is a random walk satisfying the same conditions as S_k and

$$B = \#\big(\{\widetilde{S}_k : 0 \le k < b-a\} \cap \{\widetilde{S}_k : b-a \le k \le b\}\big).$$

By Lemma 3.1,

$$E[B^l] \le c_2((b-a)\log(b-a))^{l/2}.$$

Since $(EB)^l \le E[B^l]$ by Jensen's inequality, combining with the estimate for $\overline{A}$ proves (a).

Let $D = b - a$ and

$$G_k = \frac{(R_{k+a} - ER_{k+a}) - (R_a - ER_a)}{(D\log D)^{1/2}}.$$

To show (b) we need to show

$$P\left(\max_{k\le D} |G_k| > \lambda\right) \le \frac{c_3}{\lambda^l}. \tag{3.39}$$

Note from (a) that

$$E|G_k - G_j|^l \le c_4(|k-j|/D)^{l/2}. \tag{3.40}$$

For each k let k_j be the largest element of $\{\langle mD/2^j\rangle : m \le 2^j\}$ that is less than or equal to k. We have

$$G_k = G_{k_0} + (G_{k_1} - G_{k_0}) + (G_{k_2} - G_{k_1}) + \cdots.$$

The sum is actually finite because from some point on all the k_j are equal to k. Thus, in order for $|G_k|$ to be larger than λ for some $k \le D$ there must be a $j \ge 0$ and an $m \le 2^j$ such that

$$\big|G_{\langle (m+1)D/2^j\rangle} - G_{\langle mD/2^j\rangle}\big| \ge \frac{\lambda}{40(j+1)^2}.$$

Therefore, using (3.40),

$$\begin{aligned} P\Big(\max_{k\le D}|G_k|>\lambda\Big) &\le \sum_{j=0}^{\infty}\sum_{m=0}^{2^j} P\bigg(\big|G_{\langle (m+1)D/2^j\rangle}-G_{\langle mD/2^j\rangle}\big|\ge \frac{\lambda}{40(j+1)^2}\bigg)\\ &\le c_5\sum_{j=0}^{\infty}2^j\frac{(1/2^j)^{l/2}(j+1)^{2l}}{\lambda^l}\\ &\le \frac{c_6}{\lambda^l} \end{aligned}$$

as long as $l>2$. This proves (3.39). □

4. Proof: two-dimensional case. We split the proof of Theorem 2.5 into two parts. The first is the following.

PROPOSITION 4.1. *Suppose $d=2$. There exists $c_{4.1}>0$ such that*

$$\limsup_{n\to\infty}\frac{\sup_{j\le n}(R_j-ER_j)}{n\log_3 n/(\log n)^2}\le c_{4.1}\qquad a.s. \tag{4.1}$$

We do not require Assumption 2.4 here.

Using (2.6), it is enough to prove the theorem when we replace ER_n with $\kappa n/\log n$. We fix n and for each $j\in\mathbf{N}$ with $j\le n$, set

$$\begin{aligned} \varphi_j &= \begin{cases} 1, & j=1,\\ \dfrac{j}{\log j}, & j>1,\end{cases}\\ F_j &= R_j-\kappa\varphi_j,\\ G_j &= G_j^n = F_j\frac{(\log n)^2}{n},\\ K &= [\log_2 n]+1. \end{aligned}$$

We will show that $\max_{j\le k}G_j$ is almost subadditive. If it had been subadditive, we could have used the technique in [2], Section 3. Here we must modify the ideas in [2] appropriately.

