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GENERAL GAUGE AND CONDITIONAL GAUGE THEOREMS

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General gauge and conditional gauge theorems are established for a large class of (not necessarily symmetric) strong Markov processes, including Brownian motions with singular drifts and symmetric stable processes. Furthermore, new classes of functions are introduced under which the general gauge and conditional gauge theorems hold. These classes are larger than the classical Kato class when the process is Brownian motion in a bounded $C^{1,1}$ domain.

1. Introduction. Given a strong Markov process X and a potential q, the conditional expectation $u(x, y)$ of the Feynman–Kac transform of X by q is called the conditional gauge function. (The precise definition will be given later.) The function u is important in studying the potential theory of the Schrödinger-type operator $\mathcal{L} + q$, as it is the ratio of the Green's function of $\mathcal{L} + q$ and that of $\mathcal{L}$, where $\mathcal{L}$ is the infinitesimal generator of X. The conditional gauge theorem says that under suitable conditions on X and q, either u is identically infinite or u is bounded between two positive numbers. The conditional gauge theorem was first proved for Brownian motions (see [12] for a history). Very recently it was established in [8] for symmetric stable processes. The proofs of the conditional gauge theorem for symmetric stable processes in [8] and [10] are quite different from that for Brownian motion, due to the complication that the sample paths of symmetric stable processes are discontinuous. See also [7].

A few years ago, Professor Kai Lai Chung suggested to one of the authors that the conditional gauge theorem for Brownian motion might be proved via the gauge theorem for the conditional processes. In this paper, we show that it is indeed possible to prove the conditional gauge theorem via the gauge theorem. This new approach not only simplifies the proof but also yields a quite general conditional gauge theorem that is applicable to a large class of strong Markov processes having strong duals, including Brownian motions with singular drifts and symmetric stable processes. Furthermore, we introduce new classes of functions $\mathbf{K}_1(X)$ and $\mathbf{S}_1(X)$ so that the gauge and conditional gauge theorems hold for q in $\mathbf{K}_1(X)$ and in $\mathbf{S}_1(X)$, respectively. The classes $\mathbf{K}_1(X)$ and $\mathbf{S}_1(X)$ are larger than the (classical) Kato class when X is Brownian motion in a bounded $C^{1,1}$ domain.

Now let us lay out the setting of this paper carefully.

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Let E be a Lusin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space), let $\mathcal{B}(E)$ be the Borel σ-algebra on E and let m be a σ-finite measure on $\mathcal{B}(E)$ with $\mathrm{supp}[m] = E$. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \mathbf{P}_x, x \in E)$ be a Borel right process on E having left limits on $(0, \zeta)$ which is transient in the sense of [19]. Here a Borel right process on a Lusin space E is a right-continuous, strong Markov process with no branching points and with a Borel-measurable resolvent. The shift operators θ_t, $t \geq 0$, satisfy $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geq 0$. Adjoined to the state space E is an isolated point $\partial \notin E$; the process X retires to ∂ at its "lifetime" $\zeta := \inf\{t \geq 0 : X_t = \partial\}$. Denote $E \cup \{\partial\}$ by E_∂. Throughout this paper, the process X is assumed to be *m-irreducible* in the sense that if a measurable set A has positive m-measure then $\mathbf{P}_x[T_A < \infty] > 0$ for all $x \in E$, where $T_A = \inf\{t > 0,\ X_t \in A\}$ is the first hitting time of A.

The transition operators P_t, $t \geq 0$, are defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t); t < \zeta].$$

(Here and in the rest of the paper, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂.) We assume that there is a Borel function $G(x, y)$ on $E \times E$ such that

$$\mathbf{E}_x\left[\int_0^\infty f(X_s)\, ds\right] = \int_E G(x, y) f(y) m(dy)$$

for all measurable $f \geq 0$. Note that $G(x, y)$ is called the Green's function of X.

Now we suppose that we have another transient Borel right process $\widehat{X} = (\widehat{\Omega}, \widehat{\mathcal{M}}, \widehat{\mathcal{M}}_t, \widehat{X}_t, \widehat{\mathbf{P}}_x, x \in E)$ on the same state space E which is a strong dual of X with respect to the measure m. That is, the semigroup $\{\widehat{P}_t\}_{t \geq 0}$ of $\widehat{X}$ is the dual in $L^2(E, m)$ to the semigroup $\{P_t\}_{t \geq 0}$ of X:

$$\int_E f(x) P_t g(x) m(dx) = \int_E g(x) \widehat{P}_t f(x) m(dx) \qquad \text{for all } f, g \in L^2(E, m)$$

and the resolvents $\{U_\alpha\}$ and $\{\widehat{U}_\alpha\}$ satisfy the following conditions: for each $\alpha > 0$, a $\mathcal{B}(E) \times \mathcal{B}(E)$-measurable potential density $G_\alpha(x, y)$ can be chosen so that:

(a) $U_\alpha(x, dy) = G_\alpha(x, y) m(dy)$, $\widehat{U}_\alpha(x, dy) = G_\alpha(y, x) m(dy)$;
(b) $x \to G_\alpha(x, y)$ is α-excessive for X, $y \to G_\alpha(x, y)$ is α-excessive for $\widehat{X}$.

When $\alpha = 0$, we will drop the subscript and write G for G_0. Under this strong duality assumption, the dual process $\widehat{X}$ also has left limits on $(0, \widehat{\zeta})$; more precisely, $\widehat{X}_{t-}$ exists in E for all $t \in (0, \widehat{\zeta})$.

For any Borel-measurable excessive function h of X, let $E_h = \{x \in E : 0 < h(x) < \infty\}$ and

$$p^h(t, x, dy) = \frac{h(y) p(t, x, dy)}{h(x)}, \qquad t > 0,\ x, y \in E.$$

Then p^h is a transition probability and determines a Borel right process X^h on E_h (cf. [21]), which is called Doob's h-transformed process of X or the h-conditioned process. We are going to use ζ^h to denote the lifetime of the h-conditioned process. The process X^h has left limits on $(0, \zeta^h)$. For any $x \in E$, we are going to use $\mathbf{P}_x^h$ and $\mathbf{E}_x^h$ to denote, respectively, the probability and expectation for the h-conditioned process starting from x. When $h(\cdot) = G(\cdot, y)$ for some $y \in E$, we will use $\mathbf{P}_x^y$ and $\mathbf{E}_x^y$ to denote, respectively, the probability and expectation for the h-conditioned process starting from x. In this case, the lifetime ζ^h will be denoted as ζ^y.

Throughout this paper, we assume that the Borel function $q\colon E \to [-\infty, \infty]$ is finite m-almost everywhere. For convenience, we set

$$e_q(t) = \exp\left(\int_0^t q(X_s)\,ds\right), \qquad t > 0.$$

We define the gauge function $g\colon E \to [0, \infty]$ and the conditional gauge function $u\colon E \times E \to [0, \infty]$ by

$$g(x) := \mathbf{E}_x[e_q(\zeta)], \qquad u(x, y) := \mathbf{E}_x^y[e_q(\zeta^y)].$$

It is understood that suitable hypotheses must be imposed on X and q to ensure that $e_q(\zeta)$ and $e_q(\zeta^y)$ are well defined almost surely with respect to $\mathbf{P}_x$ and $\mathbf{P}_x^y$, respectively. The gauge theorem takes the following form.

GAUGE THEOREM. *Under suitable hypotheses on the process X and the function q, if g is finite at some point $x \in E$, then g is bounded on E.*

The gauge theorem has been proved for quite general Markov processes in Chung and Rao [11]. See also [28].

The conditional gauge theorem is a result of the following type.

CONDITIONAL GAUGE THEOREM. *Under suitable hypotheses on X and q, if u is finite at some point $(x, y) \in (E \times E) \setminus d$, then u is bounded on $(E \times E) \setminus d$, where $d = \{(x, y) \in E \times E : G(x, y) = 0$ or $\infty\}$.*

As we mentioned earlier, unlike the gauge theorem, the conditional gauge theorem had been proved only for a very limited class of *symmetric* Markov processes, mainly for Brownian motion and symmetric stable processes in bounded Lipschitz domains. In the conditional gauge theorems proved so far, q is assumed to be in the classical Kato class or some smaller class of functions. We remark here that the proof of the conditional gauge theorem is more difficult than that of the gauge theorem.

In this paper, we obtain a general conditional gauge theorem by first establishing a general gauge theorem that is applicable to conditional processes. This new approach not only simplifies the proof but also yields a quite general conditional gauge theorem that is applicable to a large class of strong Markov processes having strong duals, including Brownian motions with singular drifts and symmetric stable processes. Furthermore, we introduce new classes of functions $\mathbf{K}_1(X)$ and $\mathbf{S}_1(X)$ so that the gauge and conditional gauge theorems hold for q in $\mathbf{K}_1(X)$ and in $\mathbf{S}_1(X)$, respectively. We point out here when the conditional gauge theorem proved in this paper is applied to discontinuous symmetric stable processes in bounded Lipschitz domains, it not only extends but also refines the conditional gauge theorem obtained in [7], [8] and [10]. In [7], [8] and [10], the conditional gauge theorem for discontinuous symmetric stable processes was proved under the condition that the corresponding gauge function is bounded.

