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A GROWTH MODEL IN A RANDOM ENVIRONMENT[1]

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We consider a model of interface growth in two dimensions, given by a height function on the sites of the one-dimensional integer lattice. According to the discrete time update rule, the height above the site x increases to the height above $x-1$, if the latter height is larger; otherwise the height above x increases by 1 with probability p_x. We assume that p_x are chosen independently at random with a common distribution F and that the initial state is such that the origin is far above the other sites. We explicitly identify the asymptotic shape and prove that, in the pure regime, the fluctuations about that shape, normalized by the square root of time, are asymptotically normal. This contrasts with the quenched version: conditioned on the environment, and normalized by the cube root of time, the fluctuations almost surely approach a distribution known from random matrix theory.

1. Introduction. Processes of random growth and deposition have a long history in the physics literature, typically as models of systems far from equilibrium (e.g., [18] and the more than 1300 references listed therein). They made their appearance in probabilistic research about 35 years ago, with arguably the most basic growth rule, *first passage percolation* [13]. The fundamental asymptotic result is an ergodic theorem: scaled by time t, the growing set of sites approaches a deterministic limiting shape. As these early successes were based on nonconstructive subadditivity arguments, they posed two natural questions: (1) can the asymptotic shape be identified analytically and (2) how large are fluctuations about the limit? While there has been no resolution of the first issue, ingenious probabilistic and geometric arguments have yielded much progress on the second [1], although the matter is still far from settled. It is therefore of some importance to be able to provide a complete answer on some other simple, but nontrivial, interacting growth process. It turns out that several two-dimensional oriented models with a *last passage* property [3, 10, 16, 17, 21, 24, 25] are most convenient, as they can be represented, on the one hand, as particle systems related to asymmetric exclusion and, on the other hand, as increasing paths in random matrices and associated Young diagrams. This allows explicit answers to both questions (1) and (2) above.

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In this paper we continue to study *oriented digital boiling* (ODB) (see [12], Feb. 12, 1996, Recipe; see also [9, 10]), perhaps one of the simplest models for a coherent growing interface in the two-dimensional lattice $\mathbf{Z}^2$. The occupied set, which changes in discrete time $t = 0, 1, 2, \dots$, is given by $\mathcal{A}_t = \{(x, y) : x \in \mathbf{Z}, y \le h_t(x)\}$, and the height function h_t evolves according to the following rule:

$$h_{t+1}(x) = \max\{h_t(x-1), h_t(x) + \varepsilon_{x,t}\}.$$

Here $\varepsilon_{x,t}$ are independent Bernoulli random variables, with $P(\varepsilon_{x,t} = 1) = p_x$. Thus the probability of a random increase depends on the spatial location. It remains to specify the initial state, which will be

$$h_0(x) = \begin{cases} 0, & \text{if } x = 0, \\ -\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

In [10] we analyzed the homogeneous case $p_x \equiv p$, identifying the following four asymptotic regimes:

1. *Finite x GUE regime*—if x is fixed and $t \to \infty$, then $(h_t(x) - pt)/\sqrt{p(1-p)t} \xrightarrow{d} M_x$, a Brownian functional whose law can be computed explicitly as the largest eigenvalue of an $(x+1) \times (x+1)$ Hermitian matrix from the Gaussian unitary ensemble (GUE).
2. *GUE universal regime*—if x is a positive multiple of t, and if $x/t < 1 - p$, then there exist constants c_1 and c_2 so that $(h_t(x) - c_1 t)/(c_2 t^{1/3})$ converges weakly to a distribution F_2 known from random matrix theory [14].
3. *Critical regime*—if $x = (1-p)t + o(\sqrt{t})$, then $P(h_t(x) - (t - x) \le -k)$ converges to a $k \times k$ determinant.
4. *Deterministic regime*—if x is a positive multiple of t, and if $x/t > 1 - p$, $P(h_t(x) = t - x) \to 1$ exponentially fast.

The focus of this paper is ODB in a random environment, in which p_x are initially chosen at random, with common distribution given by $P(p_x \le s) = F(s)$. We also assume that p_x are independent, although in several instances this assumption can be considerably weakened. In statistical physics, processes in a random environment are often called *disordered systems*, or, especially in the Ising-type models, *spin glasses*. In this context, the random environment (choice of p_x) is referred to as *quenched* randomness, as opposed to the dynamic (*thermal*) fluctuations induced by the coin flips $\varepsilon_{x,t}$. In general, rigorous research in this area has been a notoriously difficult enterprise; for some recent breakthroughs (as well as reviews of the literature) we refer the reader to [19, 20, 26, 27].

We now state our main results. Throughout, we denote by $\langle \cdot \rangle$ integration with respect to dF and by p a generic random variable with distribution F.

Construct a random $m \times n$ matrix $A = A(F)$, with independent Bernoulli entries $\varepsilon_{i,j}$ and such that $P(\varepsilon_{i,j} = 1) = p_j$, where, again, $p_j \stackrel{d}{=} p$ are i.i.d. Label columns as usual, but with rows started at the bottom. We call a sequence of 1's in A whose positions have column index nondecreasing and row index strictly increasing an *increasing path* in A. Let $H = H(m, n)$ be the length of the longest increasing path. [Sometimes, to emphasize dependence on F, we write $H = H(F) = H(m, n, F)$.] The following lemma is then easy to prove [10].

LEMMA 1.1. *Under a simple coupling*, $h_t(x) = H(t - x, x + 1)$.

We therefore concentrate our attention on the random matrix A from now on, switching to the height function only occasionally to interpret the results. We also note that Lemma 1.1 demonstrates that ODB is equivalent to the Seppäläinen–Johansson model [17, 25].

Our first theorem identifies the time constant. In the sequel, we present two completely different methods for proving these limits, a variational approach and a determinantal approach. The first method (which is similar to the one in [7]) is based on the crucial symmetry property of H (Lemma 2.2) and provides some information on the longest increasing path itself; the second one is deeper and more precise and thus able also to determine fluctuations. Seppäläinen and Krug [26] study a related model, present yet another technique, based on an exclusion process representation, and observe similar phase transitions. Throughout this paper, we let

$$b = b(F) = \min\{s : F(s) = 1\}$$

be the right edge of the support of dF and assume that $n = \alpha m$ for some $0 < \alpha < \infty$. (Actually, $n = \lfloor \alpha m \rfloor$, but we drop the integer part as it is obvious where it should be used and to avoid complicating expressions.) We also define the critical values

$$\alpha_c = \left\langle \frac{p}{1-p} \right\rangle^{-1}, \qquad \alpha_c' = \left\langle \frac{p(1-p)}{(b-p)^2} \right\rangle^{-1} \tag{1.2}$$

and define $c = c(\alpha, F)$ to be the time constant

$$c = c(\alpha, F) = \lim_{m \to \infty} \frac{H}{m}. \tag{1.3}$$

Note that c determines the limiting shape of $\mathcal{A}_t$, namely $\lim \mathcal{A}_t/t$, as $t \to \infty$ for the corner initialization given by (1.1). By virtue of the Wulff transform, it then also gives the speeds of *some* half-planes, that is, $\lim \mathcal{A}_t/t$ when $\mathcal{A}_0$ comprises points below a fixed line. See [26] for much more on this issue.

THEOREM 1. *The limit in* (1.3) *exists almost surely*. *If* $b = 1$, *then* $c(\alpha, F) = 1$ *for all* α, *while if* $b < 1$, *then*

$$c(\alpha, F) = \begin{cases} b + \alpha(1-b)\left\langle \dfrac{p}{b-p} \right\rangle, & \text{if } \alpha \le \alpha_c', \\ a + \alpha(1-a)\left\langle \dfrac{p}{a-p} \right\rangle, & \text{if } \alpha_c' \le \alpha \le \alpha_c, \\ 1, & \text{if } \alpha_c \le \alpha. \end{cases}$$

Here $a = a(\alpha, F) \in [b, 1]$ *is the unique solution to*

$$\alpha \left\langle \frac{p(1-p)}{(a-p)^2} \right\rangle = 1.$$

Note that that $\langle (b-p)^{-2} \rangle = \infty$ iff $\alpha_c' = 0$ iff there is only one critical value.

Next we turn our attention to fluctuations. In this paper we present complete results for the *pure* regime $\alpha_c' < \alpha < \alpha_c$ and for the (easy) *deterministic* regime $\alpha_c < \alpha$. The *composite* regime $\alpha < \alpha_c'$ is addressed in [11], while both *critical* cases when α equals either critical value currently remain unresolved. To explain the results, and to connect with the spin-glass terminology we have just used, we turn to a simulation. For an example, we use $F(s) = 1 - (1-2s)^3$ so that $b = 1/2$, $\alpha_c \approx 6.3$ and $\alpha_c' \approx 0.5$ and run the simulation until time $t = 40{,}000$ (with a single realization of the environment and the coin flips). When x is close to the origin, it is clear from the picture that the interface mostly consists of sheer walls followed by flat pieces. The walls correspond to the rare sites with update probability p_x close to $1/2$. Those are much faster than the other sites so they pull ahead of their left neighbors, creating walls, and dominate their right neighbors by "feeding" them at nearly the largest possible rate. In fact, this state of affairs persists up to about $x = t/3$ although close to $x = t/3$ these effects are less pronounced. In the pure regime, when x/t ranges approximately from 0.333 to approximately 0.863, the fluctuations are much more regular, and in fact, as we will demonstrate, asymptotically normal. For larger x/t the shape has slope -1 and no fluctuations.

For comparison, consider the case when p is uniform on $[0, 1/2]$, the case that has $\alpha_c = 1/(\ln 4 - 1) \approx 2.59$ and $\alpha_c' = 0$. The fluctuations are normal up to $x/t \approx 0.72$. Figure 1 depicts the results of simulations, first complete boundaries of two occupied sets (the top curve is the uniform case), then two details (the right curve is the uniform case) for $x \in [1000, 5000]$.