LEMMA 4.2. *There exists $c_{4.2}$ such that if $A,B\in\mathbf{N}$, $C=A+B$ and $\alpha=(A\wedge B)/C$, then*

$$|\varphi_C-\varphi_A-\varphi_B|\le c_{4.2}\frac{C}{(\log C)^2}\alpha^{1/2}. \tag{4.2}$$

PROOF. The cases where A or B equal 1 are easy, so we suppose $A, B > 1$. We start with the identity

$$\varphi_C - \varphi_A - \varphi_B = \frac{C}{\log C}\left[-\frac{A}{C}\frac{\log C - \log A}{\log A} - \frac{B}{C}\frac{\log C - \log B}{\log B}\right].$$

If $2 \le A \le C^{1/2}$, then $\log A \ge \frac{1}{3}$ and

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le 3\left(\frac{A}{C}\right)^{1/2}\frac{1}{\log C}\frac{(\log C)^2}{C^{1/4}} \le \frac{c_1}{\log C}\left(\frac{A}{C}\right)^{1/2}.$$

If $C^{1/2} \le A \le C/2$, then

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le 2\frac{A}{C}\frac{\log(C/A)}{\log C} \le \frac{c_2}{\log C}\left(\frac{A}{C}\right)^{1/2}.$$

If $A \ge C/2$, then

$$0 \le \frac{A}{C}\frac{\log C - \log A}{\log A} \le \frac{c_3}{\log A}|\log(1-(B/C))| \le \frac{c_4}{\log C}\left(\frac{B}{C}\right)^{1/2}.$$

We similarly bound $(B/C)((\log C - \log B)/\log B)$. $\square$

The following lemma is similar to Lemma 3.3, but here there are no absolute values and the estimates are one-sided.

LEMMA 4.3. (a) *There exists $M > 0$ not depending on n such that*

$$P\left(\max_{1\le j\le n} G_j > M\right) < \tfrac{1}{2}. \tag{4.3}$$

(b) *There exist $c_{4.3}, c_{4.4} > 0$ not depending on n such that*

$$E\left[\exp\left(c_{4.3}\max_{1\le j\le n} G_j\right)\right] \le c_{4.4}. \tag{4.4}$$

PROOF. Let θ_j be the usual shift operators. Since $R_n - R_m \le R_{n-m}\circ\theta_m$, then by Lemma 4.2,

$$G_n - G_m \le G_{n-m}\circ\theta_m + c_1\left(\frac{m}{n}\wedge\frac{n-m}{n}\right)^{1/2}. \tag{4.5}$$

By the Markov property, (2.6) and (2.7),

$$E[(G_j\circ\theta_m)^2] = E^{S_m}G_j^2 = EG_j^2 \le c_2(j/n)^2\left(\frac{\log n}{\log j}\right)^4 \le c_3(j/n)^{3/2}. \tag{4.6}$$

In particular

$$EG_j^2 \le c_3 (j/n)^{3/2}. \tag{4.7}$$

For each k let k_j be the largest element of $\{\langle mn/2^j \rangle : m \le 2^j\}$ that is less than or equal to k. We have

$$G_k = G_{k_0} + (G_{k_1} - G_{k_0}) + (G_{k_2} - G_{k_1}) + \cdots,$$

where the sum is a finite one. If $\max_{k \le n} G_k \ge M$, then for some $j \ge 0$ the following must hold:

$$G_{\langle (m+1)n/2^j \rangle} - G_{\langle mn/2^j \rangle} > \frac{M}{40(j+1)^2} \qquad \text{for some } m \le 2^j. \tag{4.8}$$

Let $I(m, j) = \langle (m+1)n/2^j \rangle - \langle mn/2^j \rangle$. If $m \le 2^{j/8}$, then by (4.7),

$$\begin{aligned} P\biggl(G_{\langle (m+1)n/2^j \rangle} - G_{\langle mn/2^j \rangle} > \frac{M}{40(j+1)^2}\biggr) &\le \frac{3200(j+1)^4}{M^2} \bigl(EG^2_{\langle (m+1)n/2^j \rangle} + EG^2_{\langle mn/2^j \rangle}\bigr) \\ &\le \frac{c_4 (j+1)^4 (m/2^j)^{3/2}}{M^2} \\ &\le \frac{c_5}{2^{5j/4} M^2}. \end{aligned}$$