The class $\mathbf{S}_1(X)$ also extends the class $\mathbf{S}(X)$ of functions that are "G_D-small near infinity" used in [25] and [26] when X is a Brownian motion in a domain D. In their papers, Murata and Pinchover showed that if q is G_D-small near infinity and the operator $\frac{\Delta}{2}+q$ with Dirichlet boundary conditions is subcritical, that is, it admits a positive Green's function G_q, then G_q is comparable with G_D. Applying our results to the Brownian motion case recovers and extends their results. Moreover, our results hold for nonlocal operators as well.

The rest of the paper is organized as follows. In Section 2 we prove a gauge theorem that is tailored to be applicable to the conditional processes. The conditional gauge theorem and its consequences are proved in Section 3. Examples of the class $\mathbf{S}_\infty(X)$ are given in the last section. In this paper, we use ":=" as a way of definition, which is always read as "is defined to be." For functions f and g, the notation "$f \approx g$" means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \le f \le c_2 g$.

2. Gauge theorem. Our approach to the general gauge theorem is strongly influenced by the approach in Chung and Rao [11] and Section 5.6 of Chung and Zhao [12]. But it is modified and extended in some directions and tailored to a form so that it can be applied to the conditional processes to yield the conditional gauge theorem for a large class of Markov processes in the next section. In this section, X is an irreducible transient Borel right process on a Lusin space E having left limits on $(0,\zeta)$ with Green's function $G(x,y)$, as is specified at the beginning of Section 1. We do not need to assume that X has a strong dual in this section.

DEFINITION 2.1. (i) A function q is said to be in the Kato class $\mathbf{K}(X)$ if

$$\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x\left[\int_0^t |q(X_s)|\,ds\right]=0.$$

(ii) A function q is said to be in the class $\mathbf{K}_\infty(X)$ if, for any $\varepsilon>0$, there is a set $K=K(\varepsilon)$ of finite m-measure and a constant $\delta=\delta(\varepsilon)>0$ such that

$$\sup_{x\in E}\int_{E\setminus K} G(x,y)|q(y)|m(dy)<\varepsilon \tag{1}$$

and, for all measurable sets $B \subset E$ with $m(B) < \delta$,

$$\sup_{x\in E}\int_B G(x,y)|q(y)|m(dy) < \varepsilon. \tag{2}$$

(iii) A function q is said to be in the class $\mathbf{K}_1(X)$ if there is a set K of finite m-measure and a constant $\delta > 0$ such that

$$\begin{aligned}\beta := \sup_{x\in E}\int_{E\setminus K} G(x,y)|q(y)|m(dy) \\ + \sup_{B\subset K : m(B)<\delta}\sup_{x\in E}\int_B G(x,y)|q(y)|m(dy) < 1.\end{aligned} \tag{3}$$

Clearly, $\mathbf{K}_\infty(X) \subset \mathbf{K}_1(X)$. If a function q is Green's bounded, that is, if

$$\sup_{x\in E}\int_E G(x,y)|q(y)|m(dy) < \infty,$$

then $M^{-1}q \in \mathbf{K}_1(X)$ when $M > 0$ is large enough. The next proposition tells us that functions in $\mathbf{K}_1(X)$ must be Green's bounded.

PROPOSITION 2.1. *If $q \in \mathbf{K}_1(X)$, then q is Green's bounded.*

PROOF. It follows from the definition of $\mathbf{K}_1(X)$ that we need only to show that, for any set K of finite m-measure,

$$\sup_{x\in E}\int_K G(x,y)|q(y)|m(dy) < \infty. \tag{4}$$

Let δ be the constant in Definition 2.1(iii). The set K contains at most finitely many points $\{w_1,\dots,w_k\}$ such that $m(\{w_i\}) \ge \delta/2$. As q is finite m-almost everywhere and X is transient, we have, by Proposition 2.2(iv) of [19],

$$\sup_{x\in E}\sum_{i=1}^k G(x,w_i)|q(w_i)| < \infty.$$

Clearly, $K \setminus \{w_1,\dots,w_k\}$ can be written as the disjoint union of a finite number of sets B_i with $m(B_i) < \delta$ and so, by Definition 2.1(iii),

$$\sup_{x\in E}\int_{K\setminus\{w_1,\dots,w_k\}} G(x,y)|q(y)|m(dy) < \infty.$$

This proves (4). □

This proposition implies that, for $q \in \mathbf{K}_1(X)$, the function

$$x \mapsto \mathbf{E}_x\left[\int_0^\zeta q(X_t)\,dt\right]$$

is a bounded function. Thus $\int_0^\zeta q(X_t)\,dt$ and $e_q(\zeta)$ are well defined. Hence the gauge function

$$g(x) = \mathbf{E}_x[e_q(\zeta)], \qquad x \in E,$$

is well defined, nonnegative. Since ζ is in the σ-field $\sigma\{X_t,\ t \ge 0\}$, one can check easily that g is Borel measurable.

PROPOSITION 2.2. (i) *If $q \in \mathbf{K}_1(X)$, then there exists a $t_0 > 0$ such that*

$$\theta := \sup_{x\in E} \mathbf{E}_x\left[\int_0^{t_0} |q(X_s)|\,ds\right] < 1. \tag{5}$$

(ii) $\mathbf{K}_\infty(X) \subset \mathbf{K}(X)$.

PROOF. We only prove the first assertion; the second assertion can be proved similarly. So we suppose that $q \in \mathbf{K}_1(X)$. For $\varepsilon > 0$, let K and δ be as in Definition 2.1(iii). Let M be so large that $m(B) < \delta$, where $B = \{x \in K : |q(x)| > M\}$. Then

$$\begin{aligned}
&\sup_{x\in E} \mathbf{E}_x\left[\int_0^t |q(X_s)|\,ds\right] \\
&\quad \le \sup_{x\in E} \mathbf{E}_x\left[\int_0^t |q(X_s)|\mathbb{1}_{K^c}(X_s)\,ds\right] + \sup_{x\in E} \mathbf{E}_x\left[Mt + \int_0^t |q(X_s)|\mathbb{1}_B(X_s)\,ds\right] \\
&\quad \le \sup_{x\in E} \int_{E\setminus K} G(x,y)|q(y)|m(dy) + Mt + \sup_{x\in E}\int_B G(x,y)|q(y)|m(dy) \\
&\quad < \beta + Mt.
\end{aligned}$$

The first assertion now follows immediately. $\square$

For a Brownian motion X in $\mathbb{R}^n$, any domain $D \subset \mathbb{R}^n$ when $n \ge 3$ and any Green-bounded domain D in $\mathbb{R}^2$, the proof of Theorem 5.20 in [12] implies that

$$\mathbf{K}(X) \cap L^1(D, dm) \subset \mathbf{K}_\infty(X^D). \tag{6}$$

Here m stands for the Lebesgue measure in D and X^D the part of the process X killed upon leaving D. An argument similar to that of Theorem 5.20 in [12] shows that (6) holds for any symmetric α-stable process X in $\mathbb{R}^n$ with $n > \alpha$ and for any open set D in $\mathbb{R}^n$.

PROPOSITION 2.3. *For $q \in \mathbf{K}_1(X) \cup \mathbf{K}(X)$, there exist positive constants c_1, c_2 such that*

$$\sup_{x\in E} \mathbf{E}_x[e_{|q|}(t)] \le e^{c_1 t + c_2} \qquad \textit{for all } t \ge 0.$$

PROOF. It follows from the definition of $\mathbf{K}(X)$ and Proposition 2.2 that there is a $t_0 > 0$ such that (5) is valid. Thus, by Khasminskii's inequality,

$$\sup_{x \in E} \mathbf{E}_x[e_{|q|}(t_0)] \le \frac{1}{1-\theta} < \infty.$$

It follows from the above inequality, the Markov property of X and the fact

$$e_{|q|}(t+s) = e_{|q|}(t)\bigl(e_{|q|}(s) \circ \theta_t\bigr)$$

that there are constants $c_1, c_2 > 0$ such that

$$\sup_{x \in E} \mathbf{E}_x[e_{|q|}(t)] \le e^{c_1 t + c_2}. \qquad \square$$

For $q \in \mathbf{K}_1(X)$, define a semigroup $\{T_t\}_{t \ge 0}$ by

$$T_t f(x) = \mathbf{E}_x[e_q(t) f(X_t)], \qquad f \ge 0. \tag{7}$$

REMARK. It follows from (5) that, for each $q \in \mathbf{K}_1(X)$, there is a constant $\beta > 0$ such that

$$c_\beta(q) := \sup_{x \in E} \mathbf{E}_x\left[\int_0^\zeta e^{-\beta t} |q|(X_t)\, dt\right] < \infty.$$

Furthermore,

$$c(q) = \lim_{\beta \to \infty} c_\beta(q) \le \theta < 1.$$

Hence q is in the extended Kato class of X in the sense of Voigt [29] and Stollmann and Voigt [27] with $c(q) < 1$. When X is symmetric with respect to the measure m, from [27] we know that the semigroup $\{T_t\}_{t \ge 0}$ can be extended to be a semigroup on $L^p(E, m)$ for all $1 \le p \le \infty$ and that it is strongly continuous on $L^p(E, m)$ for $1 \le p < \infty$. However, we do not need this property in this paper.

