The Annals of Probability
2002, Vol. 30, No. 3, 1082–1130

STATIONARY BLOCKING MEASURES FOR ONE-DIMENSIONAL NONZERO MEAN EXCLUSION PROCESSES

BY MAURY BRAMSON[1] AND THOMAS MOUNTFORD[2]

University of Minnesota and University of California, Los Angeles

The product Bernoulli measures ρ_α with densities α, $\alpha \in [0, 1]$, are the extremal translation invariant stationary measures for an exclusion process with irreducible random walk kernel $p(\cdot)$. In $d = 1$, stationary measures that are not translation invariant are known to exist for specific $p(\cdot)$ satisfying $\sum_x xp(x) > 0$. These measures are concentrated on configurations that are completely occupied by particles far enough to the right and are completely empty far enough to the left; that is, they are blocking measures. Here, we show stationary blocking measures exist for all exclusion processes in $d = 1$, with $p(\cdot)$ having finite range and $\sum_x xp(x) > 0$.

1. Introduction. The exclusion processes constitute one of the main families of stochastic processes in the area of interacting particle systems. Introduced in Spitzer (1970), these processes have been the object of much study; numerous references are provided by Liggett (1985), Liggett (1999) and Kipnis and Landim (1999). The exclusion process $\eta_\bullet = (\eta_t)_{t\geq 0}$, with random walk kernel $p(\cdot)$, is a continuous time Markov process on $\{0, 1\}^{\mathbb{Z}^d}$. A configuration $\eta \in \{0, 1\}^{\mathbb{Z}^d}$ is said to be occupied by a particle at x if $\eta(x) = 1$, and is empty (or vacant) at x if $\eta(x) = 0$; we employ the convention of identifying η with the set of its occupied sites. A particle moves from an occupied site x to an empty site y at rate $p(y - x)$. When the site y is already occupied, such a particle remains at x; there is always at most one particle at a given site. The exclusion process $\eta_\bullet$ is formally defined as the Feller process on $\{0, 1\}^{\mathbb{Z}^d}$, with generator

$$\Omega f(\eta) = \sum_{x,y\in\mathbb{Z}^d} \big(f(\eta_{xy}) - f(\eta)\big)p(y - x)\eta(x)\big(1 - \eta(y)\big)$$

for cylinder functions f, where

$$\eta_{xy}(x) = \eta(y), \quad \eta_{xy}(y) = \eta(x) \quad \text{and} \quad \eta_{xy}(z) = \eta(z) \qquad \text{for } z \neq x, y.$$

A basic problem is the characterization of stationary measures for exclusion processes. Assume that the random walk kernel $p(\cdot)$ is *irreducible*; that is, for each $x \in \mathbb{Z}^d$, $p^{(n)}(x) > 0$ for some $n \in \mathbb{Z}^+$. It is well known that the product Bernoulli

Received February 2001; revised September 2001.
[1]Supported in part by NSF Grant DMS-99-71248.
[2]Supported in part by NSF Grant DMS-00-71471.
AMS 2000 subject classification. 60K35.
Key words and phrases. Exclusion processes, stationary measures, blocking measures.

measures ρ_α with densities α, $\alpha \in [0, 1]$, at each site are the extremal translation invariant stationary measures for the process. When the kernel $p(\cdot)$ is symmetric, there are no nontranslation invariant stationary measures. In $d = 1$, the condition on the mean $\mu \stackrel{\text{def}}{=} \sum_x xp(x) = 0$ also ensures that there are no nontranslation invariant stationary measures. [These results and those in the next paragraph are given in Liggett (1975, 1985).]

Reversible random walk kernels provide a tool for constructing nontranslation invariant stationary measures. When the kernel $p(\cdot)$ is reversible with respect to a measure $\pi(\cdot)$, the corresponding exclusion process is reversible with respect to the product measure $\rho_{\alpha(\cdot)}$, with

$$\alpha(x) = \frac{\pi(x)}{1 + \pi(x)}. \tag{1.1}$$

In particular, when $d = 1$ and $p(1) > p(-1) > 0$, with $p(x) = 0$ for $|x| \neq 1$, the measures $\rho_{\alpha(\cdot)}$, with $\pi(\cdot)$ in (1.1) given by $\pi(x) = (p(1)/p(-1))^x$, are stationary. By applying Borel–Cantelli, it follows that $\rho_{\alpha(\cdot)}$ is supported on configurations η with $\eta(x) = 0$ for small enough x and $\eta(x) = 1$ for large enough x. The existence of such stationary measures is not surprising: the drift of the underlying random walk causes particles to be typically "close" to the rightmost positions possible, with particles scattered further to the left because of random fluctuations in their motion.

It is easy to see that, for $d = 1$, the countable set of configurations

$$\Xi = \left\{ \eta : \sum_{x<0} \eta(x) = \sum_{x \geq 0} \big(1 - \eta(x)\big) < \infty \right\} \tag{1.2}$$

is invariant for any exclusion process with $\sum_x xp(x) > -\infty$. Measures on the union of Ξ and its translates are referred to as *blocking measures*. Since the evolution of $\eta_\bullet$ on different translates is analogous, we consider just Ξ. Suppose that $\eta_0 \in \Xi$ and $p(\cdot)$ is given by the example from the previous paragraph. The existence of the stationary measure $\rho_{\alpha(\cdot)}$ implies that $\eta_\bullet$ is positive recurrent when viewed as a Markov chain on Ξ. Conversely, it is not difficult to show that when $\eta_\bullet$ is positive recurrent on Ξ for a given $p(\cdot)$, then $\eta_\bullet$ has a stationary blocking measure.

The last observation provides a method for establishing the existence of nontranslation invariant stationary measures, in $d = 1$, for a given exclusion process with $\mu > 0$. (The case $\mu < 0$ follows by mapping x to $-x$.) In Ferrari, Lebowitz and Speer (2001), it was shown that when two random walk kernels $p(\cdot)$ and $\overline{p}(\cdot)$ satisfy

$$p(x) \geq \overline{p}(y) \text{ for all } x \in (0, y] \quad \text{and} \quad p(y) \leq \overline{p}(x) \text{ for all } x \in [y, 0), \tag{1.3}$$

for all $y \in \mathbb{Z}$, then the existence of a stationary blocking measure for the process $\overline{\eta}_\bullet$ corresponding to $\overline{p}(\cdot)$ implies the existence of a stationary blocking measure for

the process $\eta_\bullet$ corresponding to $p(\cdot)$. [The condition (1.3) enables one to couple $\eta_\bullet$ and $\overline{\eta}_\bullet$ so that if the particles of η_0 are to the right of the corresponding particles of $\overline{\eta}_0$, then the particles of η_t remain to the right of $\overline{\eta}_t$ for all t.] The set of kernels $\overline{p}(\cdot)$ known to have stationary blocking measures is, however, limited. [Ferrari, Lebowitz and Speer (2000) mention $\overline{p}(x) = \beta^x \overline{p}(-x)$, with $\beta > 1$, which is obtained by employing (1.1).]

For general nonreversible $p(\cdot)$, with $\mu > 0$, "bad" configurations on Ξ can induce a temporary drift of particles to the left. This occurs, for instance, when $p(2) = p(-1) > 0$ and $p(x) = 0$ otherwise, at the configurations where $\eta(x) = 0$ for $x < -J$, $\eta(x) = 1$ for $x \geq J$, and $\eta(x) = 1$ at even sites and $\eta(x) = 0$ at odd sites in $[-J, J)$, for $J \in \mathbb{Z}^+$. In order to show $\eta_\bullet$ is positive recurrent on Ξ for these $p(\cdot)$, one needs to control the effect of such bad configurations, presumably without recourse to explicit calculation. A hydrodynamic limit from Rezakhanlou (1991) will provide an important tool for such an approach.

Our goal in this paper is to demonstrate the existence of a stationary blocking measure whenever the kernel $p(\cdot)$ has finite range and $\mu > 0$.

THEOREM 1.1. *Assume that $\eta_\bullet$ is an exclusion process whose random walk kernel $p(\cdot)$ has finite range and mean $\mu > 0$. Then there exists a stationary measure for $\eta_\bullet$ which is supported on Ξ.*

If it is also assumed that $p(\cdot)$ is irreducible, then there are no other nontranslation invariant stationary measures for $\eta_\bullet$ besides those supported on Ξ and its translates. This is shown in Bramson, Liggett and Mountford (2002). Analogous results were shown in Liggett (1976) for the nearest neighbor kernel $p(\cdot)$ given after (1.1).

When $\eta_0 \in \Xi$ and $p(\cdot)$ is supported on $\mathbb{Z}^+$, η_t has a pathwise limit as $t \to \infty$, which is obviously stationary. Also, the case where $p(\cdot)$ is supported on $m\mathbb{Z}$, $m \in \mathbb{Z}^+$, reduces to the case where $p(\cdot)$ is supported on all of $\mathbb{Z}$. So, to show Theorem 1.1, it is enough to consider only irreducible $p(\cdot)$. The theorem follows quickly from the following result. Here, $\eta_\bullet^N$, $N \in \mathbb{Z}^+$, denotes the Markov chain on Ξ given by $\eta_k^N = \eta_{kN}$ for $k = 0, 1, 2, \ldots$ and $\eta_0^N \in \Xi$.

THEOREM 1.2. *Assume that $\eta_\bullet$ is an exclusion process for which*

$$\text{the random walk kernel } p(\cdot) \text{ is irreducible, with finite range and mean } \mu > 0. \tag{1.4}$$

Then, for some N, the process $\eta_\bullet^N$ is positive recurrent on Ξ.

PROOF OF THEOREM 1.1 ASSUMING THEOREM 1.2. We may assume that $p(\cdot)$ satisfies (1.4) because of the above discussion. Let ν^N be the stationary

measure on Ξ for $\eta_\bullet^N$, where N is chosen as in Theorem 1.2. Also, let $\overline{\nu}^N$ be the measure defined by

$$E^{\overline{\nu}^N}[\ell(\eta_0)] = \frac{1}{N}\int_0^N E^{\nu^N}[\ell(\eta_t)]\,dt \tag{1.5}$$

for bounded continuous functions ℓ. That is, $\overline{\nu}^N$ is the average of ν^N and its translates over $[0, N]$ according to $\eta_\bullet$. Then, $\overline{\nu}^N$ is supported on Ξ, and is stationary for $\eta_\bullet$. $\square$

In order to demonstrate Theorem 1.2, we apply Foster's Criterion with an appropriate Lyapunov function h. For $\eta \in \Xi$, let

$$L(\eta) = \min\{x : \eta(x) = 1\} \quad \text{and} \quad R(\eta) = \max\{x : \eta(x) = 0\}. \tag{1.6}$$

We choose $h = f + g$, where

$$f(\eta) = -\sum_{x<0} x\eta(x) + \sum_{x\geq 0} x\big(1 - \eta(x)\big) \tag{1.7}$$

and

$$g(\eta) = \sigma N\big((L(\eta) + \beta N)^- + (R(\eta) - \beta N)^+\big) \tag{1.8}$$

for $\eta \in \Xi$. The constants β and σ, which satisfy $0 < \beta^{-1} \ll \sigma \ll 1$, will be specified in Section 6.

THEOREM 1.3. *Assume that $\eta_\bullet$ is an exclusion process which satisfies* (1.4). *Then, for appropriate $N \in \mathbb{Z}^+$, $\beta > 0$ and $\sigma > 0$, $E[h(\eta_N)] < \infty$ for all $\eta_0 \in \Xi$. If, in addition, $\eta_0 \in G$ for an appropriate G where G^c is finite, then*

$$E[h(\eta_N)] - h(\eta_0) \leq -\sigma\mu N^2/12. \tag{1.9}$$

[The set G is given in (6.24).]

For irreducible $p(\cdot)$, all states of Ξ communicate under $\eta_\bullet$, and hence under $\eta_\bullet^N$. This is straightforward to show but is a bit tedious, so we omit the details. (The basic idea is that, with positive probability, one can move a finite number of particles as far to the left of 0 as desired, with the remaining particles forming a single block of occupied sites to infinity. One by one, these particles to the left of 0 can then be specified to move along prescribed paths until reaching their desired positions, with all other particles in the meantime remaining immobile.) Theorem 1.2 immediately follows from Theorem 1.3, Foster's Criterion and this observation. The rest of the paper will be devoted to demonstrating Theorem 1.3.

In the remainder of this section, we motivate the reasoning behind Theorem 1.3 and summarize the contents of the remaining five sections. We first note that the function f in (1.7) is a reasonable first guess for a Lyapunov function, since it will decrease when particles move freely. There are certain configurations in G,

however, for which our bound on $E[f(\eta_N)] - f(\eta_0)$ will be slightly positive. In these configurations, either the leftmost particle or rightmost hole (i.e., empty site) will be close to only a few other particles or holes, which will imply that $E[g(\eta_N)] - g(\eta_0)$ is negative. It will follow that the perturbation of f by g given by $h = f + g$ satisfies (1.9) on G.

In order to demonstrate Theorem 1.3, we partition $\mathbb{Z}$, using η_0, into intervals which have length of order N, except for the semi-infinite intervals on the left and right. These partitions include intervals of two basic types, where either

(1.10) there are both of order N particles and N empty sites

or

(1.11) either particles or holes dominate.

In Sections 2–5, we will provide the machinery for decomposing $\eta_\bullet$ into different exclusion processes $\eta_\bullet^i$, with each η_0^i corresponding to one of these intervals, and analyzing the evolution of $\eta_\bullet^i$. In Section 6, we apply these results to obtain Theorem 1.3.

Propositions 2.1 and 2.2 are the two main results of Section 2. Proposition 2.1 says, in essence, that one can change η_0 at a relatively small number of sites without affecting η_t very much if t is not too large. (The two processes will typically remain the same except within a linearly increasing distance of the changes.) In particular, this enables us to change η_0 on the "boundaries" of the intervals in (1.10) and (1.11) to all 1's or all 0's, without affecting η_N too much. [These "boundaries" will be intervals of length of order N, which are, however, comparatively short relative to the intervals in (1.10) and (1.11).] The exclusion process $\eta'_\bullet$ thus obtained from $\eta_\bullet$ can be decomposed into exclusion processes $\eta_\bullet^i$ corresponding to these different intervals, with $\eta_0^i(x) = \eta_0(x)$ on the corresponding interval and $\eta_0^i(x)$ constant on each side outside the interval. It follows from Proposition 2.2 that the error introduced by doing this is small at time N. (The "boundaries" are long enough to typically prevent the "influence" of sites inside an interval from spreading outside the interval, and to prevent the "influence" of sites outside the interval from spreading inside the interval.) The results in Propositions 2.1 and 2.2 are based on elementary large deviation bounds of random walks.

Proposition 4.1 provides the main estimates required for the exclusion processes $\eta_\bullet^i$ corresponding to the intervals in (1.10) possessing both of order N particles and N empty sites. It gives lower bounds on the average movement to the right of particles by time N, when low density subintervals lie immediately to the right of high density subintervals. Sections 3 and 4 are devoted to the demonstration of Proposition 4.1. The proposition relies heavily on a hydrodynamic limit from Rezakhanlou (1991), which says that, in the scaling limit, the density of particles $u(t, x)$ for the exclusion process evolves as an entropy solution of Burger's

equation

$$\frac{\partial u}{\partial t} + \mu \frac{\partial (u(1-u))}{\partial x} = 0. \tag{1.12}$$

Estimates on solutions of (1.12) under appropriate initial data therefore provide the desired drift of particles to the right. (We will, in practice, employ a variant of this approach.) Rezakhanlou's result is stated for processes with initial states given by product measure. To apply it to our setting with deterministic initial states η_0^i, we need the large deviation bounds on pathwise coupling given in Proposition 3.1, which compare the evolution of a pair of exclusion processes ${}^1\eta_\bullet$ and ${}^2\eta_\bullet$ having different initial states. Section 3 is devoted to this result.

Propositions 5.1 and 5.2 provide the main estimates required for the exclusion processes $\eta_\bullet^i$ corresponding to the intervals in (1.11) where either particles or holes dominate. Intuitively, when there are few particles, the motions of the different particles should not interfere much with one another, and the positive drift of the underlying random walk kernel $p(\cdot)$ should be largely retained by the particles of $\eta_\bullet^i$. (Analogous behavior will hold where there are few holes, with holes drifting to the left.) Propositions 5.1 and 5.2 state such results, with Proposition 5.1 addressing the position of the leftmost particle at time N and Proposition 5.2 addressing the average drift of the particles.

Section 6 employs the results of Sections 2–5 to demonstrate Theorem 1.3. The section is divided into three parts. We first introduce heterogeneous, homogeneous and boundary intervals. The first two types of intervals correspond to the intervals described in (1.10) and (1.11). The boundary intervals are the relatively short intervals in between these intervals, where one changes η_0 to all 1's and all 0's to obtain η_0'.

