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THE EULER SCHEME WITH IRREGULAR COEFFICIENTS[1]

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Weak convergence of the Euler scheme for stochastic differential equations is established when coefficients are discontinuous on a set of Lebesgue measure zero. The rate of convergence is presented when coefficients are Hölder continuous. Monte Carlo simulations are also discussed.

1. Introduction. We consider the following stochastic differential equation (SDE) with coefficients b and σ, driven by a Brownian motion B in $\mathbb{R}^r$,

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s, \tag{1}$$

where X_0 is an $\mathbb{R}^d$-valued random variable, which is independent of B, b is a d-dimensional function of $\mathbb{R}^{d+1}$, and $\sigma = (\sigma_{ij})$ is a $d \times r$ matrix-valued function of $\mathbb{R}^{d+1}$. For background information about SDEs, we refer to Chapter 5 of Protter [19], Chapter 9 of Revuz and Yor [21] and Chapter 5 of Karatzas and Shreve [13]. In applications one often wants to solve the SDE (1) numerically, when possible. Because of simulation difficulties, it is usually advisable to solve the SDE (1) with an Euler scheme, rather than a more complicated one. See the survey paper of Talay [22] for a discussion of this issue. The continuous Euler scheme $\{X_t^n : 0 \le t \le T\}$ for the SDE (1) on the time interval $[0, T]$ is defined as follows: $X_0^n = X_0$, and

$$X_t^n = X_{\tau_k^n}^n + b\big(\tau_k^n, X_{\tau_k^n}^n\big)\big(t - \tau_k^n\big) + \sigma\big(\tau_k^n, X_{\tau_k^n}^n\big)\big(B_t - B_{\tau_k^n}\big), \tag{2}$$

for $\tau_k^n < t \le \tau_{k+1}^n$, $k = 0, 1, \dots, n$, where $0 = \tau_0^n \le \tau_1^n \le \cdots \le \tau_n^n = T$ is a sequence of random partitions of $[0, T]$. τ_k^n is not necessary a stopping time. If we define $\bar{X}_t^n = X_{\tau_k^n}^n$, for $\tau_k^n \le t < \tau_{k+1}^n$, then $\{\bar{X}_t^n : 0 \le t \le T\}$ is called a discretized Euler scheme. Our goal of this paper is to study the conditions, without assuming a continuity condition on b and σ, under which the Euler scheme converges to the solution of the SDE (1) as long as a weak solution exits and is unique. Moreover, we want to determine the rate of convergence without assuming the Lipschitz condition.

We have two motivations for this problem. The first one is inspired by a result of Englebert and Schmidt [10], who gave necessary and sufficient conditions on

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b and σ in order that the SDE (1) have a weak solution and that it be unique. These necessary and sufficient conditions do not include the Lipschitz condition (or even continuity) on b and σ. Lipschitz conditions however are standard in prior work on showing that the Euler scheme converges weakly to the strong solution. [The SDE (1) has a unique strong solution under Lipschitz conditions.] We want to know how much we can relax the conditions and still have the convergence and some rate of convergence.

Our second motivation is the computation of the expectation of functionals of solutions of SDEs arising from probabilistic models, for example, the calculation of the energy of the response of a stochastic dynamical system or the price of a capital asset. In such models, one is often interested in estimating quantities of the form $E[f(X_T)]$ or $E[\int_0^T f(X_s)\,ds]$ for a fixed nonrandom time T and some function f. For example, in capital asset pricing models (CAPM), such quantities can represent the price of a financial derivative, such as options. One wants to use a Monte Carlo technique to estimate $E[f(X_T)]$ or $E[\int_0^T f(X_s)\,ds]$, but one cannot do it immediately in general the distributions of $f(X_T)$ or of $\int_0^T f(X_s)\,ds$ are not known. Instead, one can approximate $E[f(X_T)]$ or $E[\int_0^T f(X_s)\,ds]$ by using a numerical simulation of the solution $\{X_t : 0 \le t \le T\}$ of the SDE (1). The simplest such scheme is the Euler scheme. Once we know the convergence or the rate of convergence of X^n, we will have an idea how well $E[f(X_T^n)]$ and $\frac{T}{n}\sum_{k=1}^n E[f(X_{t_k}^n)]$ can approximate $E[f(X_T)]$ and $E[\int_0^T f(X_s)\,ds]$, respectively.

Relaxing the Lipschitz condition to a Hölder continuity condition, or even dropping the continuity condition altogether is not only a mathematical extension but it also has possible applications in practice. For example, in stochastic finance theory, the Black–Scholes model is a standard CAPM. It assumes that the security follows a SDE (1) with its diffusion function being proportional to the level of the security price [i.e., $\sigma(s, X_s) = \sigma X_s$], or in finance terms, with a constant volatility σ. While practitioners find that the volatility changes constantly, randomly and with small or big jumps, sometimes, when the stock prices reach some specific levels, their volatility becomes very large due to sudden heavy trading. When using a Black–Scholes model, we always update the model daily or half-daily, in that case, we are actually using a model with the volatility function being a step function of time and its jump size depending on the price levels of its underlying assets. This example illustrates that it is desirable to have a pricing model with a discontinuous diffusion function.

An example of a model that leads to a truly discontinuous volatility, again coming from finance theory, might be as follows: imagine an asset price that has certain psychological trigger points. A current example may be the price of Euros given in U.S. dollars. Parity (€1.00 = \$1.00) has a certain appeal, it is easy to imagine a sudden change in volatility if that level is breached. A trigger point might be €1.00 = \$0.80, or €1.00 = \$1.18. A more subtle example may occur when stock prices of certain industry sectors surpass traditional psychological bounds for price/earnings ratios (P/E). The volatility of these stocks may suddenly increase. Discontinuities of volatility may also occur due to sudden external shocks, such as Federal Reserve "irrational exuberance" remarks. A last example from finance theory is that the drift function $b(s, X_s)$ in SDE (1) may also have discontinuities. For example, if one company announces a takeover of another, a sudden jump in the drift of one or both companies can occur. This also can create volatility with discontinuities.

The rate of convergence of the Euler scheme has been studied in many papers for various convergence criteria: for convergence rate of the expectation of functionals of solutions of SDEs (1) with smooth coefficients, see Talay and Tubaro [23]; for convergence rate of the distribution function, see Bally and Talay [2]; for convergence rate of the density, see Bally and Talay [3]; for error analysis, see Bally and Talay [1]; for reviews, see Talay [22] or Kloeden and Platen [14]. The case of SDEs driven by discontinuous semimartingales was studied by Kurtz and Protter [16] in weak convergence of the normalized Euler scheme error; L^p estimates of the Euler scheme error were given by Kohatsu-Higa and Protter [15]. Protter and Talay [20] also studied the Euler scheme for SDE driven by Lèvy processes. Protter and Jacod [12] obtained a celebrated result about the asymptotic error distributions for the Euler scheme for SDEs driven by a vector of semimartingales. A basic assumption of these prior works is that the coefficient function satisfies a Lipschitz condition. The case of SDEs with discontinuous coefficients has barely been investigated. Chan and Stramer [7] studied the weak convergence of the Euler scheme for SDEs with coefficients satisfying some regularity conditions.

In this paper, we prove the weak convergence of the Euler scheme by the martingale representation theorem. Efforts are directed at the quadratic variation of the limit of the Euler scheme. In Section 2 we consider the Euler scheme with uniform partitions for the SDE without drift. In Section 3 we study the Euler schemes with general partitions for a system of SDEs. In Section 4 we use an estimation of the local time of the error process of the Euler scheme to get rates of convergence by the Meyer–Tanaka formula and Gronwall's lemma. In Section 5 we discuss the Monte Carlo approximation of $E[f(X_T)]$ and $E[\int_0^T f(X_s)\,ds]$ by simulating a discrete-time Euler scheme of SDEs.