THEOREM 2.1. *For every $x \in E$ with $g(x) < \infty$, $g(X_t)$ is right continuous and has left limits in $t \in (0, \zeta)$, $\mathbf{P}_x$-a.s.*

PROOF. Let $x \in E$ be such that $g(x) < \infty$. By the strong Markov property of X, for any bounded stopping time T,

$$g(X_T) = \mathbf{E}_{X_T}[e_q(\zeta)] = e_{-q}(T)\mathbf{E}_x[e_q(\zeta) \mid \mathcal{M}_T], \qquad \mathbf{P}_x\text{-a.s. on } \{T < \zeta\}. \tag{8}$$

Here the martingale $t \to \mathbf{E}_x[e_q(\zeta)|\mathcal{M}_t]$ is taken to be the right-continuous version. As $t \to X_t$ is right continuous having left limits and g is Borel measurable, the process $t \to g(X_t)$ is optional. Hence, by an application of the optional section theorem (cf. Theorem 4.10 in [22]), we have from (8) that

$$\mathbf{P}_x\bigl(e_q(t) g(X_t) = \mathbf{E}_x[e_q(\zeta) \mid \mathcal{M}_t] \text{ for all } t \in [0, \zeta)\bigr) = 1.$$

Therefore $t \to g(X_t)$ is right continuous and has left limits in $t \in (0, \zeta)$, $\mathbf{P}_x$-a.s. $\square$

THEOREM 2.2. *Assume that* $q \in \mathbf{K}_1(X)$. *Then the gauge function* g *is finely continuous. Furthermore*, g *is either bounded on* E *or identically* ∞ *on* E.

PROOF. Define $F = \{x \in E : g(x) < \infty\}$. Let $x \in F$ and K be any closed subset of $E \setminus F$. Define $T_K = \inf\{t > 0 : X_t \in K\}$. By the strong Markov property,

$$\infty > g(x) \geq \mathbf{E}_x[T_K < \zeta; e_q(T_K)g(X_{T_K})].$$

Since K is closed, $X_{T_K} \in K$ by the right continuity of $t \to X_t$. Thus $g(X_{T_K}) = \infty$ on $\{T_K < \zeta\}$. On the other hand, $e_q(T_K) > 0$ on $\{T_K < \zeta\}$, $\mathbf{P}_x$-a.s. It follows that $\mathbf{P}_x(T_K < \zeta) = 0$. This being true for all closed subsets $K \subset F$, we have

$$\mathbf{P}_x(T_{F^c} < \zeta) = 0.$$

Thus F is absorbing.

Next, let K, δ and β be as in Definition 2.1(iii). Choose M large enough so that the set $K \cap \{x \in E : M < g(x) < \infty\}$ has m-measure less than δ. Let $B = K^c \cup \{x \in K : M < g(x) < \infty\}$. By Khasminskii's inequality, for any $x \in E$,

$$\begin{aligned} \mathbf{E}_x[e_q(\tau_B)] &\leq \frac{1}{1 - \sup_{x\in E} \mathbf{E}_x\bigl[\int_0^{\tau_B} q(X_t)\,dt\bigr]} \\ &\leq \frac{1}{1-\beta} := \gamma, \end{aligned}$$

where $\tau_B := T_{B^c} = \inf\{t > 0 : X_t \notin B\}$. Thus, for $x \in E$, we have

$$\begin{aligned} g(x) &= \mathbf{E}_x[\tau_B = \zeta; e_q(\tau_B)] + \mathbf{E}_x[\tau_B < \zeta; e_q(\tau_B)g(X_{\tau_B})] \\ &\leq \gamma + \mathbf{E}_x[\tau_B < \zeta; e_q(\tau_B)g(X_{\tau_B})]. \end{aligned} \tag{9}$$

Note that, for $x \in B \cap F$, $\mathbf{P}_x$-a.s. on $\{\tau_B < \zeta\}$, X_{τ_B} does not belong to $E \setminus F$ because F is absorbing. So $g(X_{\tau_B}) \leq M$ as, by Theorem 2.1, $t \to g(X_t)$ is right continuous on $[0, \zeta)$. Therefore the second term on the right-hand side of (9) is bounded by γM. It follows that, on $B \cap F$, g is bounded by $\gamma(1+M)$; it is bounded by M on $F \setminus B$ by the definition of B. Thus $F = \{x \in E : g(x) \leq \gamma(1 + M)\}$.

We now show that the gauge function is finely continuous. It is equivalent to show that $t \to g(X_t)$ is right continuous on $[0, \zeta)$, $\mathbf{P}_x$-a.s. for all $x \in E$. Define $T = \inf\{t > 0 : g(X_t) < \infty\}$ with the convention $\inf \varnothing = \zeta$. Clearly, $g(X_t) = \infty$ for $t < T$. It follows from Theorem 2.1 that $t \to g(X_t)$ is finite and right continuous for $t \in (T, \zeta)$, $\mathbf{P}_x$-a.s. Hence it suffices to show that $g(X_T) < \infty$, $\mathbf{P}_x$-a.s. on $\{T < \zeta\}$ and apply Theorem 2.1. For this, observe that, for each bounded stopping time S,

$$\begin{aligned} g(X_S)e_q(S) &= \mathbf{E}_x[e_q(\zeta) \mid \mathcal{M}_S] \\ &= \lim_{n\to\infty} \uparrow \mathbf{E}_x[e_q(\zeta) \wedge n \mid \mathcal{M}_S], \qquad \mathbf{P}_x\text{-a.s. on } \{S < \zeta\}, \end{aligned}$$

where the symbol $\uparrow$ indicates increasing convergence. Here the martingale

$$s \to \mathbf{E}_x[e_q(\zeta) \wedge n \mid \mathcal{M}_s]$$

is automatically taken to be the right-continuous version. As $t \to X_t$ is right continuous with left limits and g is Borel measurable, so $t \to g(X_t)$ is optional. By the optional section theorem again (cf. Theorem 4.10 of [22]), we have, $\mathbf{P}_x$-a.s.,

$$g(X_t)e_q(t) = \lim_{n\to\infty} \uparrow \mathbf{E}_x[e_q(\zeta) \wedge n \mid \mathcal{M}_t] \qquad \text{for all } t \in [0, \zeta). \tag{10}$$

On the other hand, $\mathbf{P}_x$-a.s. on $\{T < \zeta\}$, as $g(X_{T+s}) \le \gamma(1+M)$ for $s \in (0, \zeta \circ \theta_T)$, we have

$$g(X_{T+s})e_q(T+s) \le \gamma(1+M)e_{|q|}(T+1) \qquad \text{for all } 0 < s < 1 \wedge \zeta \circ \theta_T.$$

By (10) and the optional sampling theorem, $\mathbf{P}_x$-a.s. on $\{T < \zeta\}$,

$$\mathbf{E}_x[e_q(\zeta) \wedge n \mid \mathcal{M}_{T+s}] \le \gamma(1+M)e_{|q|}(T+1)$$

holds for each $n \ge 1$, every $s \in (0, (\zeta \circ \theta_T) \wedge 1)$ and hence for $s = 0$ almost surely. Thus, by (10) again,

$$g(X_T)e_q(T) \le \gamma(1+M)e_{|q|}(T+1), \qquad \mathbf{P}_x\text{-a.s. on } \{T < \zeta\}.$$

In view of Proposition 2.1, this implies that $g(X_T) < \infty$. Now, by Theorem 2.1,

$$\lim_{r\downarrow T} g(X_r) = \lim_{s\downarrow 0} g(X_s) \circ \theta_T = g(X_T), \qquad \mathbf{P}_x\text{-a.s. on } \{T < \zeta\}.$$

This proves the fine continuity of g.

Since $F^c = \{x \in E : g(x) > \gamma(1+M)\}$ is finely open, if F^c is not empty, then, for $x \in F^c$,

$$\int_{F^c} G(x,y)m(dy) = \mathbf{E}_x\left[\int_0^\infty \mathbb{1}_{F^c}(X_s)\,ds\right] > 0$$

and so $m(F^c) > 0$. This would imply by the m-irreducibility of X that F cannot be absorbing unless F is empty. This says that either F or F^c is empty, and therefore g is either identically infinity or bounded on E. □

REMARK. It is not difficult to see that the condition on the potential q in Theorem 2.2 can be relaxed. In fact, Theorem 2.2 holds, for example, when $q^- := \max\{-q, 0\}$ is locally in Kato class $\mathbf{K}_1(X)$ and $q^+ = \max\{q, 0\}$ in $\mathbf{K}_1(X)$. Here a function f is said to be locally in $\mathbf{K}_1(X)$ if there is an increasing sequence of relatively compact open sets O_n with $\bigcup_{n=1}^\infty O_n = E$ and a sequence of functions f_n in $\mathbf{K}_1(X)$ such that $f = f_n$ on O_n.