We next provide upper bounds for $E[f(\eta_N)] - f(\eta_0)$. In Proposition 6.1, this is done for the case where η_0 contains at least two heterogeneous intervals, and in Proposition 6.2, for the case where η_0 contains a unique heterogeneous interval. (Since $\eta_0 \in \Xi$, η_0 will always contain at least one heterogeneous interval.) The bound for the first case will be negative; that for the second case may or may not be, depending on the location of the heterogeneous interval.

We then provide upper bounds for $E[g(\eta_N)] - g(\eta_0)$. The elementary bound in Lemma 6.2 always holds, and suffices for our bound in (1.9) on $E[h(\eta_N)] - h(\eta_0)$, except when the bound on $E[f(\eta_N)] - f(\eta_0)$ in Proposition 6.2 is positive. Proposition 6.3 is instead employed in the latter case, and also produces the bound in (1.9). These last steps are combined in Proposition 6.4, which is a more explicit version of Theorem 1.3.

2. Bounds on $\sum_x x(\eta_t(x) - \eta_0(x))$. Pairs of exclusion processes that have the same initial state at "most" sites should evolve similarly. Comparisons of this nature will be employed in Section 6, where the exclusion process is broken into a number of "pieces," each of which has been modified at its ends. Propositions 2.1

and 2.2 provide the needed tools. In this section we also demonstrate elementary inequalities in Lemmas 2.1 and 2.2. We begin with certain preliminaries.

An exclusion process can be constructed from a Harris system consisting of a system of independent Poisson point processes $\mathcal{N}^{x,y}$, $x, y \in \mathbb{Z}$, with rates $p(y-x)$ corresponding to the underlying random walk. One stipulates that if at $t \in \mathcal{N}^{x,y}$, $\eta_{t-}(x) = 1$ and $\eta_{t-}(y) = 0$, then $\eta_t(x) = 0$ and $\eta_t(y) = 1$, with there otherwise being no change in $\eta_\bullet$. That is, at $t \in \mathcal{N}^{x,y}$, "a particle tries to move from site x to site y." The filtration $\{\mathcal{F}_t,\ t \geq 0\}$ for the process will be the natural filtration associated with the whole Harris system, together with any information about initial configurations of processes we consider.

Different priority schemes among particles can be employed when a particle tries to move to a site already occupied by another particle. Unless specified otherwise, particles will be assumed not to move to occupied sites (and displace other particles). In some settings, particles will be assigned a priority, such as when they are coupled with particles of another exclusion process (as in Section 3) or based on their initial location (as in Sections 4 and 5). The priority scheme does not affect $\eta_\bullet$ since it involves just a reidentification of particles. In Section 3, the labelling of particles will require a larger filtration $\{\mathcal{G}_t,\ t \geq 0\}$, where $\mathcal{G}_t$ is generated by $\mathcal{F}_t$ and certain independent random variables. Stopping times τ will be with respect to $\mathcal{F}_t$ or $\mathcal{G}_t$, depending on the context.

We will need to extend the state space Ξ considered in the introduction to $\Xi_\infty = \bigcup_J \Xi_J$, $J \in \mathbb{Z}^+$, where Ξ_J consists of the configurations $\eta \in \{0,1\}^{\mathbb{Z}}$ for which η is constant to the left of $[-J, J]$ and is constant to the right of $[-J, J]$. Unless stated otherwise in the paper, the initial configuration η_0 will be assumed to be nonrandom. Results in the paper will typically be phrased in terms of the motion of particles. A standard trick is to interchange 0's and 1's in η_t. The process thus obtained is an exclusion process with random walk kernel $\tilde{p}(x) = p(-x)$. Hence, analogous results will also hold for the motion of holes, and will be employed when appropriate.

In this section we will employ processes $X_\bullet$, with $X_t \in \mathbb{Z}$, having the property that

$$X_\bullet \text{ can move from } x \text{ to } y \text{ at time } t \text{ only if } t \in \mathcal{N}^{x,y} \text{ or } t \in \mathcal{N}^{y,x} \tag{2.1}$$

(although $X_\bullet$ is not obligated to move at these times). This property will be satisfied by certain labelled particles and holes. The following elementary lemma holds for such processes; its proof is immediate. We set $\bar{p}(x) = p(x) + p(-x)$.

LEMMA 2.1. *Assume that the process $X_\bullet$ satisfies* (2.1). *Then, there exists a nondecreasing random walk $Z_\bullet$ on $\mathbb{Z}$, with $Z_0 = 0$, which jumps from x to y, $y > x$, at rate $\bar{p}(y-x)$, so that $|X_t - X_0| \leq Z_t$ for all t.*

A similar inequality allows us to compare two exclusion processes which are initially equal on a half line. For this, we set $\nu = \sum_{x>0}(\bar{p}(x)\sum_{y=1}^{x} y)$ and choose

the integer M so that $M > (2\nu) \vee 1$; note that $M > 2|\mu|$. This choice of M will remain fixed in the paper. Also, throughout the remainder of the paper, we will implicitly assume that the random walk kernel $p(\cdot)$ has finite range. We will frequently employ $c > 0$ and C to denote constants that can vary from line to line.

LEMMA 2.2. *Let ${}^1\eta_\bullet$ and ${}^2\eta_\bullet$ be two exclusion processes generated by the same Harris system, with ${}^1\eta_0(x) = {}^2\eta_0(x)$ for $x \in (0, \infty)$. Then, $D_t \stackrel{\text{def}}{=} \sup\{x : {}^1\eta_t(x) \neq {}^2\eta_t(x)\}$ is dominated by a nondecreasing random walk $Z_\bullet$, with $Z_0 = 0$, which jumps from site x to site y, $y > x$, at time t exactly when $t \in \mathcal{N}^{w,y}$ or $t \in \mathcal{N}^{y,w}$ for some $w \leq x$. Moreover, there exist $c > 0$ and C so that for $l \geq 0$ and all t,*

$$P\left(\sup_{s \leq t} D_s \geq \frac{Mt}{2} + l\right) \leq Ce^{-c(t+l)}. \tag{2.2}$$

PROOF. The first part of the lemma, that $D_t \leq Z_t$, is immediate. Since $Z_\bullet$ has bounded increments and drift ν, (2.2) follows from the first part and standard large deviations result. $\square$

When $\eta_0(x) = 1$ for $x \in (0, \infty)$, one has as a special case of (2.2), that for $\ell \geq 0$ and all t,

$$P\left(\max_{s \in [0,t]} R(\eta_s) \geq \frac{Mt}{2} + l\right) \leq Ce^{-c(t+l)}. \tag{2.3}$$

This, of course, implies that $E[[R(\eta_t)]^+] \leq Mt$ for such η_0 and large enough t. Analogs of Lemma 2.2 and (2.3) hold when ${}^1\eta_0(x) = {}^2\eta_0(x)$ for $x \in (-\infty, 0)$ instead of $x \in (0, \infty)$, and when $\eta_0(x) = 0$ for $x \in (-\infty, 0)$ instead of $\eta_0(x) = 1$ for $x \in (0, \infty)$. These results are obtained from Lemma 2.2 and (2.3) by interchanging 0's and 1's for η_t. We also note that the same exponential estimates as in (2.2) hold for the random walk $Z_\bullet$ from Lemma 2.1, since this random walk has drift at most ν.

Let $S_1, S_2, \dots, S_m$ be a finite sequence of finite disjoint intervals on $\mathbb{Z}$ which are ordered from left to right, let $q(\cdot)$ be a function with $q : \{1, \dots, m\} \to \{0, 1\}$, and $\eta_0 \in \Xi_\infty$. We define η_0^q such that

$$\eta_0^q(x) = \begin{cases} \eta_0(x), & \text{for } x \notin \bigcup_i S_i, \\ q(i), & \text{for } x \in S_i, \end{cases} \tag{2.4}$$

and denote by $\eta_\bullet^q$ the exclusion process with this initial state. Of course, $\eta_0 \in \Xi_\infty$ implies that $\eta_0^q \in \Xi_\infty$.

In Proposition 2.1, we compare $E[\sum_x x(\eta_t(x) - \eta_0(x))]$ with $E[\sum_x x(\eta_t^q(x) - \eta_0^q(x))]$. Using Lemma 2.1, we show that the difference can only increase linearly in time and proportionally to the cardinality of $\{x : \eta_0(x) \neq \eta_0^q(x)\}$.

PROPOSITION 2.1. *For a given $\eta_0 \in \Xi_\infty$, let η_0^q be as in* (2.4). *Then, the exclusion processes $\eta_\bullet$ and $\eta_\bullet^q$ satisfy*

$$\begin{aligned}\left|E\left[\sum_x x\big(\eta_t(x)-\eta_0(x)\big)\right]-E\left[\sum_x x\big(\eta_t^q(x)-\eta_0^q(x)\big)\right]\right| \\ \le \frac{Mt}{2}\big|\{x:\eta_0(x)\neq\eta_0^q(x)\}\big|\end{aligned} \tag{2.5}$$

for all t.

PROOF. Assume that the exclusion processes $\eta_\bullet$ and $\eta_\bullet^q$ are generated by the same Harris system. At time 0, we refer to those sites where $\eta_0^q(x) < \eta_0(x)$ as positive discrepancies and those sites where $\eta_0^q(x) > \eta_0(x)$ as negative discrepancies. One can check that, as time evolves, a discrepancy moves from x to y at $t \in \mathcal{N}^{x,y}$, if the corresponding process does not already occupy y, and a discrepancy moves from x to y at $t \in \mathcal{N}^{y,x}$ if both processes already occupy y. When two discrepancies meet, they disappear. Note that at sites x where there is no discrepancy, $\eta_t^q(x) = \eta_t(x)$.

Denote by X_t^k, $t \ge 0$, the process corresponding to the discrepancy initially at k, if it exists; we continue $X_\bullet^k$ after the discrepancy disappears by keeping its position fixed. Also, let K^+ and K^- denote the index sets of the positive and negative discrepancies. It is easy to check that for $s \ge 0$,

$$\sum_x x\big(\eta_s(x)-\eta_s^q(x)\big)=\sum_{k\in K^+}X_s^k-\sum_{k\in K^-}X_s^k.$$

Substituting in $s = 0$ and $s = t$, it follows that

$$\begin{aligned}\sum_x x\big(\eta_t(x)-\eta_0(x)\big)-\sum_x x\big(\eta_t^q(x)-\eta_0^q(x)\big) \\ =\sum_{k\in K^+}(X_t^k-X_0^k)-\sum_{k\in K^-}(X_t^k-X_0^k).\end{aligned} \tag{2.6}$$

(Since $\eta_0, \eta_0^q \in \Xi_\infty$, each summand on the left-hand side has only finitely many nonzero terms.) Each $X_\bullet^k$ satisfies (2.1) and the random walk $Z_\bullet$ in Lemma 2.1 has drift at most ν. The bound in (2.5) therefore follows upon taking expectations in (2.6) and applying the lemma. □

In Proposition 2.2, we estimate $E[\Sigma_x x(\eta_t(x) - \eta_0(x))]$ by the sum of the expectations of a finite number of exclusion processes $\eta_\bullet^i$, each having an appropriately restricted initial state. In Lemmas 2.3 and 2.4, we first handle simpler η_0, which we then combine in Proposition 2.2.

For integers x_1 and x_2 with $x_1 \le x_2 - 2MN$, $N \in \mathbb{Z}^+$, set $H = [x_1, x_2)$, and let $\eta_0 \in \Xi_\infty$. We introduce the process $\zeta_\bullet^H$ generated by the same Harris system as $\eta_\bullet$, such that $\zeta_t^H(x) = 1$ when a particle of $\eta_\bullet$, which was originally in H, is at site x

at time t. Clearly, $\zeta_t^H \subset \eta_t$ for all t. We index such particles by k, for $k \in \zeta_0^H$, and write $X_t^{\zeta,k}$ for the position of such a particle at time t. Then, $\zeta_t^H(x) = 1$ exactly when $X_t^{\zeta,k} = x$ for some $k \in \zeta_0^H$. Note that the process $\zeta_\bullet^H$ is not an exclusion process, or, typically, even Markov.

We also introduce the exclusion process $\eta_\bullet^H$ generated by the same Harris system as $\eta_\bullet$, with $\eta_0^H(x) = \eta_0(x)$ for $x \in H$, $\eta_0^H(x) = v_1$ for $x < x_1$ and $\eta_0^H(x) = v_2$ for $x > x_2$, where v_1 and v_2 are the values taken by η_0 at x_1 and x_2. We index the corresponding particles by k, for $k \in \eta_0^H$, and write $X_t^{\eta,k}$ for their positions at time t. Note that $\zeta_0^H \subset \eta_0^H$. Of course, $\zeta_\bullet^H$ and $\eta_\bullet^H$ are typically different processes. However, they behave similarly in the sense of (2.9) of Lemma 2.3 below, when it is assumed that η_0 is constant on each of the intervals $[x_1 - MN, x_1 + MN)$ and $[x_2 - MN, x_2 + MN)$ (although η_0 is not necessarily constant on their union).

Since the proof of Lemma 2.3 is rather long, we present the basic idea first. On $[x_1 - MN, x_2 + MN)$, $\eta_0^H(x) = \eta_0(x)$. So, by Lemma 2.2, $\eta_t^H(x) = \eta_t(x)$ on (Y_t, Z_t), where $Y_\bullet$ and $Z_\bullet$ are random walks which start from the boundaries of the interval $[x_1 - MN, x_2 + MN)$ and drift in. On (Y_t, Z_t), particles $X_\bullet^{\zeta,k}$ and $X_\bullet^{\eta,k}$, $k \in \zeta_0^H$, see the same environment, and so

$$X_N^{\zeta,k} = X_N^{\eta,k} \qquad \text{for } k \in \zeta_0^H, \tag{2.7}$$

if $X_t^{\zeta,k} \in (Y_t, Z_t)$ for $t \le N$. Since $\zeta_0^H \subset [x_1, x_2)$, one has that $|k - Y_0| \ge MN$ and $|k - Z_0| \ge MN$, and so this holds with overwhelming probability. On the other hand, for $k \notin H$, all sites of η_0^H within distance MN are either all occupied or all vacant. Therefore, with overwhelming probability,

$$X_N^{\eta,k} = X_0^{\eta,k} \qquad \text{for } k \in \eta_0^H \backslash \zeta_0^H. \tag{2.8}$$

Employing (2.7) and (2.8) in conjunction with (2.10) below, we will obtain (2.9).

LEMMA 2.3. *Let H, η_0, $\zeta_\bullet^H$ and $\eta_\bullet^H$ be chosen as above. Then,*

$$\left| E\left[\sum_x x\big(\zeta_N^H(x) - \zeta_0^H(x)\big)\right] - E\left[\sum_x x\big(\eta_N^H(x) - \eta_0^H(x)\big)\right]\right| \le Ce^{-cN} \tag{2.9}$$

for some $c > 0$ and C not depending on N, η_0 or H.

PROOF. We first verify that

$$\begin{aligned} &\left| E\left[\sum_x x\big(\zeta_N^H(x) - \zeta_0^H(x)\big)\right] - E\left[\sum_x x\big(\eta_N^H(x) - \eta_0^H(x)\big)\right]\right| \\ &\quad \le \sum_{k \in \zeta_0^H} E\big[|X_N^{\zeta,k} - X_N^{\eta,k}|\big] + \sum_{k \in \eta_0^H \backslash \zeta_0^H} E\big[|X_N^{\eta,k} - X_0^{\eta,k}|\big]. \end{aligned} \tag{2.10}$$

To see this, note that

$$\sum_x x\big(\zeta_N^H(x) - \zeta_0^H(x)\big) = \sum_{k\in\zeta_0^H} (X_N^{\zeta,k} - X_0^{\zeta,k})$$

and

$$\sum_x x\big(\eta_N^H(x) - \eta_0^H(x)\big) = \sum_{k\in\eta_0^H} (X_N^{\eta,k} - X_0^{\eta,k}).$$

Since $\zeta_0^H \subset \eta_0^H$, it follows that

$$\begin{aligned}\sum_x x\big(\zeta_N^H(x) - \zeta_0^H(x)\big) &- \sum_x x\big(\eta_N^H(x) - \eta_0^H(x)\big)\\ &= \sum_{k\in\zeta_0^H} (X_N^{\zeta,k} - X_N^{\eta,k}) - \sum_{k\in\eta_0^H\setminus\zeta_0^H} (X_N^{\eta,k} - X_0^{\eta,k}).\end{aligned}$$

Taking expectations implies (2.10).