2. SDEs without drift.

In this section we consider the following SDE driven by a Brownian motion B in $\mathbb{R}^1$,

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dB_s, \tag{3}$$

where X_0 is an $\mathbb{R}^1$-valued random variable, which is independent of B, and σ is a measurable function in $\mathbb{R}^1$. Its Euler scheme with uniform partitions is defined as follows: $X_0^n = X_0$, and

$$X_t^n = X_{t_k}^n + \sigma(X_{t_k}^n)(B_t - B_{t_k}), \tag{4}$$

for $t_k < t \le t_{k+1}$, where $t_k = kT/n$, $k = 0, 1, \dots, n$. If we define $\eta_n(t) = t_k$ for $t_k < t \le t_{k+1}$, then this Euler scheme can be written as

$$X_t^n = X_0 + \int_0^t \sigma(X_{\eta_n(s)}^n)\, dB_s. \tag{5}$$

Under the assumption that the SDE (3) has a unique weak solution, we study the conditions under which the Euler scheme $\{X_t^n : 0 \le t \le T\}$ converges weakly to the weak solution $\{X_t : 0 \le t \le T\}$ of the SDE (3). In order to obtain the weak convergence of the Euler scheme, it is necessary to have tightness. For this purpose, we assume that $\sigma(\cdot)$ has at most linear growth, that is, there exist two constants c_1 and c_2 such that $|\sigma(x)| \le c_1 + c_2|x|$ for all $x \in \mathbb{R}^1$. From this we obtain the uniform boundedness of the fourth moment of X^n in Lemma 2.1, which implies that $\{X^n : n \ge 1\}$ is tight in $C[0, T]$, the space of all continuous functions on $[0, T]$ with the uniform topology. To ensure that the weak limit of the Euler scheme is the weak solution of SDE (3), we assume that $\sigma(\cdot)$ is continuous almost everywhere with respect to Lebesgue measure, and that the limit inferior of $\sigma^2(\cdot)$ is not zero at its discontinuity points, which is illustrated in Theorem 2.2.

LEMMA 2.1. *If $E(X_0)^4 < \infty$ and $\sigma(\cdot)$ has at most linear growth, then*

$$\sup_{n\ge 1} E|(X^n)_T^*|^4 < \infty \quad \textit{and} \quad \sup_{n\ge 1} E[X^n, X^n]_T^2 < \infty,$$

where $(X)_T^* = \max_{0\le t\le T} |X_s|$, $[X, X]_T$ is the quadratic variation of X.

PROOF. Since $X_{t_{k+1}}^n = X_{t_k}^n + \sigma(X_{t_k}^n)(B_{t_{k+1}} - B_{t_k})$, by induction on k we know that $E(X_{t_k}^n)^4 < \infty$ for any k. Taking the fourth moment of $X_{t_{k+1}}^n$,

$$E(X_{t_{k+1}}^n)^4 = E(X_{t_k}^n)^4 + \frac{3T^2}{n^2} E\sigma^4(X_{t_k}^n) + \frac{6T}{n} E(X_{t_k}^n \sigma(X_{t_k}^n))^2.$$

Because $\sigma(\cdot)$ has at most linear growth, there exits a constant c such that $E(X_{t_{k+1}}^n)^4 \le (1 + c/n)E(X_{t_k}^n)^4 + c/n$. Recursively, $E(X_T^n)^4 = E(X_{t_n}^n)^4 \le (1 + c/n)^n(E(X_0)^4 + 1) \le (E(X_0)^4 + 1)e^c$. Since $\{X_t^n : 0 \le t \le T\}$ is a martingale for every fixed n, the lemma follows by Doob's L^4 maximal inequality and Burkholder's inequality. $\square$

LEMMA 2.2. *If the conditions of Lemma* 2.1 *are satisfied*, *then the Euler scheme $\{X^n : n \ge 1\}$ is tight in $C[0, T]$.*

PROOF. We will use the criterion in Theorem 8.3 on page 56 of Billingsley [4] to prove the tightness. First, since $E(X_0)^4 < \infty$, $\{X_0^n = X_0, n \geq 1\}$ is tight. Second, let $c = \sup_{n\geq 1} E(\sigma^4(X_{\eta_n}^n))_T^*$, since $\sigma(\cdot)$ has at most linear growth, c is a finite constant by Lemma 2.1. For fixed $t \in [0, T]$, let

$$Z_s^n = X_{t+s}^n - X_t^n = \int_t^{t+s} \sigma\big(X_{\eta_n(r)}^n\big)\, dB_r.$$

By the Cauchy–Schwarz inequality,

$$E([Z^n, Z^n]_s)^2 = E\bigg(\int_t^{t+s} \sigma^2\big(X_{\eta_n(r)}^n\big)\, dr\bigg)^2 \leq s E \int_t^{t+s} \sigma^4\big(X_{\eta_n(r)}^n\big)\, dr \leq cs^2.$$

Since $Z_s^n$is a continuous martingale, by Burkholder's inequality there exists a constant c_4 such that $E((Z^n)_\delta^*)^4 \leq c_4 E([Z^n, Z^n]_\delta)^2 \leq c_4 c\delta^2$ for $\delta > 0$. Now, for $\forall \varepsilon > 0$, $\eta > 0$, $\exists \delta = \varepsilon^4 \eta (c_4 c)^{-1}$, which does not depend on t and n, such that

$$P\Big(\sup_{t\leq s\leq t+\delta} |X_s^n - X_t^n| \geq \varepsilon\Big) \leq \varepsilon^{-4} E\big((Z^n)_\delta^*\big)^4 \leq \varepsilon^{-4} c_4 c\delta^2 = \delta\eta.$$

Therefore $\{X^n, n \geq 1\}$ is tight in $C[0, T]$. $\square$

Since $\{X^n, n \geq 1\}$ is tight in $C[0, T]$, which is a separable and complete space, by Prohorov's Theorem, $\{X^n, n \geq 1\}$ is relatively compact in $C[0, T]$. Thus for any subsequence n' there exists a subsubsequence n'_k of n' and a process X in $C[0, T]$ such that $X^{n'_k}$ converges to X weakly. By the almost sure representation Theorem 1.10.4 on page 59 of van der Vaart and Wellner [25] there exist a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ and a sequence of processes Y^k and Y, defined on $\overline{\Omega}$ taking values in $C[0, T]$ with $\mathcal{L}(Y^k) = \mathcal{L}(X^{n'_k})$ for all $k \geq 1$, $\mathcal{L}(X) = \mathcal{L}(Y)$, and $\lim_{k\to\infty} Y^k = Y$almost surely in $C[0, T]$. Furthermore, we can choose Y^k and $Y = Y^\infty$as follows:

$$Y^k(\overline{\omega}) = X^{n'_k}\big(\phi_k(\overline{\omega})\big), \tag{6}$$

with the maps ϕ_k measurable, and $P^k = \overline{P} \circ \phi_k^{-1}$, for $k = 1, 2, \ldots, \infty$, where P^k is the probability measure on the original probability space where $X^{n'_k}$ lives on. Since we build up our Euler scheme $\{X^n : n \geq 1\}$ on the same probability space, P^k actually does not depend on k. If we define

$$W^k(\overline{\omega}) = B\big(\phi_k(\overline{\omega})\big),$$

then W^k is a Brownian motion on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. This is because for any Borel set D,

$$\begin{aligned}\overline{P}(W^k \in D) &= \overline{P}\{B\big(\phi_k(\overline{\omega})\big) \in D\} = \overline{P}\{\phi_k^{-1}\big(B^{-1}(D)\big)\} \\ &= P^k\big(B^{-1}(D)\big) = P(B \in D).\end{aligned}$$

Therefore for every $n \geq 1$, Y^n satisfies the following equation:

$$Y_t^n = Y_0 + \int_0^t \sigma\big(Y_{\eta_n(s)}^n\big)\, dW_s^n, \tag{7}$$

which implies that $[Y^n, Y^n]_t = \int_0^t \sigma^2(Y_{\eta_n(s)}^n)\, ds$. By Lemma A.1 in the Appendix, we can claim that Y_t is a continuous martingale with respect to its natural filtration $\overline{\mathcal{F}_t} = \sigma(Y_s, s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ is the class of all null sets of $\overline{\mathcal{F}}$ under the probability measure $\overline{P}$, because Y_t is the strong limit of Y_t^n, which is a martingale for each n by (7) and which is uniformly integrable due to the uniform boundedness of its second moment by Lemma 2.1. In order to use the martingale representation theorem to show that Y is the unique weak solution of (3), we will first show that $[Y, Y]_t = \int_0^t \sigma^2(Y_s)\, ds$.

LEMMA 2.3. *If the conditions of Lemma* 2.1 *are satisfied, then for* $0 \leq t \leq T$,

$$[Y^n, Y^n]_t \xrightarrow{L^1} [Y, Y]_t, \tag{8}$$

$$\int_0^t \sigma^2\big(Y_{\eta_n(s)}^n\big)\mathbb{1}(Y_s \notin D_\sigma)\, ds \xrightarrow{L^1} \int_0^t \sigma^2(Y_s)\mathbb{1}(Y_s \notin D_\sigma)\, ds, \tag{9}$$

where D_σ is the set of discontinuous points of $\sigma(\cdot)$, $\mathbb{1}(\cdot)$ is an indicator function.