THEOREM 2. *Assume that* $b < 1$ *and* $\alpha_c' < \alpha < \alpha_c$. *Let* a *be as in Theorem* 1 *and let*

$$\tau^2 = \mathrm{Var}\left(\frac{(1-a)p}{a-p}\right).$$

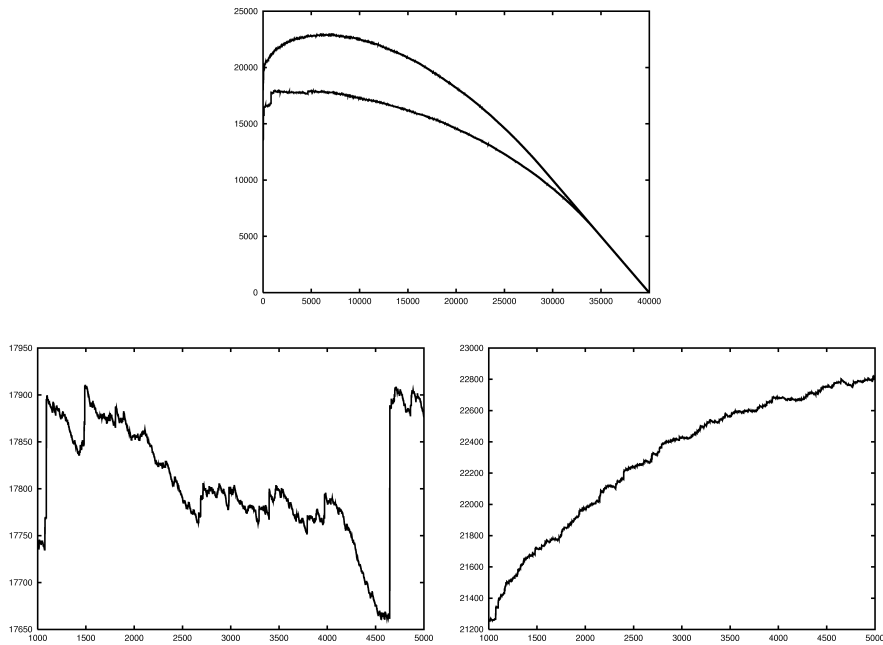

FIG. 1. *Two ODB simulations, as explained in the text.*

Then, *as* $m \to \infty$,

$$\frac{H - cm}{\tau\sqrt{\alpha}m^{1/2}} \xrightarrow{d} N(0,1).$$

Assume that p is uniform $[0, 1/2]$ to illustrate Theorems 1 and 2. Together they imply that there exist c_1 and c_2 so that $(h_t(x) - c_1 t)/(c_2 t)^{1/2} \xrightarrow{d} N(0,1)$, where c_1 determines the limiting shape and c_2 is the variance. These two quantities are presented in Figure 2; c_1 is the top curve and c_2 is the bottom curve. For comparison, the shape of homogeneous ODB with $p_x \equiv \langle p \rangle = 1/4$ is also drawn (middle curve). Note that c_1 and c_2 approach $1/2$ and $1/4$, respectively, as $\alpha \to 0$, indicating that for small x/t the interface growth is governed by the largest update probability, which is close to $1/2$. Finally, we do the same computation for the other example in Figure 1. The variance is now drawn only on $[\alpha_c', 1]$.

We note that both a.s. convergence to the limiting shape [which is equivalent to a.s. convergence in (1.3)] and its convexity follow from subadditivity, which in turn is a consequence of the fact that this is an oriented model in which influences only travel in one direction. To be more precise, fix integer sites $(x_1, y_1), (x_2, y_2) \in \mathbf{Z}_+ \times \mathbf{Z}_+$ and define times $T_{(x_1,y_1),(x_2,y_2)}$ as follows. First wait until time $T_{(0,0),(x_1,y_1)}$ when the dynamics reaches (x_1, y_1). Then restart the

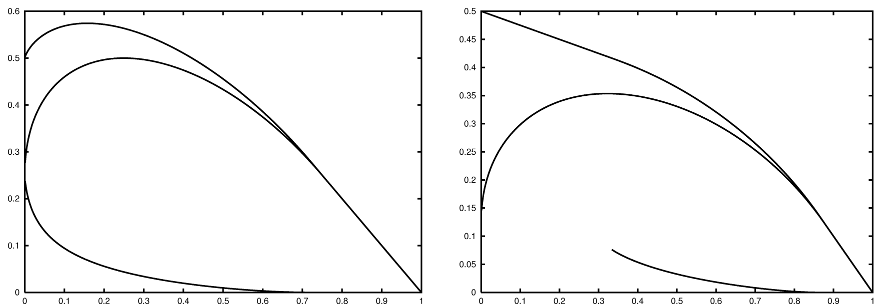

FIG. 2. c_1 *(top)*, c_2 *(bottom) and the shape for* $p_x \equiv \langle p \rangle$ *(middle) versus* x/t. *The two distributions are uniform* $[0, 1/2]$ *(left) and* $F(s) = 1 - (1-2s)^3$.

dynamics from the initial state

$$h_0(x) = \begin{cases} y_1, & \text{if } x = x_1, \\ -\infty, & \text{otherwise,} \end{cases}$$

and let $T_{(x_1,y_1),(x_2,y_2)}$ be the time at which the occupied set reaches (x_2, y_2). This random variable is independent of p_x for $x \le x_1 - 1$ and $T_{(0,0),(x_2,y_2)} \le T_{(0,0),(x_1,y_1)} + T_{(x_1,y_1),(x_2,y_2)}$. Therefore, the subadditive ergodic theorem can be applied as in the first chapter of [8].

The main step in the proof of Theorem 2 establishes a limit law for fluctuations conditioned on the state of the environment. In many ways, such a result is more pertinent to understanding physical processes modeled by simple growth models such as ODB.

THEOREM 3. *Assume that* $b < 1$ *and* $\alpha_c' < \alpha < \alpha_c$. *Then there exists a sequence of random variables* $G_n \in \sigma\{p_1, \dots, p_n\}$ *and a constant* $g_0 \neq 0$ *(both depending on* α*) such that, as* $m \to \infty$,

$$P\left(\frac{H - G_n}{g_0^{-1} m^{1/3}} \le s \,\Big|\, p_1, \dots, p_n\right) \to F_2(s),$$

almost surely, for any fixed s.

The random variables $G_n = c_n m$ are given in terms of the solution of an algebraic equation in which $p_1, \dots, p_n$ appear as parameters [see (3.4) and (3.5)], while the deterministic constant g_0 is specified before the statement of Lemma 3.5. The limiting distribution function F_2 first arose in connection with eigenvalues of random matrices ([28]; see [29] for a review). Since then it has been observed in many other contexts, including growth processes [2, 10, 16, 17, 21, 22]. Most suitable for computations is the identity

$$F_2(s) = \exp\left(-\int_s^\infty (x-s) q(x)^2 \, dx\right),$$

where q is the unique solution of the Painlevé II equation

$$q'' = sq + 2q^3,$$

which is asymptotic to the Airy function, $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$. When proving limit laws, it is more useful that F_2 can be represented as a Fredholm determinant (see, e.g., [10] and Section 3 below).

In Theorem 3 the environment is assumed as given, H is approximated by the quenched shape G_n, with the fluctuations about this shape of the order $m^{1/3}$ and given by the F_2 distribution. As we prove in Section 3, $(\alpha m)^{-1/2}(G_n - cm)$ converges to the standard normal, making it clear why Theorem 2 holds: the environmental noise eventually drowns out the more interesting quenched fluctuations of Theorem 3. An illustration is provided in Figure 3, in which p is again uniform on $[0, 1/2]$ and h_t (solid curve), a deterministic approximation based on Theorem 2 (dotted curve) and the much better random approximation based on Theorem 3 (dashed curve) are all depicted at times $t = 100, 200, \dots, 1000$.

In conclusion, we note that the connection between random matrix theory and random combinatorial objects, which has become the key to rigorous understanding of random interface fluctuations, made its initial appearance in [3], while an inhomogeneous model of ODB type was first studied in [14]. This last paper, together with its companion [15], extends the study of random words from the homogeneous case in [30] in a somewhat analogous way as the present paper builds on the work in [10]. In particular, connections with operator determinants (from the beginning of Section 3) are very similar (see also [23], which features

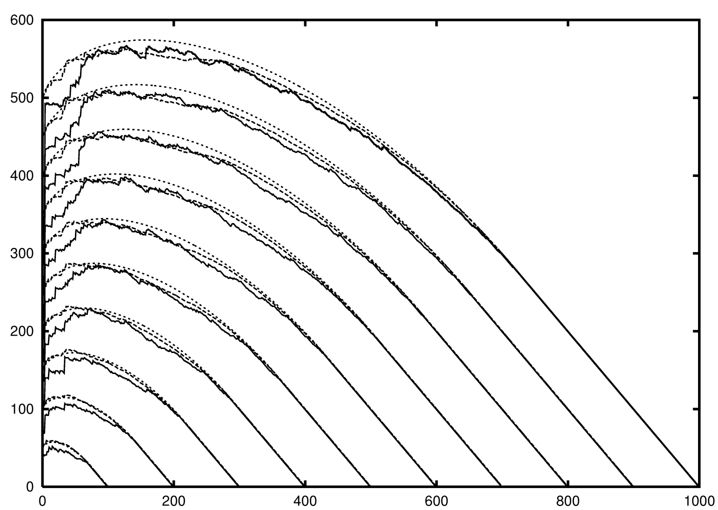

FIG. 3. *Approximations to h_t (solid curve) based on Theorems 2 (dotted curve) and 3 (dashed curve).*

a general inhomogeneous setup). However, randomness of the environment, which seems to be a new feature in rigorous analysis of explicitly solvable models, then forces our techniques to take a novel turn.

2. A variational characterization of the time constant. We start with a remark on constructing the random matrix A. The most convenient design uses as the probability space (Ω, P) a countably infinite product of unit intervals $[0, 1]$ with Lebesgue measure. A copy of the unit interval (and thus a factor in the product) is associated with each point in $\mathbf{N} \times \mathbf{N}$ and, in addition, with each positive integer in $\mathbf{N}$. (The former factors correspond to matrix entries, and the latter to its columns.) If $\omega = (m_{ij}, c_j) \in \Omega$ is a generic realization, we define the following random variables: $p_j = F^{-1}(c_j)$ [where $F^{-1}(x) = \sup\{y : F(y) < x\}$ as usual] and $\varepsilon_{ij} = \mathbb{1}_{\{m_{ij} < p_j\}}$. By restricting to the $m \times n$ rectangle at the lower right corner of $\mathbf{N} \times \mathbf{N}$, this constructs the random matrices A for all m and n simultaneously. The following useful lemma also follows immediately.