If $m > 2^{j/8}$, then using (4.5),

$$\begin{aligned} G_{\langle (m+1)n/2^j \rangle} - G_{\langle mn/2^j \rangle} &\le G_{I(m,j)} \circ \theta_{\langle mn/2^j \rangle} + c_1 (m+1)^{-1/2} \\ &\le G_{I(m,j)} \circ \theta_{\langle mn/2^j \rangle} + \frac{M}{80(j+1)^2} \end{aligned}$$

if M is large enough. In this case, using (4.6),

$$\begin{aligned} P\biggl(G_{\langle (m+1)n/2^j \rangle} - G_{\langle mn/2^j \rangle} > \frac{M}{40(j+1)^2}\biggr) &\le P\biggl(G_{I(m,j)} \circ \theta_{\langle mn/2^j \rangle} > \frac{M}{80(j+1)^2}\biggr) \\ &\le c_6 \frac{(j+1)^4}{M^2} \frac{1}{2^{3j/2}} \\ &\le \frac{c_7}{2^{5j/4} M^2}. \end{aligned}$$

We thus have

$$P\Big(\max_{j\le n} G_j > M\Big) \le \sum_{j=0}^{\infty}\sum_{m=1}^{2^j} P\Big(G_{\langle (m+1)n/2^j\rangle} - G_{\langle mn/2^j\rangle} > \frac{M}{40(j+1)^2}\Big)$$
$$\le \sum_{j=0}^{\infty} c_8 \frac{2^j}{M^2}\frac{1}{2^{5j/4}}$$
$$\le \frac{c_8}{M^2} \le \frac{1}{2}$$

if M is large enough.

We next prove (4.4). Note that by (4.5), we have

$$G_n - G_m \le G_{n-m}\circ\theta_m + c_9. \tag{4.9}$$

Now, choose c_{10} large so that $c_{10}/2 > c_9$ and

$$P\Big(\max_{1\le j\le n} G_j > (c_{10}/2) - c_9\Big) < 1/2 \qquad \text{for all } n \in \mathbf{N}, \tag{4.10}$$

which is possible by (4.3). Let $T_k = \min\{j : G_j > c_{10}k\}$. Then

$$P\Big(\max_{j\le n} G_j > c_{10}(k+1)\Big) = P(T_{k+1}\le n)$$
$$\le P\Big(T_k \le n, \max_{T_k\le j\le n}(G_j - G_{T_k}) > c_{10}/2\Big)$$
$$= E\Big[P\Big(\max_{T_k\le j\le n}(G_j - G_{T_k}) > c_{10}/2 \,\Big|\, \mathcal{F}_{T_k}\Big); T_k\le n\Big]$$
$$\le E\Big[P\Big(\max_{j\le n} G_j > (c_{10}/2) - c_9\Big); T_k \le n\Big]$$
$$\le \tfrac{1}{2}P(T_k\le n),$$

where the second inequality follows by (4.9) and the third inequality by (4.10). By induction we obtain $P(T_k \le n) \le 2^{-n}$, which yields (4.4). $\square$

PROOF OF PROPOSITION 4.1. Let

$$C_j = \max_{\langle jn/K\rangle \le i < \langle (j+1)n/K\rangle}\big[R_i - R_{\langle jn/K\rangle} - \kappa\varphi_{i-\langle jn/K\rangle}\big]$$

and

$$D_j = \frac{C_j}{(n/K)/(\log(n/K))^2}.$$

By Lemma 4.3 there exist c_1, c_2 such that $Ee^{c_1 D_j} \le c_2$. Moreover, the D_j are independent. Let

$$e_{K,n} = \frac{|\varphi_n - K\varphi_{\langle n/K\rangle}|}{n/(\log n)^2}.$$

An elementary computation shows that

$$e_{K,n} \le c_3 \log K.$$

Since

$$\max_{m\le n} \frac{(R_m - \kappa\varphi_m)}{n/(\log n)^2} \le \frac{c_4}{K}\sum_{j=1}^{K} D_j + \kappa e_{K,n}$$