PROOF. By (7) and Lemma 2.1, $E([Y^n, Y^n]_t)^2$ are uniformly bounded for all n and t. Since Y^n is a continuous martingale with respect to its own filtration and it converges to Y almost surely in $C[0, T]$, by Theorem 2.2 of Kurtz and Protter [17], $\int Y^n\, dY^n$ converges to $\int Y\, dY$ in probability. Since $[X, X] = X^2 - 2\int X\, dX$ for any continuous semimartingale X, $[Y^n, Y^n]$ converges to $[Y, Y]$ in probability. The uniform boundedness of $E([Y^n, Y^n]_s)^2$ and $E\sigma^4(Y_{\eta_n(s)}^n)$ implies that $\{[Y^n, Y^n]_t,\ n \geq 1\}$ and $\{\int_0^t \sigma^2(Y_{\eta_n(s)}^n)\mathbb{1}(Y_s \notin D_\sigma)\, ds,\ n \geq 1\}$ are uniformly integrable, from which L^1 convergence follows. □

In the next theorem we give a necessary and sufficient condition under which the Euler scheme converges weakly to the unique weak solution. For the necessary and sufficient condition to establish the existence and uniqueness of the weak solution for the SDE (3), we refer to Englebert and Schmidt [10].

THEOREM 2.1. *If $\sigma(\cdot)$ has at most linear growth with D_σ of Lebesgue measure zero and $E(X_0)^4 < \infty$, then the Euler scheme $\{X_t^n : 0 \leq t \leq T\}$ defined in* (4) *converges weakly to the unique weak solution of the SDE* (3) *if and only if the following conditions*,

$$\int_0^T \mathbb{1}(Y_s \in D_\sigma \cap N^c)\, ds = 0 \qquad a.s., \tag{10}$$

$$\lim_{n\to\infty} E\int_0^T \sigma^2\big(Y_{\eta_n(s)}^n\big)\mathbb{1}(Y_s \in D_\sigma \cap N)\, ds = 0, \tag{11}$$

hold for all sequences Y^n, which are chosen as (6), *where Y is the almost sure limit process of Y^n and $N = \{x \in \mathbb{R}^1 : \sigma(x) = 0\}$.*

PROOF. We prove the necessity first. If X is the unique weak solution of the SDE (3), then there exists a Brownian motion B on some probability space such that the SDE (3) holds. By the occupation time formula,

$$\int_0^T \sigma^2(X_s)\mathbb{1}(X_s \in D_\sigma)\,ds = \int_0^T \mathbb{1}(X_s \in D_\sigma)\,d[X,X]_s = \int_{D_\sigma} L_T^x\,dx = 0,$$

which implies that $\int_0^T \mathbb{1}(X_s \in D_\sigma \cap N^c)\,ds = 0$. If the Euler scheme X^n converges weakly to X, then X and Y have the same law. We can conclude (10) and

$$\begin{aligned} E[Y,Y]_T = E[X,X]_T &= E\int_0^T \sigma^2(X_s)\,ds \\ &= E\int_0^T \sigma^2(X_s)\mathbb{1}(X_s \notin D_\sigma)\,ds = E\int_0^T \sigma^2(Y_s)\mathbb{1}(Y_s \notin D_\sigma)\,ds. \end{aligned}$$

On the other hand, since $[Y^n, Y^n]_T = \int_0^T \sigma^2(Y^n_{\eta_n(s)})\,ds$ by (7),

$$\begin{aligned} &\lim_{n\to\infty} E\int_0^T \sigma^2(Y^n_{\eta_n(s)})\mathbb{1}(Y_s \in D_\sigma)\,ds \\ &\quad= \lim_{n\to\infty} E[Y^n, Y^n]_T - \lim_{n\to\infty} E\int_0^T \sigma^2(Y^n_{\eta_n(s)})\mathbb{1}(Y_s \notin D_\sigma)\,ds \qquad (12) \\ &\quad= E[Y,Y]_T - E\int_0^T \sigma^2(Y_s)\mathbb{1}(Y_s \notin D_\sigma)\,ds = 0 \end{aligned}$$

by Lemma 2.3.

Combining (10) and (12) we can get (11). Next we prove the sufficiency. When (10) and (11) hold, we have

$$\int_0^T \sigma^2(Y_s)\mathbb{1}(Y_s \in D_\sigma)\,ds = 0,$$

$$\lim_{n\to\infty} E\int_0^T \sigma^2(Y^n_{\eta_n(s)})\mathbb{1}(Y_s \in D_\sigma)\,ds = 0.$$

It follows from this and Lemma 2.3 that for $0 \le t \le T$

$$\int_0^t \sigma^2(Y^n_{\eta_n(s)})\,ds \xrightarrow{L^1} \int_0^t \sigma^2(Y_s)\,ds. \qquad (13)$$

Since the left-hand side of (13) is equal to $[Y^n, Y^n]$ and it converges to $[Y,Y]$ in L^1 by Lemma 2.3, we have $[Y,Y]_t = \int_0^t \sigma^2(Y_s)\,ds$. Since Y is a square integrable continuous martingale, by Theorem 7.1$'$ on page 90 of Ikeda and Watanabe [11], there exists a Brownian motion W on a possibly enlarged probability space such that

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\,dW_s.$$

That is, Y is the weak solution of the SDE (3). Since $Y^k \overset{\text{a.s.}}{\to} Y$ as $k \to \infty$ and $\mathcal{L}(Y^k) = \mathcal{L}(X^{n'_k})$, $X^{n'_k}$ converges weakly to Y, that is, for any subsequence X^{n_k} of the Euler scheme X^n, there is always a subsubsequence $X^{n'_k}$ converging weakly to the unique weak solution, which implies that the Euler scheme (4) converges weakly to the unique weak solution of SDE (3). $\square$

Next we give a sufficient condition under which the Euler scheme (4) converges weakly to the weak solution of SDE (3). Let $\sigma_1^2(y) = \liminf_{x\to y} \sigma^2(x)$.

THEOREM 2.2. *Suppose that $\sigma(\cdot)$ has at most linear growth with D_σ of Lebesgue measure zero and that $E(X_0)^4 < \infty$. If $\sigma_1^2(y) > 0$ for $y \in D_\sigma$, then the Euler scheme* (4) *converges weakly to the unique weak solution of SDE* (3).

PROOF. By Theorem 2.1 it suffices to prove that $\int_0^T \mathbb{1}(Y_s \in D_\sigma)\,ds = 0$ a.s. For $0 \le r \le s \le t \le T$,

$$\begin{aligned} E([Y,Y]_t - [Y,Y]_r) &= \lim_{n\to\infty} E([Y^n,Y^n]_t - [Y^n,Y^n]_r) \\ &= \lim_{n\to\infty} E\int_r^t \sigma^2\big(Y^n_{\eta_n(s)}\big)\,ds \ge E\int_r^t \liminf_{n\to\infty} \sigma^2\big(Y^n_{\eta_n(s)}\big)\,ds \\ &= E\int_r^t \sigma_1^2(Y_s)\,ds. \end{aligned}$$

By the occupation time formula,

$$\begin{aligned} E\int_0^T \mathbb{1}(Y_s \in D_\sigma)\sigma_1^2(Y_s)\,ds &\le E\int_0^T \mathbb{1}(Y_s \in D_\sigma)\,d[Y,Y]_s \\ &= E\int_{D_\sigma} L_T^x(Y)\,dx = 0, \end{aligned}$$

because D_σ has Lebesgue measure zero and $L_T^x(Y) < \infty$ for almost all x. Since $\sigma_1^2(\cdot) > 0$ on D_σ, $\int_0^T \mathbb{1}(Y_s \in D_\sigma)\,ds = 0$ a.s. $\square$

Next, we extend our results to the Euler schemes with a general partitions for a system of SDEs. Since we use the local time technique in one dimensional case, we will put some conditions on the projections of the sets of discontinuity points of the coefficients onto each time and space axis in the multi-dimensional case.

3. A system of SDEs. In this section we study the conditions under which the Euler scheme defined in (2) converges weakly to the weak solution of the SDE (1) driven by a Brownian motion in $\mathbb{R}^r$.

LEMMA 3.1. *If $E|X_0|^4 < \infty$, and the coefficients b and σ have at most linear growth, that is, there exist two constants c_1 and c_2 such that, for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, $|b(t,x)| + |\sigma(t,x)| \le c_1 + c_2|x|$, then*

$$\sup_{n\ge 1} E|(X^n)^*_T|^4 < \infty \quad \textit{and} \quad \sup_{n\ge 1} E|[X^n, X^n]_T|^2 < \infty,$$

where $|\cdot|$ stands for Euclidean norm in the appropriate space.