LEMMA 2.1. *If $F_1 \le F_2$ are two distribution functions, the two corresponding random matrices $A(F_1)$ and $A(F_2)$ can be coupled so that $H(F_2) \le H(F_1)$.*

Next we state the crucial property for the variational approach to work: conditioned on the environment, H is a symmetric function of flip probabilities.

LEMMA 2.2. *A regular conditional distribution*

$$P(H \le h | p_1, \dots, p_n)$$

is a symmetric function of $p_1, \dots, p_n$.

See [10], Section 2.2, for the proof of Lemma 2.2.

Somewhat loosely, we denote by H_n the random variable H obtained by fixing $p_1, \dots, p_n$. In fact this is nothing more than a shorthand notation, for example, $E(\varphi(H_n)) = E(\varphi(H) \mid p_1, \dots, p_n)$ for any bounded measurable function φ.

The time constant $c(\alpha, x) = c(\alpha, \delta_x)$ for the case $p_j \equiv x$ is given in [9]. The next lemma summarizes the relevant conclusions.

LEMMA 2.3. *Assume that $dF = \delta_x$. Then*

$$c = c(\alpha, x) = \begin{cases} 2\sqrt{\alpha}\sqrt{x(1-x)} + (1-\alpha)x, & (1-x)/x > \alpha, \\ 1, & (1-x)/x \le \alpha. \end{cases}$$

Moreover, for every $\varepsilon > 0$ there exists a constant $\gamma = \gamma(\varepsilon) > 0$ so that

$$P(|H/m - c| > \varepsilon) < e^{-\gamma m} \tag{2.1}$$

for $m \ge m_0(\varepsilon, \alpha, x)$.

PROOF. The formula for c follows from (3.1) of [10], while the large deviation estimate can be proved by the method of bounded differences as in Lemma 5.4 of [9]. $\square$

It turns out the following function is more convenient than c:

$$\zeta(y,x) = yc(1/y,x) = \begin{cases} 2\sqrt{y}\sqrt{x(1-x)} + (y-1)x, & x/(1-x) < y, \\ y, & x/(1-x) \geq y. \end{cases}$$

Note that the partial derivative

$$\zeta_y(y,x) = \begin{cases} y^{-1/2}\sqrt{x(1-x)} + x, & x/(1-x) < y, \\ 1, & x/(1-x) \geq y, \end{cases}$$

is decreasing in y (obviously) and increasing in x (easily checked). In particular, $\zeta(\cdot, x)$ is a convex function.

We now derive a variational problem for c, initially without paying attention to rigor. Start by a nice distribution function F and approximate it by the discrete distribution function given by

$$P\left(p_j = \frac{i}{k}\right) = \Delta F_k(i) = F\left(\frac{i}{k}\right) - F\left(\frac{i-1}{k}\right), \qquad i = 1, \dots, k.$$

Let $\psi : [0,\alpha] \to [0,1]$, $\psi(0) = 0$, $\psi(\alpha) = 1$ be a nondecreasing function, with $\Delta\psi_k(i) = \psi(\alpha F(i/k)) - \psi(\alpha F((i-1)/k))$. Define the functionals

$$\mathcal{F}(\psi) = \int_0^1 \zeta\big(\psi'(\alpha F(x)), x\big)\alpha\, dF(x)$$

and

$$\mathcal{F}_k(\psi) = \sum_{i=1}^{k} \zeta\left(\frac{\Delta\psi_k(i)}{\alpha\,\Delta F_k(i)}, \frac{i}{k}\right)\alpha\,\Delta F_k(i).$$

Generate the p_j's and denote by N_i the number of p_j equal to i/k. By Lemma 2.2, we can assume flip probability $1/k$ in the first N_1 columns, $2/k$ in the next N_2 columns etc. Moreover, the strong law suggests that the identity $N_i = \Delta F_k(i)n$ nearly holds. As we know the asymptotics for the longest increasing paths in the slivers of widths N_i in which the probabilities are constant, the longest increasing path in A is determined by the most advantageous choice of transition points between the slivers. These transition points are specified by a function ψ as described above. If we approximate the differences with derivatives, we obtain

$$\begin{aligned} c(\alpha, F) &= \lim_{k\to\infty} c(\alpha, F_k) \\ &= \lim_{k\to\infty} \max_{\psi} \sum_{i=1}^{k} c\left(\frac{\alpha\,\Delta F_k(i)}{\Delta\psi_k(i)}, \frac{i}{k}\right)\Delta\psi_k(i) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \max_{\psi} \mathcal{F}_k(\psi) \\
&= \lim_{k\to\infty} \max_{\psi} \sum_{i=1}^{k} \zeta\left(\psi'\left(\alpha F\left(\frac{i}{k}\right)\right), \frac{i}{k}\right)\alpha F'\left(\frac{i}{k}\right)\frac{1}{k} \\
&= \max_{\psi} \mathcal{F}(\psi).
\end{aligned}
$$

At this point, we remark that a connection between longest increasing paths and variational problems has appeared before in the literature. The result closest to ours is by Deuschel and Zeitouni [7], who used a variational approach to study a variant of Ulam's problem. In their case, a number of points in the unit square is chosen independently according to some distribution with a density, then a longest sequence, increasing in both coordinates, is extracted from this sample. The Deuschel–Zeitouni functional is different from ours as the length of the longest increasing path has a nontrivial dependence on α (i.e., through c) in our case.

The (integrated) Euler functional for the variational problem is

$$\zeta_y\big(\psi'(x), F^{-1}(x/\alpha)\big) = a$$

or, writing $g(x) = \psi'(\alpha F(x))$,

$$\zeta_y(g(x), x) = a. \tag{2.2}$$

Since $\zeta_y \le 1$ and equal to 1 if and only if $x/(1-x) \ge y$, the integration constant $a \in [0, 1]$. If $a = 1$, then $g(x) \le x/(1-x)$, $\zeta(g(x), x) = g(x)$ and

$$c(\alpha, F) = \int_0^1 \psi'(\alpha F(x))\alpha\, dF(x) = 1.$$

Assume now that $b < 1$. In this case, it is necessary to specify g only on $[0, b)$. However, (2.2) gives

$$g(x) = \frac{x(1-x)}{(a-x)^2}. \tag{2.3}$$

The constant a is given by the boundary conditions. Assuming that (2.3) holds on $[0, b]$,

$$1 = \alpha \int_0^b g(x)\, dF(x) = \alpha \int_0^b \frac{x(1-x)}{(a-x)^2}\, dF(x). \tag{2.4}$$

The smallest the last integral can be is when $a = 1$, which yields the condition

$$1 > \alpha \int_0^b \frac{x}{1-x}\, dF(x) = \frac{\alpha}{\alpha_c}.$$

On the other hand, the largest that the integral in (2.4) can be is when $a = b$. Therefore, if $\alpha \in (\alpha_c', \alpha_c)$, we have found the minimizer and

$$c(\alpha, F) = \int_0^b \zeta(g(x), x)\alpha\, dF(x) = \alpha\left\langle \frac{-p^2 - a^2 p + 2ap}{(a-p)^2}\right\rangle,$$

which reduces, upon using the defining equation for a, to the formula in Theorem 1.

If $\alpha < \alpha_c'$, the minimizer ψ has to make a jump of size $1-\alpha/\alpha_c'$ at α. The natural interpretation for this is that the minimizer given by (2.3) is used in the lower left part of A with dimensions $(\alpha/\alpha_c')m \times (n-1)$. To the resulting increasing path in this submatrix one needs to add the number of 1's in the upper segment of length $(1-\alpha/\alpha_c')m$ in the last column, in which nearly the largest probability b is used. Therefore,

$$c(\alpha, F) = c(\alpha_c', F)\frac{\alpha}{\alpha_c'} + b\left(1 - \frac{\alpha}{\alpha_c'}\right),$$

which again reduces to the appropriate formula in Theorem 1.

We now proceed to give a proof Theorem 1, the heart of which is a somewhat involved multistage approximation scheme.

PROOF OF THEOREM 1 WHEN $b = 1$. This follows simply by observing that, for any $\varepsilon > 0$, $\max_j p_j \to b$ a.s. as $m \to \infty$. Since a trivial lower bound is obtained by using only the column with the largest p_j, one concludes that $\liminf H/m \ge b$ a.s. $\square$

PROOF OF THEOREM 1 WHEN $\alpha \in (\alpha_c', \alpha_c)$. We begin with the following lemma.

LEMMA 2.4. *Assume that a sequence of distribution functions F_N converges to F in the usual sense* (*i.e.*, *the induced measures converge weakly*). *Assume also that $b(F_N) \to b(F)$ and that $\alpha_N \to \alpha$. Then $c(\alpha_N, F_N) \to c(\alpha, F)$* (*as given in Theorem* 1).

PROOF. If $a' > b(F)$ and

$$\alpha \int x(1-x)(a'-x)^{-2}\, dF(x) > 1,$$

then, for a large N, $a' > b(F_N)$ and, since the integrand is bounded,

$$\alpha_N \int x(1-x)(a'-x)^{-2}\, dF_N(x) > 1.$$

Hence $a_N = a(\alpha_N, F_N) > a'$. If $a_N \to a_0$, then $x(1-x)(a_N - x)^{-2}$ converges to $x(1-x)(a_0 - x)^{-2}$ uniformly for $x \in [0, a']$ and so

$$1 = \alpha_N \int x(1-x)(a_N - x)^{-2}\, dF_N(x) \to \alpha \int x(1-x)(a_0 - x)^{-2}\, dF(x).$$

Therefore $a_0 = a(\alpha, F)$ and consequently $a_N \to a(\alpha, F)$. As $x(a_N - x)^{-1}$ also converges uniformly on $[0, a']$,

$$c(\alpha_N, F_N) = a_N + \alpha_N(1 - a_N) \int x(a_N - x)^{-1}\, dF_N$$

$$\to a + \alpha(1 - a) \int x(a - x)^{-1}\, dF = c(\alpha, F). \qquad \square$$

First we assume that F is *nice*, that is, a one-to-one function on $[\beta, b] \subset (0, 1)$, with $F(\beta) = 0$ and $F(b) = 1$, and continuously differentiable on $(0, 1)$. We also assume that Ψ is the class of nondecreasing convex functions $\psi \in \mathcal{C}^2[0, \alpha]$, with $\psi(0) = 0$, $\psi(\alpha) = 1$, $\psi'(0) \geq \beta/2$. This last assumption is necessary because $\zeta(y, x)$ is not Lipshitz near $y = 0$.