3. Conditional gauge theorem. In addition to the assumptions on X made in the previous section, we assume that the process X has a strong dual Borel right process $(\widehat{X}, \widehat{\mathbf{P}}_x, x \in E)$ on E with respect to measure m. Under our assumption, the dual process $\widehat{X}$ has Green's function $\widehat{G}(x,y) = G(y,x)$. Let $d := \{(x,y) \in E \times E : G(x,y) = 0 \text{ or } \infty\}$. For each fixed $z \in E$, set $E_z := \{x \in E : 0 < G(x,z) < \infty\}$.

We first define the class of potentials we are going to work with in this section.

DEFINITION 3.1. (i) A function q is said to be in the class **semi-**$\mathbf{S}_\infty(X)$ if, for any $\varepsilon > 0$ and $z \in E$, there is a Borel subset $K = K(\varepsilon, z)$ of finite m-measure and a constant $\delta = \delta(\varepsilon, z) > 0$ such that

$$\sup_{x \in E_z} \int_{E \setminus K} \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \le \varepsilon \tag{11}$$

and, for all measurable sets $B \subset E$ with $m(B) < \delta$,

$$\sup_{x \in E_z} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \le \varepsilon. \tag{12}$$

(ii) A function q is said to be in the class $\mathbf{S}_\infty(X)$ if, for any $\varepsilon > 0$, there is a Borel subset $K = K(\varepsilon)$ of finite m-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,z) \in (E \times E) \setminus d} \int_{E \setminus K} \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \le \varepsilon \tag{13}$$

and, for all measurable sets $B \subset E$ with $m(B) < \delta$,

$$\sup_{(x,z) \in (E \times E) \setminus d} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \le \varepsilon. \tag{14}$$

(iii) A function q is said to be in the class **semi-**$\mathbf{S}_1(X)$ if, for each $z \in E$, there is a Borel set $K = K(z)$ of finite m-measure and a constant $\delta = \delta(z) > 0$ such that

$$\begin{aligned} \beta_1 := {} & \sup_{x \in E_z} \int_{E \setminus K} \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \\ & + \sup_{B \subset K : m(B) < \delta} \sup_{x \in E_z} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) < 1. \end{aligned} \tag{15}$$

(iv) A function q is said to be in the class $\mathbf{S}_1(X)$ if there is a Borel set K of finite m-measure and a constant $\delta > 0$ such that

$$\begin{aligned} \beta_2 := {} & \sup_{B \subset K : m(B) < \delta} \sup_{(x,z) \in (E \times E) \setminus d} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) \\ & + \sup_{(x,z) \in (E \times E) \setminus d} \int_{E \setminus K} \frac{G(x,y)G(y,z)}{G(x,z)} |q(y)| m(dy) < 1. \end{aligned} \tag{16}$$

Clearly, **semi-**$\mathbf{S}_\infty(X) \subset$ **semi-**$\mathbf{S}_1(X)$ and $\mathbf{S}_\infty(X) \subset \mathbf{S}_1(X)$. Also $\mathbf{S}_1(X) = \mathbf{S}_1(\widehat{X})$ and $\mathbf{S}_\infty(X) = \mathbf{S}_\infty(\widehat{X})$.

PROPOSITION 3.1. (i) *A function q is in $\mathbf{S}_\infty(X)$ if and only if, for every $\varepsilon > 0$, there is a Borel subset $K = K(\varepsilon)$ of finite m-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that, for any excessive function f of X,*

$$\int_{E\setminus K} G(x,y)f(y)|q(y)|m(dy) \le \varepsilon f(x) \qquad \textit{for all } x \in E \tag{17}$$

and, for all measurable sets $B \subset E$ with $m(B) < \delta$,

$$\int_B G(x,y)f(y)|q(y)|m(dy) < \varepsilon f(x) \qquad \textit{for all } x \in E. \tag{18}$$

(ii) *A function q is in $\mathbf{S}_1(X)$ if and only if, for every $\varepsilon > 0$, there is a set K of finite m-measure, a constant $\delta > 0$ and a positive constant $\beta_1 < 1$ such that, for any excessive function f of X,*

$$\begin{aligned}\int_{E\setminus K} G(x,y)f(y)|q(y)|m(dy) + \sup_{B\,:\,m(B)<\delta}\int_B G(x,y)f(y)|q(y)|m(dy)&\\ \le \beta_1 f(x)&\end{aligned} \tag{19}$$

for all $x \in E$.

PROOF. (i) For any $z \in E$, the function $y \to G(y,z)$ is an excessive function of X, so (17) and (18) imply (13) and (14) and therefore q is in $\mathbf{S}_\infty(X)$. Conversely, suppose that (13) and (14) hold. Then (17) and (18) are valid when f is the potential of some measure ν. Now the conclusion follows because any excessive function is the increasing limit of a sequence of potentials of the form Gh_n, where h_n are nonnegative functions.

(ii) can be proved similarly. □

Since the constant function 1 is an excessive function of X, we can take $f = 1$ in the proposition above and get:

COROLLARY 3.1. $\mathbf{S}_1(X) \subset \mathbf{K}_1(X)$ *and* $\mathbf{S}_\infty(X) \subset \mathbf{K}_\infty(X)$.

For each $z \in E$, let $X^{\cdot,z} = (X, \mathbf{P}_x^z, x \in E_z)$ be the h-conditioned process of X with $h(\cdot) = G(\cdot,z)$; that is, $X^{\cdot,z}$ has transition probability $q(t,x,dy) = p(t,x,dy)G(y,z)/G(x,z)$. The state space for $X^{\cdot,z}$ is E_z. It follows from Proposition 5.4, Theorem 6.5 and, in particular, Example 6.14 in [21] that $X^{\cdot,z}$ is a transient Borel right process with left limits on $(0,\zeta^z)$. Clearly, the conditional process $X^{\cdot,z}$ is irreducible. Note that the Green's function for $X^{\cdot,z}$ with respect to the measure m is

$$\frac{G(x,\cdot)G(\cdot,z)}{G(x,z)}.$$

THEOREM 3.1. *Let q be in the class* **semi-S**$_1(X)$. *Then for each $z \in E$, either* $\mathbf{E}_x^z[e_q(\zeta)] \equiv \infty$ *or* $x \to \mathbf{E}_x^z[e_q(\zeta^z)]$ *is bounded on* E_z.

PROOF. Note that $\bigcap_{z\in E} \mathbf{K}_1(X^{\cdot,z}) = \textbf{semi-S}_1(X)$. The theorem follows from Theorem 2.2. $\square$

The following result is proved in [21] (see Theorem 6.5 and Example 6.14 there). Here we give a slick way of proving it under an extra assumption that X has a transition density function $p(t, x, y)$ with respect to the measure m.

PROPOSITION 3.2. *Fix $x, z \in E$ with $0 < G(x, z) < \infty$. Reversing the conditional process $(X, \mathbf{P}_x^z)$ at its lifetime ζ^z and taking a right-continuous version, the time-reversed process has the same distribution as the conditional process $(\widehat{X}, \widehat{\mathbf{P}}_z^x)$. Consequently,* $\mathbf{E}_x^z[e_q(\zeta^z)] = \widehat{\mathbf{E}}_z^x[e_q(\zeta^x)]$.

PROOF. Note that

$$\mathbf{P}_x^z(\zeta^z > t) = \int_t^\infty p(s, x, z)\, ds / G(x, z). \tag{20}$$

By identifying the finite-dimensional distributions, it is easy to see that, conditioned on $\{\zeta^z = t\}$, the process $(X, \mathbf{P}_x^z, x \in E_z)$ has the same law as the process X conditioned on $\{X_t = z\}$. In other words, conditioning on $\{\zeta^z = t\}$, $X^{\cdot,z}$ has transition density function $p(s, x, y)p(t - s, y, z)/p(t, x, z)$ with respect to the measure m. Therefore the conditional process $X^{\cdot,z}$ can be constructed in the following way:

1. for each fixed $T > 0$, construct a process Y_s with $Y_0 = x$ for $s \in [0, T]$ from X by conditioning on $X_T = z$ (i.e., construct a process Y_s with density function $p(s, x, y)p(T - s, y, z)/p(T, x, z)$;
2. randomize T according to the distribution

$$P(T > t) = \int_t^\infty p(s, x, z)\, ds / G(x, z).$$

From this construction, it is clear that if one reverses $(X, \mathbf{P}_x^z)$ at its lifetime ζ^z, the time-reversed process has the same distribution as the conditional process $(\widehat{X}, \widehat{\mathbf{P}}_z^x)$. This, in particular, implies that $\mathbf{E}_x^z[e_q(\zeta^z)] = \widehat{\mathbf{E}}_z^x[e_q(\zeta^x)]$. $\square$

REMARK. Note that (cf. the proof of Proposition 2.1), for $q \in \mathbf{S}_1(X)$,

$$\mathbf{E}_x^z\left[\int_0^{\zeta^z} |q(X_s^{\cdot,z})|\, ds\right] = \int_E \frac{G(x, y)G(y, z)}{G(x, z)} |q(y)| m(dy)$$

is bounded on $(E \times E) \setminus d$, where $d = \{(x, w) \in E \times E : G(x, w) = 0 \text{ or } \infty\}$. Hence $e_q(\zeta^z)$ is well defined and, by Jensen's inequality,

$$\inf_{(x,z)\in(E\times E)\setminus d} \mathbf{E}_x^z[e_q(\zeta^z)] > 0.$$

THEOREM 3.2. *Suppose q is in the class $\mathbf{S}_1(X)$. If $\mathbf{E}_x^z[e_q(\zeta^z)]$ is finite for some $(x_0, z_0) \in (E \times E) \setminus d$, then $\mathbf{E}_x^z[e_q(\zeta^z)]$ is bounded on $(E \times E) \setminus d$.*