for $A \ge 2c_3\kappa$, we have

$$\begin{aligned} P\left(\max_{m\le n} \frac{R_m - \kappa\varphi_m}{n/(\log n)^2} > A\log K\right) &\le P\left(\frac{c_4}{K}\sum_{j=1}^{K} D_j > A\log K - \kappa e_{K,n}\right) \\ &\le P\left(\sum_{j=1}^{K} D_j > AK(\log K)/(2c_4)\right) \\ &\le e^{-c_1 AK(\log K)/(2c_4)} Ee^{c_1 \sum D_j} \\ &\le e^{-c_5 AK(\log K)/2} c_2^K \\ &\le e^{-c_5 AK(\log K)/2} \end{aligned}$$

if K is large enough. Using this inequality for $n = n_i = 2^i$ and $K = \langle \log_2 n\rangle$, the right-hand side is summable in i, and we can apply Borel–Cantelli. Since

$$\frac{\sup_{j\le n}(R_j - \kappa\varphi_j)}{n\log_3 n/(\log n)^2} \le 2\frac{\sup_{j\le n_{i+1}}(R_j - \kappa\varphi_j)}{n_{i+1}\log_3 n_{i+1}/(\log n_{i+1})^2}$$

for $n_i \le n < n_{i+1}$ if i is large, we obtain (4.1). □

We next work on the lower bound.

PROPOSITION 4.4. *Suppose* $d = 2$. *Under Assumption* 2.4, *there exists* $c_{4.5} > 0$ *such that*

$$\limsup_{n\to\infty} \frac{\sup_{j\le n}(R_j - ER_j)}{n\log_3 n/(\log n)^2} \ge c_{4.5} \qquad a.s. \tag{4.11}$$

The idea of the proof of Proposition 4.4 is to split $S_0, S_1, \dots, S_n$ into about $\log_2 n$ blocks of approximately equal length. We show that there is sufficiently large probability that the jth and kth blocks will not overlap if $|j-k| > 1$. If J_j is the range of the jth block and H_j is the cardinality of the overlap of the $(j-1)$st and jth blocks, we can then write

$$R_n - ER_n = \sum_j (J_j - EJ_j) + \left(\sum_j EJ_j - ER_n\right) - \sum_j H_j.$$

We estimate the first term on the right-hand side using the central limit theorem of [18]. The second term is a straightforward computation. To bound the final term, we first need to develop some estimates for the intersections of two random walks.

In order to prove Proposition 4.4, we need two lemmas. Let I_n be the cardinality of $\{S_k : k \in [0,n]\} \cap \{S'_k : k \in [0,n]\}$ where S_k and S'_k are two independent random walks with $S_0 = y$, $S'_0 = y'$ for some $y, y' \in \mathbf{Z}^2$. Note that the initial points y, y' can be chosen arbitrarily in $\mathbf{Z}^2$. Denote by α_t the intersection local time of two independent two-dimensional Brownian motions up to time t.

LEMMA 4.5. *Under Assumption* 2.4, *there exists* $c_{4.6} > 0$ *such that*

$$E\left[\left(\frac{I_n}{n/(\log n)^2}\right)^p\right] \le c_{4.6}^p E\alpha_1^p \qquad \textit{for all } p \in \mathbf{N}. \tag{4.12}$$

PROOF. Let $T_y = \inf\{k > 0 : S_k = y\}$. We will show

$$\bigl(\log(|y|^2 T)\bigr) P^0(T_y \le |y|^2 T) \le c_1 \nu([0,T]) \qquad \text{for all } y \in \mathbf{Z}^2, \tag{4.13}$$

where ν is the measure on $\mathbf{R}$ with density $(1/t)\exp(-1/(c_2 t))$, with some $c_2 > 0$ to be chosen later. We first prove the lemma assuming (4.13). Using (5.a), (5.b) and (5.c) of Le Gall [18],

$$\frac{(\log n)^{2p}}{n^p} E[I_n^p] \le \sum_{\sigma,\sigma' \in \mathcal{S}_p} \int_{(R^2)^p} du_1 \cdots du_p \theta_n(u_1,\ldots,u_p)\theta'_n(u_1,\ldots,u_p),$$

for all $p \in \mathbf{N}$ where $\mathcal{S}_p$ is the set of permutations of $\{1,\ldots,p\}$ and θ_n is defined by