LEMMA 2.5. *Assume that* $\alpha \in (\alpha_c', \alpha_c)$. *Among all* $\psi \in \Psi$, *the functional* $\mathcal{F}(\psi)$ *is uniquely maximized by*

$$\psi(x) = \int_0^x g\big(F^{-1}(u/\alpha)\big)^2\, du,$$

where g *is given by* (2.3).

PROOF. This follows from standard calculus of variations. Both $\psi'(0) \geq \beta/2$ and convexity of ψ are easily checked. $\square$

We now justify the approximation steps in the heuristic argument, using the same notation. First, if $\varepsilon > 0$ is fixed, then with probability exponentially (in n) close to 1,

$$(1 - \varepsilon)\, \Delta F_k(i) n \leq N_i \leq (1 + \varepsilon)\, \Delta F_k(i) n$$

for every $i = 1, \dots, k$. By obvious monotonicity, the longest increasing path in A is then bounded above by the longest increasing path in A' in which all $N_i = (1 + \varepsilon)\, \Delta F_k(i) n$, and therefore we can get an upper bound by increasing α to $\alpha(1 + 2\varepsilon)$ and assuming $N_i = \Delta F_k(i) n$. A lower bound is obtained similarly. As our final characterization of c is continuous with respect to α (Lemma 2.4), we can, and will, assume that $N_i = \Delta F_k(i) n$ from now on.

The above paragraph eliminates randomness of p_j's; we now proceed to replace the coin flips with deterministic quantities. Again, fix an $\varepsilon > 0$ and let $M = \varepsilon m$. For $j_1 \leq j_2$ and $i = 1, \dots, n$, consider the longest increasing paths $\pi_{j_1, j_2, i}$ between $(F_k(i - 1)n, j_1)$ (noninclusive) and $(F_k(i)n, j_2)$ (inclusive). Then, with probability exponentially close to 1, the length of any $\pi_{j_1, j_2, i}$ is at most

$$(1 + \varepsilon) c\left(\frac{\Delta F_k(i) n}{M\lceil (j_2 - j_1)/M \rceil}, \frac{i}{k}\right) M\lceil (j_2 - j_1)/M \rceil$$

$$= (1 + \varepsilon)\zeta\left(\frac{M\lceil (j_2 - j_1)/M \rceil}{\Delta F_k(i) n}\right) \Delta F_k(i) n.$$

(This uses Lemma 2.3 when $j_2 - j_1$ is divisible by M and fills the rest by monotonicity. Note that Lemma 2.3 is therefore only applied finitely many times for fixed ε and k.) The lower bound is obtained by rounding down instead of up. It follows that the length of any $\pi_{j_1,j_2,i}$ is bounded above (resp. below) by

$$\zeta\left(\frac{j_2 - j_1}{\Delta F_k(i)n}\right)\Delta F_k(i)n \tag{2.5}$$

computed on the matrix of size $(m+M)\times n$ [resp. $(m-M)\times n$]. Once again we can use continuity to assume that the length of any $\pi_{j_1,j_2,i}$ is given by (2.5).

It remains to show that the discrete deterministic optimization problem $\max_\psi \mathcal{F}_k(\psi)$ is for large k close to its continuous counterpart $\max_\psi \mathcal{F}(\psi)$. To this end, we first prove that we can indeed restrict the set of function ψ to those in Ψ, that is, those that are convex and have a large enough derivative. Let $\Delta x_1 = \alpha\,\Delta F_k(i)$, $\Delta x_2 = \alpha\,\Delta F_k(i+1)$, $\Delta y_1 = \Delta\psi_k(i)$, $\Delta y_2 = \Delta\psi_k(i+1)$, $\Delta y = \Delta y_1 + \Delta y_2$, $p_1 = i/n$, $p_2 = (i+1)/n$. Then

$$\zeta\left(\frac{\Delta y_1}{\Delta x_1}, p_1\right)\Delta x_1 + \zeta\left(\frac{\Delta y - \Delta y_1}{\Delta x_2}, p_2\right)\Delta x_2 \tag{2.6}$$

is nondecreasing with decreasing Δy_1 as soon as $p_1 \le p_2$ and $\Delta y_1/\Delta x_1 \ge \Delta y_2/\Delta x_2$. This means that the maximum is achieved at a convex ψ. Similarly, the expression (2.6) is nondecreasing with increasing Δy_1 if $p_1 \ge \beta$ and $\Delta y_1/\Delta x_1 < \delta/(1-\delta)$, and therefore the maximum is achieved at a $\psi \in \Psi$.

Next we note that

$$\mathcal{F}_k(\psi) \le \sum_{i=1}^{k} \zeta\big(\psi'(\alpha F(i/k)), i/k\big)\alpha\,\Delta F_k(i),$$

while

$$\mathcal{F}(\psi) \ge \sum_{i=1}^{k} \zeta\big(\psi'(\alpha F(i/k)), i/k\big)\alpha\,\Delta F_{k+1}(i).$$

Therefore, $\max_\psi \mathcal{F}_k(\psi) \le \max_\psi \mathcal{F}(\psi) + \mathcal{O}(1/k)$. As a lower bound is obtained similarly, this concludes the proof for nice distribution functions F.

To prove the general case, we again use Lemmas 2.1 and 2.4. For an arbitrary distribution function, choose nice $F_N^{\pm}$ so that $F_N^- \le F$ and $F \cdot \mathbb{1}_{(1/N,1]} \le F_N^+$ and $F_N^{\pm} \to F$ and $b(F_N^{\pm}) \to b(F)$. Then $c(\alpha, F_N^{\pm}) \to c(\alpha, F)$. By Lemma 2.1, it immediately follows that $\limsup H/m \le c(\alpha, F)$ a.s.

The lower bound, however, does not immediately follow as F is not below F_N^+. The remedy for this is to assume that $F(1/N) < 1/2$, replace α with $\alpha' < \alpha$ and observe that the distribution F will induce, with probability exponentially close to 1, at least $(\alpha - \alpha')m/4$ probabilities $p_j \ge 1/N$. Therefore the length of the longest increasing path in an $m \times \alpha m$ matrix using F is eventually above the length of the longest increasing path in an $m \times \alpha' m$ matrix using F_N^+. By Lemma 2.4, $\liminf H/m \ge c(\alpha, F)$ a.s. $\square$

PROOF OF THEOREM 1 WHEN $\alpha \le \alpha_c'$. Applying the same strategy as before we construct sequences $\{F_N^\pm\}$ of distribution functions which satisfy $F_N^- \le F \le F_N^+$ and for which Theorem 1 already holds, and such that $c(F_N^-)$ and $c(F_N^+)$ approach the same limit as $N \to \infty$. Lemma 2.1 will then complete the proof. (We suppress α from the notation, since it is the same throughout this proof.)

Take a sequence $\eta_N \searrow 0$ such that $b - \eta_N$ are points of continuity of F. Let $F_N^\pm$ agree with F outside $[b - \eta_N, b)$, while on $[b - \eta_N, b)$ the two functions are constant: $F_N^- \equiv F(b - \eta_N)$ and $F_N^+ \equiv 1$. Let $\varepsilon_N = 1 - F(b - \eta_N)$; note that $\varepsilon_N \to 0$ and $dF_N^- = \mathbb{1}_{(0,\, b-\eta_N)}\, dF + \varepsilon_N\, \delta_b$ and $dF_N^+ = \mathbb{1}_{(0,\, b-\eta_N)}\, dF + \varepsilon_N\, \delta_{b-\eta_N}$. Clearly the already proved part of Theorem 1 applies to both F_N^+ and F_N^-.

We proceed to show that $a(F_N^-) \to b$. If this does not hold, the fact that $a(F_N^-) > b(F_N^-) = b$ implies that there exists an $\eta > 0$ so that $a(F_N^-) \ge b + \eta$ along a subsequence. Then $\delta = \langle p(1-p)[(b-p)^{-2} - (b+\eta-p)^{-2}]\rangle > 0$ and

$$\begin{aligned}
1 &= \alpha \int_0^{b-\eta_N} \frac{x(1-x)}{(a(F_N^-)-x)^2}\, dF + \alpha\varepsilon_N \frac{b(1-b)}{(a(F_N^-)-b)^2} \\
&\le \alpha \int_0^{b} \frac{x(1-x)}{(b+\eta-x)^2}\, dF + \alpha\varepsilon_N \frac{b(1-b)}{\eta^2} \qquad (2.7)\\
&\le -\delta + \frac{\alpha}{\alpha_c'} + \alpha\varepsilon_N \frac{b(1-b)}{\eta^2},
\end{aligned}$$

along the same subsequence. As $N \to \infty$, this yields a contradiction with $\alpha \le \alpha_c'$.