The remainder of the proof consists of bounding the summands in (2.10) by reasoning along the lines of the above summary. For this, we introduce boundary processes $Y_\bullet$ and $Z_\bullet$ as follows. Let $Y_\bullet$ be the nondecreasing random walk, with $Y_0 = x_1 - MN - 1$, that jumps from w to y, for $y > w$, at time t when $t \in \mathcal{N}^{x,y}$ or $t \in \mathcal{N}^{y,x}$ for some $x \le w$. Similarly, $Z_\bullet$ is the nonincreasing random walk, with $Z_0 = x_2 + MN$, that jumps from w to y, for $y < w$, at time t when $t \in \mathcal{N}^{x,y}$ or $t \in \mathcal{N}^{y,x}$ for some $x \ge w$.

By (2.7), for any $k \in \zeta_0^H$,

$$\big\{X_t^{\zeta,k} \in (Y_t, Z_t) \text{ for all } t \in [0, N]\big\} \subset \big\{X_N^{\zeta,k} = X_N^{\eta,k}\big\}. \tag{2.11}$$

Suppose that k is distance r from H^c and hence distance $MN + r$ from $[x_1 - MN, x_2 + MN)^c$. Lemma 2.2, with $l = r/2$, gives upper bounds on the probability of either $Y_\bullet$ or $Z_\bullet$ reaching a site in $[x_1 - MN, x_2 + MN)$, by time N, which is distance greater than $(M/2)N + r/2$ from its initial position. But, the probability that the displacement of the random walk $X_\bullet^{\zeta,k}$ exceeds $(M/2)N + r/2$ over $[0, N]$ also satisfies these bounds. It follows from this and (2.11), that for $k \in \zeta_0^H$,

$$P(X_N^{\zeta,k} \ne X_N^{\eta,k}) \le 4Ce^{-c(N+r/2)}$$

for $c > 0$ and C not depending on k, r or N. Setting $\tau_k = \inf\{t : X_t^{\zeta,k} \ne X_t^{\eta,k}\}$, the above inequality may be written as

$$P(\tau_k \le N) \le 4Ce^{-c(N+r/2)}. \tag{2.12}$$

We need to bound how far apart $X_N^{\zeta,k}$ and $X_N^{\eta,k}$ on the average are on the event $\tau_k \le N$. For this we apply the strong Markov property to the coupled processes $(\eta_\bullet^H, \zeta_\bullet^H)$ at time τ_k. After this time, either $X_\bullet^{\zeta,k}$ or $X_\bullet^{\eta,k}$ can only move from site x

at time $t-$ if $t \in \mathcal{N}^{x,y}$ or $t \in \mathcal{N}^{y,x}$ for some site y. So, applying Lemma 2.1 to the motions of $X_\bullet^{\eta,k}$ and $X_\bullet^{\zeta,k}$ on the interval $[\tau_k, N]$, we find that

$$E\big[|X_N^{\zeta,k} - X_N^{\eta,k}| \,|\, \mathcal{F}_{\tau_k}\big] \le M(N-\tau_k) + |X_{\tau_k}^{\zeta,k} - X_{\tau_k}^{\eta,k}| \le MN + d \le 2MN$$

on the event $\{\tau_k \le N\}$, for $N \ge d$, where d is the magnitude of the largest jump of $p(\cdot)$. It follows from this and (2.12), that

$$E\big[|X_N^{\zeta,k} - X_N^{\eta,k}|\big] \le 8CMNe^{-c(N+r/2)}. \tag{2.13}$$

Summing over $k \in \zeta_0^H$ gives

$$E\left[\sum_{k\in\zeta_0^H} |X_N^{\zeta,k} - X_N^{\zeta,k}|\right] \le Ce^{-cN}$$

for a new choice of $c > 0$ and C, which do not depend on N, η_0 or H. This bounds the first sum on the right-hand side of (2.10).

To bound the second sum on the right-hand side of (2.10), we use a similar approach. First, suppose $X_0^{\eta,k}$ is to the right of interval H. In this case, we consider the nondecreasing random walk $Y_\bullet$, with $Y_0 = x_2 - MN - 1$, that jumps from w to y, for $y > w$, at time t when either $t \in \mathcal{N}^{x,y}$ or $t \in \mathcal{N}^{y,x}$ for some $w \ge x$. Then, as in (2.11),

$$\big\{X_0^{\eta,k} > Y_t \text{ for all } t \in [0,N]\big\} \subset \big\{X_N^{\eta,k} = X_0^{\eta,k}\big\}.$$

Suppose that site k is r sites to the right of H. By the comment after (2.3) on the random walk $Z_\bullet$,

$$P(X_N^{\eta,k} \ne X_0^{\eta,k}) \le Ce^{-c(N+r)}$$

for some $c > 0$ and C. Reasoning as through (2.13), one obtains from this, that for $k \ge x_2$,

$$E\big[|X_N^{\eta,k} - X_0^{\eta,k}|\big] \le 2CMNe^{-c(N+r)}.$$

We get the same bounds for particles starting to the left of H. So, summing over $k \in \eta_0^H \backslash \zeta_0^H$ gives

$$\sum_{k\in\eta_0^H\backslash\zeta_0^H} E\big[|X_N^{\eta,k} - X_0^{\eta,k}|\big] \le Ce^{-cN}$$

for a new choice of $c > 0$ and C. This bounds the last sum in (2.10), which concludes the proof of the lemma. □

In the material leading up to Lemma 2.3, we assumed that H is finite. We now deal with semi-infinite intervals H, where $H = (-\infty, x_1]$ or $H = [x_1, \infty)$. Suppose that $\eta_0 \in \Xi_\infty$ is constant on $[x_1 - MN, x_1 + MN)$. We define $\zeta_\bullet^H$ and $\eta_\bullet^H$ analogously to the case where H is finite, with $\eta_0^H(x) = \eta_0(x)$ for $x \in H$ and $\eta_0^H(x) = \eta_0(x_1)$ for $x \notin H$. The same arguments used to prove Lemma 2.3 can also be used to show the following result.

LEMMA 2.4. *Let the semi-infinite interval H, η_0, $\zeta_\bullet^H$ and $\eta_\bullet^H$ be chosen as above. Then,*

$$\left|E\left[\sum_x x\big(\zeta_N^H(x)-\zeta_0^H(x)\big)\right]-E\left[\sum_x x\big(\eta_N^H(x)-\eta_0^H(x)\big)\right]\right|\le Ce^{-cN} \tag{2.14}$$

for some $c>0$ and C, not depending on N, η_0 or H.

Employing Lemmas 2.3 and 2.4, we estimate $E[\sum_x x(\eta_N(x)-\eta_0(x))]$ in Proposition 2.2 for certain η_0, by using exclusion processes $\eta_\bullet^i$ with simpler initial states. We start with a fixed sequence of disjoint intervals $S_1, S_2, \dots, S_m$ on $\mathbb{Z}$ which are ordered from left to right, where each S_i has length $2MN$. Assume that η_0 is constant on each S_i, that is, for some $q:\{1,2,\dots,m\}\to\{0,1\}$,

$$\eta_0(x)=q(i) \qquad \text{for } x\in S_i,\ i=1,2,\dots,m. \tag{2.15}$$

Define intervals $H_2,\dots,H_m$ such that H_i consists of the rightmost MN sites of S_{i-1}, the leftmost MN sites of S_i and all sites in between these intervals; H_1 and H_{m+1} are the corresponding semi-infinite intervals. Along the lines of the discussion preceding Lemma 2.3, we introduce the processes $\zeta_\bullet^i$ and $\eta_\bullet^i$ for the same Harris system as $\eta_\bullet$. We set $\zeta_t^i(x)=1$ whenever some particle of $\eta_\bullet$, which was originally in H_i, is at site x at time t; $\eta_\bullet^i$ denotes the exclusion process where $\eta_0^i(x)=\eta_0(x)$ for $x\in H_i$, $\eta_0^i(x)=q(i-1)$ for x to the left of H_i and $\eta_0^i(x)=q(i)$ for x to the right of H_i. The processes $\zeta_\bullet^i$ are not exclusion processes, but satisfy

$$\sum_x x\big(\eta_N(x)-\eta_0(x)\big)=\sum_i\left(\sum_x x\big(\zeta_N^i(x)-\zeta_0^i(x)\big)\right). \tag{2.16}$$

Together with Lemmas 2.3 and 2.4, (2.16) immediately implies the following result. It will be used in conjunction with Proposition 2.1 in Section 6. [For such applications, we note that since η_0 is constant on S_{i-1} and on S_i, an equivalent definition for η_0^i is that $\eta_0^i(x)=\eta_0(x)$ for sites in the interval J_i between S_{i-1} and S_i, and $\eta_0^i(x)$ takes the values $q(i-1)$ and $q(i)$ everywhere to the left and to the right of J_i.]

PROPOSITION 2.2. *Let $\eta_0\in\Xi_\infty$ satisfy* (2.15), *and choose the exclusion processes $\eta_\bullet^i$, $i=1,2,\dots,m+1$, as above. For some $c>0$ and C not depending on η_0, N or S_i,*

$$\left|E\left[\sum_x x\big(\eta_N(x)-\eta_0(x)\big)\right]-\sum_{i=1}^{m+1}\left(E\left[\sum_x x\big(\eta_N^i(x)-\eta_0^i(x)\big)\right]\right)\right|\le C(m+1)e^{-cN}.$$

3. Bounds on $\sum_{x\geq y}({}^1\eta_t(x) - {}^2\eta_t(x))$. In this section, we compare two exclusion processes, ${}^1\eta_\bullet$ and ${}^2\eta_\bullet$, which are generated by the same Harris system and have deterministic initial states. Setting $\phi_t(x) = {}^1\eta_t(x) - {}^2\eta_t(x)$, our main result, Proposition 3.1, says that pathwise, $\sup_y \sum_{x\geq y}\phi_t(x)$ will not typically increase much over time. For the exclusion processes ${}^j\eta_\bullet$ which are examined in Section 4, this quantity will typically be small at $t = 0$. It will then follow from Proposition 3.1 that this quantity will remain small over all times. Recall that the kernel of the underlying random walk is assumed to have finite range. In this section, we also assume it is irreducible.

PROPOSITION 3.1. *Let ${}^j\eta_\bullet$, $j = 1, 2$, be exclusion processes generated by the same Harris system. Assume that ${}^j\eta_0 \in \Xi_{KN}$, with ${}^1\eta_0(x) = {}^2\eta_0(x)$ for $x \notin [-KN, KN]$, and $K, N \in \mathbb{Z}^+$. Then, for each $\gamma > 0$,*

$$P\left(\sup_y \sum_{x\geq y}\phi_t(x) - \sup_y \sum_{x\geq y}\phi_0(x) > \gamma N\right) \leq Ce^{-cN} \tag{3.1}$$

for all t and N, and appropriate $c > 0$ and C, depending on K but not on N or ${}^j\eta_0$.

Lemmas 3.2 and 3.3 are the main tools that are used to demonstrate Proposition 3.1. For Lemma 3.2, we use the following elementary inequality. The processes ${}^j\eta^m_\bullet$ employed here are exclusion processes on the interval $[0, m]$. (That is, the state is given by $\{0, 1\}^{[0,m]}$, and particle jumps between $[0, m]$ and $[0, m]^c$ are suppressed.)

LEMMA 3.1. *Let ${}^j\eta^m_\bullet$, $j = 1, 2$, be exclusion processes on $[0, m]$ which are generated by the same Harris system. Assume that*

$${}^1\eta^m_0(0) = 1 - {}^2\eta^m_0(0) = 1 \quad \textit{and} \quad {}^1\eta^m_0(m) = 1 - {}^2\eta^m_0(m) = 0.$$

Then, there exists m_0 so that for $m \geq m_0$,

$$P\left(\sum_{x=0}^m |{}^1\eta^m_1(x) - {}^2\eta^m_1(x)| \leq \sum_{x=0}^m |{}^1\eta^m_0(x) - {}^2\eta^m_0(x)| - 2\right) > 0.$$

The lemma says, in effect, that the Hamming distance between ${}^1\eta^m_t$ and ${}^2\eta^m_t$ has a positive probability of decreasing by time 1 for corresponding realizations. (As pointed out in the following discussion, the number of uncoupled particles will never increase.) This result is proved in Mountford (2000). The argument relies on the existence of a random walk path from 0 to m which lies entirely within $[0, m]$, if m is taken large enough. For the remainder of the section, we set $m_1 = m_0 + d$, where d is the magnitude of the largest jump of $p(\cdot)$.

Consider exclusion processes ${}^1\eta_\bullet$ and ${}^2\eta_\bullet$ which are generated by the same Harris system. We say that a particle of ${}^1\eta_\bullet$ (${}^2\eta_\bullet$) is *coupled* at a given time if there is an ${}^2\eta_\bullet$ (${}^1\eta_\bullet$) particle currently at the same site; otherwise, we say it is *uncoupled*. At each $t \in \mathcal{N}^{x,y}$, $x, y \in \mathbb{Z}$, we will, when necessary, reidentify particles so that coupled particles have priority over uncoupled particles; that is, coupled particles can displace uncoupled particles, but not vice versa. It is easy to see that once a particle is coupled, it remains so forever with the same companion.

For the remainder of the section, we will find the following labelling scheme of ${}^1\eta_\bullet$ particles useful. (We will not label ${}^2\eta_\bullet$ particles; of course, one can reverse the roles of the two processes if desired.) At time 0, each ${}^1\eta_\bullet$ particle will be assigned the label equal to its position. As time evolves, uncoupled particles will exchange labels according to the rules given below, but coupled particles will keep their labels. We will also talk in terms of coupled and uncoupled labels. Given a label k, we will denote its position at time t by Y_t^k.

Relabelling occurs when: (a) An uncoupled ${}^1\eta_\bullet$ particle moves from a site x to a site y at time t, and it does not couple then. At this time, we reassign labels for uncoupled particles on the interval $[x, y]$ (or $[y, x]$, if $y < x$), so that the positions of these labels are in the same order at time t as at $t-$. (So, movement of ${}^1\eta_\bullet$ particles without coupling does not change the relative order of uncoupled labels.) Note that an uncoupled ${}^1\eta_\bullet$ particle may move because a coupled particle jumps to its site; this may cause its label to change. (b) Coupling occurs at a site y at time t. This can occur through the motion of either an uncoupled ${}^1\eta_\bullet$ particle or ${}^2\eta_\bullet$ particle. The coupling proceeds in two stages. First, a label is chosen uniformly from the labels of all uncoupled particles in $[y - m_1, y + m_1]$ at time $t-$, and is moved to y, where it is henceforth associated with the coupled ${}^1\eta_\bullet$ particle presently there. Second, after this choice, the remaining uncoupled labels in $[y - m_1, y + m_1]$ are reassigned to uncoupled ${}^1\eta$ particles there, so that the positions of these labels are in the same order at time t as at $t-$. [So, coupling does not change the relative order of the remaining uncoupled labels. Together with the corresponding comment in (a), this implies uncoupled labels are always increasing from left to right.] In order to include the information needed for relabelling, we will employ the filtration $\{\mathcal{G}_t,\ t \geq 0\}$ generated by $\{\mathcal{F}_t,\ t \geq 0\}$ and the uniform random variables introduced in (b), when working with labelled exclusion processes. We note that a label can only move from a site z to a site w at time t, if, when no coupling occurs at time t,

(3.2) there are sites x and y with $z, w \in [x, y]$ and $t \in \mathcal{N}^{x,y}$ or $t \in \mathcal{N}^{y,x}$,

or, when coupling occurs at time t at y,

$$z, w \in [y - m_1, y + m_1]. \tag{3.3}$$

For each label k, we will employ the following sequence of stopping times, $0 = S_1 \leq T_1 \leq S_2 \leq T_2 \leq \cdots$. If k is coupled at time S_n (T_n), set $T_n = S_n$

$(S_{n+1} = T_n)$; that is, these times are equal from this point on. If k is uncoupled at such a time, set

$$T_n = \inf\{t > S_n : \text{ at time } t, \text{ a coupling occurs in } [Y^k_{t-} - m_1, Y^k_{t-} + m_1] \\ \text{or } \exists \text{ an uncoupled } {}^2\eta_\bullet \text{ particle in } [Y^k_t + m_0, Y^k_t + m_1]\}$$

and

$$S_{n+1} = \inf\{t > T_n : \text{at time } t, \text{ a coupling occurs in } [Y^k_{t-} - m_1, Y^k_{t-} + m_1], \\ \text{or either } t \in \mathcal{N}^{x,y} \text{ or } t \in \mathcal{N}^{y,x} \text{ for } x \in [Y^k_{T_n}, Y^k_{T_n} + m_1] \\ \text{and } y \in [Y^k_{T_n}, Y^k_{T_n} + m_1]^c\}.$$

Note that if k is uncoupled, then $S_{n+1} > T_n$, but $T_n = S_n$ is possible.