$$\theta_n(u_1,\ldots,u_p) = (\log n)^p P^0\bigl(T_{[n^{1/2}u_{\sigma(1)}]} \le \cdots \le T_{[n^{1/2}u_{\sigma(p)}]} \le n\bigr)$$

for each $u_1,\ldots,u_p \in \mathbf{R}^2$. Similarly to the proof just before equation (5.d) in [18], and using (4.13), we have

$$\begin{aligned}\theta_n(u_1,\ldots,u_p) \le c_1^p &\int_0^1 \frac{dt_1}{t_1} \exp\left(-\frac{|u_{\sigma(1)}|^2}{c_2 t_1}\right) \\ &\times \int_0^{1-t_1} \frac{dt_2}{t_2} \exp\left(-\frac{|u_{\sigma(2)} - u_{\sigma(1)}|^2}{c_2 t_2}\right) \times \cdots \\ &\times \int_0^{1-(t_1+\cdots+t_{p-1})} \frac{dt_p}{t_p} \exp\left(-\frac{|u_{\sigma(p)} - u_{\sigma(p-1)}|^2}{c_2 t_p}\right).\end{aligned}$$

As in (5.d) in [18], this is less than

$$c_3^p \int \cdots \int_{0 \le s_1 \le \cdots \le s_p \le 1} ds_1 \cdots ds_p \prod_{i=1}^{p} q_{s_i - s_{i-1}}(u_{\sigma(i-1)}, u_{\sigma(i)}),$$

where q_s be the transition density of a two-dimensional Brownian motion, with the variance at time t not tI but $c_4 tI$ (where $c_4 = c_2/2$); in other words, a speeded up Brownian motion. Also we set $\sigma(0) = 0$. Following Le Gall's argument on page 495 in [18], this in turn is less than $c_5^p E\ell_{c_6}^p$, but now ℓ_{c_6} is the intersection local time of two independent Brownian motions that each have covariance matrix $c_6 tI$. By scaling, this will be less than $c_5^p c_6^p E\alpha_1^p$, which completes the proof of the lemma.

It remains to show (4.13). For the proof of this, we observe two facts. One is Theorem 3.6 in [18]:

$$(\log n) P^y(T_0 \le n) \le c_7 \biggl\{ (\log \sqrt{n}/|y|)^+ + \frac{n}{|y|^2} \mathbb{1}_{\{|y|/\sqrt{n} \ge 1/2\}} \biggr\}. \tag{4.14}$$

The other is Bernstein's inequality,

$$P\Bigl(\max_{k \le n} |S_k| > \lambda\Bigr) \le \exp\biggl(-\frac{\lambda^2}{c_8(n + \Lambda\lambda)}\biggr),$$

where Λ is given in Assumption 2.4(b); see [3].

Using (4.14) with $n = |y|^2 T$, (4.13) is clear for $T > (32\Lambda^2)^{-1}$ [note that replacing y by $-y$ in (4.14) and using translation invariance, (4.14) holds for $P^0(T_y \le n)$]. If $T \le (32\Lambda^2)^{-1}$ and $|y| \le 4\Lambda$, then $|y|^2 T \le \frac{1}{2}$ and (4.13) follows trivially. So we look at the case where $T \le (32\Lambda^2)^{-1}$ and $|y| > 4\Lambda$. It is easy to see that $\nu([0,T]) \ge c_9 \exp(-1/(c_{10}T))$ for $T \le (32\Lambda^2)^{-1}$. Denote by $\tau_{B(0,r)}$ the first exit time from the ball centered at 0 and radius r. By the strong Markov property we have

$$\begin{aligned} P^0(T_y \le |y|^2 T) &= P^0\bigl(\tau_{B(0,|y|/2)} \le |y|^2 T, T_y \le |y|^2 T\bigr) \\ &\le E^0\Bigl[P^{S_{\tau_{B(0,|y|/2)}}}\bigl(T_{y - S_{\tau_{B(0,|y|/2)}}} \le |y|^2 T\bigr); \tau_{B(0,|y|/2)} \le |y|^2 T\Bigr]. \end{aligned} \tag{4.15}$$