PROOF. Let $u(x,z) = \mathbf{E}_x^z[e_q(\zeta^z)]$ and $\hat{u}(x,z) = \widehat{\mathbf{E}}_x^z[e_q(\zeta^z)]$ for $x \in E$ and $z \in E_x$. Applying Theorem 3.1 to the process $X^{\cdot, z_0}$ and using Proposition 3.2, we get

$$\sup_{x \in E_{z_0}} \hat{u}(z_0, x) = \sup_{x \in E_{z_0}} u(x, z_0) < \infty.$$

This implies, by applying Theorem 3.1 to the process $\widehat{X}^{\cdot, x}$, that for any $x \in E$,

$$\sup_{z \in \widehat{E}_x} u(x,z) = \sup_{z \in \widehat{E}_x} \hat{u}(z,x) < \infty,$$

where $\widehat{E}_x = \{z \in E : 0 < G(x,z) < \infty\}$. Let the set K and the constant $\delta > 0$ be as in Definition 3.1(iv) for the class $\mathbf{S}_1(X)$. Note that $u(x,z)$ is in $\mathcal{B}(E \times E)$ if we set $u(x,z) = 1$ for $(x,z) \in d$. Hence $\{x \in K : \sup_{z \in E} u(x,z) > M\}$ is $\mathcal{B}(E)$-measurable since it is the x-projection of the set $\{(x,z) \in K \times E : u(x,z) > M\}$. As $\bigcap_{M=2}^{\infty} \{x \in K : \sup_{w \in E} u(x,w) > M\} = \varnothing$, we can choose M large enough so that the set $\{x \in K : \sup_{z \in E_x} u(x,z) \geq M\}$ has m measure less than δ. Let $B = K^c \cup \{x \in K : \sup_{z \in E_x} u(x,z) \geq M\}$. Note that by Khasminskii's inequality, for any $(x,z) \in (E \times E) \setminus d$,

$$\mathbf{E}_x^z[e_q(\tau_B)] \leq \frac{1}{1 - \sup_{x \in E_z} \mathbf{E}_x^z\big[\int_0^{\tau_B} q(X_t^{\cdot,z})\,dt\big]} \leq \frac{1}{1 - \beta_2} := \gamma_2.$$

Thus, for any $(x,z) \in (E \times E) \setminus d$,

$$\begin{aligned} u(x,z) &= \mathbf{E}_x^z[\tau_B = \zeta^z; e_q(\tau_B)] + \mathbf{E}_x^z[\tau_B < \zeta^z; e_q(\tau_B) u(X_{\tau_B}^{\cdot,z}, z)] \\ &\leq \gamma_2 + \mathbf{E}_x^z[\tau_B < \zeta^z; e_q(\tau_B) u(X_{\tau_B}^{\cdot,z}, z)]. \end{aligned} \tag{21}$$

Observe also that, for $x \in B \cap E_z$, $\mathbf{P}_x^z$-a.s. on $\{\tau_B(X^{\cdot,z}) < \zeta^z\}$, $X_{\tau_B}^{\cdot,z}$ is not equal to z. So $u(X_{\tau_B}^{\cdot,z}, z) \leq M$ since, by Theorem 2.1, $t \to u(X_t^{\cdot,z}, z)$ is right continuous on $[0, \zeta^z)$. Therefore the second term on the right-hand side of (21) is bounded by $\gamma_2 M$. It follows that, for $x \in B \cap E_z$, $u(x,z) \leq \gamma_2(1+M)$. By the definition of B, we know that, for $x \in B^c \cap E_z$, $u(x,z) \leq M$. Hence, for any $z \in E$,

$$\sup_{x \in E_z} u(x,z) \leq \gamma_2(1+M). \qquad \square$$

REMARK. Similar to the remark at the end of the previous section, Theorem 3.2 holds if q^- is locally in $\mathbf{S}_1(X)$ and $q^+ \in \mathbf{S}_1(X)$. Here a function f is said to be locally in $\mathbf{S}_1(X)$ if there is an increasing sequence of relatively compact open sets O_n with $\bigcup_{n=1}^{\infty} O_n = E$ and a sequence of functions f_n in $\mathbf{S}_1(X)$ such that $f = f_n$ on O_n.

THEOREM 3.3. *Let q be a function in $\mathbf{S}_1(X)$ such that the conditional gauge function $u(x,y) := \mathbf{E}_x^y[e_q(\zeta^y)]$ is not identically infinite on $(E\times E)\setminus d$. Define $G_q(x,y) = u(x,y)G(x,y)$ for $(x,y)\in (E\times E)\setminus d$. Then*

$$G_q(x,y) = G(x,y) + \int_E G(x,z)q(z)G_q(z,y)m(dz). \tag{22}$$

Furthermore, for any Borel function ϕ with $G|\phi|$ being finite, we have

$$G_q\phi = G\phi + G[q(G_q\phi)].$$

PROOF. From Theorem 3.2 we know that, under the assumptions of the theorem, $u(x,y)$ is bounded for $(x,y)\in(E\times E)\setminus d$. Hence

$$\sup_{(x,y)\in(E\times E)\setminus d}\int_E \frac{G(x,z)|q(z)|G(z,y)}{G(x,y)}u(z,y)m(dz) < \infty.$$

Therefore we can apply Fubini's theorem to get

$$\begin{aligned}
\mathbf{E}_x^y[e_q(\zeta^y)] - 1 &= \mathbf{E}_x^y\left[\exp\left(\int_0^{\zeta^y} q(X_s^{\cdot,y})\,ds\right) - 1\right]\\
&= \mathbf{E}_x^y\left[\int_0^{\zeta^y} q(X_t^{\cdot,y})\exp\left(\int_t^{\zeta^y} q(X_s^{\cdot,y})\,ds\right)dt\right]\\
&= \mathbf{E}_x^y\left[\int_0^{\zeta^y} q(X_t^{\cdot,y})\mathbf{E}^y_{X^{\cdot,y}(t)}[e_q(\zeta^y)]\,dt\right]\\
&= \int_E \frac{G(x,z)q(z)G(z,y)}{G(x,y)}u(z,y)m(dz).
\end{aligned}$$

In other words, (22) is valid. The last assertion is an immediate consequnce of (22). □

REMARK. (1) If the Harnack inequality holds for positive $\mathcal{L}$-harmonic functions, then the above theorem holds for q in **semi**-$\mathbf{S}_1(X)$ as well.

(2) Let $\mathcal{L}$ denote the the extended generator of X introduced in Getoor [20]. It is possible to show that G_q in Theorem 3.3 is the Green's function of $\mathcal{L}+q$, which is the extended generator for the perturbed semigroup $\{T_t\}_{t\ge 0}$ defined by (7) (see Theorem 5.10 and Remark 5.15 of [20]). We omit the details here.

Now we give some examples.

EXAMPLE 1. Suppose that X is a symmetric α-stable process on $\mathbb{R}^n$, with $\alpha\in(0,2]$ and $n>\alpha$. It is known that X is transient with the Green's function $G(x,y) = c|x-y|^{\alpha-n}$. Let D be a bounded domain in $\mathbb{R}^n$ and denote by $G_D(x,y)$ the Green's function of X in D. More precisely, G_D is the Green's function of X^D, the part process of X killed upon leaving the domain D. Assume that the following $3G$ inequality holds: there is a constant $c > 0$ such that

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le c(|x-y|^{\alpha-n} + |y-z|^{\alpha-n}), \qquad x, y, z \in D. \tag{23}$$

The above $3G$ inequality holds, for example, when D is a bounded Lipschitz domain. It is known from [31] that

$$\mathbf{K}(X) = \left\{ q : \lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|y-x|<r} |x-y|^{\alpha-n} |q(y)|\, dy = 0 \right\}. \tag{24}$$

For a function q that vanishes outside D, it is easy to see that $q \in \mathbf{K}(X)$ if and only if

$$\lim_{\delta \downarrow 0} \sup_{B : m(B) \le \delta} \sup_{x \in \mathbb{R}^n} \int_B |x-y|^{\alpha-n} |q(y)|\, dy = 0.$$

By (23), we see that $\mathbf{K}(X) \subset \mathbf{S}_\infty(X^D)$. Therefore the conditional gauge theorem holds for any $q \in \mathbf{K}(X)$ when D is a bounded Lipschitz domain. So when $\alpha < 2$, we have established a refinement of the conditional gauge theorem obtained in [7], [8] and [10].