In Lemma 3.2, we will show that a given uncoupled label couples with at least a certain probability over each interval $[T_n, S_{n+1}]$, and in Lemma 3.3, that $\sum_{x \geq Y^k_t} \phi_t(x)$ cannot grow by much over $[S_n, S_{n+1}]$. The first event in the definitions of T_n and S_{n+1} is thus a "good" event, since it is connected with this coupling. It will follow from Lemma 3.1 that the second event for T_n also allows coupling with a certain probability, provided the second event in S_{n+1} does not occur too quickly.

LEMMA 3.2. *Let ${}^j\eta_\bullet$, $j = 1, 2$, be exclusion processes generated by the same Harris system, and with arbitrary initial data ${}^j\eta_0$. Let S_n and T_n be defined as above, and suppose that the label k has not coupled by time S_n, $n \geq 1$. Then, for some fixed $c_1 > 0$ not depending on k, n or ${}^j\eta_0$,*

$$P(\textit{label } k \textit{ couples in } [T_n, S_{n+1}] \mid \mathcal{G}_{S_n}) \geq c_1 P(T_n < \infty \mid \mathcal{G}_{S_n}). \tag{3.4}$$

PROOF. We set $D = \{T_n < \infty\}$ and consider the behavior on the two events defining T_n separately. The first event, which we denote by A, is easy to treat. On $A \cap D$, there is a first time t, $t > S_n$, at which coupling occurs at some $y \in [Y^k_{t-} - m_1, Y^k_{t-} + m_1]$. This $t = T_n$; the label k is uncoupled until then. Let V, $V \leq 2m_1 + 1$, denote the number of uncoupled ${}^1\eta_\bullet$ particles in $[y - m_1, y + m_1]$ at T_n-. Since the label is chosen uniformly from uncoupled ${}^1\eta_\bullet$ particles in the interval,

$$\begin{aligned} P(k \text{ couples at } T_n; A \cap D \mid \mathcal{G}_{S_n}) &= E[1/V; A \cap D \mid \mathcal{G}_{S_n}] \\ &\geq \frac{1}{2m_1 + 1} P(A \cap D \mid \mathcal{G}_{S_n}). \end{aligned} \tag{3.5}$$

On $A^c \cap D$, there is an uncoupled ${}^2\eta_\bullet$ particle at $y \in [Y^k_{T_n} + m_0, Y^k_{T_n} + m_1]$ at time T_n. The event B, where the second event in the definition of S_{n+1} does not occur by $T_n + 1$, has probability $\exp\{-2\sum_x |x| p(x)\}$, and is independent of $\mathcal{G}_{T_n}$. We claim that, conditioned on $A^c \cap B \cap D$, the probability that a coupling occurs within distance m_1 of label k over $[T_n, T_n + 1]$ is at least c_2, $c_2 > 0$; we demonstrate

this in the next paragraph. We also observe that, under the first event in S_{n+1}, coupling automatically occurs within distance m_1 of k. So, if coupling occurs within distance m_1 of k over $[T_n, T_n+1]$, it will also occur on $[T_n, S_{n+1}]$. Together with the above claim, a repetition of the reasoning leading to (3.5) will therefore show that

$$\begin{aligned} &P(k \text{ couples by } S_{n+1}; A^c \cap B \cap D \mid \mathcal{G}_{S_n}) \\ &\quad \geq \frac{c_2}{2m_1+1} \exp\left\{-2\sum_x |x| p(x)\right\} P(A^c \cap D \mid \mathcal{G}_{S_n}). \end{aligned} \tag{3.6}$$

Together with (3.5), this will imply (3.4).

We now show the claim. Conditional on the event B and on $\mathcal{G}_{T_n}$, the pair $({}^1\eta_\bullet, {}^2\eta_\bullet)$ restricted to the interval $[Y^k_{T_n}, Y^k_{T_n}+m_1]$ is a finite coupled exclusion process on $[T_n, T_n+1]$. Applying Lemma 3.1, with $y=m$, it follows that, for a given configuration at time T_n, a coupling occurs by time T_n+1 with positive probability. Since there are only finitely many configurations on such an interval, given $\mathcal{G}_{T_n}$ and B, the probability of a coupling there is at least c_2, for some $c_2>0$.

This coupling will be within distance m_1 of label k if the label is still in $[Y^k_{T_n}, Y^k_{T_n}+m_1]$ at this later time. To finish the proof, we therefore consider the possibility that the label is outside the interval at the time of the coupling. Under B, if k leaves the interval by time T_n+1, the event in (3.2) cannot occur, and so (3.3) must occur. However, this means that a coupling within distance m_1 of k has already occurred, which is the "good" first event of S_{n+1}. $\square$

The events in T_n and S_{n+1} also produce the bound in (3.7) on the growth of $\sum_{x\geq Y^k_t} \phi_t(x)$ over $[S_n, S_{n+1}]$. The basic idea is that the second event in the definition of T_n restricts the number of uncoupled ${}^2\eta_\bullet$ particles that can cross to the left of $Y^k_\bullet$ over $[S_n, T_n)$, whereas the second event in the definition of S_{n+1} restricts the number of particles that can cross $Y^k_\bullet$ over (T_n, S_{n+1}). This provides the bounds m_0 and m_1 on the growth of the left-hand side of (3.7) over these times; the growth at the times T_n and S_{n+1} is bounded by $2m_1$. Putting these bounds together produces the bound in (3.7).

LEMMA 3.3. *Let ${}^j\eta_\bullet$, $j=1,2$, be exclusion processes generated by the same Harris system. Assume that ${}^j\eta_0 \in \Xi_J$ for some J, with ${}^1\eta_0(x) = {}^2\eta_0(x)$ for $x \notin [-J, J]$, and that the label k has not coupled by time S_n. Then, for all realizations and all $t \in [S_n, S_{n+1}]$,*

$$\sum_{x\geq Y^k_t} \phi_t(x) - \sum_{x\geq Y^k_{S_n}} \phi_{S_n}(x) \leq 6m_1. \tag{3.7}$$

PROOF. Checking all possibilities, it is not difficult to see that the most $Y^k_\bullet$ can change at any point in time is $m_1+d<2m_1$. This will occur when the particle

corresponding to k makes a maximal size jump d which results in coupling, and then k moves an additional distance m_1 due to relabelling. The integrand of the first sum in (3.7) can increase (or decrease) at only one site due to a single jump. The first sum in (3.7) can therefore increase by at most $2m_1$ at any point in time.

We treat the intervals $[S_n, T_n]$ and $[T_n, S_{n+1}]$ separately. We first consider the behavior of the left-hand side of (3.7) on $[S_n, T_n]$. By the definition of T_n, $T_n = S_n$ if there is an uncoupled ${}^2\eta_\bullet$ particle in $[Y^k_{S_n} + m_0, Y^k_{S_n} + m_1]$. So, we can assume there are no such particles in this interval at time S_n. There are at most $m_0 - 1$ uncoupled ${}^2\eta_\bullet$ particles in $[Y^k_{S_n}, Y^k_{S_n} + m_0)$, and hence, under the above assumption, at most $m_0 - 1$ such particles in $[Y^k_{S_n}, Y^k_{S_n} + m_1]$ then.

One can check that, until time T_n, all uncoupled ${}^2\eta_\bullet$ particles to the right of $Y^k_{S_n} + m_1$ at time S_n will remain to the right of the label k until time T_n. This is because $m_1 \geq m_0 + d$, and hence no such particles can leap over $[Y^k_t + m_0, Y^k_t + m_1]$, or, by (3.2), reappear on the left of the interval after the label k moves, for $t \in [S_n, T_n)$. (Since $t < T_n$, no coupling within distance m_1 of k has occurred yet.) It follows from this and the previous paragraph that, over $[S_n, T_n)$, at most $m_0 - 1$ uncoupled ${}^2\eta_\bullet$ particles which are to the right of label k at S_n can cross to the left over $[S_n, T_n)$. Also, since the movement of ${}^1\eta_\bullet$ particles without coupling does not change the relative order of labels, uncoupled labels which are to the left of label k at S_n will remain to the left over $[S_n, T_n)$. Together, these last two observations imply that for $t \in [S_n, T_n)$, the left-hand side of (3.7) is at most $m_0 - 1$. Since this difference can increase by at most $2m_1$ at T_n, one gets that

$$\sum_{x \geq Y^k_t} \phi_t(x) - \sum_{x \geq Y^k_{S_n}} \phi_{S_n}(x) \leq 3m_1 \tag{3.8}$$

for $t \in [S_n, T_n]$.

We next consider $t \in (T_n, S_{n+1}]$. By (3.2) and the definition of S_{n+1}, $Y^k_t \in [Y^k_{T_n}, Y^k_{T_n} + m_1]$ for $t \in (T_n, S_{n+1})$. It also follows from the definition of S_{n+1}, that no ${}^1\eta_\bullet$ or ${}^2\eta_\bullet$ particles have entered or left this interval over this time (although labels other than k may leave, because of nearby coupling). So, at most $m_1 - 1$ uncoupled particles can cross Y^k_t over (T_n, S_{n+1}). This implies that

$$\sum_{x \geq Y^k_t} \phi_t(x) - \sum_{x \geq Y^k_{T_n}} \phi_{T_n}(x) < m_1 \tag{3.9}$$

for $t \in (T_n, S_{n+1})$. Since this difference can increase by at most $2m$ at S_{n+1}, the left-hand side of (3.9) is at most $3m_1$ for $t \in (T_n, S_{n+1}]$. Together with (3.8), this implies (3.7). $\square$

Combining Lemmas 3.2 and 3.3, we obtain the following result. Here, we let $F_k(t)$ denote the set where label k has not coupled by time t.

PROPOSITION 3.2. *Let ${}^j\eta_\bullet$, $j = 1, 2$, be exclusion processes generated by the same Harris system. Assume that ${}^j\eta_0 \in \Xi_J$ for some J, with ${}^1\eta_0(x) = {}^2\eta_0(x)$ for $x \notin [-J, J]$. Then, for each $\gamma > 0$,*

$$P\left(\sum_{x \geq Y_t^k} \phi_t(x) - \sum_{x \geq Y_0^k} \phi_0(x) > \gamma N; F_k(t)\right) \leq Ce^{-cN} \tag{3.10}$$

for some $c > 0$ and C, not depending on k, t, N or ${}^j\eta_0$.

PROOF. Suppose that the label has not coupled by time S_n. By Lemma 3.3,

$$\sum_{x \geq Y_t^k} \phi_t(x) - \sum_{x \geq Y_0^k} \phi_0(x) > 6m_1 n$$

can only occur for $t > S_n$. But, by repeatedly applying Lemma 3.2,

$$P\big(F_k(t);\ t > S_n\big) \leq (1 - c_1)^{n-1}$$

for each k and n, where $c_1 > 0$. The proposition follows immediately from these two observations. □

Proposition 3.1 follows quickly from Proposition 3.2.

PROOF OF PROPOSITION 3.1. Suppose that for some t, γ and N,

$$\sup_y \sum_{x \geq y} \phi_t(x) - \sup_y \sum_{x \geq y} \phi_0(x) > \gamma N$$

for a given realization. Then, for some y, $\sum_{x \geq y} \phi_t(x) - \sup_z \sum_{x \geq z} \phi_0(x) > \gamma N$. If there are no uncoupled labels in an interval $[y, y')$ at time t for $y' > y$, then there are at least as many ${}^2\eta_\bullet$ particles as ${}^1\eta_\bullet$ particles there, and so $\sum_{x \geq y'} \phi_t(x) - \sup_z \sum_{x \geq z} \phi_0(x) > \gamma N$ as well. Let k be the first uncoupled label to the right of or at y at a time t. Since $\lim_{y \to \infty} \sum_{x \geq y} \phi_t(x) = 0$, such a label always exists, and

$$\sum_{x \geq Y_t^k} \phi_t(x) > \sup_z \sum_{x \geq z} \phi_0(x) + \gamma N \geq \sum_{x \geq Y_0^k} \phi_0(x) + \gamma N.$$

There are initially at most $2KN + 1$ labels. So, by Proposition 3.2, the expected number of uncoupled labels k, with $\sum_{x \geq Y_t^k} \phi_t(x) > \sum_{x \geq Y_0^k} \phi_0(x) + \gamma N$, is at most $C(2KN + 1)e^{-cN}$, for appropriate $c > 0$ and C. With a new choice of $c > 0$ and C, it follows from this and the previous paragraph, that for given $\gamma > 0$,

$$P\left(\sup_y \sum_{x \geq y} \phi_t(x) - \sup_y \sum_{x \geq y} \phi_0(x) > \gamma N\right) \leq Ce^{-cN}$$

for any t and N, which is the same as (3.1). □

4. Growth of $\sum_x x(\eta_t(x) - \eta_0(x))$. As discussed in Section 1, an important ingredient in showing Theorem 1.3 is to show that for configurations η_0 with large numbers of both particles and holes locally, $\sum_x x(\eta_t(x) - \eta_0(x))$, on the average, increases at least at a certain rate. This is made precise in Proposition 4.1, the main result of this section. As in Section 3, the kernel of the underlying random walk is assumed to have finite range and to be irreducible in this section.

PROPOSITION 4.1. *Let $\eta_\bullet$ be an exclusion process, with $\mu > 0$ and $\eta_0 \in \Xi_{KN}$, where $K, N \in \mathbb{Z}^+$. Assume that $[-KN, KN]$ includes disjoint intervals I_1, $I_2, \dots, I_{2n}$ (ordered from left to right), each of length εN, such that I_{2i-1} and I_{2i} are adjacent, with the number of particles, under η_0, being at least $\varepsilon\delta N$ in each interval I_{2i-1}, and the number of holes being at least $\varepsilon\delta N$ in each interval I_{2i}, $i = 1, 2, \dots, n$. Then, for given K, $\varepsilon \in (0, \mu/8]$, $\delta \in (0, 1/2]$ and $\varepsilon_1 > 0$, and for N sufficiently large,*

$$E\left[\sum_x x\big(\eta_N(x) - \eta_0(x)\big)\right] \geq (n\varepsilon^2\delta^2/2 - \varepsilon_1)N^2 \tag{4.1}$$

for all $n \geq 0$ and all η_0 as specified above.

As in Section 3, $\eta_\bullet$ is employed to denote exclusion processes with deterministic initial data. The symbol $\xi_\bullet$ will be used for exclusion processes whose initial data are given by some product measure. To provide the background needed to demonstrate Proposition 4.1, our approach will be to analyze the asymptotic behavior of $\xi_\bullet$, and then to use Proposition 3.1 to compare it with $\eta_\bullet$. The analysis of $\xi_\bullet$ relies heavily on the following result from Rezakhanlou (1991).

THEOREM 4.1. *Let $u(0, x) : \mathbb{R} \to [0, 1]$ be a piecewise constant function, and let the random configurations ξ_0^N, $N \in \mathbb{Z}^+$, have independent components $\xi_0^N(x)$, $x \in \mathbb{Z}$, such that*

$$P(\xi_0^N(x) = 1) = u\left(0, \frac{x}{N}\right). \tag{4.2}$$

Let $\xi_\bullet^N$ denote the corresponding exclusion processes. Then, for any finite interval $J \subset \mathbb{R}$, $t \geq 0$ and $\varepsilon > 0$,

$$P\left(\left|\frac{1}{N}\sum_{x \in NJ} \xi_{tN}^N(x) - \int_J u(t, x)\,dx\right| > \varepsilon\right) \to 0 \tag{4.3}$$

as $N \to \infty$, where $u(t, x)$ is the entropy solution of

$$\frac{\partial u}{\partial t} + \mu\frac{\partial(u(1-u))}{\partial x} = 0 \tag{4.4}$$

with initial data $u(0, x)$.

Theorem 4.1 is a restriction of Rezakhanlou's Theorem 1.3, which holds on $\mathbb{R}^d$ and allows arbitrary measurable initial data taking values in [0, 1]. It also applies to certain other conservative particle systems, such as attractive zero range processes. General references for Burger's equation (4.4) are Evans (1998) and Smoller (1993).

The remainder of the section is structured as follows. In Proposition 4.2, we give a version of Theorem 4.1 for expectations. Using Proposition 3.1, we show in Proposition 4.3 that

$$E\left[\left|\sum_x x\big(\eta_N(x) - \xi_N(x)\big)\right|\right] = o(N^2)$$

for suitably chosen ξ_0. Together, these results will show, in Proposition 4.4, that the limiting behavior of $E[|\sum_x x(\eta_t^N(x) - \eta_0^N(x))|]$ only depends on the random walk kernel $p(\cdot)$ through its mean. To show Proposition 4.1, it therefore suffices to analyze the case where $p(\cdot)$ is nearest neighbor with only jumps to the right. This is done with the aid of Proposition 4.5.