PROOF. If we define $\eta_n(s) = \tau^n_k$ for $\tau^n_k < s \le \tau^n_{k+1}$, then

$$X^n_t = X_0 + \int_0^t b\big(\eta_n(s), X^n_{\eta_n(s)}\big)\, ds + \int_0^t \sigma\big(\eta_n(s), X^n_{\eta_n(s)}\big)\, dB_s.$$

By the inequality $(a+b+c)^4 \le 3^3(a^4 + b^4 + c^4)$ for any real numbers a, b, c and Hölder's inequality and Burkholder's inequality, there exist two positive constants a and b such that

$$E|X^n_t|^4 \le a + b\int_0^t E\big|X^n_{\eta_n(s)}\big|^4\, ds. \tag{14}$$

Let $f_n(t) = E|X^n_{\eta_n(t)}|^4$, since $\eta_n(t) \le t$, $f_n(t) \le a + b\int_0^t f_n(s)\, ds$. By Gronwall's lemma, $f_n(t) \le a\exp(bt)$ for $0 \le t \le T$. By (14) and Burkholder's inequality we can conclude the lemma. $\square$

By the same argument as used in the proof of Lemma 2.2, we can show that $\{X^n,\ n \ge 1\}$ is tight in $C[0,T]$. Therefore it is relatively compact in $C[0,T]$. By the arguments used in Section 2, for any subsequence n' of n there exists a subsubsequence n'_k of n', and a process X in $C[0,T]$ such that $X^{n'_k}$ converges weakly to X. By the almost sure representation theorem there exist a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ and a sequence of processes Y^k and Y, defined on $\overline{\Omega}$ taking values in $C[0,T]$ with $\mathcal{L}(Y^k) = \mathcal{L}(X^{n'k})$ for all $k \ge 1$, $\mathcal{L}(X) = \mathcal{L}(Y)$, and $\lim_{k\to\infty} Y^k = Y$ almost surely in $C[0,T]$. Furthermore, we can choose Y^k and a Brownian motion W^n on $\overline{\Omega}$ such that

$$Y^n_t = Y_0 + \int_0^t b\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, ds + \int_0^t \sigma\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, dW^n_s, \tag{15}$$

which implies that

$$[Y^n(i), Y^n(j)]_t = \sum_{k=1}^r \int_0^t \sigma_{ik}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\sigma_{jk}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, ds. \tag{16}$$

Since Y^n converges to Y almost surely and $E|(Y^n)^*_T|^4 < \infty$, by Theorem 2.2 of Kurtz and Protter [17], for $0 \le t \le T$

$$[Y^n(i), Y^n(j)]_t \xrightarrow{L^1} [Y(i), Y(j)]_t. \tag{17}$$

In order that the weak limit of X^n be the solution of the SDE (1), we make the following assumptions on the discontinuity points of b and σ:

H1: For every i from 1 to d, we have $\lambda(D^0_{b_i}) = 0$ or there exists a k such that $\lambda(D^k_{b_i}) = 0$, and $\sigma^2_k(t, y) > 0$ for $(t, y) \in D_{b_i}$.
H2: For every pair of (i, j) from 1 to d, we have $\lambda(D^0_{\sigma_{ij}}) = 0$ or there exists a k such that $\lambda(D^k_{\sigma_{ij}}) = 0$, and $\sigma^2_k(t, y) > 0$ for $(t, y) \in D_{\sigma_{ij}}$.

Here D_{b_i} and $D_{\sigma_{ij}}$ are the sets of discontinuity points of b_i and σ_{ij}, respectively; λ is Lebesgue measure on $\mathbb{R}^1$; $D^0_{b_i}$ and $D^0_{\sigma_{ij}}$ are the projection of D_{b_i} and $D_{\sigma_{ij}}$ onto the t axis respectively; that is, $D^0_{b_i} = \{0 \le t \le T : \exists (x_1, \dots, x_d) \ni (t, x_1, \dots, x_d) \in D_{b_i}\}$, $D^k_{b_i}$ and $D^k_{\sigma_{ij}}$ are the projection of D_{b_i} and $D_{\sigma_{ij}}$ onto the x_k axis, respectively. $\sigma^2_k(t, y) = \liminf_{s \to t, x \to y} (\sigma^2_{k1}(s, x) + \sigma^2_{k2}(s, x) + \cdots + \sigma^2_{kr}(s, x))$.

LEMMA 3.2. *Under the conditions of Lemma* 3.1, *if hypotheses* H1 *and* H2 *hold*, *then for all* i, j *and* t,

$$\text{(i)} \int_0^t \mathbb{1}\big((s, Y_s) \in D_{b_i}\big)\,ds = 0 \quad \textit{and} \quad \text{(ii)} \int_0^t \mathbb{1}\big((s, Y_s) \in D_{\sigma_{ij}}\big)\,ds = 0 \qquad \textit{a.s.}$$

PROOF. Since (i) and (ii) can be proved by the similar arguments, we prove only (i) here. For fixed i, if $\lambda(D^0_{b_i}) = 0$ then

$$\int_0^t \mathbb{1}\big((s, Y_s) \in D_{b_i}\big)\,ds \le \int_0^t \mathbb{1}\big(s \in D^0_{b_i}\big)\,ds \le \lambda\big(D^0_{b_i}\big) = 0. \tag{18}$$

If there exists a k such that $\lambda(D^k_{b_i}) = 0$ and $\sigma^2_k(t, y) > 0$ on D_{b_i}, then we prove (i) by using the local time formula. For $0 \le t_1 \le s \le t_2 \le T$,

$$\begin{aligned}
&E\big([Y(k), Y(k)]_{t_2} - [Y(k), Y(k)]_{t_1}\big) \\
&\quad = \lim_{n \to \infty} E\big([Y^n(k), Y^n(k)]_{t_2} - [Y^n(k), Y^n(k)]_{t_1}\big) \\
&\quad = \lim_{n \to \infty} E \sum_{j=1}^r \int_{t_1}^{t_2} \sigma^2_{kj}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\,ds \\
&\quad \ge E \int_{t_1}^{t_2} \sigma^2_k(s, Y_s)\,ds,
\end{aligned}$$

and hence, by the occupation time formula,

$$\begin{aligned}E\int_0^t \mathbb{1}\big((s,Y_s)\in D_{b_i}\big)\sigma_k^2(s,Y_s)\,ds &\le E\int_0^t \mathbb{1}\big((s,Y_s)\in D_{b_i}\big)\,d[Y(k),Y(k)]_s\\ &\le E\int_0^t \mathbb{1}\big(Y_s(k)\in D_{b_i}^k\big)\,d[Y(k),Y(k)]_s\\ &= E\int_{D_{b_i}^k} L_t^x\big(Y(k)\big)\,dx = 0,\end{aligned}$$

because $D_{b_i}^k$ has Lebesgue measure zero and $L_t^x(Y(k))<\infty$ for x a.s. We conclude (i) by the assumption that $\sigma_k^2(\cdot)>0$ on D_{b_i}. $\square$

LEMMA 3.3. *Let* $\Delta^n=\max_{0\le k\le n}|\tau_{k+1}^n-\tau_k^n|$. *If the conditions in Lemma* 3.2 *are satisfied*, *and* $\lim_{n\to\infty}\Delta^n=0$, *then for all* i, j *and* $0\le t\le T$, *as* n *goes to* ∞,

$$\int_0^t b_i\big(\eta_n(s),Y_{\eta_n(s)}^n\big)\,ds \overset{L^2}{\to} \int_0^t b_i(s,Y_s)\,ds, \tag{19}$$

$$\int_0^t \sigma_{ik}\big(\eta_n(s),Y_{\eta_n(s)}^n\big)\sigma_{jk}\big(\eta_n(s),Y_{\eta_n(s)}^n\big)\,ds \overset{L^1}{\to} \int_0^t \sigma_{ik}(s,Y_s)\sigma_{jk}(s,Y_s)\,ds. \tag{20}$$

PROOF. We prove only (19) here, we can use similar arguments to prove (20). By the Cauchy–Schwarz inequality and Lemma 3.2,

$$\begin{aligned}&E\bigg(\int_0^t b_i\big(\eta_n(s),Y_{\eta_n(s)}^n\big)-b_i(s,Y_s)\,ds\bigg)^2\\ &\quad\le tE\int_0^t \big(b_i\big(\eta_n(s),Y_{\eta_n(s)}^n\big)-b_i(s,Y_s)\big)^2\,ds\\ &\quad= t\int_0^t E\big(b\big(\eta_n(s),Y_{\eta_n(s)}^n\big)-b(s,Y_s)\big)^2\mathbb{1}\big((s,Y_s)\notin D_{b_i}\big)\,ds\\ &\quad=: (I).\end{aligned}$$