EXAMPLE 2 (Brownian motion with singular drift). Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 3$ and $b(x) = (b_1(x), \dots, b_n(x))$ an $\mathbb{R}^n$-valued function on D such that $|b| \in L^p(D)$ with $p > n$ and the distributional derivative $\sum_{k=1}^n (\partial b_k / \partial x_k) \ge 0$. Then the diffusion X^D in D given by

$$dX_t^D = dW_t + b(X_t^D)\, dt, \qquad t \le \tau_D := \inf\{t > 0 : X_t^D \notin D\},$$

has a strong dual with respect to the Lebesgue measure in D. Here W is a Brownian motion in $\mathbb{R}^n$. Let $G_D(x,y)$ be the Green's function of X^D, and $G_D^\Delta(x,y)$ the Green's function of the Brownian motion killed upon leaving the domain D. It is known from Ancona [2] that there is a constant $c > 1$ such that

$$\frac{1}{c} G_D^\Delta(x,y) \le G_D(x,y) \le c G_D^\Delta(x,y) \qquad \text{for } x, y \in D.$$

Hence it follows from $3G$ inequality (23) for G_D^Δ that

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le c(|x-y|^{2-n} + |y-z|^{2-n}), \qquad x, y, z \in D.$$

Therefore $\mathbf{K}(W) \subset \mathbf{S}_\infty(X^D)$ and the conditional gauge theorem for X^D holds for any $q \in \mathbf{K}(W)$.

4. The class $\mathbf{S}_\infty(X)$. In this section we continue to look at some concrete examples. Throughout the rest of this paper, X is a symmetric α-stable process in $\mathbb{R}^n$ with $\alpha \in (0, 2]$ and $n > \alpha$, and $m(dx) = dx$ is the Lebesgue measure on $\mathbb{R}^n$. Let D be a domain in $\mathbb{R}^n$, and X^D the part of the process X killed upon leaving the domain D, whose Green's function is denoted by G_D.

We say that a function q defined on D is locally in $\mathbf{K}(X^D)$, written as $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$, if, for any compact subset K of D, $\mathbb{1}_K q$ is in $\mathbf{K}(X^D)$.

DEFINITION 4.1. A function $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$ is said to be G_D-small at infinity if, for any $\varepsilon > 0$, there is a compact subset $K = K(\varepsilon)$ of D such that

$$\sup_{x,y\in D\setminus K} \frac{1}{G_D(x,y)} \int_{D\setminus K} G_D(x,z)|q(z)|G_D(z,y)\,dz \le \varepsilon \tag{25}$$

The collection of functions which are G_D-small at infinity is denoted by $\mathbf{S}(X^D)$.

When X is a Brownian motion, the above definition was first introduced in Pinchover [26] but renamed to the current one in Murata [25]. It can be shown by using the maximum principle that the function $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$ is G_D-small at infinity if and only if, for any $\varepsilon > 0$, there is a compact subset K of D such that

$$\sup_{x,y\in D} \frac{1}{G_D(x,y)} \int_{D\setminus K} G_D(x,z)|q(z)|G_D(z,y)\,dz \le \varepsilon$$

(cf. Lemma 2.1 of [25]).

PROPOSITION 4.1. *If $q \in \mathbf{S}(X^D)$, then the family of functions*

$$\left\{\frac{G_D(x,\cdot)|q(\cdot)|G_D(\cdot,y)}{G_D(x,y)} : x, y \in D\right\}$$

is uniformly integrable in D; that is, for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that, for any set $A \subset D$ with $m(A) < \delta$,

$$\sup_{x,y\in D} \frac{1}{G_D(x,y)} \int_A G_D(x,z)|q(z)|G_D(z,y)\,dz \le \varepsilon.$$

PROOF. For any $\varepsilon > 0$, by the definition of $\mathbf{S}(X^D)$ there exists a relatively compact open subset O of D such that

$$\sup_{x,y\in D} \frac{1}{G_D(x,y)} \int_{D\setminus O} G_D(x,z)|q(z)|G_D(z,y)\,dz \le \frac{\varepsilon}{2}. \tag{26}$$

For any relatively compact open subsets O_1 and O_2 of D with $\overline{O} \subset O_1 \subset \overline{O_1} \subset O_2$, $y \to G_D(x,y)$ is harmonic (for the process X^D) in O_2 for each fixed $x \in D \setminus O_2$. Hence, by the Harnack inequality, there is a constant $C_1 > 1$ such that

$$\sup_{x\in D\setminus O_2, y,z\in O_1} \frac{G_D(x,z)}{G_D(x,y)} = C_1 < \infty. \tag{27}$$

Note that

$$G_D(x,y) = G(x,y) - \mathbf{E}_x[G(X_{\tau_D}, y)] \qquad \text{for } x, y \in D,$$

where $G(x,y) = c(n,\alpha)|x-y|^{\alpha-n}$ is the Green's function of X. From this, we see that there is a constant $c > 1$ such that

$$c^{-1}|x-y|^{\alpha-n} \le G_D(x,y) \le c|x-y|^{\alpha-n} \qquad \text{for } x, y \in O_2. \tag{28}$$

Therefore there exists a constant $C_2 > 1$ such that

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \le C_2\big(G_D(x,z) + G_D(z,y)\big), \qquad x, y, z \in O_2. \tag{29}$$

Since $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$, by (24) and (28) there exists a $\delta > 0$ such that, for any set $A \subset O_2$ with $m(A) < \delta$,

$$\sup_{x \in \mathbb{R}^n} \int_A G_D(x,z)|q(z)|\,dz \le \frac{\varepsilon}{2C_1C_2}.$$

Hence, for such a set A,

$$\int_{A\cap O} \frac{G_D(x,z)|q(z)|G_D(z,y)}{G_D(x,y)}\,dz \le \frac{\varepsilon}{2} \tag{30}$$

for all $(x,y) \in ((D \setminus O_2) \times O_1) \cup (O_2 \times O_2)$. In the above inequality, we used (27) for $(x,y) \in (D \setminus O_2) \times O_1$ and (29) for $(x,y) \in O_2 \times O_2$. In particular, inequality (30) holds for $(x,y) \in D \times O_1$, and by the symmetry of the Green's function G_D, it holds for $(x,y) \in O_1 \times D$ as well. Fix $y \in D \setminus O_1$. The function

$$x \to \frac{\varepsilon}{2} G_D(x,y) - \int_{A\cap O} G_D(x,z)|q(z)|G_D(z,y)\,dz$$

is positive on O_1 and is a superharmonic function in $D \setminus \overline{O}$ for the process X^D. Hence, by the definition of superharmonicity, $\frac{\varepsilon}{2} G_D(x,y) \ge \int_{A\cap O} G_D(x,z)|q(z)| \times G_D(z,y)\,dz$ for every $x \in D$, and therefore (30) holds for any $(x,y) \in D \times D$. Combining the above with (26), we have

$$\sup_{x,y \in D} \int_A \frac{G_D(x,z)|q(z)|G_D(z,y)}{G_D(x,y)}\,dz \le \varepsilon,$$

which proves the proposition. □

The above proposition shows that $\mathbf{S}(X^D) \subset \mathbf{S}_\infty(X^D)$. In fact, we have:

THEOREM 4.1. $\mathbf{S}(X^D) = \mathbf{S}_\infty(X^D)$.

PROOF. It remains to show $\mathbf{S}_\infty(X^D) \subset \mathbf{S}(X^D)$. Let $q \in \mathbf{S}_\infty(X^D)$. By Proposition 2.2 and Corollary 3.1, $q \in \mathbf{K}_\infty(X^D) \subset \mathbf{K}(X^D)$. For any $\varepsilon > 0$, there is a Borel-measurable set $K = K(\varepsilon/2)$ and a constant $\delta = \delta(\varepsilon/2) > 0$ such that (13) and (14) hold with $\varepsilon/2$ in place of ε. As one can always find a compact set $\tilde{K} \subset K$ such that $m(K \setminus \tilde{K}) < \delta$, it follows that

$$\begin{aligned}
&\sup_{x,z\in D} \int_{D\setminus\tilde{K}} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|\,dy \\
&\qquad \le \sup_{x,z\in D} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|\,dy \\
&\qquad\quad + \sup_{x,z\in D} \int_{K\setminus\tilde{K}} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|\,dy \\
&\qquad < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

This proves that $q \in \mathbf{S}(X^D)$. □

It is now easy to see that the following holds.

COROLLARY 4.1. *$\mathbf{S}_\infty(X)$ are those functions $q \in \mathbf{K}(X)$ such that*

$$\lim_{M\uparrow\infty} \sup_{x\in\mathbb{R}^n} \int_{|y|>M} \frac{|q(y)|\,dy}{|x-y|^{n-\alpha}} = 0.$$

Hence, when $\alpha = 2$, $\mathbf{S}_\infty(X)$ is exactly the space of Green's-tight functions introduced in Zhao [32]. In the rest of the paper, we will use $\delta_D(x)$ to denote the distance from x to the Euclidean boundary ∂D of D. We will drop the subscript D from $\delta_D(x)$ when there is no danger of confusion.