PROPOSITION 4.2. *Let u and $\xi_\bullet^N$ be as in Theorem* 4.1 *with, in addition, $u(0,x) \equiv 0$ or $u(0,x) \equiv 1$ on $(-\infty, -K)$, and $u(0,x) \equiv 0$ or $u(0,x) \equiv 1$ on (K, ∞), for some $K \in \mathbb{Z}^+$. Then,*

$$\frac{1}{N^2} E\left[\sum_x x\big(\xi_N^N(x) - \xi_0^N(x)\big)\right] \to \int_{-\infty}^{\infty} x\big(u(1,x) - u(0,x)\big)\,dx \tag{4.5}$$

as $N \to \infty$.

PROOF. The argument relies on Theorem 4.1 together with appropriate truncations. Set

$$R_N = \max\{x : \xi_{tN}^N(x) \neq \xi_0^N(x) \text{ for some } t \le 1\},$$
$$L_N = \min\{x : \xi_{tN}^N(x) \neq \xi_0^N(x) \text{ for some } t \le 1\},$$

and let $A_N(\ell)$ be the event $\{R_N \ge (K + \ell M)N\} \cup \{L_N \le -(K + \ell M)N\}$ for $\ell \in \mathbb{Z}^+$, where M is chosen as in Section 2. Note that $A_N(1) \supset A_N(2) \supset \cdots$, and that by (2.3), $P(A_N(\ell)) \le Ce^{-c\ell N}$ for appropriate $c > 0$ and C. One can check that, on $A_N(\ell)^c$,

$$\left|\sum_x x\big(\xi_N^N(x) - \xi_0^N(x)\big)\right| \le (K + \ell M)^2 N^2. \tag{4.6}$$

It follows from these last two inequalities that

$$E\left[\left|\sum_x x\big(\xi_N^N(x) - \xi_0^N(x)\big)\right|; A_N(\ell) \setminus A_N(\ell+1)\right] \le C\big(K + (\ell+1)M\big)^2 N^2 e^{-c\ell N}.$$

Summing over $\ell = 1, 2, \dots$, we conclude that

$$E\left[\left|\sum_x x\big(\xi_N^N(x) - \xi_0^N(x)\big)\right|; A_N(1)\right] \le Ce^{-cN} \tag{4.7}$$

for a new choice of $c > 0$ and C.

We partition $[-(K+M), K+M]$ into intervals $J_1, J_2, \dots, J_q$ of equal length $\varepsilon > 0$. By Theorem 4.1, for each i and t,

$$P\left(\left|\frac{1}{N}\sum_{x\in NJ_i} \xi_{tN}^N(x) - \int_{J_i} u(t,x)\,dx\right| > \varepsilon^2\right) \to 0$$

as $N \to \infty$, where $u(t,x)$ is the entropy solution of (4.4). Using this, one can check that

$$P\left(\left|\frac{1}{N^2}\sum_{x\in NJ_i} x\xi_{tN}^N(x) - \int_{J_i} xu(t,x)\,dx\right| > C'\varepsilon^2\right) \to 0$$

as $N \to \infty$, for appropriate C'. Summing over i gives

$$P\left(\left|\frac{1}{N^2}\sum_{x=-(K+M)N}^{(K+M)N} x\xi_{tN}^N(x) - \int_{-(K+M)}^{K+M} xu(t,x)\,dx\right| > C'\varepsilon\right) \to 0, \tag{4.8}$$

for a new choice of C'. On $A_N(1)^c$, $[L_N, R_N] \subset [-(K+M)N, (K+M)N]$, and so $\xi_{tN}^N(x) = \xi_0^N(x)$ for $|x| \ge K+M$ and $t \le 1$. Also, since $M > |\mu|$ and the absolute value of the slope of the characteristics of (4.4) is at most $|\mu|$, $u(t,x) = u(0,x)$ for $|x| > K+M$. (This follows from Theorem 4.1 as well.) Recall from (4.6) that on $A_N(1)^c$, $\frac{1}{N^2}\sum_x x(\xi_N^N(x) - \xi_0^N(x))$ is bounded. The limit (4.5) therefore follows from (4.7), (4.8) with $t = 0$ and $t = 1$, and bounded convergence, since ε is arbitrary. $\square$

To show Proposition 4.1, we will not require the full statement in Proposition 4.2, but rather that the limiting behavior of the left side depends only on the mean of the underlying random walk kernel $p(\cdot)$. This version is given below.

COROLLARY 4.1. *Let u and ${}^j\xi_\bullet^N$, $j = 1, 2$, be as in Theorem 4.1, with, in addition, $u(0,x) \equiv 0$ or $u(0,x) \equiv 1$ on $(-\infty, -K)$, and $u(0,x) \equiv 0$ or $u(0,x) \equiv 1$ on (K, ∞), for some $K \in \mathbb{R}^+$. Assume that the random walk kernels ${}^jp(\cdot)$ underlying ${}^j\xi_\bullet^N$ have the same mean. Then,*

$$\frac{1}{N^2}E\left[\sum_x x\big({}^1\xi_N^N(x) - {}^1\xi_0^N(x)\big)\right] - \frac{1}{N^2}E\left[\sum_x x\big({}^2\xi_N^N(x) - {}^2\xi_0^N(x)\big)\right] \to 0 \tag{4.9}$$

as $N \to \infty$.

We will employ the analog of (4.9), but for exclusion processes $\eta_\bullet$ with deterministic initial data. To replace $\xi_\bullet$ by $\eta_\bullet$ in (4.9), we need to approximate $\eta_\bullet$ by appropriate $\xi_\bullet$. For $K, N \in \mathbb{Z}^+$ and $\varepsilon_1 > 0$, with $2K/\varepsilon_1 \in \mathbb{Z}^+$, denote by $I_1, \dots, I_{2K/\varepsilon_1}$ the partition of $[-KN, KN]$ into $2K/\varepsilon_1$ intervals of equal lengths $\varepsilon_1 N$. Given $\eta_0 \in \Xi_{KN}$, we choose ξ_0 so that

$$\xi_0(x) = \eta_0(x) \qquad \text{on } [-KN, KN]^c \tag{4.10}$$

and

$$\xi_0 \text{ has product measure with constant density } [\rho_i/\varepsilon_1^2]\varepsilon_1^2 \text{ on } I_i, \tag{4.11}$$

where $\rho_i = \frac{1}{\varepsilon_1 N}\sum_{x\in I_i}\eta_0(x)$ and [z] denotes the integer part of z. Thus, at each $x \in I_i$, we assign a density to ξ_0 which is close to the average density ρ_i of η_0 over I_i. The reason for taking the integer part of ρ_i/ε_1^2 is that we wish to compare different $\eta_0 \in \Xi_{KN}$ with only a finite number of such ξ_0. This will allow us to show (4.1) holds uniformly over such η_0.

The following lemma employs Proposition 3.1 and elementary large deviation estimates to show that η_t and the above ξ_t are typically close. As in Section 3, we abbreviate by setting $\phi_t(x) = \eta_t(x) - \xi_t(x)$.

LEMMA 4.1. *Let $\eta_\bullet$ and $\xi_\bullet$ be exclusion processes generated by the same Harris system. For given K, assume that $\eta_0 \in \Xi_{KN}$, $\varepsilon_1 \in (0, 1/8K)$, and that ξ_0 is chosen as in* (4.10) *and* (4.11). *Then, for appropriate $c > 0$ and C not depending on η_0,*

$$P\left(\left|\sum_{x\ge y}\phi_t(x)\right| > 4\varepsilon_1 N \textit{ for some } y\right) \le Ce^{-cN} \tag{4.12}$$

for all $N \ge 1/\varepsilon_1$ and t.

PROOF. Setting ${}^1\eta_\bullet = \eta_\bullet$, ${}^2\eta_\bullet = \xi_\bullet$ and $\gamma = \varepsilon_1$ in Proposition 3.1, and integrating over the initial states there, it follows that

$$P\left(\sup_y \sum_{x\ge y}\phi_t(x) - \sup_y \sum_{x\ge y}\phi_0(x) > \varepsilon_1 N\right) \le Ce^{-cN} \tag{4.13}$$

for appropriate $c > 0$ and C not depending on η_0. The analog of (4.13) also holds with the roles of $\eta_\bullet$ and $\xi_\bullet$ reversed. So, in order to demonstrate (4.12), it suffices to show that

$$P\left(\left|\sum_{x\ge y}\phi_0(x)\right| > 3\varepsilon_1 N \text{ for some } y\right) \le Ce^{-cN}.$$

As there are only $2KN + 1$ sites y where $\phi_0(x) \ne 0$ is possible, it suffices to show the above bound for fixed y. The interval I_j, with $y \in I_j$, has at most

$\varepsilon_1 N + 1 \le 2\varepsilon_1 N$ sites. Since there are $2K/\varepsilon_1$ intervals I_i, it is therefore enough to show that for each $i > j$,

$$P\left(\left|\sum_{x\in I_i} \phi_0(x)\right| > \varepsilon_1^2 N/2K\right) \le Ce^{-cN} \tag{4.14}$$

for appropriate $c > 0$ and C. But, the $[\cdot]$ in (4.11) moves the mean of $\sum_{x\in I_i} \phi_0(x)$ at most $2\varepsilon_1^3 N \le \varepsilon_1^2 N/4K$ away from 0. The bound in (4.14) is therefore a standard large deviation estimate. $\square$

In Proposition 4.3, we give a version of Lemma 4.1 for expectations. Its proof is similar to that of Proposition 4.2, but uses the above lemma instead of Theorem 4.1.

PROPOSITION 4.3. *Let $\eta_\bullet$ and $\xi_\bullet$ be exclusion processes generated by the same Harris system. Assume that $\eta_0 \in \Xi_{KN}$ with $K, N \in \mathbb{Z}^+$, $\varepsilon_1 \in (0, 1/8K]$, and that ξ_0 is chosen as in* (4.10) *and* (4.11). *Then, for large enough N not depending on η_0 and appropriate C not depending on η_0 or ε_1,*

$$E\left[\left|\sum_x x\phi_N(x)\right|\right] \le C\varepsilon_1 N^2. \tag{4.15}$$

PROOF. As in the proof of Proposition 4.2, we truncate, this time using

$$R_N = \max\{x : \eta_N(x) \ne \eta_0(x) \text{ or } \xi_N(x) \ne \xi_0(x)\},$$
$$L_N = \min\{x : \eta_N(x) \ne \eta_0(x) \text{ or } \xi_N(x) \ne \xi_0(x)\}.$$

Note that

$$\phi_N(x) = 0 \qquad \text{for } x \in [L_N \wedge (-KN), R_N \vee (KN)]^c. \tag{4.16}$$

As before, we set $A_N(\ell) = \{R_N \ge (K + \ell M)N\} \cup \{L_N \le -(K + \ell M)N\}$. Proceeding precisely as through (4.7), one obtains

$$E\left[\left|\sum_x x\phi_N(x)\right|; A_N(1)\right] \le CN^2 e^{-cN} \tag{4.17}$$

for appropriate $c > 0$ and C. These constants do not depend on η_0.

Let B be the event that for some y, $|\sum_{x\ge y} \phi_N(x)| > 4\varepsilon_1 N$. By Lemma 4.1, $P(B) \le Ce^{-cN}$ for appropriate $c > 0$ and C depending on ε_1, but not on η_0. Together with (4.16), this implies that

$$E\left[\left|\sum_x x\phi_N(x)\right|; A_N(1)^c \cap B\right] \le C(K + M)^2 N^2 e^{-cN}. \tag{4.18}$$

To obtain (4.15), it remains to bound

$$E\left[\left|\sum_x x\phi_N(x)\right|; A_N(1)^c \cap B^c\right]. \tag{4.19}$$

Abbreviating $(K+M)N$ by H, one has, by (4.16), that on $A_N(1)^c$,

$$\sum_x x\phi_N(x) = \sum_{x=-H}^{H} x\phi_N(x).$$

By Abel partial summation ("summation by parts") and then (4.16), this equals

$$\left(\sum_{y=-H+1}^{H}\sum_{x=y}^{H}\phi_N(x)\right) - \left(H\sum_{x=-H}^{H}\phi_N(x)\right)$$
$$= \left(\sum_{y=-H+1}^{H}\sum_{x\geq y}\phi_N(x)\right) - \left(H\sum_{x\geq -H}\phi_N(x)\right),$$

which, on B^c, is at most $12\varepsilon_1 HN = 12\varepsilon_1(K+M)N^2$. This is an upper bound on (4.19) and, together with (4.17) and (4.18), implies (4.15). □

Corollary 4.1 compares two exclusion processes ${}^j\xi_\bullet$, $j=1,2$, with random walk kernels having the same mean and starting from the same (product) random configuration. Proposition 4.3 compares the exclusion process $\eta_\bullet$, with a given deterministic initial configuration, with the exclusion process $\xi_\bullet$ having the same random walk kernel and with the random initial configuration chosen in (4.10) and (4.11). Putting these two results together immediately implies Proposition 4.4, which compares two exclusion processes ${}^j\eta_\bullet$ with random walk kernels having the same mean and deterministic initial configuration. Note that the bound in (4.20) is uniform over $\eta_0 \in \Xi_{KN}$. This does not cause difficulties when employing (4.9) to derive the bound, since only finitely many processes $\xi_\bullet$ are needed for a given ε_1 because of the construction in (4.10) and (4.11).

PROPOSITION 4.4. *Let ${}^j\eta_\bullet$, $j=1,2$, be exclusion processes with ${}^1\eta_0 = {}^2\eta_0 \in \Xi_{KN}$, for given K. Assume that the random walk kernels ${}^jp(\cdot)$ underlying ${}^j\eta_\bullet$ have the same mean. Then, for all $\varepsilon_1 > 0$ and large enough N not depending on ${}^j\eta_0$,*

$$E\left[\left|\sum_x x\left({}^1\eta_N(x) - {}^2\eta_N(x)\right)\right|\right] \leq \varepsilon_1 N^2. \tag{4.20}$$

We may consider Proposition 4.4 as a sort of invariance principle—to analyze an exclusion process $\eta_\bullet$, it suffices to analyze a simpler exclusion process whose underlying random walk has the same mean. We will apply this when showing Proposition 4.1. An alternative approach for Proposition 4.1 would be to employ

Propositions 4.2 and 4.3, together with lower bounds on the integral on the right-hand side of (4.5). We prefer the present approach since it only requires the limit of the exclusion process given in (4.26), rather than familiarity with solutions of Burger's equation.

Assume now that the random walk kernel of the given exclusion process has mean $\mu > 0$. Applying Proposition 4.4, we choose to instead work with the exclusion process with the deterministic kernel $p(\cdot)$; that is, nearest neighbor with only jumps to the right; that is,

$$p(1) = \mu \quad \text{and} \quad p(x) = 0 \text{ for } x \neq 1. \tag{4.21}$$

Two simple consequences of this property are that

(4.22) the motion of particles is not affected by particles to their left

and

(4.23) for ${}^1\eta_\bullet$ and ${}^2\eta_\bullet$ generated by the same Harris system and satisfying $\sum_{x \le y}({}^1\eta_t(x) - {}^2\eta_t(x)) \ge 0$ for all y at $t = 0$, this inequality persists for all t.

Most of the remaining work to show Proposition 4.1 is devoted to showing Proposition 4.5. There, we set the number of pairs of intervals n in Proposition 4.1 equal to 1, and work with the exclusion process with $p(\cdot)$ given by (4.21). We will then "glue together" such solutions in the proof of Proposition 4.1. First, we demonstrate the following lemma. It gives lower bounds on the extent to which the particles of $\eta_\bullet$, under certain specific initial data, move to the right by a given time.

LEMMA 4.2. *Let $\eta_\bullet^N$ be the exclusion process with $p(\cdot)$ satisfying* (4.21) *and*

$$\eta_0^N(x) = \begin{cases} 1, & \text{on } [-2\varepsilon N, (\gamma - 2\varepsilon)N] \cup [0, \infty), \\ 0, & \text{otherwise}, \end{cases} \tag{4.24}$$

where $0 \le \gamma < 2\varepsilon \le \mu/4$. *Then*,

$$\frac{1}{N} \sum_{x \le -\gamma N} \eta_N^N(x) \to 0 \quad \text{and} \quad \frac{1}{N} \sum_{x > -\gamma N} \big(1 - \eta_N^N(x)\big) \to 0 \tag{4.25}$$

in probability as $N \to \infty$.