Now

$$\begin{aligned}
c(F_N^-) &= a(F_N^-) + \alpha\big(1 - a(F_N^-)\big) \\
&\qquad \times \left(\int_0^{b-\eta_N} \frac{x}{a(F_N^-) - x}\, dF + \varepsilon_N \frac{b}{a(F_N^-) - b} \right). \qquad (2.8)
\end{aligned}$$

By (2.7),

$$\varepsilon_N \frac{b}{a(F_N^-) - b} \le \frac{a(F_N^-) - b}{\alpha(1-b)} \to 0.$$

To show that

$$\big\langle \mathbb{1}_{\{p \le b-\eta_N\}}\, p/\big(a(F_N^-) - p\big)\big\rangle \to \langle p/(b-p)\rangle \qquad (2.9)$$

we note that the integrand on the left-hand side of (2.9) is uniformly integrable [as it is bounded by $p/(b-p)$, which is square-integrable] and converges to the integrand on the right-hand side a.s. By (2.8) and (2.9),

$$c(F_N^-) \to b + \alpha(1-b)\langle p/(1-p)\rangle.$$

The argument for $c(F_N^+)$ is very similar and hence omitted. $\square$

PROOF OF THEOREM 1 WHEN $\alpha \geq \alpha_c$. If $\alpha \nearrow \alpha_c$, then $a(\alpha, F) \nearrow 1$ and hence $c(\alpha, F) \nearrow 1$. $\square$

We note that the above proof of Theorem 1 actually shows exponential convergence to c, that is, (2.1) in Lemma 2.3 holds in a random environment as well. Also, once probabilities are ordered using Lemma 2.1, one could investigate convergence, in the sense of [7] and [24], of the longest increasing path in A to the maximizer of $\mathcal{F}(\psi)$. This is easy to prove if F is nice (cf. Lemma 2.5), but it actually holds whenever the maximizer is unique.

We conclude this section by showing that the deterministic case indeed has no fluctuations.

PROPOSITION 2.6. *Assume that $b < 1$ and $\alpha > \alpha_c$. Then $P(H = m)$ converges to* 1 *exponentially fast* [*and therefore $P(H = m$ eventually$) = 1$*].

PROOF. We begin by modifying the construction from Section 3.3.1 of [9]. Recall that a random $m \times n$ matrix is the lower left corner of an infinite random matrix. For an $(i, j) \in \mathbf{N} \times \mathbf{N}$, let $\eta_{(i,j)} = \inf\{k \geq 1 : \varepsilon_{(i+k,j)} = 0\}$ be the relative position of the first 0 above (i, j) and let $\xi_{(i,j)} = \inf\{k \geq 1 : \varepsilon_{(i,j+k)} = 1\}$ be the relative position of the first 1 to the right of (i, j).

Now define i.i.d. two-dimensional random vectors $X_1 = (\xi_1, \eta_1)$, $X_2 = (\xi_2, \eta_2), \dots$ as follows:

$$\begin{aligned}
\xi_1 &= \xi_{(0,1)}, & \eta_1 &= \eta_{(\xi_1,1)},\\
\xi_2 &= \xi_{(\xi_1,1+\eta_1)}, & \eta_2 &= \eta_{(\xi_1+\xi_2,1+\eta_1)},\\
\xi_3 &= \xi_{(\xi_1+\xi_2,1+\eta_1+\eta_2)}, & \eta_2 &= \eta_{(\xi_1+\xi_2+\xi_3,1+\eta_1+\eta_2)},\\
&\cdots.
\end{aligned}$$

Let $S_k = (0, 1) + X_1 + \cdots + X_k$ be the corresponding random walk, and let T_m (resp. T'_n) be the first time S_k is in $\{(x, y) : x > n\}$ [resp. $\{(x, y) : y > m\}$]. If $T'_m < T_n$, then there is an increasing path of 1's inside the $m \times n$ rectangle which goes through its "roof" without skipping a row; thus

$$\{H < m\} \subset \{T_n \leq T'_m\}.$$

Therefore, we need to show that $P(T_n \leq T'_m)$ goes to 0 exponentially fast. To this end, note that, for any $\varepsilon > 0$,

$$\begin{aligned}
P(T_n \leq T'_m) &\leq P(T_n \wedge T'_m \leq \varepsilon m)\\
&\quad + \sum_{k=\varepsilon m}^{\infty} P\big(\xi_1 + \cdots + \xi_k \geq \alpha(\eta_1 + \cdots + \eta_k)\big).
\end{aligned} \tag{2.10}$$

If we show that ξ_1 and η_1 have exponential tails, and that $E(\xi_1) - \alpha E(\eta_1) < 0$, then we can choose a small enough $\varepsilon > 0$ so that the upper bound in (2.10) decays exponentially. First, $P(\xi_1 \geq k) = \langle 1-p\rangle^{k-1}$ and so $E(\xi_1) = 1/\langle p\rangle$. Moreover, the conditional distribution of p given that a single coin flip gives 1 is

$$dF_1(x) = \frac{1}{\langle p\rangle} x\, dF(x);$$

therefore

$$P(\eta_1 \geq k) = \int_0^1 x^{k-1}\, dF_1(x) = \frac{\langle p^k\rangle}{\langle p\rangle},$$

and so $E(\eta_1) = \langle p/(1-p)\rangle/\langle p\rangle$. $\square$

3. The saddle-point method and fluctuations. Throughout this section, we assume that $b < 1$ and that $\alpha = n/m$ is fixed (but see Remark 3 at the end). In addition, our standing assumption is that

$$\alpha_c' < \alpha < \alpha_c.$$

We investigate the limiting behavior of $P(H \leq h)$ without using results proved in Section 2. An asymptotic analysis of this quantity when $\alpha < \alpha_c'$ is carried out in [11].

We begin with deterministic inhomogeneous ODB, in which the jth column is assigned a fixed *deterministic* probability p_j. At first, our derivation will use a fixed n and no particular properties of the eventual random choice of the environment. For notational convenience, we therefore drop the subscript n, which practically every quantity would otherwise have. See the discussion preceding the key formula (3.6), where the random environment is reintroduced.

As explained in [10, Section 2.2], we have

$$P(H \leq h) = \prod (1-p_j)^m\, D_h(\varphi),$$

where D_h is the $h \times h$ Toeplitz determinant with symbol

$$(1-z^{-1})^{-m} \prod_{j=1}^{n} (1 + r_j z)$$

and $r_j = p_j/(1-p_j)$. Applying an identity of Borodin and Okounkov [6] (see also [4]) this becomes

$$P(H \leq h) = \det(I - K_h),$$

where K_h is the infinite matrix acting on $\ell^2(\mathbf{Z}^+)$ with j,k entry

$$K_h(j,k) = \sum_{\ell=0}^{\infty} (\varphi_-/\varphi_+)_{h+j+\ell+1} (\varphi_+/\varphi_-)_{-h-k-\ell-1}.$$

The subscripts here denote Fourier coefficients and the functions $\varphi_\pm$ are the Wiener–Hopf factors of φ, so

$$\varphi_+(z) = \prod_{j=1}^{n}(1 + r_j z), \qquad \varphi_-(z) = (1 - z^{-1})^{-m}.$$

The matrix K_h is the product of two matrices, with j, k entries given by

$$\left(\frac{\varphi_+}{\varphi_-}\right)_{-h-j-k-1} = \frac{1}{2\pi i}\int \prod(1 + r_j z)\,(z-1)^m\, z^{-m+h+j+k}\,dz$$

and

$$\left(\frac{\varphi_-}{\varphi_+}\right)_{h+j+k+1} = \frac{1}{2\pi i}\int \prod(1 + r_j z)^{-1}(z-1)^{-m}\, z^{m-h-j-k-2}\,dz.$$

The contours for both integrals go around the origin once counterclockwise; in the second integral 1 is on the inside and all the $-r_j^{-1}$ are on the outside.

Eventually we let $m, n \to \infty$ and take $h = cm + sm^{1/3}$, where c, as yet to be determined, gives the transition between the limiting probability being 0 and the limiting probability being 1. In [10] we considered the case where all the p_j were the same. We found that with c chosen as in Lemma 2.3 we could do a steepest descent analysis. The conclusion was that the product of the two matrices scaled, by means of the scaling $j \to m^{1/3}x,\ k \to m^{1/3}y$, to the square of the integral operator on $(0, \infty)$ with kernel $\mathrm{Ai}(gs + x + y)$, where g is another explicitly determined constant. This gave the limiting result

$$\lim_{n\to\infty} P(H \le cm + sm^{1/3}) = F_2(gs),$$

where $F_2(s)$ is the Fredholm determinant of the Airy kernel on (s, ∞). We can do very much the same here. If $h = cm + sm^{1/3}$ and we set

$$\psi(z) = \prod(1 + r_j z)\,(z-1)^m\, z^{-(1-c)m},$$

then

$$\left(\frac{\varphi_+}{\varphi_-}\right)_{-h-j-k-1} = \frac{1}{2\pi i}\int \psi(z)\, z^{sm^{1/3}+j+k}\,dz, \tag{3.1}$$

$$\left(\frac{\varphi_-}{\varphi_+}\right)_{h+j+k+1} = \frac{1}{2\pi i}\int \psi(z)^{-1}\, z^{-sm^{1/3}-j-k-2}\,dz. \tag{3.2}$$

To do an eventual steepest descent we define

$$\sigma(z) = \frac{1}{m}\log\psi(z) = \frac{\alpha}{n}\sum_{j=1}^{n}\log(1 + r_j z) + \log(z-1) + (c-1)\log z,$$

and look for zeros of

$$\sigma'(z) = \frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j}{1+r_j z} + \frac{1}{z-1} + \frac{c-1}{z}. \tag{3.3}$$

The number of zeros equals 1 plus the number of distinct r_j. There is a zero between two consecutive $1/r_j$ and, in general, two other zeros which are either unequal reals or a pair of complex conjugates. In the exceptional case there is a single real zero of multiplicity 2. We choose c so that we are in this exceptional case. If the double zero is at $z = u$, then u and c must satisfy the pair of equations

$$\frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j}{1+r_j u} + \frac{1}{u-1} + \frac{c-1}{u} = 0,$$

$$\frac{\alpha}{n}\sum_{j=1}^{n}\left(\frac{r_j}{1+r_j u}\right)^2 + \frac{1}{(u-1)^2} + \frac{c-1}{u^2} = 0.$$

If we multiply the second equation by u and subtract, we get

$$\frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j}{(1+r_j u)^2} = \frac{1}{(u-1)^2}. \tag{3.4}$$

The first equation gives

$$c = \frac{1}{1-u} - \frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j u}{1+r_j u}. \tag{3.5}$$

Conversely, if the second pair of equations is satisfied, then so is the first.

LEMMA 3.1. *Assume that* $\alpha n^{-1}\sum r_j < 1$ *and set* $\bar{u} = \max\{-1/r_j\}$. *Then* (3.4) *has a unique solution* $u \in (\bar{u}, 0)$ *and if* c *is then defined by* (3.5), *we have* $c \in (0, 1)$.