PROPOSITION 4.2. *Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$. Then $q \in \mathbf{S}_\infty(X^D)$ if and only if, for any $\varepsilon > 0$, there is a compact subset $K = K(\varepsilon)$ of D such that*

$$\sup_{x\in D\setminus K} \int_{D\setminus K} \frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}} G_D(x,y)|q(y)|\,dy \le \varepsilon.$$

PROOF. It is known (see [6] for the $n = 1$ case, [30] for the Brownian case $\alpha = 2$ and [9] for $0 < \alpha < 2$) that, on $D \times D$,

$$G_D(x,y) \approx |x-y|^{\alpha-n} \min\left\{1, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^{\alpha}}\right\}. \tag{31}$$

Thus

$$
\begin{aligned}
G_D(x,y) &\approx \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^{n-\alpha}(|x-y|^{\alpha}+\delta(x)^{\alpha/2}\delta(y)^{\alpha/2})} \\
&\approx \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^{n-\alpha}(|x-y|^{\alpha}+\delta(x)^{\alpha}+\delta(y)^{\alpha})}.
\end{aligned} \tag{32}
$$

If we put

$$\rho(x,y)=|x-y|^{n-\alpha}\bigl(|x-y|^{\alpha}+\delta(x)^{\alpha}+\delta(y)^{\alpha}\bigr),$$

it is easy to check [cf. (38)–(40) below] that there is some constant $C=C(D)>0$ such that

$$\rho(x,z)\le C\bigl(\rho(x,y)+\rho(y,z)\bigr), \qquad x,y,z\in D.$$

This is equivalent to

$$
\begin{aligned}
&\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \\
&\quad \le C\biggl(\frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}}G_D(x,y)+\frac{\delta(y)^{\alpha/2}}{\delta(z)^{\alpha/2}}G_D(y,z)\biggr), \qquad x,y,z\in D.
\end{aligned}
$$

Thus our condition is sufficient for $q\in \mathbf{S}_\infty(X^D)$.

To prove the necessity, let $\phi\ge 0$ be the eigenfunction corresponding to the first eigenvalue $\lambda<0$ of the infinitesimal generator of X^D. As $\phi(x)=-\lambda^{-1}G_D\phi(x)$, one can easily deduce from (31) that $\phi(x)\approx\delta(x)^{\alpha/2}$ on D. The necessity of our condition now follows from Proposition 3.1. $\square$

The proposition above implies that, for a bounded $C^{1,1}$ domain D, $q\in\mathbf{S}_\infty(X^D)$ if and only if the family of functions

$$\biggl\{\frac{\delta(\cdot)^{\alpha/2}}{\delta(x)^{\alpha/2}}G_D(x,\cdot)|q(\cdot)|;\ x\in D\biggr\}$$

is uniformly integrable in D.

COROLLARY 4.2. *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and that q is a function in $\mathbf{K}_{\mathrm{loc}}(X^D)$. If there exist constants $C>0$, $\beta\in(0,\alpha)$ and a compact subset K of D such that*

$$|q(x)|\le C\delta(x)^{-\beta}, \qquad x\in D\setminus K,$$

then $q\in\mathbf{S}_\infty(X^D)$.

PROOF. It follows from (32) that

$$\frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}} G_D(x,y)\delta(y)^{-\beta} \le |x-y|^{\alpha-\beta-n}.$$

As $\{|x-\cdot|^{\alpha-\beta-n}; x \in D\}$ is uniformly integrable, the corollary is established. $\square$

For $1 < \alpha \le 2$, the function $\delta(x)^{-\beta}$ is not integrable on D when $\beta \in [1,\alpha)$, so the function $\mathbb{1}_D(x)\delta^{-\beta}(x)$ cannot be in the Kato class $\mathbf{K}(X)$. Thus the above corollary shows that, at least when D is a bounded $C^{1,1}$ domain, the class $\mathbf{S}_\infty(X^D)$ is strictly larger than the classical Kato class $\mathbf{K}(X)$. The class $\mathbf{S}_\infty(X^D)$ contains functions which are singular near the boundary of D.

Using the Kelvin transform, one can similarly prove the following.

PROPOSITION 4.3. *Suppose that D is an unbounded $C^{1,1}$ domain in $\mathbb{R}^n$ with compact boundary, $q \in \mathbf{K}_{\mathrm{loc}}(X^D)$ and $\mathbb{1}_{B(0,r)^c} q \in \mathbf{K}_\infty(X)$ for some large $r > 0$. If there exist a $C > 0$, a $\beta \in (0,\alpha)$ and an $r_0 > 0$ such that*

$$|q(x)| \le C\delta(x)^{-\beta}, \qquad x \in D \text{ with } \delta(x) \le r_0,$$

then q belongs to $\mathbf{S}_\infty(X^D)$.

PROPOSITION 4.4. *Let $n \ge 3$ and let W be a Brownian motion in $\mathbb{R}^n$. Suppose that $D = \{x \in \mathbb{R}^n : x_1 > 0\}$ is the upper half space in $\mathbb{R}^n$ and $q \in \mathbf{K}_{\mathrm{loc}}(W^D)$. Then $q \in \mathbf{S}_\infty(W^D)$ if and only, if for any $\varepsilon > 0$, there is a compact subset K of D such that*

$$\sup_{x \in D\setminus K} \int_{D\setminus K} \frac{y_1}{x_1} G_D(x,y)|q(y)|\,dy \le \varepsilon.$$

PROOF. It suffices to show the following inequality

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le C\left(\frac{y_1}{x_1}G_D(x,y) + \frac{y_1}{z_1}G_D(y,z)\right), \qquad x,y,z \in D, \tag{33}$$

for some $C = C(D) > 0$, as the "if" part follows immediately from it, while the "only if" part follows from Proposition 3.1 and the fact that the function $x \mapsto x_1$ is a positive harmonic function on D.

The Green's function G_D is given by

$$G_D(x,y) = \frac{C_n}{|x-y|^{n-2}} - \frac{C_n}{|x-y'|^{n-2}}, \qquad x,y \in D,$$

where $C_n = \Gamma(\frac{n}{2}-1)/(2\pi^{n/2})$ and $y' = (-y_1, y_2, \ldots, y_n)$ is the reflection of x with respect to the hyperplane $x_1 = 0$. If we set

$$\gamma(x,y) = \frac{4x_1y_1}{|x-y|^2},$$

then, for any $x, y \in D$,

$$\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y'|^{n-2}} = \left(1 - \left(1+\gamma(x,y)\right)^{-(n-2)/2}\right)\frac{1}{|x-y|^{n-2}}, \tag{34}$$

Using monotonicity and the facts

$$\lim_{r\downarrow 0} \frac{1-(1+r)^{-(n-2)/2}}{r} = \frac{n-2}{2}$$

and

$$\lim_{r\uparrow\infty}\left(1-(1+r)^{-(n-2)/2}\right) = 1,$$

we see that

$$1-(1+r)^{-(n-2)/2} \approx \min\{1, r\}.$$

So we have, on $D \times D$,

$$G_D(x,y) \approx |x-y|^{2-n}\min\left\{1, \frac{x_1 y_1}{|x-y|^2}\right\}. \tag{35}$$

Since

$$\min\left\{1, \frac{x_1 y_1}{|x-y|^2}\right\} \approx \frac{x_1 y_1}{|x-y|^2 + x_1 y_1}$$

and

$$|x-y|^2 + x_1 y_1 \approx |x-y|^2 + x_1^2 + y_1^2,$$

we get

$$G_D(x,y) \approx \frac{x_1 y_1}{|x-y|^{n-2}(|x-y|^2 + x_1^2 + y_1^2)}. \tag{36}$$

If we put

$$\rho(x,y) = |x-y|^{n-2}(|x-y|^2 + x_1^2 + y_1^2),$$

then (33) is equivalent to the following inequality:

$$\rho(x,z) \le C\left(\rho(x,y) + \rho(y,z)\right), \qquad x, y, z \in D, \tag{37}$$

for some $C = C(D) > 0$. We are going to prove (37). Obviously, we have

$$\begin{aligned} |x-z|^n &\le 2^{n-1}(|x-y|^n + |y-z|^n) \\ &\le 2^{n-1}\left(\rho(x,y) + \rho(y,z)\right), \qquad x, y, z \in D. \end{aligned} \tag{38}$$

When $x_1 < z_1$, we have

$$\begin{aligned} x_1^2|x-z|^{n-2} &\le 2^{n-3}x_1^2(|x-y|^{n-2}+|y-z|^{n-2}) \\ &\le 2^{n-3}x_1^2|x-y|^{n-2}+2^{n-3}z_1^2|z-y|^{n-2} \\ &\le 2^{n-3}\big(\rho(x,y)+\rho(y,z)\big). \end{aligned}$$

When $x_1 \ge z_1$, using the fact that $x_1 \le |x-z|+z_1$ and the displays above, we get

$$\begin{aligned} x_1^2|x-z|^{n-2} &\le 2|x-z|^n+2z_1^2|x-z|^{n-2} \\ &\le 2^n\big(\rho(x,y)+\rho(y,z)\big)+2^{n-1}\big(\rho(x,y)+\rho(y,z)\big) \\ &\le 2^{n+1}\big(\rho(x,y)+\rho(y,z)\big). \end{aligned}$$

Combining the two cases above, we arrive at

$$x_1^2|x-z|^{n-2} \le 2^{n+1}\big(\rho(x,y)+\rho(y,z)\big), \qquad x,y,z \in D. \tag{39}$$

Similarly, we have

$$z_1^2|x-z|^{n-2} \le 2^{n+1}\big(\rho(x,y)+\rho(y,z)\big), \qquad x,y,z \in D. \tag{40}$$

Equation (37) now follows from (38), (39) and (40). $\square$

The proposition above implies that, for the upper half space $D = \{x \in \mathbb{R}^n : x_1 > 0\}$, $q \in \mathbf{S}_\infty(W^D)$ if and only if the family of functions

$$\left\{y \to \frac{y_1}{x_1} G_D(x,y)|q(y)|;\ x \in D\right\}$$

is uniformly integrable in D.