PROOF. Before analyzing $\eta_\bullet^N$, we first consider two exclusion processes with simpler initial conditions. Let ${}^1\eta_\bullet$ be the exclusion process with ${}^1\eta_0(x) = 1$ on $(-\infty, 0]$ and ${}^1\eta_0(x) = 0$ on $(0, \infty)$ [and satisfying (4.21)]. It is well known that for $|\beta| \le \mu$,

$$\frac{1}{N} \sum_{x \ge \beta N} {}^1\eta_N(x) \to \frac{1}{4\mu}(\mu - \beta)^2 \qquad \text{in probability} \tag{4.26}$$

as $N \to \infty$ [see, e.g., Liggett (1985), page 407].

We compare ${}^1\eta_\bullet$ with the exclusion processes ${}^2\eta_\bullet^N$, with ${}^2\eta_0^N(x) = 1$ on $[-\gamma N, 0]$ and ${}^2\eta_0^N(x) = 0$ elsewhere. By (4.22), the motion of the particles of ${}^2\eta_\bullet^N$ is the same as the motion of the particles of ${}^1\eta_\bullet$ which begin in $[-\gamma N, 0]$. Since these particles remain to the right of all other particles of ${}^1\eta_\bullet$, it follows from (4.26) that, for $\beta \le \mu - 2\sqrt{\gamma\mu}$,

$$\frac{1}{N}\sum_{x<\beta N} {}^2\eta_N^N(x) \to 0 \qquad \text{in probability} \tag{4.27}$$

as $N \to \infty$.

We now compare ${}^2\eta_\bullet^N$ with $\eta_\bullet^N$. We classify those particles of $\eta_\bullet^N$, which begin in $[-2\varepsilon N, (\gamma - 2\varepsilon)N]$, as first class particles, and those particles on $[0, \infty)$ as second class particles. First class particles are assumed to have priority over second class particles; that is, they can displace second class particles, but not vice versa. Since $[-2\varepsilon N, (\gamma - 2\varepsilon)N]$ is a translate of $[-\gamma N, 0]$, we can compare the motion of the first class particles with that of the particles of ${}^2\eta_\bullet^N$. Off of the exceptional sets given by (4.27), only $o(N)$ of these first class particles are to the left of $((\sqrt{\mu} - \sqrt{\gamma})^2 - 2\varepsilon)N$ at time N, which is at least 0 under our assumptions $\gamma \le 2\varepsilon \le \mu/4$.

Reverse the role of particles and holes of $\eta_\bullet^N$. Because of (4.21), none of the holes can ever jump to the right of any particle. This includes the second class particles, and so holes always remain in $(-\infty, 0)$. Hence, by the previous paragraph, no hole is to the right of more than $o(N)$ first class particles, and therefore to the right of $o(N)$ particles of any class.

One can also label particles in the standard manner, so that they all move without priority. Under this scheme, particles starting in $[0, \infty)$ never move, and because of the above behavior of holes, all except for $o(N)$ of the particles starting in $[-2\varepsilon N, (\gamma - 2\varepsilon)N]$ have moved to $(-\gamma N, 0)$, with no holes lying to their right. This implies both limits in (4.25). $\square$

As mentioned earlier, Proposition 4.5 is one of the main steps in showing Proposition 4.1. The idea behind its proof is that, since the particles of $\eta_\bullet$ start to the right of the corresponding particles of each of the processes $\eta_\bullet^N$ in Lemma 4.2, by (4.23), they always remain to the right of these particles, and so (4.25) can be applied to $\eta_\bullet$. This will imply that typically $\varepsilon\delta N(1+o(1))$ particles will each move at least $\varepsilon\delta N(1+o(1))$ to the right, which gives the bound in (4.30).

PROPOSITION 4.5. *Let $\eta_\bullet$ be an exclusion process with $p(\cdot)$ satisfying* (4.21), *such that*

$$\eta_0(x) = \begin{cases} 1, & \text{on } [0, \infty), \\ 0, & \text{on } (-\infty, -2\varepsilon N), \end{cases} \tag{4.28}$$

and

$$\sum_{x\in I_1}\eta_0(x)\ge \varepsilon\delta N,\qquad \sum_{x\in I_2}\big(1-\eta_0(x)\big)\ge \varepsilon\delta N, \tag{4.29}$$

where $I_1=[-2\varepsilon N,-\varepsilon N)$ *and* $I_2=[-\varepsilon N,0)$, *for given* ε *and* δ, *with* $\varepsilon\in(0,\mu/8]$ *and* $\delta\in(0,1/2]$. *Then, for large enough* N *not depending on* η_0,

$$E\left[\sum_x x\big(\eta_N(x)-\eta_0(x)\big)\right]\ge \tfrac{1}{2}\varepsilon^2\delta^2N^2. \tag{4.30}$$

PROOF. For a given N, we compare $\eta_\bullet$ with the exclusion process $\eta_\bullet^N$ generated by the same Harris system, with initial data satisfying (4.24), and γ chosen so that $\sum_{x\in I}\eta_0(x)=\sum_{x\in I}\eta_0^N(x)$, where $I=I_1\cup I_2$. This last condition implies that

$$\sum_{x\le y}\big(\eta_0^N(x)-\eta_0(x)\big)\ge 0$$

for all y, and hence by (4.23),

$$\sum_{x\le y}\big(\eta_N^N(x)-\eta_N(x)\big)\ge 0$$

for all y. Applying the first limit in (4.25) to $\eta_\bullet^N$, it follows that for given $\delta_1>0$ and large enough N,

$$P\left(\sum_{x\le-\gamma N}\eta_N(x)>\delta_1N\right)<\delta_1 \tag{4.31}$$

for all η_0 satisfying (4.28) and (4.29). Analogous reasoning shows that for large enough N,

$$P\left(\sum_{x>-\gamma N}\big(1-\eta_N(x)\big)>\delta_1N\right)<\delta_1 \tag{4.32}$$

for such η_0. We claim that, together with (4.28) and (4.29), (4.31) and (4.32) will imply (4.30).

We consider two cases. For $\gamma\le\varepsilon$, off the exceptional set A in (4.31), all except for δ_1N of the at least $\varepsilon\delta N$ particles starting in I_1 are to the right of $-\varepsilon N$ at time N. Since the order of these particles is preserved over time, each of these particles must move at least $(\varepsilon\delta-\delta_1)N$ to the right. None of the other particles can move to the left. So, on A^c,

$$\sum_x x\big(\eta_N(x)-\eta_0(x)\big)\ge(\varepsilon\delta-\delta_1)^2N^2. \tag{4.33}$$

Since $\delta_1>0$ is arbitrary, this implies (4.30) for $\gamma\le\varepsilon$.

The reasoning for $\gamma > \varepsilon$ is analogous, except that the role of particles and holes is reversed. One applies (4.32) to at least $\varepsilon\delta N$ holes starting in I_2. This implies that, off the exceptional set in (4.32), at least $(\varepsilon\delta - \delta_1)N$ holes have moved $(\varepsilon\delta - \delta_1)N$ to the left by time N. This also implies (4.33), and hence (4.30) for $\gamma > \varepsilon$ as well. $\square$

We note that the lengths of the intervals I_1 and I_2 in (4.29) can easily be modified without affecting (4.30). Smaller I_1 and I_2 are included by extending them to the left or right since (4.29) will continue to hold for these larger intervals, whereas extension of I_1 or I_2 by a fixed length J corresponds to a new choice of δ in (4.29) for the original intervals, and will change the lower bound in (4.33) by at most $2\varepsilon\delta JN$, which can be absorbed into the right side of (4.30).

We now demonstrate Proposition 4.1. Because of Proposition 4.4, we can restrict $p(\cdot)$ to the kernel given in (4.21). Under this setting, we compare $\eta_\bullet$ with the process $\tilde{\eta}_\bullet$ obtained by not permitting particles to move from one pair of intervals (I_{2i-1}, I_{2i}) to the next. Because of our choice of $p(\cdot)$, the expectation on the left-hand side of (4.1) is decreased by replacing $\eta_\bullet$ by $\tilde{\eta}_\bullet$. This new expectation can then be broken into pieces, with Proposition 4.5 being applied to each piece.

PROOF OF PROPOSITION 4.1. Assume that the exclusion process $\eta_\bullet$ has the random walk kernel $p(\cdot)$ given in (4.21). Let ${}^i\eta_\bullet$, $i = 1, \dots, n$, denote the exclusion processes generated by the same Harris system as $\eta_\bullet$, but with

$$
{}^i\eta_0(x) = \begin{cases} \eta_0(x), & \text{on } I_{2i-1} \cup I_{2i}, \\ 0, & \text{to the left of } I_{2i-1}, \\ 1, & \text{to the right of } I_{2i}. \end{cases} \tag{4.34}
$$

Since particles can move only to the right, no particles of ${}^i\eta_\bullet$ ever enter I_{2i-1} from the right or leave I_{2i} on the left. Let $\tilde{\eta}_\bullet$ denote the exclusion process generated by the same Harris system as $\eta_\bullet$, and with $\tilde{\eta}_0 = \eta_0$, but where jumps from the left of I_{2i-1} into I_{2i-1}, and from I_{2i} to the right of I_{2i} are suppressed. For all t,

$$
\sum_x x\big(\tilde{\eta}_t(x) - \tilde{\eta}_0(x)\big) = \sum_{i=1}^n \sum_x x\big({}^i\eta_t(x) - {}^i\eta_0(x)\big). \tag{4.35}
$$

It is easy to see that, since $p(\cdot)$ is nearest neighbor, particles in η_t always lie to the right of the corresponding particles in $\tilde{\eta}_t$. So, for all t,

$$
\sum_x x\big(\eta_t(x) - \eta_0(x)\big) \geq \sum_x x\big(\tilde{\eta}_t(x) - \tilde{\eta}_0(x)\big).
$$

Together with (4.35), this implies that for all t,

$$
\sum_x x\big(\eta_t(x) - \eta_0(x)\big) \geq \sum_{i=1}^n \sum_x x\big({}^i\eta_t(x) - {}^i\eta_0(x)\big). \tag{4.36}
$$

We apply Proposition 4.5 to each of the processes ${}^i\eta_\bullet$ at time N. (εN is not assumed to be an integer and so I need not be of constant length, but the comment after the proposition compensates for this.) Together with (4.36), the proposition implies that

$$E\left[\sum_x x\big(\eta_N(x) - \eta_0(x)\big)\right] \geq \tfrac{1}{2} n\varepsilon^2\delta^2 N^2 \tag{4.37}$$

for large enough N, not depending on η_0, and all n. The bound in (4.37) holds for $p(\cdot)$ given by (4.21). Application of Proposition 4.4 generalizes this to all $p(\cdot)$ having the same mean and gives (4.1). $\square$

5. Drift to the right for low density configurations. In this section, we show that particles of "low density" configurations of exclusion processes, with $\mu > 0$, tend to drift to the right. As always, the underlying random walk kernel is assumed to have finite range. Proposition 5.3 is the main result in this section. Proposition 5.1, its corollary and Proposition 5.2 follow quickly from Proposition 5.3, and will be employed in Section 6.

Proposition 5.1 says that if there are few enough particles, then they will all drift to the right off a set of negligible probability. We recall the notation $L(\eta_t) = \min\{x : \eta_t(x) = 1\}$.

PROPOSITION 5.1. *Assume that $\eta_\bullet$ is an exclusion process with $\mu > 0$, and that η_0 has at most δN particles, with $L(\eta_0) \geq 0$. Then, for $\delta > 0$ chosen small enough, and appropriate $c > 0$ and C,*

$$P\big(L(\eta_N) \leq \tfrac{1}{4}\mu N\big) \leq Ce^{-cN} \tag{5.1}$$

for all N and all such η_0.

The following corollary of Proposition 5.1 will be employed in Section 6 to obtain bounds on $E[g(\eta_N)]$, where $g(\cdot)$ is given by (1.8).

COROLLARY 5.1. *Assume that $\eta_\bullet$ is an exclusion process with $\mu > 0$ and $L(\eta_0) > -\infty$, and that η_0 has at most δN particles in the interval $[L(\eta_0), L(\eta_0) + MN]$. Then, for $\delta > 0$ sufficiently small and N sufficiently large,*

$$E\big[\big(L(\eta_N) + \beta N\big)^-\big] - \big(L(\eta_0) + \beta N\big)^- \leq 1$$

for all β and all such η_0. If, in addition, $L(\eta_0) + \beta N \leq -\frac{1}{4}\mu N$, then

$$E\big[\big(L(\eta_N) + \beta N\big)^-\big] - \big(L(\eta_0) + \beta N\big)^- \leq -\tfrac{1}{5}\mu N.$$

Proposition 5.2 says that if the local density is always sufficiently low in a system with a finite number of particles, then the mean position of the particles of $\eta_\bullet$ will drift to the right.

PROPOSITION 5.2. *Assume that $\eta_\bullet$ is an exclusion process with $\mu > 0$, and that η_0 has $\Gamma < \infty$ particles, with*

$$\sum_{x=[i\mu N]}^{[(i+1)\mu N]} \eta_0(x) \le \delta N \tag{5.2}$$

for each $i \in \mathbb{Z}$. Then, for $\delta > 0$ chosen small enough,

$$E\left[\sum_x x\big(\eta_N(x) - \eta_0(x)\big)\right] \ge \tfrac{1}{4}\mu\Gamma N \tag{5.3}$$

for all such η_0 and large enough N, not depending on η_0.

Replacement of particles by empty sites and empty sites by particles immediately implies the analogs of Proposition 5.1, Corollary 5.1 and Proposition 5.2, but for high density configurations instead of low density configurations. Inequality (5.3) again holds, but the lower bounds on $L(\eta_N)$ in Proposition 5.1 and its corollary are replaced by upper bounds on $R(\eta_N)$. These versions of the above results will also be employed in Section 6. We remark that the results in (5.1) and (5.3) still hold if the coefficient $\frac{1}{4}$ there is replaced by any coefficient strictly less than 1, and similarly, that the second display in Corollary 5.1 holds if the cofficients $\frac{1}{4}$ and $\frac{1}{5}$ are replaced by ε_1 and ε_2 with $\varepsilon_2 < \varepsilon_1 < 1$.

In order to show Propositions 5.1 and 5.2, we argue inductively. We first follow the rightmost particle of η_0 as time evolves, and then successively include additional particles to its left. For this, we order the particles as $X_\bullet^1, X_\bullet^2, \dots, X_\bullet^\Gamma$, according to their initial positions, with a smaller index indicating an initial position farther to the right. As $\eta_\bullet$ evolves, we employ the rule that particles with lower index always have priority; that is, a lower-indexed particle can displace a particle with a higher index, but not vice versa. Consequently, the evolution of $(X_\bullet^1, X_\bullet^2, \dots, X_\bullet^k)$ does not depend on $X_\bullet^{k'}$ for $k' > k$. We set $L^k(\eta_t) = \min\{X_t^1, \dots, X_t^k\}$.

Propositions 5.1 and 5.2 will follow quickly from the following result. Here and later on, we employ the function $h(\cdot)$ which is obtained from η_0, where $h(1) = X_0^1$, and $h(k)$, $k \le \Gamma$, is given inductively by $h(k) = X_0^k \wedge (h(k-1) - b)$ for some fixed b, with $b \ge d$. (Recall that d is the magnitude of the largest jump of the underlying random walk.)

PROPOSITION 5.3. *Assume that $\eta_\bullet$ is an exclusion process with $\mu > 0$, and that η_0 has only finitely many particles. Then, for b chosen large enough and $c > 0$ small enough,*

$$E\left[\int_0^\infty \exp\{c\big(\tfrac{1}{2}\mu t - L^k(\eta_t) + h(k)\big)\}\,dt\right] \le C \tag{5.4}$$

for appropriate C, all k and all such η_0. Equivalently,

$$E\left[\sum_{n=1}^{\infty}\exp\left\{c\sup_{t\in[n-1,n]}\{\tfrac{1}{2}\mu t - L^k(\eta_t)+h(k)\}\right\}\right]\le C. \tag{5.5}$$

The bound (5.5) implies that

$$E\big[\exp\{c\big(\tfrac{1}{2}\mu t - L^k(\eta_t)+h(k)\big)\}\big]\le C \tag{5.6}$$

for any t. Proposition 5.1 follows quickly from this bound.

PROOF OF PROPOSITION 5.1. By assumption, $L(\eta_0)\ge 0$, and so $h(k)>-bk$ for all k. Since there are at most δN particles, it follows that $h(k)>-b\delta N$. Substitution of this bound for $h(k)$ and L for L^k, in (5.6), implies that

$$E\big[\exp\{c\big((\tfrac{1}{2}\mu - b\delta)N - L(\eta_N)\big)\}\big]\le C.$$

Together with Markov's inequality, this implies (5.1) for small enough δ. □

Corollary 5.1 follows from Proposition 5.1 and bounds from Section 2.