PROOF. The left-hand side of (3.4) decreases from ∞ to $\alpha n^{-1}\sum r_j$ as u runs over the interval $(\bar{u}, 0)$ whereas the right-hand side increases and has the value 1 at $u = 0$. Our assumption implies the first statement of the lemma. As for the second, $c > 0$ since $u < 0$ and each $1 + r_j u > 0$. Moreover, Schwarz's inequality, our assumption and (3.4) give

$$\frac{\alpha}{n}\sum\frac{r_j}{1+r_j u} \le \left\{\frac{\alpha}{n}\sum r_j\right\}^{1/2}\left\{\frac{\alpha}{n}\sum\frac{r_j}{(1+r_j u)^2}\right\}^{1/2} < \frac{1}{1-u}.$$

Hence

$$c < \frac{1}{1-u} - \frac{u}{1-u} = 1. \qquad \square$$

To derive the asymptotics using steepest descent we have to compute $\sigma'''(u)$ and understand the steepest descent curves. For the first we multiply (3.3) by z, differentiate twice and use the fact that $\sigma'(u) = \sigma''(u) = 0$ to obtain

$$u\sigma'''(u) = -\frac{2\alpha}{n} \sum_{j=1}^{n} \frac{r_j^2}{(1 + r_j u)^3} + \frac{2}{(u-1)^3}.$$

Note that $\sigma'''(u) > 0$ since $u < 0$.

There are three curves emanating from $z = u$ on each of which $\Im\sigma$ is constant. One is $\Im z = 0$, which is of no interest. The other two come into u at angles $\pm\pi/3$ and $\pm 2\pi/3$. Call the former C^+ and the latter C^-. Approximate shapes of these curves are illustrated in Figure 4. For the integral involving $\psi(z)$ we want $|\psi(z)|$ to have a maximum at the point u on the curve and for the integral involving $\psi(z)^{-1}$ we want $|\psi(z)|$ to have a minimum at u. Since $\sigma'''(u) > 0$ the curve for $\psi(z)$ must be C^+ and the curve for $\psi(z)^{-1}$ must be C^-.

As for the global natures of the curves, $C^\pm$ can only end at a zero of $\psi(z)^{\pm 1}$, at a zero of $\sigma'(z)$ or at infinity. The two curves are simple and cannot intersect since $|\psi(z)|$ is decreasing on C^+ as we move away from $z = u$ while $|\psi(z)|$ is increasing on C^-. It follows that C^+ closes at $z = 1$, while the two branches of C^- go to infinity. From the fact that

$$\Im\sigma(z) = \frac{\alpha}{n} \sum_{j=1}^{n} \arg(1 + r_j z) + \arg(z - 1) + (c - 1)\arg z$$

is constant on C^- we can see that the two branches go to infinity in the directions $\arg z = \pm c\pi/(c + \alpha(1 - \nu))$, where ν is the fraction of r_j equal to zero (which is the same as the fraction of the p_j equal to zero). Observe that in the integral

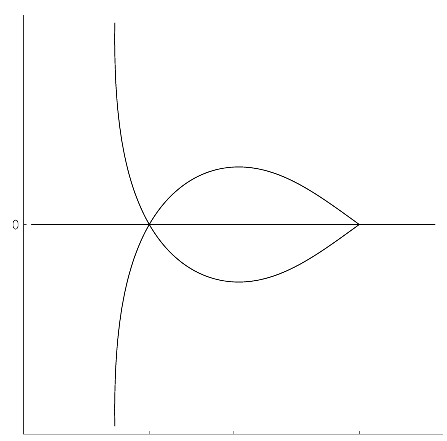

FIG. 4. *The steepest descent curves $C^\pm$ as described in the text.*

in (3.1) the path can be deformed into C^+ and in the integral in (3.2) the path can be deformed into C^-. Both contours will be described downward near u.

To see formally what steepest descent gives, we replace our matrices $M(j,k)$ depending on the parameter m and acting on $\ell^2(\mathbf{Z}^+)$ by kernels $m^{1/3}M(m^{1/3}x, m^{1/3}y)$ acting on $L^2(0, \infty)$. Thus (3.1) becomes the operator with kernel

$$\frac{1}{2\pi i}m^{1/3}\int e^{m\sigma(z)}\, z^{m^{1/3}(s+x+y)}\, dz.$$

If steepest descent worked, the main contribution would come from the immediate neighborhood of $z = u$. We would set $z = u + \zeta$, make the replacements

$$\sigma(z) \to \sigma(u) + \tfrac{1}{6}\sigma'''(u)\zeta^3 \quad \text{and} \quad z \to u\, e^{\zeta/u}$$

in the integral and integrate (downward) on the rays $\arg\zeta = \pm\pi/3$. The above integral becomes

$$e^{m\sigma(u)}\, u^{m^{1/3}(s+x+y)}\frac{1}{2\pi i}m^{1/3}\int e^{(m/6)\sigma'''(u)\zeta^3+m^{1/3}(s+x+y)\zeta/u}\, d\zeta,$$

and we can then replace the rays by the imaginary axis (downward). The variable change $\zeta \to -i\zeta/m^{1/3}$ replaces this by (recall that the Airy function is defined by $\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta^3/3+ix\zeta}\, d\zeta$)

$$\begin{aligned}
&-e^{m\sigma(u)}\, u^{m^{1/3}x}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(i/6)\sigma'''(u)\zeta^3-i(s+x+y)\zeta/u}\, d\zeta\\
&\qquad = -e^{m\sigma(u)}\, u^{m^{1/3}(s+x+y)}\, |u| g\, \mathrm{Ai}(g(s+x+y)),
\end{aligned}$$

where we have set

$$g = |u|^{-1}\{\tfrac{1}{2}\sigma'''(u)\}^{-1/3}.$$

Thus, if we multiply the matrix entries on the left-hand side of (3.1) by

$$-e^{-m\sigma(u)}\, u^{-m^{1/3}s-j-k},$$

then the result has as scaling limit the operator on $L^2(0, \infty)$ with kernel

$$|u| g\, \mathrm{Ai}(g(s+x+y)).$$

Similarly if we multiply the matrix entries on the left-hand side of (3.2) by

$$-e^{m\sigma(u)}\, u^{m^{1/3}s+j+k},$$

then the result has as scaling limit the operator on $L^2(0, \infty)$ with kernel

$$|u|^{-1} g\, \mathrm{Ai}(g(s+x+y)).$$

It follows that the product of the two matrices has in the limit the same Fredholm determinant as the operator with kernel

$$g^2 \int_0^\infty \mathrm{Ai}(g(s+x+z))\,\mathrm{Ai}(g(s+z+y))\,dz$$
$$= g\int_0^\infty \mathrm{Ai}(g(s+x)+z)\,\mathrm{Ai}(g(s+y))\,dz,$$

which in turn has the same Fredholm determinant as the kernel

$$\int_0^\infty \mathrm{Ai}(gs+x+z)\,\mathrm{Ai}(gs+z+y)\,dz.$$

This Fredholm determinant equals $F_2(gs)$.

Assuming the argument we sketched above goes through we will have shown that, in some sense,

$$\lim_{n\to\infty} P(H \le cm + sm^{1/3}) = F_2(gs),$$

where c and g are as above and are determined once we know the p_j and α.

We begin the rigorous justification by introducing some notation. Recall that we consider a random environment in which the probabilities p_j are chosen independently with distribution function F. We explained the notation H_n after Lemma 2.2; in addition, we give the subscript n to the quantities $\sigma_n(z)$, u_n, g_n and curves $C_n^\pm$ to emphasize that they are functions of $p_1, \dots, p_n$. Therefore

$$P(H \le h) = \langle P(H_n \le h)\rangle,$$

where $\langle\cdot\rangle$ is the expected value with respect to $p_1, \dots, p_n$.

Our object is to show that with probability 1, for each fixed s,

$$P(H_n \le c_n m + sm^{1/3}) = F_2(g_n s) + o(1) \tag{3.6}$$

as $n \to \infty$. We will demonstrate these asymptotics by pointing out the necessary modifications to the argument in [10].

All the $-1/r_j$ in our previous discussion are contained in the interval $(-\infty, \xi]$, where $\xi = 1 - 1/b$. (Recall that b is the maximum of the support of dF.) Let F_n be the empirical distribution function given by

$$dF_n = n^{-1}\sum \delta_{p_j}$$

and let $\langle\cdot\rangle_{F_n}$ denote the integration with respect to dF_n. Recall the Glivenko–Cantelli theorem, which says that, with probability 1, F_n converges uniformly to F as $n \to \infty$.

We first show that, under our standing assumptions, the quantities c_n and u_n of Lemma 3.1 converge almost surely as $n \to \infty$ to the corresponding quantities associated with the distribution function F. Recall that we set $r = p/(1-p)$, $p = r/(1+r)$. We remark that c_0 in the following lemma is the same as c in Theorem 1, and $u_0 = (a-1)/a$. The notation has changed to conform to (3.4) and (3.5), which are in turn chosen to connect with the saddle-point approach in [10].

LEMMA 3.2. *The equation*

$$\alpha\left\langle\frac{r}{(1+ru_0)^2}\right\rangle=\frac{1}{(u_0-1)^2}$$

has a unique solution $u_0 \in (\xi, 0)$ *and if* c_0 *is then defined by*

$$c_0=\frac{1}{1-u_0}-\alpha\left\langle\frac{ru_0}{1+ru_0}\right\rangle,$$

we have $c_0 \in (0, 1)$.

PROOF. The argument goes almost exactly as for Lemma 3.1. The assumption $\alpha < \alpha_c$ is equivalent to $\alpha\langle r/(1+ru)^2\rangle > 1/(u-1)^2$ when $u = \xi$, while $\alpha > \alpha_c'$ yields the opposite inequality when $u = 0$. $\square$

Note that one obtains u_n as u_0, except that the expectation $\langle\cdot\rangle$ is replaced by the expectation $\langle\cdot\rangle_{F_n}$.

LEMMA 3.3. *Almost surely*, $u_n \to u_0$ *and* $c_n \to c_0$ *as* $n \to \infty$.