Since b has at most linear growth and $\mathcal{L}(Y^n)=\mathcal{L}(X^n)$, by Lemma 3.1 we know that $Eb_i^4(\eta_n(s),Y_{\eta_n(s)}^n)$ is uniformly bounded for all $t\in[0,T]$ and $n\ge 1$. Therefore $\{b_i(\eta_n(s),Y_{\eta_n(s)}^n)\mathbb{1}(Y_s\notin D_{b_i}); n\ge 1\}$ is uniformly integrable. Since for any $s\in[0,T]$, $b_i(\eta_n(s),Y_{\eta_n(s)}^n)\mathbb{1}(Y_s\notin D_{b_i})\overset{\text{a.s.}}{\to} b_i(s,Y_s)\mathbb{1}(Y_s\notin D_{b_i})$ as $n\to\infty$, we have $E(b_i(\eta_n(s),Y_{\eta_n(s)}^n)-b_i(s,Y_s))^2\mathbb{1}((s,Y_s)\notin D_{b_i})\to 0$. By the dominated convergence theorem, (i) goes to zero. $\square$

Now let's denote $Z_t=(Z_t(1),\dots,Z_t(d))$ and $Z_t^n=(Z_t^n(1),\dots,Z_t^n(d))$, where

$$Z_t(i) = Y_t(i) - \int_0^t b_i(s, Y_s)\, ds,$$

$$\begin{aligned} Z_t^n(i) &= Y_t^n(i) - \int_0^t b_i\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, ds \\ &= Y_0(i) + \sum_{j=1}^r \int_0^t \sigma_{ij}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, dB_s^{n,j} \qquad \text{by (15).} \end{aligned}$$

LEMMA 3.4. *If the conditions in Lemma* 3.2 *are satisfied*, *then*

$$[Z(i), Z(j)]_t = \sum_{k=1}^r \int_0^t \sigma_{ik}(s, Y_s)\sigma_{jk}(s, Y_s)\, ds, \tag{21}$$

for all i, j *and* $0 \le t \le T$.

PROOF. Since $[Z^n, Z^n] = [Y^n, Y^n]$ and $[Z, Z] = [Y, Y]$, by (17) for any pairs of (i, j), $[Z^n(i), Z^n(j)] \xrightarrow{L^1} [Z(i), Z(j)]$. However, by (16) and Lemma 3.3,

$$\begin{aligned} [Z^n(i), Z^n(j)]_t &= \sum_{k=1}^r \int_0^t \sigma_{ik}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\sigma_{jk}\big(\eta_n(s), Y^n_{\eta_n(s)}\big)\, ds \\ &\xrightarrow{L^1} \sum_{k=1}^r \int_0^t \sigma_{ik}(s, Y_s)\sigma_{jk}(s, Y_s)\, ds \qquad \text{for every } t. \end{aligned} \tag{22}$$

Both limits must be equal, so the lemma follows. □

THEOREM 3.1. *If* $E|X_0|^4 < \infty$, $\lim_{n\to\infty} \Delta^n = 0$ *and* b *and* σ *have at most linear growth with the assumptions* H1 *and* H2 *satisfied*, *then the Euler scheme defined in* (2) *weakly converges to the unique weak solution of SDE* (1) *as* $n \to \infty$.

PROOF. Because Z^n is a martingale with respect to its own natural filtration and Z^n is uniformly integrable by Lemma 3.1, and Z^n converges to Z in probability by Lemma 3.3, by Lemma A.1 in the Appendix, we know that Z_t *is a continuous martingale in* $\mathbb{R}^d$ with the quadratic covariation

$$[Z(i), Z(j)]_t = \sum_{k=1}^r \int_0^t \sigma_{ik}(s, Y_s)\sigma_{ik}(s, Y_s)\, ds,$$

which is from Lemma 3.4. By Theorem 7.1′ on page 90 of Ikeda and Watanabe [11], there is a standard Brownian motion B_t in $\mathbb{R}^d$ on a possibly enlarged probability space such that

$$Z_t(i) = Z_0(i) + \sum_{j=1}^r \int_0^t \sigma_{ij}(s, Y_S)\, dB_s^j.$$

Since $Z_t(i) = Y_t(i) - \int_0^t b_i(s, Y_s)\,ds$, we have

$$Y_t(i) = Y_0(i) + \int_0^t b_i(s, Y_s)\,ds + \sum_{j=1}^{r} \int_0^t \sigma_{ij}(s, Y_S)\,dB_s^j;$$

that is, Y is the unique weak solution of SDE (1). Since $Y^k \overset{\text{a.s.}}{\to} Y$ as $k \to \infty$, and $\mathcal{L}(Y^k) = \mathcal{L}(X^{n'_k})$, we have $X^{n'_k}$ converges weakly to Y, where $X^{n'_k}$ is a subsubsequence of any subsequence of the Euler scheme, which implies that the Euler scheme converges weakly to the unique weak solution of SDE (1). □

4. The rate of convergence of the Euler scheme. In this section we will consider the rate of convergence of the Euler scheme for the following SDE driven by a Brownian motion $B \in \mathbb{R}^1$ on the time interval $[0, T]$:

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s, \tag{23}$$

where b and σ are measurable functions on $[0, T] \times \mathbb{R}^1$, and the initial point X_0 is independent of B. Suppose that the SDE (23) has a unique weak solution, that is, there exists a probability space $(\Omega, \mathcal{F}, P)$ on which we can define a standard Brownian motion B and a process X such that (23) holds. We can build up its Euler scheme on this probability space as follows: $X_0^n = X_0$,

$$X_t^n = X_{t_k}^n + b(t_k, X_{t_k}^n)(t - t_k) + \sigma(t_k, X_{t_k}^n)(B_t - B_{t_k}), \tag{24}$$

for $t_k < t \le t_{k+1}$, where $t_k = kT/n$ and $k = 0, 1, \dots, n$. If we define $\eta_n(t) = t_k$ for $t_k < t \le t_{k+1}$, then this Euler scheme can be written as

$$X_t^n = X_0 + \int_0^t b(\eta_n(s), X_{\eta_n(s)}^n)\,ds + \int_0^t \sigma(\eta_n(s), X_{\eta_n(s)}^n)\,dB_s. \tag{25}$$

Jacod and Protter [12] proved that when the coefficient functions are in C^1, the Euler scheme (24) converges weakly to the solution of the SDE (23) at the rate $1/\sqrt{n}$. That is, $\sqrt{n}(X_t^n - X_t)$ weakly converges to a process U_t, which satisfies a linear SDE. However, it seems to be not easy to obtain the rate of convergence of $X_t^n - X_t$ without assuming that the coefficient functions satisfy the Lipschitz conditions. For example, we consider the rate of convergence of $E(X_t^n - X_t)^2$, taking expectation on both sides after we apply Itô's formula for $(X_t^n - X_t)^2$, we have

$$\begin{aligned} E(X_t^n - X_t)^2 = 2\int_0^t E(X_s^n - X_s)\{b(\eta_n(s), X_{\eta_n(s)}^n) - b(s, X_s)\}\,ds \\ + \int_0^t E\{\sigma(\eta_n(s), X_{\eta_n(s)}^n) - \sigma(s, X_s)\}^2\,ds. \end{aligned}$$

It is hard to estimate $E(X_t^n - X_t)^2$ without the Lipschitz conditions of b and σ. Fortunately in the one dimensional case we can use the Meyer–Tanaka formula to decompose the error $|X_t^n - X_t|$ as a summation of a finite variation process, a martingale and a local time of $(X^n - X)$. By taking expectations we can obtain the rate of convergence of $E|X_t^n - X_t|$ by removing the martingale. A key step is to estimate the expectation of the local time of $(X^n - X)$. We do that first.

4.1. *A local time inequality.* Let X be a continuous semimartingale with $X_0 = 0$. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = 0$, $\tau_1 = \inf(t > 0 : X_t = \varepsilon)$, $\sigma_n = \inf(t > \tau_{n-1} : X_t = 0)$, $\tau_n = \inf(t > \sigma_n : X_t = \varepsilon)$. Let $U_t(X) = \sup\{n \in N : \tau_n < t\}$ be the number of upcrossings of X through $[0, \varepsilon]$ before time t. We denote $n(t) = t \wedge \sigma_{U_t(X)+1}$.

LEMMA 4.1. *Let X be a continuous semimartingale with $X_0 = 0$, for any $\varepsilon > 0$ and any real function $F(\cdot) \in C^2$ with $F(0) = 0$, we have*

$$\begin{aligned} L_t^0(X)\big(F(\varepsilon) - \varepsilon F'(0)\big) &= 2F(\varepsilon)\big(X_t^+ - X_{n(t)}^+\big) - 2\varepsilon\big(F(X_t^+) - F\big(X_{n(t)}^+\big)\big) \\ &\quad - 2\int_0^t \theta_s(X)\big(F(\varepsilon) - \varepsilon F'(X_s^+)\big)\,dX_s \\ &\quad + \int_0^t \theta_s(X)\varepsilon F''(X_s^+)\,d[X, X]_s, \end{aligned} \tag{26}$$

where $\theta_s(X) = \sum_{n=1}^{\infty} \mathbb{1}(\sigma_n < s \le \tau_n, 0 < X_s \le \varepsilon)$.