PROOF OF COROLLARY 5.1. Let $\eta'_\bullet$ denote the exclusion process generated by the same Harris system as $\eta_\bullet$, and with $\eta'_0(x)=\eta_0(x)$ for $x\le L(\eta_0)+MN$ and $\eta'_0(x)=0$ otherwise. Apply Proposition 5.1 to η'_N and compare η_N with η'_N, using Lemma 2.2, to obtain

$$P\big(L(\eta_N)-L(\eta_0)\le \tfrac{1}{4}\mu N\big)\le Ce^{-cN}$$

for appropriate $c>0$ and C, and all N. By (2.3),

$$P\big(L(\eta_N)-L(\eta_0)\le -\tfrac{1}{2}MN-\ell\big)\le Ce^{-c(N+\ell)}$$

for $\ell\in\mathbb{Z}^+$ and appropriate $c>0$ and C. The first bound in the corollary follows easily from these two inequalities. Suppose now that $L(\eta_0)+\beta N\le -\frac{1}{4}\mu N$ for our choice of β. Then, off the exceptional set in the first inequality,

$$\big(L(\eta_N)+\beta N\big)^- - \big(L(\eta_0)+\beta N\big)^-\le -\tfrac{1}{4}\mu N.$$

Together with the second inequality, this implies the second bound in the corollary. □

Proposition 5.2 follows from (5.6) and the definition of h.

PROOF OF PROPOSITION 5.2. For η_0 satisfying (5.2) and $\delta<\mu/16b$,

$$h(k)\ge X_0^k - 2b\delta N\ge X_0^k-\tfrac{1}{8}\mu N, \tag{5.7}$$

for all k. To see the first inequality, we note that it is immediate for $k=1$. For $1<k\le\Gamma$ with $X_0^k\in[[i\mu N],[(i+1)\mu N])$, we argue by induction, considering

first the case where for some $k' < k$ with $X_0^{k'} \le (i+2)\mu N$, $h(k') = X_0^{k'}$. Let k_1 be the largest such k'. Then,

$$h(k) \ge X_0^{k_1} - 2b\delta N > X_0^k - 2b\delta N.$$

On the other hand, if there is no such k', let k_2 be the largest index k' with $X_0^{k'} \ge (i+2)\mu N$. Then,

$$h(k) \ge h(k_2) - 2b\delta N \ge (i+1)\mu N - 2b\delta N \ge X_0^k - 2b\delta N,$$

where the second inequality follows from the induction hypothesis.

Substitution into (5.6) of the bound for $h(k)$ given in (5.7), together with Markov's inequality, implies that for small enough $c > 0$,

$$P\big(X_N^k \le X_0^k + \tfrac{1}{3}\mu N - \ell\big) \le Ce^{-c(N+\ell)}$$

for all k and N, $\ell \in \mathbb{Z}^+$, appropriate $c > 0$ and C, and all η_0 satisfying (5.2). So, for large enough N,

$$E[X_N^k] - E[X_0^k] \ge \tfrac{1}{4}\mu N$$

for each k. Summation over k implies (5.3). $\square$

Inequalities (5.4) and (5.5) are equivalent (when allowing different choices of C in the displays). Clearly, (5.5) implies (5.4). The other direction also holds since, over $[n-1, n]$, $L(\eta_t) - L(\eta_{n-1})$ is bounded below by the exponential bounds given in (2.3), and is also at most 0 with at least a fixed positive probability. [The term c, in (5.4) and (5.5), needs to be chosen smaller than that in (2.3).] For the sake of readability, we choose to demonstrate (5.4) since the estimates are somewhat messier for (5.5).

The basic idea behind the inductive argument we will use to show (5.4) is that, as long as $X_\cdot^k$ lies at least distance $d+1$ below $L^{k-1}(\eta_\cdot)$, the higher priority particles $X_\cdot^1, \dots, X_\cdot^{k-1}$ will not impede its movement. Since this occurs most of the time when the particle density is low, $X_\cdot^k$ will move similarly to a (continuous time) random walk with drift almost μ. We will actually show, using induction, that $L^k(\eta_\cdot)$ will have a long term drift at least $\mu/2$.

Rather than work directly with $X_\cdot^k$, it is more convenient to employ $Y_\cdot^k$, the stochastic process on $\mathbb{Z}$; that is, coupled to $X_\cdot^k$ so that $Y_\cdot^k$ jumps together with $X_\cdot^k$, except we require that

$$Y_t^k \le L^{k-1}(\eta_t) - d \qquad \text{for all } t. \tag{5.8}$$

[We set $L^0(\eta_t) = \infty$.] To ensure this when $Y_\cdot^k$ attempts to jump above $L^{k-1}(\eta_\cdot) - d$, we set $Y_t^k = L^{k-1}(\eta_t) - d$. Similarly, when $L^{k-1}(\eta_\cdot)$ decreases so as to violate (5.8), we decrease $Y_\cdot^k$ so that equality again holds. Thus, $Y_\cdot^k$ is a finite range random walk on $y \le L^{k-1}(\eta_t) - d$, with "reflection" at $L^{k-1}(\eta_\cdot) - d$. The initial state Y_0^k is the largest value for which $Y_0^k \le X_0^k$ and (5.8) holds at $t = 0$; it follows

from the definition of h that $Y_0^k \geq h(k)$. Note that since the jumps of $X_\cdot^k$ are only suppressed when $X_t^k \geq L^{k-1}(\eta_t) - d$, one has $Y_t^k \leq X_t^k$ for all t, and hence,

$$Y_t^k \leq L^k(\eta_t) \qquad \text{for all } t. \tag{5.9}$$

So, in order to obtain a lower bound on $X_\cdot^k$ or $L^k(\eta_\cdot)$, it suffices to instead analyze $Y_\cdot^k$.

We find it convenient to shift coordinates in order to induce an appropriate drift on $Y_\cdot^k$. We set

$$\varphi_t^k = \tfrac{1}{2}\mu t + h(k+1) + d - L^k(\eta_t) \tag{5.10}$$

and

$$Z_t^k = \tfrac{1}{2}\mu t + h(k) - Y_t^k. \tag{5.11}$$

Then, by (5.8) and the discussion following it, $Z_\cdot^k$ is a finite range random walk translated upward at rate $\mu/2$, with net drift $-\mu/2$ and with reflection at $\varphi_\cdot^{k-1}$; the last property implies that

$$Z_t^k \geq \varphi_t^{k-1} \qquad \text{for all } t. \tag{5.12}$$

By (5.11), $Z_0^k \leq 0$. We also note that, from (5.9)–(5.11) and the definition of $h(k)$,

$$Z_t^k \geq \varphi_t^k + h(k) - h(k+1) - d \geq \varphi_t^k + b - d. \tag{5.13}$$

This recursion relating $\varphi_\cdot^k$ and $Z_\cdot^k$ will be an important ingredient in the proof of Proposition 5.3.

It suffices to bound $E[\int_0^\infty \exp\{cZ_t^k\}\,dt]$ in order to show (5.4). If $\varphi_\cdot^{k-1}$ were given by a line with a negative slope, this would be easy to do using standard large deviation estimates. However, $\varphi_\cdot^{k-1}$ can, in fact, increase quickly, which can cause $Z_\cdot^k$ to increase. To control the effect $\varphi_\cdot^{k-1}$ has on $Z_\cdot^k$, we introduce stopping times $S^k(0) = 0$, $S^k(1), \dots, S^k(j), \dots$ and corresponding processes $W_\cdot^k(j)$. We inductively set

$$S^k(j) = \min\Big\{t > S^k(j-1) : \varphi_t^{k-1} \geq \varphi_{S^k(j-1)}^{k-1} + 1\Big\} \wedge \bar{S}^k(j-1), \tag{5.14}$$

where $\bar{S}^k(j-1)$ is the smallest integer strictly greater than $S^k(j-1)$. In particular, the amount $\varphi_\cdot^{k-1}$ can increase between these times is bounded and $S^k(j) - S^k(j-1) \leq 1$ for all j. We also set

$$W_t^k(j) = \begin{cases} \varphi_{S^k(j)}^{k-1}, & \text{for } t = S^k(j), \\ -\infty, & \text{for } t < S^k(j), \end{cases} \tag{5.15}$$

and on $t > S^k(j)$, couple $W_\cdot^k(j)$ to $Z_\cdot^k$ so that $W_\cdot^k(j)$ evolves according to the same translated random walk, except that (a) there is no boundary which reflects $W_\cdot^k(j)$ and (b) on the "initial" interval $[S^k(j), \bar{S}^k(j))$, negative jumps of $W_\cdot^k(j)$

are suppressed. We will show in Lemma 5.3, that the processes $W_\bullet^k(j)$, $j = 0, 1, 2, \dots$, together provide an upper bound for $Z_\bullet^k$.

To analyze each $W_\bullet^k(j)$, we consider the translated random walk $W_\bullet$, with $W_0 = 0$, and which evolves according to the same transition law as each $W_\bullet^k(j)$, except that negative jumps at times $t \in [0, 1)$, instead of at $t \in [S^k(j), \bar{S}^k(j))$, are suppressed. Since at times $t \in [1, \infty)$, the kernel of $W_\bullet$ is finite range and $W_\bullet$ has drift $-\mu/2 < 0$, $W_t \to -\infty$ linearly off a large deviation set as $t \to \infty$. The following elementary bound suffices for our purposes. We set $w_u = E[\int_0^u e^{cW_t}\,dt]$.

LEMMA 5.1. *For $c > 0$ chosen small enough, $w_\infty < \infty$.*

PROOF. Choose $a > 1$, and set $W_t' = W_{t+a} - W_a$; this defines a random walk with $W_0' = 0$ (and no suppression of jumps). Therefore,

$$w_u \le w_a + E[e^{cW_a}]w_{u-a} \le w_a + E[e^{cW_a}]w_u \tag{5.16}$$

for $u \ge a$. Expanding $E[\exp\{c(W_a - W_1)\}]$, one can check that, for small $c > 0$, this expectation is less than γ^{a-1} for appropriate $\gamma \in (0, 1)$. So

$$E[e^{cW_a}] \le \gamma^{a-1} E[e^{cW_1}] < 1$$

for large enough a. Plugging this into (5.16) implies that

$$w_u \le w_a/\bigl(1 - E[e^{cW_a}]\bigr).$$

Letting $u \to \infty$ implies $w_\infty < \infty$. $\square$

Let $j_0, j_1, j_2, \dots$ be the indices at which $S^k(j_n) = n$ for a given k, and set $T_n = S^k(j_n - 1)$. Applying (5.14) and Lemma 5.1, one obtains the following bound for the integrals of the moment generating functions of $W_\bullet^k(j)$ summed over j for which $S^k(j) \in [n-1, n)$, in terms of the moment generating function for $\varphi_{T(n)}^{k-1}$.

LEMMA 5.2. *Let $S^k(0), S^k(1), S^k(2), \dots$ denote the stopping times in* (5.14). *Then, for $c > 0$ chosen small enough,*

$$E\left[\sum_{j=j_{n-1}}^{j_n - 1} \int_0^\infty \exp\{cW_t^k(j)\}\,dt\right] \le CE\bigl[\exp\{c\varphi_{T(n)}^{k-1}\}\bigr] \tag{5.17}$$

for all n, $1 < k \le \Gamma$ and appropriate C.

PROOF. Define the process $\hat{W}_\bullet^k(j)$ by

$$\hat{W}_t^k(j) = W_{t+S^k(j)}^k(j) - W_{S^k(j)}^k(j),$$

for $t \geq 0$. Then,

$$E\left[\int_{S^k(j)}^{\infty} \exp\{cW_t^k(j)\}\,dt\right] = E\left[\exp\{cW_{S^k(j)}^k(j)\}E\left[\int_0^{\infty} \exp\{c\hat{W}_t^k(j)\}\,dt \,\middle|\, \mathcal{F}_{S^k(j)}\right]\right]. \tag{5.18}$$

Let $W_\bullet$ denote the translated random walk obtained from $\hat{W}_\bullet^k(j)$ by suppressing all negative jumps on $[0, 1)$. Since $\hat{W}_t^k(j) \leq W_t$ for all t, and $W_\bullet$ is independent of $\mathcal{F}_{S^k(j)}$, the right-hand side of (5.18) is less than or equal to

$$E\left[\exp\{cW_{S^k(j)}^k(j)\}E\left[\int_0^{\infty} \exp\{cW_t\}\,dt \,\middle|\, \mathcal{F}_{S^k(j)}\right]\right] = w_\infty E\left[\exp\{cW_{S^k(j)}^k(j)\}\right].$$

Also, by (5.14), $\varphi_{S^k(j+1)}^{k-1} \geq \varphi_{S^k(j)}^{k-1} + 1$ for $S^k(j), S^k(j+1) \in [n-1, n)$ and any n. So, the right-hand side of the above display is less than or equal to

$$w_\infty e^{-c(j_n-1-j)} E\left[\exp\{cW_{T(n)}^k(j_n-1)\}\right]$$

for such j. By Lemma 5.1, $w_\infty < \infty$. Summing these inequalities from $j = j_{n-1}$ to $j = j_n - 1$ implies that

$$E\left[\sum_{j=j_{n-1}}^{j_n-1} \int_{S^k(j)}^{\infty} \exp\{cW_t^k(j)\}\,dt\right] \leq CE\left[\exp\{cW_{T(n)}^k(j_n-1)\}\right];$$

by (5.15), this is equivalent to (5.17). □

Let U denote the first time at which $Z_\bullet^k$ hits $\varphi_\bullet^{k-1}$ for a given k. The following lemma shows that for $t \geq U$, Z_t^k is dominated by $\max_j W_t^k(j) + 1$. Together with Lemma 5.2, which bounds the integrals of the moment generating function for $W^k(j)$, this will enable us to do the same for the integrals of the moment generating function of $Z_\bullet^k$.

LEMMA 5.3. *Let $Z_\bullet^k$ be the process defined in* (5.11) *and $W_\bullet^k(j)$, $j = 0, 1, 2, \dots$, be the processes defined in* (5.15). *Then, for $t \geq U$,*

$$Z_t^k < \max_j W_t^k(j) + 1. \tag{5.19}$$

PROOF. Suppose that $Z_V^k = \varphi_V^{k-1}$ at a given time V. We claim that

$$Z_V^k < W_V^k(j) + 1 \qquad \text{for some } j. \tag{5.20}$$

If $V = S^k(i)$ for some i, then by (5.15), $Z_V^k = W_V^k(i)$. When this assumption does not hold, then by (5.14), V is not an integer and $\varphi_{S^k(i)}^{k-1} + 1 > \varphi_V^{k-1}$ for some $S^k(i) \in [[V], V)$. Since $W_\bullet^k(i)$ cannot decrease over $[S^k(i), V]$, this implies that

$$W_V^k(i) + 1 \geq W_{S^k(i)}^k(i) + 1 = \varphi_{S^k(i)}^{k-1} + 1 > \varphi_V^{k-1} = Z_V^k. \tag{5.21}$$

Hence, (5.20) holds.

At times after V, the process $W_\bullet^k(j)$ evolves according to the same law as $Z_\bullet^k$, except that it is not restricted by the boundary $\varphi_\bullet^{k-1}$, and negative jumps on $[S^k(j), \bar{S}^k(j))$ are suppressed. So, until the next time V' at which $Z_\bullet^k$ is restricted by $\varphi_\bullet^{k-1}$, $Z_t^k < W_t^k(j) + 1$. After a finite amount of time, the finite exclusion process $\eta_\bullet$ considered here has only a finite number of changes of state, and so $Z_\bullet^k$ attempts to cross $\varphi_\bullet^{k-1}$ only a finite number of times. By induction, (5.19) will therefore hold for all $t \geq U$. $\square$

We now demonstrate (5.4) by using the previous lemmas.

PROOF OF (5.4). We will show by induction that for all k,

$$E\left[\int_0^\infty e^{cZ_t^k}\,dt\right] \leq C \tag{5.22}$$

for large enough b, small enough $c > 0$ and appropriate C, which do not depend on k. Together with (5.9) and (5.11), this implies (5.4).

The case $k = 1$ is simple: $Z_\bullet^1$ is a translated finite range random walk with drift $-\mu/2$ and $Z_0^1 \leq 0$. Comparison with $W_\bullet$, together with Lemma 5.1, implies that

$$E\left[\int_0^\infty e^{cZ_t^1}\,dt\right] \leq C_1 \tag{5.23}$$

for small enough $c > 0$ and appropriate C_1. [One can also show (5.23) directly.]