PROOF. Integration by parts gives

$$\left\langle\frac{r}{(1+rz)^2}\right\rangle=\int_0^{b/(1-b)}(1-F(p))\frac{d}{dr}\frac{r}{(1+rz)^2}\,dr.$$

The derivative in the integrand is uniformly bounded for z in any compact subset of the complement of $(-\infty, \xi]$. Hence the expected value is continuous in F and differentiable for $z \notin (-\infty, \xi]$. Moreover,

$$\frac{\partial}{\partial z}\left(\alpha\left\langle\frac{r}{(1+rz)^2}\right\rangle-\frac{1}{(z-1)^2}\right)$$

is negative, hence nonzero, at $z = u_0$. The statement concerning u_n therefore follows from the fact that $F_n \to F$ uniformly and the implicit function theorem. The assertion for c_n then follows by a similar integration by parts. $\square$

LEMMA 3.4. *There exists a* (*deterministic*) *wedge* W *with vertex* $v > \xi$, *bisected by the real axis to the left of* v, *such that*, *almost surely*, *the curves* $C_n^{\pm}$ *lie outside* W *for sufficiently large* n.

PROOF. First we show that, if ε is small enough, C_n^- is disjoint from the disc

$$D(\xi,\varepsilon)=\{z:|z-\xi|\le\varepsilon\}.$$

From the facts that $\sigma_n'(u_n)=\sigma_n''(u_n)=0$, $\sigma_n'''(u_n)>0$ and $\sigma_n'(z)\neq 0$ for $z \in (\xi, u_n)$, it follows that σ_n is strictly increasing in the interval (ξ, u_n). Therefore,

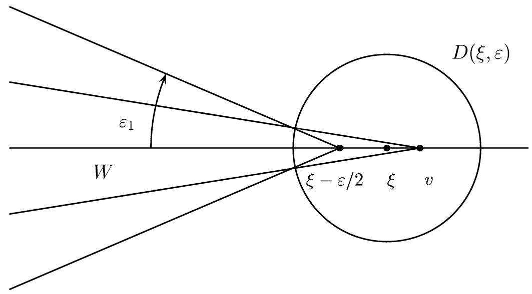

FIG. 5. *Wedge W, angle ε_1 and disk $D(\xi,\varepsilon)$ as described in the proof of Lemma* 3.4.

we can choose small enough $\varepsilon > 0$ and $\delta > 0$ so that $\sigma_n(\xi+\varepsilon) < \sigma_n(u_n) - 2\delta$ for all sufficiently large n. In addition, if ε is small enough,

$$\log|z-1| + (c_n-1)\log|z| < \log|\xi+\varepsilon-1| + (c_n-1)\log|\xi+\varepsilon| + \delta$$

for all $z \in D(\xi,\varepsilon)$. Now each $|1+r_jz|$, and so its logarithm, achieves its maximum on $D(\xi,\varepsilon)$ at the point $z = \xi+\varepsilon$. By combining the last three observations, we see that everywhere on $D(\xi,\varepsilon)$ we have

$$\Re\,\sigma_n(z) < \sigma_n(\xi+\varepsilon) + \delta < \sigma_n(u_n) - \delta.$$

Since $\Re\,\sigma$ achieves its minimum on C_n^- at $z = u_n$, the curve must be disjoint from $D(\xi,\varepsilon)$.

For a small $\varepsilon_1 > 0$ (possibly much smaller than ε), denote by W' the wedge with vertex $\xi - \varepsilon/2$ bounded by the real axis to the left of $\xi-\varepsilon/2$ and the ray $\arg(z-\xi+\varepsilon/2) = \pi - \varepsilon_1$. Our next step is to show that C_n^- is disjoint from W' if ε_1 is small enough. As $\Im\,\sigma_n(z)$ is constant on the portion of C_n^- in the upper half-plane,

$$\frac{\alpha}{n}\sum_{j=1}^{n}\arg(1+r_jz) + \arg(z-1) + (c_n-1)\arg z = c_n\pi,$$

where all arguments lie in $[0,\pi]$. For $z \in W'$,

$$\arg(z-1) + (c_n-1)\arg z \ge c_n \arg z \ge c_n(\pi-\varepsilon_1).$$

Since b is in the support of dF, the strong law implies that, almost surely, at least a positive fraction η of the $-1/r_j$ lie in the interval $[\xi-\varepsilon/2,\xi]$ for sufficiently large n. The contribution of these terms (and nonnegativity of the others) in the following sum provides a lower bound valid for $z \in W'$:

$$\frac{\alpha}{n}\sum_{j=1}^{n}\arg(1+r_jz) \ge \alpha\eta(\pi-\varepsilon_1).$$

Hence, for $z \in W'$,

$$\Im\,\sigma_n(z) \ge \alpha\eta(\pi-\varepsilon_1) + c_n(\pi-\varepsilon_1) = (\alpha\eta+c_n)(\pi-\varepsilon_1) > c_n\pi,$$

if ε_1 is chosen to be small enough. Therefore, C_n^- is disjoint from the wedge W' for sufficiently small ε_1. By symmetry, C_n^- is disjoint from the reflection of W' over the imaginary axis. We have shown that the curve is also disjoint from $D(\xi, \varepsilon)$ and the union of the disc and the two wedges contains a wedge of the form described in the statement of the lemma.

This establishes the statement of the lemma concerning C_n^-. Since C_n^+ is to the "right" of C_n^- (it begins to the right and they cannot cross), the statement for C_n^+ follows automatically. $\square$

In the following lemma σ_0 denotes the function σ associated with the distribution F,

$$\sigma_0(z) = \alpha\langle\log(1+rz)\rangle + \log(z-1) + (c_0 - 1)\log z$$

and

$$g_0 = |u_0|^{-1}\left\{\tfrac{1}{2}\sigma_0'''(u_0)\right\}^{-1/3}.$$

LEMMA 3.5. *Almost surely*, $z\sigma_n'(z) \to z\sigma_0'(z)$ *uniformly outside the wedge W of Lemma* 3.4.

PROOF. We have

$$\begin{aligned} z\sigma_n'(z) - z\sigma_0'(z) &= \alpha \int \frac{rz}{1+rz}\, d(F_n(p) - F(p)) + c_n - c_0 \\ &= \alpha \int_0^{b/(1-b)} (F_n(p) - F(p))\, \frac{z}{(1+rz)^2}\, dr + c_n - c_0. \end{aligned}$$

The last term goes to 0 by Lemma 3.3. The last factor in the integrand is uniformly bounded for $r \in (0, b/(1-b))$, $z \notin W$ and z bounded. Thus $z\sigma_n'(z) \to z\sigma_0'(z)$ uniformly on bounded subsets of the complement of W. If z is sufficiently large and outside W, then it is outside some wedge with vertex 0 bisected by the negative real axis, and on the complement of any such wedge

$$\int_0^{b/(1-b)} \frac{|z|}{|1+rz|^2}\, dr$$

is uniformly bounded. Thus $z\sigma_n'(z) \to z\sigma_0'(z)$ uniformly throughout the complement of W. $\square$

The preceding lemmas show that the curves $C_n^\pm$ are uniformly smooth, as we now argue. The function $\sigma_0'(z)$ can have no other zero in the complement of $(-\infty, \xi)$ than at $z = u_0$. This follows from uniform convergence and the fact that the corresponding statement holds for the $\sigma_n(z)$. Thus the functions $\sigma_n'(z)$ are uniformly bounded away from zero on compact subsets not containing u_0. As we move outward (i.e., away from u_n) along $C_n^\pm$, $\Im\sigma$ is constant and $\Re\sigma$ is increasing on C_n^- and decreasing on C_n^+. It follows that if s measures arc length on the curves, then, for $z \in C_n^\pm$,

$$\frac{dz}{ds} = \mp \frac{|\sigma_n'(z)|}{\sigma_n'(z)}. \tag{3.7}$$

This shows that the $C_n^\pm$ are uniformly smooth on compact sets (to be more precise, the portions in the upper and lower half-planes are). Moreover, they are uniformly close on compact sets to the corresponding curves $C_0^\pm$ for the distribution function F. In particular, the length of C_n^+ is $O(1)$.

To see what happens for large z on C_n^- observe that

$$\lim_{z\to\infty} z\sigma_n'(z) = \alpha + c_n > 0$$

uniformly in n. This and (3.7) show that $|z|$ is increasing as we move far enough out along C_n^-. If Γ is an arc of C_n^- going from a to b, then

$$\int_\Gamma |\sigma_n'(z)|\, ds = \int_\Gamma \sigma_n'(z)\, dz = \sigma_n(b) - \sigma_n(a).$$

Hence the length of Γ is at most $|b-a|$ times

$$\frac{\max_{z\in[a,b]} |\sigma_n'(z)|}{\min_{z\in\Gamma} |\sigma_n'(z)|},$$

where $[a, b]$ is the line segment joining a and b, as long as this segment does not meet $(-\infty, \xi)$. It follows from the above, for example, that the L^1 norm of the function $(1+|z|^2)^{-1}$ on C_n^- is $O(1)$.

In [10] we needed asymptotics with error bounds for all $j, k \le h$ and this required a more careful analysis of the integrals in (3.1) and (3.2) than we indicated; instead of the steepest descent curves passing through the same point they pass through different, but nearby, points. With what we now know we can show that these curves are uniformly smooth with uniformly regular behavior near infinity, and this is what is needed to see that in our case the asymptotics hold uniformly in n.

LEMMA 3.6. *Almost surely*, $g_n \to g_0 \neq 0$ *and* (3.6) *holds*.

PROOF. The first statement follows from Lemmas 3.3 and 3.5 and the fact that the $\sigma_n'(z)$ have only two zeros outside $(-\infty, \xi]$ counting multiplicity, and therefore $\sigma'''(u_0) \neq 0$.

To establish (3.6), one now has to go through the steepest descent argument in [10], Section 3.1.2, and make some obvious changes, justified by the results of this section. For the analogue of Lemma 3.1 there, for example, we would add the phrase "and all sufficiently large n" to the end of the statement. At the end of the second sentence of the proof we would add the phrase "since $\sigma_n'''(u_n)$ is uniformly bounded away from zero the length of C_n^+ is $O(1)$." After the last sentence we would add "again since the length of C_n^+ is $O(1)$." Analogous changes need to be made throughout the argument and we skip further details. $\square$

PROOF OF THEOREM 3. By Lemma 3.6, we can take $G_n = c_n m$. $\square$

PROOF OF THEOREM 2. Note first that

$$\tau^2 = \mathrm{Var}\left(\frac{ru_0}{1+ru_0}\right),$$

where u_0 is as in Lemma 3.2 (and, as we remarked earlier, $c_0 = c$).