PROOF. By the Meyer–Tanaka formula (see page 169 of Protter [19]),

$$X_{\tau_n \wedge t}^+ - X_{\sigma_n \wedge t}^+ = \int_{\sigma_n \wedge t}^{\tau_n \wedge t} \mathbb{1}(X_s > 0)\,dX_s + \tfrac{1}{2}\big(L_{\tau_n \wedge t}^0(X) - L_{\sigma_n \wedge t}^0(X)\big). \tag{27}$$

Since X does not vanish on $[\tau_n, \sigma_{n+1})$, $L_{\sigma_{n+1}\wedge t}^0(X) = L_{\tau_n \wedge t}^0(X)$. Summing up (27) for all n,

$$\sum_{n=1}^{\infty}\big(X_{\tau_n \wedge t}^+ - X_{\sigma_n \wedge t}^+\big) = \int_0^t \theta_s(X)\,dXs + \tfrac{1}{2}L_t^0(X). \tag{28}$$

By the definition of the sequence of stopping times, the left-hand side of (28) is equal to $\varepsilon U_t(X) + X_t^+ - X_{n(t)}^+$, As a result,

$$\varepsilon U_t(X) = \int_0^t \theta_s(X)\,dXs + \tfrac{1}{2}L_t^0(X) - X_t^+ + X_{n(t)}^+. \tag{29}$$

By Itô's formula for a function $F \in C^2$,

$$F(X_t^+) - F(X_0^+) = \int_0^t F'(X_s^+)\,dX_s^+ + \frac{1}{2}\int_0^t F''(X_s^+)\,d[X^+, X^+]_s.$$

By the Meyer–Tanaka formula, we have $dX_s^+ = \mathbb{1}(X_s > 0)\,dX_s + \frac{1}{2}\,dL_s^0(X)$, $d[X^+, X^+]_s = \mathbb{1}(X_s > 0)\,d[X, X]_s$. Therefore,

$$\begin{aligned} F(X_t^+) - F(X_0^+) &= \int_0^t F'(X_s^+)\mathbb{1}(X_s > 0)\,dX_s \\ &\quad + \tfrac{1}{2}\int_0^t F'(X_s^+)\,dL_s^0(X) \\ &\quad + \tfrac{1}{2}\int_0^t F''(X_s^+)\mathbb{1}(X_s > 0)\,d[X, X]_s. \end{aligned}$$

For the sequence of stopping times $\tau_n \wedge t$ and $\sigma_n \wedge t$, we have

$$\begin{aligned} F\big(X_{\tau_n\wedge t}^+\big) - F\big(X_{\sigma_n\wedge t}^+\big) &= \int_{(\sigma_n\wedge t,\tau_n\wedge t]} F'(X_s^+)\mathbb{1}(X_s > 0)\,dX_s \\ &\quad + \tfrac{1}{2}\int_{(\sigma_n\wedge t,\tau_n\wedge t]} F'(X_s^+)\,dL_s^0(X) \\ &\quad + \tfrac{1}{2}\int_{(\sigma_n\wedge t,\tau_n\wedge t]} F''(X_s^+)\mathbb{1}(X_s > 0)\,d[X, X]_s, \end{aligned}$$

Summing over n and noting that $F(0) = 0$, and $\int_0^t F'(X_s^+)\,dL_s^0(X) = F'(0)L_t^0(X)$ since the measure $dL_t^0(X)$ is almost surely carried by the set $\{t : X_t = 0\}$, we have

$$\begin{aligned} F(\varepsilon)U_t(X) + F(X_t^+) - F\big(X_{n(t)}^+\big) &= \int_0^t \theta_s(X)F'(X_s^+)\,dX_s + \tfrac{1}{2}F'(0)L_t^0(X) \\ &\quad + \tfrac{1}{2}\int_0^t \theta_s(X)F''(X_s^+)\,d[X^+, X^+]_s. \end{aligned} \tag{30}$$

We complete our proof by canceling $U_t(X)$ in (29) and (30). □

Based on the above lemma we introduce a local time inequality, which is used in the estimation of the error of the Euler scheme.

LEMMA 4.2. *Under the assumption of Lemma* 4.1, *if* $F'(0) = 0$ *and* $F(\cdot) > 0$ *on* $(0, \varepsilon_0)$ *for some* $\varepsilon_0 > 0$, *then for any* $0 < -\varepsilon < \varepsilon_0$ *we have*

$$\begin{aligned} 0 \le L_t^0(X) &\le 2\varepsilon - \frac{2}{F(\varepsilon)}\int_0^t \theta_s(X)\big(F(\varepsilon) - \varepsilon F'(X_s^+)\big)\,dX_s \\ &\quad + \frac{1}{F(\varepsilon)}\int_0^t \theta_s(X)\varepsilon F''(X_s^+)\,d[X, X]_s. \end{aligned}$$

PROOF. By the definitions of the double sequence of stopping times and of $U_t(X)$, we observe that if $\tau_{U_t(X)} \le t < \sigma_{U_t(X)+1}$, then $n(t) = t$; if $\sigma_{U_t(X)+1} \le t < \tau_{U_t(X)+1}$, then $n(t) = \sigma_{U_t(X)+1}$, $X_{n(t)}^+ = 0$ and $X_t^+ \le \varepsilon$. Therefore, the first term in the right-hand side of (26) is less than $2\varepsilon F(\varepsilon)$ and the second term is less than zero. The inequality follows from Lemma 4.1. □

4.2. *The rate of convergence of the Euler scheme.* In order to get the rate of convergence of the Euler scheme, we need some smoothness conditions on b and σ. There are many different criteria for the rates of convergence. In this section we prove a rate of convergence in L^1. We assume that b satisfies the Lipschitz condition in the space variable and that b is Hölder continuous in the time variable. We also assume that σ is Hölder continuous in both space and time variables. Specifically, there exists a constant c, $0 \le \alpha, \beta_1 \le 1$, $0 \le \beta_2 \le 2$, such that

$$|b(s,x) - b(t,y)| \le c|x-y| + c|s-t|^{\beta_1}, \tag{31}$$

$$|\sigma(s,x) - \sigma(t,y)|^2 \le c|x-y|^{1+\alpha} + c|s-t|^{\beta_2}, \tag{32}$$

for all $x, y \in \mathbb{R}^1$ and $s, t \in [0,T]$. Since this smoothness condition implies that b, σ have at most linear growth, by Lemma 3.1 we know that $\sup_{n\ge 1} E((X^n)^*_T)^4 < \infty$. We will denote m as the upper bound of $\sup_{n\ge1} E((X^n)^*_T)^2$, $\sup_{n\ge1} E(b(\cdot, X^n)^*_T)^2$ and $\sup_{n\ge1} E(\sigma(\cdot, X^n)^*_T)^2$. We present our result in the following theorem and give its proof at the end of this section.

THEOREM 4.1. *Under the smoothness conditions of* (31) *and* (32), *there exists a constant c and* $\gamma = \beta_1 \wedge \frac{\alpha}{2} \wedge \frac{\alpha}{1+\alpha}\beta_2$ *such that for* $n > T$ *and* $0 \le t \le T$,

$$E|X^n_t - X_t| \le cn^{-\gamma}.$$

LEMMA 4.3. *Let* $\delta = T/n \le 1$, *for* $\alpha \in [0,1]$ *and* $0 \le t \le T$,

$$E\int_0^t \left|X^n_s - X^n_{\eta_n(s)}\right|^{1+\alpha} ds \le 3mt\delta^{(1+\alpha)/2}.$$

PROOF. By (25), for $0 \le s \le T$,

$$X^n_s - X^n_{\eta_n(s)} = b\big(\eta_n(s), X^n_{\eta_n(s)}\big)\big(s - \eta_n(s)\big) + \sigma\big(\eta_n(s), X^n_{\eta_n(s)}\big)\big(B_s - B_{\eta_n(s)}\big).$$

Since $|a+b|^{1+\alpha} \le 2(|a|^{1+\alpha} + |b|^{1+\alpha})$ for any real numbers a, b and $\alpha \in [0,1]$,

$$\begin{aligned} E\left|X^n_s - X^n_{\eta_n(s)}\right|^{1+\alpha} &\le 2E\left|b\big(\eta_n(s), X^n_{\eta_n(s)}\big)\big(s-\eta_n(s)\big)\right|^{1+\alpha} \\ &\quad + 2E\left|\sigma\big(\eta_n(s), X^n_{\eta_n(s)}\big)\big(B_s - B_{\eta_n(s)}\big)\right|^{1+\alpha} \\ &\le 2m\Big(\big(s-\eta_n(s)\big)^{1+\alpha} + E\left|B_s - B_{\eta_n(s)}\right|^{1+\alpha}\Big) \\ &\le 4m\big(s-\eta_n(s)\big)^{(1+\alpha)/2}. \end{aligned}$$

Our lemma follows from that $\int_0^t (s-\eta_n(s))^\beta\, ds \le t\delta^\beta/(1+\beta)$ for any $\beta \ge 0$. □

LEMMA 4.4. *Under the smoothness conditions of* (31) *and* (32), *we have*

$$EL_t^0(X^n - X) \le 2\varepsilon + 2ct\big(\varepsilon + 3m\delta^{1/2} + \delta^{\beta_1} + 2\varepsilon^\alpha + 6m\delta^{(1+\alpha)/2}\varepsilon^{-1} + \delta^{\beta_2}\varepsilon^{-1}\big),$$

for any $\varepsilon > 0$ *and* $\delta = T/n \le 1$.