By (4.14),

$$P^{S_{\tau_{B(0,|y|/2)}}}\bigl(T_{y - S_{\tau(B(0,|y|/2))}} \le |y|^2 T\bigr) \le c_{11}/\log(|y|^2 T), \tag{4.16}$$

when $|y|/2 > 2\Lambda$. By Assumption 2.4(b) the random walk cannot go a distance more than $\Lambda|y|^2 T$ in time $|y|^2 T$. So if $T < 1/(4\Lambda|y|)$, then $\Lambda|y|^2 T < |y|/4$ and $P^0(\tau_{B(0,|y|/2)} < |y|^2 T) = 0$. If $T > 1/(4\Lambda|y|)$, then by Bernstein's inequality,

$$P^0(\tau_{B(0,|y|/2)} \le |y|^2 T) \le \exp\biggl(-\frac{|y|^2}{4c_8(|y|^2 T + \Lambda|y|/2)}\biggr) \le e^{-1/(c_{12}T)}$$

if c_{12} is chosen sufficiently large. Putting this and (4.16) in (4.15) yields (4.13). □

LEMMA 4.6. *There exists* $c_{4.7} > 0$ *such that*

$$E\alpha_1^p \le c_{4.7}^p p! \qquad \textit{for all } p \in \mathbf{N}. \tag{4.17}$$

Further, *there exists* $c_{4.8}, c_{4.9} > 0$ *such that*

$$E\left[\exp\left(\frac{c_{4.8} I_n}{n/(\log n)^2}\right)\right] < c_{4.9} \qquad \textit{for all } n \in \mathbf{N}. \tag{4.18}$$

PROOF. (4.17) is proved in Lemma 2 of [20]. Inequality (4.18) is then deduced by combining (4.17) with Lemma 4.5. □

PROOF OF PROPOSITION 4.4. Fix n. Let $K = \langle \beta \log_2 n \rangle$, where β is a number we will choose later. Let $M = n/K$. Set

$$J_j = \#\{S_k : k \in [\langle jM \rangle, \langle (j+1)M \rangle)\},$$
$$H_j = \#(\{S_k : k \in [\langle jM \rangle, \langle (j+1)M \rangle)\} \cap \{S_k : k \in [\langle (j-1)M \rangle, \langle jM \rangle)\}).$$

Let $\mathbf{e}$ be a vector of length $\sqrt{M}$. We denote by $B(x, r)$ the ball of radius r centered at x. Let A_j be the event that $S_{\langle jM \rangle} \in B(j\mathbf{e}, \frac{1}{8}\sqrt{M})$ and

$$\{S_k : k \in [\langle jM \rangle, \langle (j+1)M \rangle)\} \subset B((j + \tfrac{1}{2})\mathbf{e}, \sqrt{M}).$$

Let B_j be the event that $\overline{J}_j (\log M)^2/M$ is greater than some number $-c_1$. By the usual central limit theorem we know $P(A_j) \ge c_2$ if n is large. Thanks to (2.8), if we take c_1 sufficiently large, then $P(A_j \cap B_j) > c_2/2$. By the Markov property,

$$P(F) \ge (c_2/2)^K, \tag{4.19}$$

where $F = \bigcap_{j=1}^K (A_j \cap B_j)$.

On the set F we have that $\{S_k : k \in [\langle jM \rangle, \langle (j+1)M \rangle)\}$ is disjoint from $\{S_k : k \in [\langle iM \rangle, \langle (i+1)M \rangle)\}$ if $|i - j| > 1$, and so

$$\overline{R}_n = \sum_{j=1}^K \overline{J}_j + \left(\sum_{j=1}^K EJ_j - ER_n\right) - \sum_{j=1}^K H_j. \tag{4.20}$$

On the set F the event B_j holds for each j, and so

$$\sum_{j=1}^K \overline{J}_j \ge -\frac{c_1 KM}{(\log M)^2} \ge -\frac{c_3 n}{(\log n)^2}. \tag{4.21}$$