The proposition above can be generalized as follows.

PROPOSITION 4.5. *Let $1 \le k \le n$, $D = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0\}$ and $q \in \mathbf{K}_{\mathrm{loc}}(W^D)$. Then $q \in \mathbf{S}_\infty(W^D)$ if and only if, for any $\varepsilon > 0$, there is a compact subset K of D such that*

$$\sup_{x \in D\setminus K} \int_{D\setminus K} \frac{y_1\cdots y_k}{x_1\cdots x_k} G_D(x,y)|q(y)|\,dy \le \varepsilon.$$

PROOF. Similar to the proof of the proposition above, it suffices to show that the inequality

$$\begin{aligned} &\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \\ &\quad \le C\left(\frac{y_1\cdots y_k}{x_1\cdots x_k} G_D(x,y) + \frac{y_1\cdots y_k}{z_1\cdots z_k} G_D(y,z)\right), \qquad x,y,z \in D, \end{aligned} \tag{41}$$

holds for some $C = C(D) > 0$, as the "if" part follows immediately from it, while the "only if" part follows from Proposition 3.1 and the fact that the function $x \mapsto x_1 \cdots x_k$ is a positive harmonic function on D. We are only going to show (41) in the case $k = 2$.

The Green's function G_D is given by

$$G_D(x, y) = C_n\big((|x - y|^{2-n} - |x - y'|^{2-n}) \\ - (|x - \tilde{y}|^{2-n} - |x - \overline{y}|^{2-n})\big), \qquad x, y \in D,$$

where $C_n = \Gamma(\frac{n}{2} - 1)/(2\pi^{n/2})$, $y' = (-y_1, y_2, \ldots, y_n)$, $\tilde{y} = (y_1, -y_2, y_3, \ldots, y_n)$ and $\overline{y} = (-y_1, -y_2, y_3, \ldots, y_n)$. Note that, by (34),

$$\begin{aligned} G_D(x, y) &= C_n\left(1 - \left(1 + \frac{4x_1y_1}{|x - y|^2}\right)^{-(n-2)/2}\right)|x - y|^{2-n} \\ &\quad - C_n\left(1 - \left(1 + \frac{4x_1y_1}{|x - \tilde{y}|^2}\right)^{-(n-2)/2}\right)|x - \tilde{y}|^{2-n} \\ &= C_n\left(\left(1 + \frac{4x_1y_1}{|x - \tilde{y}|^2}\right)^{-(n-2)/2} - \left(1 + \frac{4x_1y_1}{|x - y|^2}\right)^{-(n-2)/2}\right)|x - y|^{2-n} \\ &\quad + C_n\left(1 - \left(1 + \frac{4x_1y_1}{|x - \tilde{y}|^2}\right)^{-(n-2)/2}\right) \\ &\quad \times \left(1 - \left(1 + \frac{4x_2y_2}{|x - y|^2}\right)^{-(n-2)/2}\right)|x - y|^{2-n} \\ &= C_n\left(1 + \frac{4x_2y_2}{|x - y|^2}\right)^{-(n-2)/2} \\ &\quad \times \left(\left(1 + \frac{4x_1y_1}{|x - \tilde{y}|^2}\right)^{-(n-2)/2} - \left(1 + \frac{4x_1y_1}{|x - y|^2}\right)^{-(n-2)/2}\right)|x - y|^{2-n} \\ &\quad + C_n\left(1 - \left(1 + \frac{4x_1y_1}{|x - y|^2}\right)^{-(n-2)/2}\right) \\ &\quad \times \left(1 - \left(1 + \frac{4x_2y_2}{|x - y|^2}\right)^{-(n-2)/2}\right)|x - y|^{2-n} \\ &= I + II. \end{aligned} \tag{42}$$

But

$$I \le C_n |x-y|^{2-n}\left(1-\left(1+\frac{4x_1y_1}{|x-\tilde{y}|^2}\right)^{(n-2)/2}\left(1+\frac{4x_1y_1}{|x-y|^2}\right)^{-(n-2)/2}\right)$$
$$= C_n |x-y|^{2-n}\left(1-\left(1+\frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2+4x_1y_1+4x_2y_2}\right)^{-(n-2)/2}\right).$$

By the same reasoning that leads to (35), we have

$$I \le c(n)|x-y|^{2-n}\min\left\{1, \frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2+4x_1y_1+4x_2y_2}\right\}.$$

We claim that

$$\begin{aligned}&\min\left\{1, \frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2+4x_1y_1+4x_2y_2}\right\}\\ &\qquad \le 16\min\left\{1, \frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1, \frac{x_2y_2}{|x-y|^2}\right\}.\end{aligned} \tag{43}$$

This is because when $x_1y_1 \ge |x-y|^2$ and $x_2y_2 \ge |x-y|^2$, clearly (43) holds as its right-hand side is 16. If $x_1y_1 \ge |x-y|^2$ but $x_2y_2 < |x-y|^2$, then

$$\begin{aligned}&\frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2+4x_1y_1+4x_2y_2}\\ &\qquad \le \frac{4x_2y_2}{|x-y|^2} \le 4\min\left\{1, \frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1, \frac{x_2y_2}{|x-y|^2}\right\}\end{aligned}$$

so (43) is satisfied. Similarly, (43) holds when $x_1y_1 < |x-y|^2$ but $x_2y_2 \ge |x-y|^2$. Finally, when $x_1y_1 < |x-y|^2$ and $x_2y_2 < |x-y|^2$,

$$\begin{aligned}&\min\left\{1, \frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2+4x_1y_1+4x_2y_2}\right\}\\ &\qquad \le \frac{4x_1y_1}{|x-y|^2}\frac{4x_2y_2}{|x-y|^2} \le 16\min\left\{1, \frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1, \frac{x_2y_2}{|x-y|^2}\right\}.\end{aligned}$$

So (43) is proved for every $x, y \in D$ and therefore we have

$$I \le c(n)|x-y|^{2-n}\min\left\{1, \frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1, \frac{x_2y_2}{|x-y|^2}\right\}.$$

By the same reasoning that leads to (35), one has

$$1-\left(1+\frac{4x_1y_1}{|x-y|^2}\right)^{-(n-2)/2} \approx \min\left\{1, \frac{x_1y_1}{|x-y|^2}\right\}$$

and

$$1-\left(1+\frac{4x_2y_2}{|x-y|^2}\right)^{-(n-2)/2}\approx\min\left\{1,\frac{x_2y_2}{|x-y|^2}\right\}$$

and so the second term in (42) becomes

$$II\approx|x-y|^{2-n}\min\left\{1,\frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1,\frac{x_2y_2}{|x-y|^2}\right\}.$$

Thus we conclude

$$\begin{aligned}G_D(x,y)&=I+II\\&\approx|x-y|^{2-n}\min\left\{1,\frac{x_1y_1}{|x-y|^2}\right\}\min\left\{1,\frac{x_2y_2}{|x-y|^2}\right\}.\end{aligned}\tag{44}$$

Similarly to the proof of Proposition 4.4, one can easily deduce from (44) that

$$G_D(x,y)\approx\frac{x_1y_1x_2y_2}{|x-y|^{n-2}(|x-y|^2+x_1^2+y_1^2)(|x-y|^2+x_2^2+y_2^2)}.\tag{45}$$

If we put

$$\rho(x,y)=|x-y|^{n-2}(|x-y|^2+x_1^2+y_1^2)(|x-y|^2+x_2^2+y_2^2),$$

then (41) is equivalent to the following inequality:

$$\rho(x,z)\le C\big(\rho(x,y)+\rho(y,z)\big),\qquad x,y,z\in D,\tag{46}$$

for some $C=C(D)>0$. Similarly to the proof of (37), we prove the above inequality by showing that each term on the right-hand side below,

$$\begin{aligned}\rho(x,z)=|x-z|^{n+2}&+|x-z|^nx_1^2+|x-z|^ny_1^2+|x-z|^nx_2^2\\&+|x-z|^ny_2^2+|x-z|^{n-2}x_1^2x_2^2+|x-z|^{n-2}x_1^2y_2^2\\&+|x-z|^{n-2}y_1^2x_2^2+|x-z|^{n-2}y_1^2y_2^2,\end{aligned}$$

is bounded by $C(\rho(x,y)+\rho(y,z))$ for some constant $C=C(n)>0$. The proofs of these are elementary and similar to what we did in proving (37). $\square$

Therefore, for $1\le k\le n$ and $D=\{x\in\mathbb{R}^n:x_1>0,\dots,x_k>0\}$, $q\in\mathbf{S}_\infty(W^D)$ if and only if the family of functions

$$\left\{y\to\frac{y_1\cdots y_k}{x_1\cdots x_k}G_D(x,y)\,\big|\,q(y);x\in D\right\}$$

is uniformly integrable in D.

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