PROOF. By (23) and (25), for $0 \le t \le T$,

$$\begin{aligned} X_t^n - X_t = &\int_0^t \big(b(\eta_n(s), X^n_{\eta_n(s)}) - b(s, X_s)\big)\, ds \\ &+ \int_0^t \big(\sigma(\eta_n(s), X^n_{\eta_n(s)}) - \sigma(s, X_s)\big)\, dB_s. \end{aligned} \tag{33}$$

Since $X_t^n - X_t$ is a continuous semimartingale with initial value 0, we can apply Lemma 4.2 with $F(x) = x^2$,

$$\begin{aligned} L_t^0(X^n - X) \le 2\varepsilon - \frac{2}{\varepsilon}\int_0^t \theta_s(X^n - X)\big(\varepsilon - 2(X^n - X)_s^+\big)\, d(X^n - X)_s \\ + \frac{2}{\varepsilon}\int_0^t \theta_s(X^n - X)\, d[X^n - X, X^n - X]_s. \end{aligned}$$

By taking expectation on both sides, we can remove the martingale part. Noting that $0 \le \theta_s(X^n - X) \le 1$, $\theta_s(X^n - X)|\varepsilon - 2(X^n - X)_s^+| \le \varepsilon$, we have

$$\begin{aligned} &EL_t^0(X^n - X) \\ &\quad\le 2\varepsilon + 2E\int_0^t \theta_s(X^n - X)\big|b(\eta_n(s), X^n_{\eta_n(s)}) - b(s, X_s)\big|\, ds \\ &\qquad + \frac{2}{\varepsilon}E\int_0^t \theta_s(X^n - X)\big(\sigma(\eta_n(s), X^n_{\eta_n(s)}) - \sigma(s, X_s)\big)^2\, ds \\ &\quad\le 2\varepsilon + 2cE\int_0^t \theta_s(X^n - X) \\ &\qquad\qquad \times \big(|X_s^n - X_s| + \big|X_s^n - X^n_{\eta_n(s)}\big| + |\eta_n(s) - s|^{\beta_1}\big)\, ds \\ &\qquad + \frac{2}{\varepsilon}cE\int_0^t \theta_s(X^n - X) \\ &\qquad\qquad \times \big(2|X_s^n - X_s|^{1+\alpha} + 2\big|X_s^n - X^n_{\eta_n(s)}\big|^{1+\alpha} + |\eta_n(s) - s|^{\beta_2}\big)\, ds \\ &\quad\le 2\varepsilon + 2cE\int_0^t \big(\varepsilon + \big|X_s^n - X^n_{\eta_n(s)}\big| + |\eta_n(s) - s|^{\beta_1}\big)\, ds \\ &\qquad + \frac{2}{\varepsilon}cE\int_0^t \big(2\varepsilon^{1+\alpha} + 2\big|X_s^n - X^n_{\eta_n(s)}\big|^{1+\alpha} + |\eta_n(s) - s|^{\beta_2}\big)\, ds \\ &\quad\le 2\varepsilon + 2ct\big(\varepsilon + 3m\delta^{1/2} + \delta^{\beta_1} + 2\varepsilon^\alpha + 6m\delta^{(1+\alpha)/2}\varepsilon^{-1} + \delta^{\beta_2}\varepsilon^{-1}\big). \end{aligned}$$

Therefore we complete the proof. □

Before proving Theorem 4.1, we do some algebra. Let

$$h_n(\varepsilon) = (2+6ct)\varepsilon^\alpha + \big(12mct\delta^{(1+\alpha)/2} + 2ct\delta^{\beta_2}\big)\varepsilon^{-1} + 9mct\delta^{1/2} + 3ct\delta^{\beta_1}.$$

For $0 \le \alpha \le 1$ and $0 \le \varepsilon \le 1$, $\varepsilon \le \varepsilon^\alpha$, by Lemma 4.4,

$$E\{L_t^0(X^n - X)\} + 3ctm\delta^{1/2} + ct\delta^{\beta_1} \le h_n(\varepsilon). \tag{34}$$

Since $h_n(\varepsilon)$ reaches its minimum $h_n(\varepsilon_0)$ at $\varepsilon = \varepsilon_0$, where

$$\begin{aligned}
\varepsilon_0 &= \left(\frac{2+6ct}{12mct\delta^{(1+\alpha)/2} + 2ct\delta^{\beta_2}}\alpha\right)^{-1/(1+\alpha)},\\
h_n(\varepsilon_0) &= c(\alpha)\big(12mct\delta^{(1+\alpha)/2} + 2ct\delta^{\beta_2}\big)^{\alpha/(1+\alpha)} + 9m\delta^{1/2} + 3\delta^{\beta_1},\\
c(\alpha) &= \big(\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)}\big)(2+6ct)^{1/(1+\alpha)}.
\end{aligned}$$

When $n \ge T$, there exits a constant c_0 such that

$$h_n(\varepsilon_0) \le c_0 n^{-\gamma}. \tag{35}$$

PROOF OF THEOREM 4.1. By (33) and the Meyer–Tanaka formula,

$$\begin{aligned}
|X_t^n - X_t| &= \int_0^t \operatorname{sgn}(X_s^n - X_s)\big(b\big(\eta_n(s), X_{\eta_n(s)}^n\big) - b(s, X_s)\big)\,ds\\
&\quad + \int_0^t \operatorname{sgn}(X_s^n - X_s)\big(\sigma\big(\eta_n(s), X_{\eta_n(s)}^n\big) - \sigma(s, X_s)\big)\,dB_s\\
&\quad + L_t^0(X^n - X).
\end{aligned} \tag{36}$$

By taking expectation on both sides of (36) to remove the martingale part,

$$\begin{aligned}
&E|X_t^n - X_t|\\
&\quad = E\int_0^t \operatorname{sgn}(X_s^n - X_s)\big(b\big(\eta_n(s), X_{\eta_n(s)}^n\big) - b(s, X_s)\big)\,ds + EL_t^0(X^n - X)\\
&\quad \le cE\int_0^t |X_s^n - X_s| + \big|X_s^n - X_{\eta_n(s)}^n\big| + |s - \eta_n(s)|^{\beta_1}\,ds + EL_t^0(X^n - X)\\
&\quad \le c\int_0^t E|X_s^n - X_s|\,ds + 3ctm\delta^{1/2} + ct\delta^{\beta_1} + EL_t^0(X^n - X)\\
&\quad \le cE\int_0^t |X_s^n - X_s|\,ds + h_n(\varepsilon_0).
\end{aligned}$$

By Gronwall's lemma and (35), we have $E|X_t^n - X_t| \le h_n(\varepsilon_0)e^{ct} \le c_0 e^{ct} n^{-\gamma}$. □

5. Monte Carlo simulations. In practice sometimes we must numerically compute $E[f(X_T)]$ or $E[\int_0^T f(X_s)\,ds]$ where $f\colon \mathbb{R}\to\mathbb{R}$ and X is the solution of SDE (23). One application is the determination of the price of a financial security. In Section 4 we get the L^1 convergence of the continuous Euler scheme X_t^n, defined in (24), to a weak solution X_t under the assumption that X_t^n lives on a probability space that supports the weak solution. Theoretically, we can use $E[f(X_T^n)]$ to approximate $E[f(X_T)]$ and use $E[\int_0^T f(X_s^n)\,ds]$ to approximate $E[\int_0^T f(X_s)\,ds]$. However, in practice it is convenient and simple to obtain a Monte Carlo approximation of $E[f(X_T)]$ or $E[\int_0^T f(X_s)\,ds]$ by simulating a discrete-time approximation of SDE (23). For example, the Euler scheme takes approximation $\bar{X}^n$ for X, where $\bar{X}^n$ is the discrete-time process with time step size $h=T/n$, defined by

$$\bar{X}^n_{t_{k+1}}=\bar{X}^n_{t_k}+b(kh,\bar{X}^n_{t_k})h+\sigma(kh,\bar{X}^n_{t_k})\sqrt{h}\varepsilon_{k+1}, \tag{37}$$

and $\bar{X}_0^n$ is a random variable with the distribution of X_0, where $\varepsilon_1,\varepsilon_2,\dots$ is an i.i.d. sequence of standard normal random variables on some probability space, which might be different from the space that supports the weak solution. Since the exact solution is defined in the weak sense, we are interested in the quantities related to the *law* of the weak solution rather than the path properties of the solution. In Section 4, we get the rate of convergence of the continuous Euler scheme X_t^n, which is defined on the probability space that supports the weak solution. Because $\bar{X}^n_{t_k}$ has the same distribution as $X^n_{t_k}$, we can obtain some rates of convergence by using $E[f(\bar{X}^n_T)]$ to approximate $E[f(X_T)]$, and by using the Riemann summation

$$\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}^n_{t_k})$$

to approximate $E[\int_0^T f(X_s)\,ds]$. A recent work by Tanré [24] shows that

$$\left|\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}^n_{t_k})-\int_0^T E[f(X_s)]\,ds\right|\le c\|f\|_\infty n^{-1},$$

with the assumption that $b,\sigma\in C^\infty$, where c is a constant and $\|f\|_\infty$ is the upper bound of. In the next lemma we give a rate of convergence under some mild conditions on f, b and σ.