The proof rests on the crucial property (3.6) and the fact that $n^{1/2}(F_n - F)$ converges in distribution to a Brownian bridge B with an appropriate covariance structure; in particular B is a Gaussian random element in $D[0,1]$ ([5], Theorem 14.3). By the Skorohod representation theorem, we can couple F_n and B on some probability space Ω_0 so that

$$n^{1/2}(F_n - F) \to B \tag{3.8}$$

in fact converges for every $\omega \in \Omega_0$ ([5], Theorem 6.7). We now prove that, under this coupling, the solution u_n of (3.4) satisfies

$$u_n = u_0 + n^{-1/2}U + o(n^{-1/2}), \tag{3.9}$$

for every ω and for some Gaussian random variable U.

To establish (3.9), define

$$\theta_n(u) = \alpha\left\langle\frac{r}{(1+ru)^2}\right\rangle_{F_n} - \frac{1}{(u-1)^2},$$
$$\theta_0(u) = \alpha\left\langle\frac{r}{(1+ru)^2}\right\rangle - \frac{1}{(u-1)^2}.$$

By Lemma 3.6 and its proof, there exists a (deterministic) neighborhood $\mathcal{U} \subset \mathbf{C}$ of u_0 in which, with probability 1, u_n (resp. u_0) is for large n the unique solution to $\theta_n(u) = 0$ [resp. $\theta_0(u) = 0$]. Therefore we can choose a fixed contour C in $\mathcal{U}$ such that u_n and u_0 are given by

$$u_n = \frac{1}{2\pi i}\int_C \frac{\theta_n'(u)}{\theta_n(u)}u\,du, \qquad u_0 = \frac{1}{2\pi i}\int_C \frac{\theta_0'(u)}{\theta_0(u)}u\,du.$$

By (3.8), we have, uniformly for $u \in C$,

$$\theta_n(u) = \theta_0(u) + n^{-1/2}\alpha\left\langle\frac{r}{(1+ru)^2}\right\rangle_B + o(n^{-1/2}).$$

Here $\langle\cdot\rangle_B$ is the expectation with respect to dB, but by integration by parts (as in the proof of Lemma 3.3) we can make B appear in the integrand. Therefore

$$\frac{\theta_n'(u)}{\theta_n(u)} = \frac{\theta_0'(u)}{\theta_0(u)} + n^{-1/2}\alpha\frac{d}{du}\frac{\langle r/(1+ru)^2\rangle_B}{\theta_0(u)} + o(n^{-1/2}).$$

If we multiply this identity by $u/2\pi i$ and integrate over C the left-hand side becomes u_n; the first term on the right-hand side becomes u_0 while the second term becomes $n^{-1/2}U$, where

$$U = -\alpha \frac{\langle r/(1+ru_0)^2\rangle_B}{\theta_0'(u_0)}$$

is a Gaussian random variable. This proves (3.9).

Let

$$\varphi_n(u) = \frac{1}{1-u} - \alpha\left\langle \frac{r}{1+ru}\right\rangle_{F_n},$$

so that $c_n = \varphi_n(u_n)$. We claim that

$$c_n = \varphi_n(u_0) + O(n^{-1}). \tag{3.10}$$

To see this, we use the fact that $\varphi_n'(u_n) = 0$ to write

$$c_n = \varphi_n(u_n) = \varphi_n(u_0) + (u_n - u_0)^2 \int_0^1 t\varphi_n''\big(tu_n + (1-t)u_0\big)\,dt.$$

Thus, (3.10) follows from (3.9) and the uniform boundedness of the $\phi_n''(u)$ near $u = u_0$.

Now, by the central limit theorem,

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{r_j u_0}{1+r_j u_0} - \left\langle \frac{ru_0}{1+ru_0}\right\rangle\right)$$

converges in distribution to a Gaussian random variable X with mean 0 and variance τ^2. Therefore,

$$\sqrt{n}(c_n - c_0) \xrightarrow{d} \alpha X. \tag{3.11}$$

Finally, (3.6) implies that, for any $\delta > 0$,

$$P(-\delta m^{1/2} \le H - c_n m \le \delta m^{1/2}) \to 1. \tag{3.12}$$

[In fact, (3.6) implies that the above statement holds with probability 1 before the expectation with respect to $p_1, \dots, p_n$ is taken, that is, if H is replaced by H_n.] It follows from (3.11) and (3.12) that

$$(H - c_0 m)/\sqrt{m} \xrightarrow{d} \sqrt{\alpha}\, X,$$

which concludes the proof. □

REMARK 1. We did not need the full force of (3.6) for the above proof to go through. Instead, a much weaker property (3.12) suffices.

REMARK 2. As mentioned in the Introduction, independence of p_n is not necessary for the results of this section to hold. Indeed, one only needs the Glivenko–Cantelli theorem for convergence in probability of H/m to the time constant; hence ergodicity of $p_1, p_2, \dots$ is enough. Furthermore, a strong enough mixing property of this sequence is sufficient for a normal fluctuation result. This follows from Billingsley's results in Section 22 of the first (1968) edition of [5].

REMARK 3. We assumed that $\alpha = n/m$ is fixed, but the proof of Theorem 2 remains valid with $n = \alpha m + o(\sqrt{m})$.

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REFERENCES

[1] ALEXANDER, K. S. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. *Ann. Probab.* **25** 30–55.

[2] BAIK, J., DEIFT, P. and JOHANSSON, K. (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** 1119–1178.

[3] BAIK, J. and RAINS, E. M. (2000). Limiting distributions for a polynuclear growth model with external sources. *J. Statist. Phys.* **100** 523–541.

[4] BASOR, E. L. and WIDOM, H. (2000). On a Toeplitz determinant identity of Borodin and Okounkov. *Integral Equations Operator Theory* **37** 397–401.

[5] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*. Wiley, New York.

[6] BORODIN, A. and OKOUNKOV, A. (2000). A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory* **37** 386–396.

[7] DEUSCHEL, J.-D. and ZEITOUNI, O. (1995). Limiting curves for i.i.d. records. *Ann. Probab.* **23** 852–878.

[8] DURRETT, R. (1988). *Lecture Notes on Particle Systems and Percolation*. Brooks/Cole, Monterey, CA.

[9] GRAVNER, J. (1999). Recurrent ring dynamics in two-dimensional excitable cellular automata. *J. Appl. Probab.* **36** 492–511.

[10] GRAVNER, J., TRACY, C. A. and WIDOM, H. (2001). Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Statist. Phys.* **102** 1085–1132.

[11] GRAVNER, J., TRACY, C. A. and WIDOM, H. (2001). Fluctuation in the composite regime of a disordered growth model. *Comm. Math. Phys.* To appear.

[12] GRIFFEATH, D. (2000). *Primordial Soup Kitchen.* Available at www.psoup.math.wisc.edu.

[13] HAMMERSLEY, J. M. and WELSH, D. J. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Bernoulli, Bayes, Laplace Anniversary Volume* (J. Neyman and L. Le Cam, eds.) 61–110. Springer, New York.

[14] ITS, A. R., TRACY, C. A. and WIDOM, H. (2001). Random words, Toeplitz determinants and integrable systems, I. In *Random Matrix Models and Their Applications* (P. Bleher and A. R. Its, eds.) 245–258. Cambridge Univ. Press.

[15] ITS, A. R., TRACY, C. A. and WIDOM, H. (2001). Random words, Toeplitz determinants and integrable systems, II. *Phys. D* **152–153** 1085–1132.

[16] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** 437–476.

[17] JOHANSSON, K. (2001). Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math.* **153** 259–296.

[18] MEAKIN, P. (1998). *Fractals, Scaling and Growth Far from Equilibrium*. Cambridge Univ. Press.

[19] NEWMAN, C. M. and STEIN, D. L. (1999). Equilibrium pure states and nonequilibrium chaos. *J. Statist. Phys.* **94** 709–722.

[20] NEWMAN, C. M. and VOLCHAN, S. B. (1996). Persistent survival of one-dimensional contact processes in random environments. *Ann. Probab.* **24** 411–421.

[21] PRÄHOFER, M. and SPOHN, H. (2000). Universal distribution for growth processes in $1+1$ dimensions and random matrices. *Phys. Rev. Lett.* **84** 4882–4885.

[22] PRÄHOFER, M. and SPOHN, H. (2001). Scale invariance of the PNG droplet and the Airy process. Preprint (ArXiv: math.PR/0105240).

[23] RAINS, E. M. (2000). A mean identity for longest increasing subsequence problems. Preprint (ArXiv: math.CO/0004082).

[24] SEPPÄLÄINEN, T. (1997). Increasing sequences of independent points on the planar lattice. *Ann. Appl. Probab.* **7** 886–898.

[25] SEPPÄLÄINEN, T. (1998). Exact limiting shape for a simplified model of first-passage percolation on the plane. *Ann. Probab.* **26** 1232–1250.

[26] SEPPÄLÄINEN, T. and KRUG, J. (1999). Hydrodynamics and platoon formation for a totally asymmetric exclusion model with particlewise disorder. *J. Statist. Phys.* **95** 525–567.

[27] TALAGRAND, M. (1998). Huge random structures and mean field models for spin glasses. *Doc. Math.* **I** 507–536. (Extra vol.)

[28] TRACY, C. A. and WIDOM, H. (1994). Level spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174.

[29] TRACY, C. A. and WIDOM, H. (2000). Universality of the distribution functions of random matrix theory, II. In *Integrable Systems*: *From Classical to Quantum* (J. Harnad, G. Sabidussi and P. Winternitz, eds.) 251–264. Amer. Math. Soc., Providence, RI.

[30] TRACY, C. A. and WIDOM, H. (2001). On the distributions of the lengths of the longest monotone subsequences in random words. *Probab. Theory Related Fields* **119** 350–380.

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