We have

$$\begin{aligned}\sum_{j=1}^{K} EJ_j - ER_n &= K\frac{n/K}{\log(n/K)} - \frac{n}{\log n} + O\Big(\frac{n}{(\log n)^2}\Big)\\ &= \frac{n}{\log n}\Big[\frac{1}{1-(\log K/\log n)} - 1\Big] + O\Big(\frac{n}{(\log n)^2}\Big)\\ &\geq c_4\frac{n\log K}{(\log n)^2} + O\Big(\frac{n}{(\log n)^2}\Big)\\ &\geq c_5\frac{n\log_3 n}{(\log n)^2}\end{aligned}\tag{4.22}$$

if n is large.

Let C_1 be the event that

$$\sum_{\{j \text{ odd}\}} H_j \geq \frac{c_5}{3}\frac{n\log_3 n}{(\log n)^2}$$

and C_2 the event that

$$\sum_{\{j \text{ even}\}} H_j \geq \frac{c_5}{3}\frac{n\log_3 n}{(\log n)^2}.$$

Set $G = F \cap C_1^c \cap C_2^c$. For j odd the H_j are independent. By Lemma 4.6,

$$\begin{aligned}P(C_1) &= P\Big(\sum_{\{j \text{ odd}\}}\frac{H_j}{M/(\log M)^2} \geq c_6 K\log K\Big)\\ &\leq e^{-c_6c_7K\log K}Ee^{c_7\sum H_j(\log M)^2/M}\\ &\leq e^{-c_8K\log K}(c_9)^K.\end{aligned}$$

When n is large, K will be large, and then $P(C_1) \leq P(F)/3$. We have a similar estimate for $P(C_2)$, so using (4.19), we have

$$P(G) \geq (c_2/2)^K/3.\tag{4.23}$$

On the event G,

$$\sum_{j=1}^{K} H_j \leq \frac{2c_5}{3}\frac{n\log_3 n}{(\log n)^2}.\tag{4.24}$$

Combining (4.20), (4.21), (4.22) and (4.24), on the event G,

$$\overline{R}_n \geq c_{10} n\log_3 n/(\log n)^2.\tag{4.25}$$

Now let $n_\ell = \langle\exp(\ell^2)\rangle$, let $K_\ell = \langle\beta\log_2(n_{\ell+1} - n_\ell)\rangle$, let $S'_k = S_{k+n_\ell} - S_{n_\ell}$, $k = 0, 1, 2, \dots, n_{\ell+1} - n_\ell$, and let $R'_\ell = \#\{S'_k : 0 \leq k < n_{\ell+1} - n_\ell\}$. If we apply the above results to the random walk S'_k, we see there exist events G'_ℓ which are independent, which have probability at least $\frac{1}{3}(c_2/2)^{K_\ell}$, and on which

$$\overline{R'}_\ell \geq c_{10} \frac{(n_{\ell+1} - n_\ell)\log_3(n_{\ell+1} - n_\ell)}{(\log(n_{\ell+1} - n_\ell))^2} \geq c_{11} \frac{n_{\ell+1}\log_3 n_{\ell+1}}{(\log n_{\ell+1})^2}. \tag{4.26}$$

If we choose β small enough, then $\sum_\ell P(G'_\ell) = \infty$, and by Borel–Cantelli G'_ℓ occurs infinitely often with probability 1. Now R'_ℓ differs from $R_{n_{\ell+1}}$ by at most $n_\ell = o(n_{\ell+1} \log_3 n_{\ell+1}/(\log n_{\ell+1})^2)$ and the same holds for the difference of their expectations. Together with (4.26) this proves the proposition. □

PROOF OF THEOREM 2.5. We use Propositions 4.1 and 4.4, and the Hewitt–Savage zero–one law (see, e.g., Theorem 2.15 of [17]). □

Acknowledgments. The authors are grateful to J.-F. Le Gall, Y. Hamana, N. Jain, H. Kesten, D. Khoshnevisan and Z. Shi for helpful advice and comments.

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