LEMMA 5.1. *If b and σ satisfy* (31) *and* (32), *and there exists a constant $\bar{c}$ such that $|f(x)-f(y)|\le\bar{c}|x-y|$ for all x and y, then*

$$\left|Ef(\bar{X}^n_T)-Ef(X_T)\right|\le c_1 n^{-\gamma}, \tag{38}$$

$$\left|\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}^n_{t_k})-\int_0^T E[f(X_s)]\,ds\right|\le c_1 n^{-\gamma}, \tag{39}$$

for some constant c_1 and $n>T$, where γ is defined in Theorem 4.1.

PROOF. For any $0 \le t_k \le T$, $E[X_{t_k}^n]^4 < \infty$ by Lemma 2.1. The linear growth of f gives $E[f(X_{t_k}^n)]^4 < \infty$. Because $\bar{X}_{t_k}^n$ has the same distribution as $X_{t_k}^n$, we have $Ef(\bar{X}_{t_k}^n) = Ef(X_{t_k}^n)$. By the Lipschitz condition of f and Theorem 4.1,

$$|Ef(\bar{X}_{t_k}^n) - Ef(X_{t_k})| \le \bar{c}E|X_{t_k}^n - X_{t_k}| \le \bar{c}cn^{-\gamma}, \tag{40}$$

which proves (38) when $t_k = T$. Since X_t is the solution of SDE (23), it is easy to see that there exists a constant c_2 such that $E|X_{t_k} - X_s| \le c_2\sqrt{t_k - s}$ for $t_{k-1} \le s \le t_k$. By (40) we have

$$\begin{aligned}
&\left|\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}_{t_k}^n) - \int_0^T E[f(X_s)]\,ds\right| \\
&\quad \le \left|\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}_{t_k}^n) - \frac{T}{n}\sum_{k=1}^n Ef(X_{t_k})\right| + \left|\sum_{k=1}^n \int_{t_{k-1}}^{t_k} E[f(X_{t_k}) - f(X_s)]\,ds\right| \\
&\quad \le T\bar{c}cn^{-\gamma} + \bar{c}c_2 n^{-1/2} \le \bar{c}(Tc + c_2)n^{-\gamma},
\end{aligned}$$

since $\gamma \le 1/2$. This completes the proof of (39). □

Of course, we can not generally calculate the approximation $\frac{T}{n}\sum_{k=1}^n Ef(\bar{X}_{t_k}^n)$, or $E[f(\bar{X}_T^n)]$, but we can estimate them by Monte Carlo simulations. Let

$$\bar{X}_{t_{k+1}}^{ni} = \bar{X}_{t_k}^{ni} + b(kh, \bar{X}_{t_k}^{ni})h + \sigma(kh, \bar{X}_{t_k}^{ni})\sqrt{h}\varepsilon_{k+1}^i, \tag{41}$$

for $1 \le i \le m$ and $1 \le k \le n$, and $\{\bar{X}_0^{ni} : 1 \le i \le m\}$ is an i.i.d. sample from the distribution of X_0 for each n, where $\{\varepsilon_k^i : 1 \le i \le m,\ 1 \le k \le n\}$ is an i.i.d. sequence of standard normal random variables. Since $\{f(\bar{X}_{t_k}^{ni}), 1 \le i \le m\}$ is also an i.i.d. sequence of random variables, the law of large numbers implies that

$$\frac{1}{m}\sum_{i=1}^m f(\bar{X}_T^{ni}) \to E[f(\bar{X}_T^n)] \qquad \text{a.s.,} \tag{42}$$

$$\frac{T}{mn}\sum_{i=1}^m\sum_{k=1}^n Ef(\bar{X}_{t_k}^{ni}) \to \frac{T}{n}\sum_{k=1}^n Ef(\bar{X}_{t_k}^n) \qquad \text{a.s.,} \tag{43}$$

as $m \to \infty$ for fixed n. In practice one can use variance reduction techniques to improve the convergence properties of Monte Carlo simulations. For survey papers on all variance reduction techniques, see Boyle, Broadie and Glasserman [6]. For control variates of Gaussian random variables, see Chorin [8]. For sampling techniques, see Wagner [26], Newton [18].

In the next theorem, we present a reasonable tradeoff between n and m, that is, $m = O(n^{2\gamma})$, which does not agree with the "optimal tradeoff" $m = O(n^2)$ in Duffie and Glynn [9]. This is because we do not know the *exact* rate of convergence (if it exists) when $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$do not satisfy the Lipschitz conditions.

THEOREM 5.1. *Under the conditions of Lemma* 5.1, *when* $m = O(n^{2\gamma})$ *there exists a constant* c *and* γ, *which is defined in Theorem* 4.1, *such that*

$$E\left|\frac{1}{m}\sum_{i=1}^{m} f(\bar{X}_T^{ni}) - Ef(X_T)\right| \le cn^{-\gamma}, \tag{44}$$

$$E\left|\frac{T}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} f(\bar{X}_{t_k}^{ni}) - \int_0^T Ef(X_s)\,ds\right| \le cn^{-\gamma}. \tag{45}$$

PROOF. We only prove (44) here. (45) can be proved similarly. By the same reason as in the proof of Lemma 5.1, $\sup_{n\ge1} E[f(\bar{X}_T^n)]^2 = \sup_{n\ge1} E[f(X_T^n)]^2 < \infty$. A triangle inequality and Lemma 5.1 give us

$$\begin{aligned} &E\left|\frac{1}{m}\sum_{i=1}^{m} f(\bar{X}_T^{ni}) - Ef(X_T)\right| \\ &\quad \le E\left|\frac{1}{m}\sum_{i=1}^{m} f(\bar{X}_T^{ni}) - E[f(\bar{X}_T^n)]\right| + \left|E[f(\bar{X}_T^n)] - Ef(X_T)\right| \\ &\quad \le m^{-1/2}\left(E[f(\bar{X}_T^n)]^2\right)^{1/2} + n^{-1/\gamma}c. \end{aligned}$$

which concludes (44) when $m = O(n^{2\gamma})$. □

APPENDIX

LEMMA A.1. *Let* $\{Y^n : n \ge 1\}$ *be a sequence of stochastic processes. For each fixed time* t, Y_t^n *is uniformly integrable; for each* n, Y_t^n *is a martingale with respect to its own filtration. If* Y^n *converges to* Y *in probability as* n *goes to* ∞, *then* Y *is also a martingale with respect to its own filtration.*

PROOF. By Exercise 3.6 on page 9 of Blumenthal and Getoor [5], it suffices to prove that for any $0 \le s_1 < s_2 < \cdots < s_m < s < t \le T$ the following equation:

$$E\left(Y_t \prod_{i=1}^{m} f_i(Y_{s_i})\right) = E\left(Y_s \prod_{i=1}^{m} f_i(Y_{s_i})\right)$$

holds for all $f_i(\cdot) \in C_b(R)$, $i = 1, 2, \ldots, m$, $m \ge 1$, where $C_b(R)$ is the space of all continuous bounded functions on the real line. The above identity holds for Y^n, since it is a martingale. Because Y^k converges to Y almost surely, and $\{Y_t^n : n \ge 1\}$ is uniformly integrable for fixed t, we have

$$EY_t \prod_{i=1}^{m} f_i(Y_{s_i}) = \lim_{n\to\infty} EY_t^n \prod_{i=1}^{m} f_i(Y_{s_i}^n) = \lim_{n\to\infty} EY_s^n \prod_{i=1}^{m} f_i(Y_{s_i}^n) = EY_s \prod_{i=1}^{m} f_i(Y_{s_i}),$$

which completes the proof. □

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