Assume now that (5.22) holds for $k - 1$, with $C = 2C_1$. On $t < U$, $Z_\bullet^k$ is a translated random walk with the same transition law as in the previous case; we denote the extension of this process to all time by $\tilde{Z}_\bullet^1$. By Lemma 5.3,

$$Z_t^k \leq \left(\max_{j\geq 0} W_t^k(j) + 1\right) \vee \tilde{Z}_t^1 \tag{5.24}$$

for all t. Consequently,

$$E\left[\int_0^\infty e^{cZ_t^k}\,dt\right] \leq e^c E\left[\sum_{j=0}^\infty \int_0^\infty e^{cW_t^k(j)}\,dt\right] + E\left[\int_0^\infty e^{c\tilde{Z}_t^1}\,dt\right]. \tag{5.25}$$

The second expectation on the right-hand side of (5.25) is bounded, as in the previous paragraph, by the constant C_1 since $Z_0^k \leq 0$.

We need to bound the first expectation on the right-hand side of (5.25). By Lemma 5.2,

$$\begin{aligned} E\left[\sum_{j=0}^\infty \int_0^\infty e^{cW_t^k(j)}\,dt\right] &= E\left[\sum_{n=1}^\infty \sum_{j=j_{n-1}}^{j_n-1} \int_0^\infty e^{cW_t^k(j)}\,dt\right] \\ &\leq C_2 E\left[\sum_{n=1}^\infty e^{c\varphi_{T(n)}^{k-1}}\right] \end{aligned} \tag{5.26}$$

for small enough $c > 0$ and appropriate C_2 not depending on k. Since $T(n) \in [n-1, n)$ and $\exp\{c\varphi_t^k\}$ (as a function of t) is a multiple of the integrand in (5.4), the equivalence of (5.4) and (5.5) implies that

$$E\left[\sum_{n=1}^{\infty} e^{c\varphi_{T(n)}^{k-1}}\right] \le C_3 E\left[\int_0^{\infty} e^{c\varphi_t^{k-1}}\, dt\right]$$

for appropriate C_3 not depending on k. By (5.13), this is less than or equal to

$$C_3 e^{c(d-b)} E\left[\int_0^{\infty} e^{cZ_t^{k-1}}\, dt\right]. \tag{5.27}$$

The bounds (5.25)–(5.27) imply that for $k > 1$,

$$E\left[\int_0^{\infty} e^{cZ_t^{k}}\, dt\right] \le C_1 + C_2 C_3 e^{c(d+1-b)} E\left[\int_0^{\infty} e^{cZ_t^{k-1}}\, dt\right], \tag{5.28}$$

where C_1, C_2, C_3 and $c > 0$ do not depend on k. Choose b large enough so that $C_2 C_3 e^{c(d+1-b)} \le 1/2$. For such b, it follows from (5.28) and the induction hypothesis, that

$$E\left[\int_0^{\infty} e^{cZ_t^{k}}\, dt\right] \le 2C_1 = C,$$

as desired. □

6. Bounds on the Lyapunov function h. In this section, we demonstrate Theorem 1.3, which implies that the function $h = f + g$ introduced in Section 1 is a Lyapunov function for the process $\eta_\bullet^N$ off of a finite set in Ξ. The work here is divided into three subsections. We first decompose the initial state η_0 into intervals of three types. Using this decomposition in the next subsection, we obtain upper bounds on the average increase from $t = 0$ to $t = N$ of f in Propositions 6.1 and 6.2. We then obtain upper bounds on the average increase from $t = 0$ to $t = N$ of g in Lemma 6.2 and Proposition 6.3. These bounds provide the desired upper bounds on h.

The function f will evolve differently over each of the three types of intervals, which we refer to as heterogeneous, homogeneous and boundary. It will typically decrease (or at least not increase) for each of the first two and not increase by too much for the last. Propositions 2.1 and 2.2 will be employed to justify the decomposition of $\eta_\bullet$ into processes corresponding to the heterogenous and homogeneous intervals, and to bound the contribution by the boundary intervals. Propositions 4.1 and 5.2 will then bound the growth of f over the processes corresponding to the heterogeneous and homogeneous intervals. The analysis of the evolution of g does not require this decomposition of $\eta_\bullet$. The demonstration of Lemma 6.2 and Proposition 6.3 is quicker; the latter result employs Proposition 5.1. In this section, the kernel of the underlying random walk of $\eta_\bullet$ is assumed to be irreducible, with finite range and $\mu > 0$.

Decomposition of η_0. Assume that $\eta_0 \in \Xi$ and choose $\varepsilon \in (0, 1/2)$ and $N \in \mathbb{Z}^+$ so that $1/\varepsilon, \varepsilon N \in \mathbb{Z}^+$. The intervals $I_i = [i\varepsilon N, (i+1)\varepsilon N)$, $i \in \mathbb{Z}$, partition $\mathbb{Z}$. By taking unions of the I_i, we will partition $\mathbb{Z}$ by using three types of intervals whose locations depend on η_0: heterogeneous, homogeneous and boundary intervals. In spirit, heterogeneous intervals will consist of sites that are not too far from pairs $\{I_i, I_{i+1}\}$, where the densities of particles on I_i and I_{i+1} are either not close to each other or are not close to 0 or 1. Boundary intervals will consist of sites within distance $2MN$ of these heterogeneous intervals. Homogeneous intervals will consist of the complement of the first two types; the density of occupied sites for all I_i, in each of these intervals, will be close to 0 or 1. In order to keep the heterogeneous intervals below a maximum length, we will need to do some "splitting" when defining them, filling in additional boundary intervals in between the resulting parts.

Set $\rho_i = \frac{1}{\varepsilon N} \sum_{x \in I_i} \eta_0(x)$; that is, ρ_i is the density of the particles of η_0 in I_i. We classify I_i as having *high density* if $\rho_i \geq 1 - \varepsilon$, *low density* if $\rho_i \leq \varepsilon$, and *middle density* if $\rho_i \in (\varepsilon, 1 - \varepsilon)$. We say I_i has *very high density* if $\rho_i \geq 1 - \delta$ and *very low density* if $\rho_i \leq \delta$, where $\delta \in (0, \varepsilon)$ and $1/\delta \in \mathbb{Z}^+$. (Later on, we will choose ε and δ so that $\delta \ll \varepsilon$.) An ε-*interface* occurs at $\{I_i, I_{i+1}\}$ if I_i has high density and I_{i+1} has low density or vice versa; an ε-interface occurs at $\{I_{i-1}, I_i, I_{i+1}\}$ if I_i has middle density. An ε-interface is *inert* if I_i has low density and I_{i+1} has high density in the first case, and if I_{i-1} has low density and I_{i+1} has high density in the latter case; otherwise, the ε-interface is *live*. One can check that

(6.1) between any two inert ε-interfaces, there is always a live ε-interface.

A δ-*interface* is defined analogously, if ε is replaced by δ, and $1 - \varepsilon$ by $1 - \delta$ for the densities. Inert and live δ-interfaces are defined in the same manner, and the analog of (6.1) holds.

Each ε-interface is contained in a *protected interval* P_i. This is the smallest interval containing the ε-interface, whose endpoints are integer multiples of εN, so that the intervals I_j lying outside P_i, but within distance $2MN$ of an endpoint of P_i, do not contain any part of a δ-interface. (All of these $2M/\varepsilon$ intervals on a given side of P_i must have very high density or all must have very low density.) We note that distinct protected intervals are always at least distance $2MN$ apart. Also, for $\eta_0 \in \Xi$, there are only a finite number of protected intervals, each with finite length.

We would like to be able to apply Proposition 4.1 to the exclusion processes $\eta^i_\bullet$ with initial states $\eta^i_0(x) = \eta_0(x)$ on P_i, $\eta^i_0(x) = q-$ to the left of P_i and $\eta^i_0(x) = q+$ to the right of i, where $q-$, respectively $q+$, is either 0 or 1 according to the majority type on the $2M/\varepsilon$ very low density or very high density intervals I_j immediately to the left, respectively, to the right, of P_i. Since P_i may be too long to apply the proposition, we split it up as follows. If $|P_i| \leq B_1 N$, where $B_1 = 70M^3/\varepsilon^2\delta^2$, we do not change P_i. If $|P_i| > B_1 N$, we partition P_i into neighboring

intervals $V_1, S_1, V_2, S_2, \ldots, S_{\ell-1}, V_\ell$, where $|S_j| = 2MN$ and $|V_j| \in (B_2N, B_1N]$ for each j, where $B_2 = 34M^3/\varepsilon^2\delta^2$. It is not difficult to check that one can always do this. The particular choice is not important, but for a given η_0, we assume this choice is fixed for each P_i.

For a given $\eta_0 \in \Xi$, we label the collection of the intervals $V_1, \ldots, V_r$ obtained from all of the protected intervals sequentially, so that $V_{i'}$ lies to the right of V_i for $i' > i$. These intervals V_i are classified as *heterogeneous*. A heterogeneous interval V_i is *short* if $|V_i| \le B_2N$; otherwise it is *long*. Note that although, in general, a heterogeneous interval need not contain an ε-interface, it must if the interval is short. We classify as *boundary* intervals the intervals of length $2MN$ that lie on either side of a heterogeneous interval. These include the intervals lying on either side of the original protected intervals, as well as those obtained when splitting the intervals. These intervals are denoted by $S_1, \ldots, S_m$, and are ordered sequentially. If the number of particles in S_i is at most J or is at least $2MN - J$, we say that S_i is within J of *unanimity*. We classify the intervals obtained by removing all of the heterogeneous and boundary intervals from $\mathbb{Z}$ as *homogeneous* intervals, which we write as $G_1, G_2, \ldots, G_n$ and order sequentially. The lengths of G_1 and G_n are both infinite.

If one places these three types of intervals together, and orders them according to their coordinates, the sequence thus obtained begins and ends with homogeneous intervals, between which it alternates between boundary, and either heterogeneous or homogeneous intervals. Each boundary interval borders at least one heterogeneous interval, which implies, in particular, that between any two homogeneous intervals there must be at least one heterogeneous interval. Although a boundary interval can have any combination of occupied and vacant sites,

(6.2) if a boundary interval is not within $2\delta MN$ of unanimity, then it borders two long heterogeneous intervals,

since it was obtained by splitting up a protected interval. Note that

(6.3) the intervals I_i contained within any given homogeneous interval and its neighboring boundary intervals all either have high density or all have low density.

Otherwise, the homogeneous interval or one of its neighbors would contain at least part of an ε-interface. This is not possible, since all ε-interfaces are inside protected intervals, and so are contained in heterogeneous intervals, or intersect boundary intervals which border two long heterogeneous intervals.

In the following subsection, we will examine the behavior of f on these three types of intervals. We will show that, under certain restrictions, f tends to decrease (or at least not increase) on heterogeneous and homogeneous intervals, and that the contribution to f on boundary intervals has reasonable bounds. From our perspective, heterogeneous intervals will be "very good," homogeneous intervals will be "good" and boundary intervals will be "satisfactory." Hence, the mnemonics: "V," "G" and "S" for intervals of these three types.

Behavior of $E[f(\eta_N)] - f(\eta_0)$. In this subsection we analyze the behavior of $E[f(\eta_N)] - f(\eta_0)$ for $\eta_0 \in \Xi$. Our two main results are Propositions 6.1 and 6.2, which consider the cases where η_0 contains at least two heterogeneous intervals and where η_0 contains just one such interval. (Each $\eta_0 \in \Xi$ contains at least one ε-interface, and so at least one heterogeneous interval.) For our computations, we will need to extend the domain of $E[f(\eta_N)] - f(\eta_0)$ to $\eta_0 \in \Xi_\infty$. Although $f(\eta)$, as defined in (1.7), need not make sense over this extension,

$$\hat{f}_N(\eta_0) \stackrel{\text{def}}{=} -E\left[\sum_x x\big(\eta_N(x) - \eta_0(x)\big)\right]$$

is well defined for all such η_0, and satisfies $\hat{f}_N(\eta_0) = E[f(\eta_N)] - f(\eta_0)$ for $\eta_0 \in \Xi$. Later on in Section 6, we also abbreviate by setting $\hat{g}_N(\eta_0) = E[g(\eta_N)] - g(\eta_0)$ and $\hat{h}_N(\eta_0) = \hat{f}_N(\eta_0) + \hat{g}_N(\eta_0)$, for $\eta_0 \in \Xi$.

The following lemma will be used in Proposition 6.1. It enables us to obtain bounds on $\hat{f}_N(\eta_0)$ over long heterogeneous intervals. The basic point is that such an interval can be divided up into many intervals of length $2(M+\varepsilon)N$, each of which contains a δ-interface. Each interval of length $4(M+\varepsilon)N$ will then contain a live δ-interface, and we can apply Proposition 4.1 to each of these intervals. Here and later on in the section, we continue to assume that εN, $1/\varepsilon$ and $1/\delta$ are all integers.

LEMMA 6.1. *Assume that* $0 < \delta < \varepsilon < \frac{1}{16} \wedge \frac{\mu}{8}$, *and let* V *be a long heterogeneous interval for a given* $\eta_0 \in \Xi$. *Set* $\tilde{\eta}_0(x) = \eta_0(x)$ *for* $x \in V$, *and assume that* $\tilde{\eta}_0$ *is constant to the left of* V *and is constant to the right of* V. *Let* $\tilde{\eta}_\bullet$ *denote the corresponding exclusion process. Then, for large enough* N (*not depending on* η_0 *or* V),

$$E\left[\sum_x x\big(\tilde{\eta}_N(x) - \tilde{\eta}_0(x)\big)\right] \geq 3M^2N^2. \tag{6.4}$$

PROOF. Choose adjacent intervals $J_1, J_2, \dots, J_{16M^2/\varepsilon^2\delta^2} \subset V$, each being the union of $2M/\varepsilon + 2$ consecutive intervals I_i, and hence each of length $2(M+\varepsilon)N$. Since V is contained in a protected interval, each such J_j must completely contain a δ-interface. (Otherwise, it would contain a boundary interval.) One can check that for each j, $J_{2j-1} \cup J_{2j}$ must contain some I_i and I_{i+1} with $\rho_i > \delta$ and $\rho_{i+1} < 1-\delta$ (and hence a live δ-interface). There are $8M^2/\varepsilon^2\delta^2$ such disjoint pairs in V. The inequality (6.4) therefore follows from Proposition 4.1, with $K = B_1$, $n = 8M^2/\varepsilon^2\delta^2$ and $\varepsilon_1 = 1$. $\square$

In order to analyze $\hat{f}_N(\eta_0)$, we first compare $\eta_\bullet$ with an exclusion process $\eta'_\bullet$ obtained by modifying η_0 on its boundary intervals, and then decompose $\eta'_\bullet$ into exclusion processes corresponding to each of the heterogeneous and homogeneous

intervals of η_0. Let $S_1, \dots, S_m$ be the boundary intervals of $\eta_0 \in \Xi$. Define $q : \{1, \dots, m\} \to \{0, 1\}$ so that $q(j) = 1$ exactly when the strict majority of sites in S_j are occupied, and define η_0' by

$$\eta_0'(x) = \begin{cases} \eta_0(x), & \text{for } x \notin \bigcup_{j=1}^{m} S_j, \\ q(j), & \text{for } x \in S_j; \end{cases} \tag{6.5}$$

$\eta_\bullet'$ denotes the corresponding exclusion process. Let γ_j be the number of sites $x \in S_j$ where $\eta_0'(x) \neq \eta_0(x)$. Note that $\eta_0'(x) = \eta_0(x)$ on all heterogeneous and homogeneous intervals of η_0.

Let $V_1, \dots, V_r$ and $G_1, \dots, G_n$ be the heterogeneous and homogeneous intervals of η_0. We set

$$\eta_0^{V_i}(x) = \begin{cases} \eta_0(x), & \text{for } x \in V_i, \\ q(j_i), & \text{for } x \text{ to the left of } V_i, \\ q(j_{i+1}), & \text{for } x \text{ to the right of } V_i, \end{cases} \tag{6.6}$$

where S_{j_i} and $S_{j_{i+1}}$ are the boundary intervals that border V_i; $\eta_\bullet^{V_i}$ denotes the corresponding exclusion process. We define $\eta_\bullet^{G_i}$ and $\eta_0^{G_i}$ analogously.

In Proposition 6.1, we derive upper bounds on $\hat{f}_N(\eta_0)$ when η_0 contains at least two heterogeneous intervals. We will employ Proposition 2.1 to compare $\eta_\bullet$ with $\eta_\bullet'$, and Proposition 2.2 to compare $\eta_\bullet'$ with the processes $\eta_\bullet^{V_i}$ and $\eta_\bullet^{G_i}$. By Proposition 5.2, $\hat{f}_N(\eta_0^{G_i}) \le 0$ for each i. By Proposition 4.1 and Lemma 6.1, $\sum_{i=1}^{r} \hat{f}_N(\eta_0^{V_i})$ will be sufficiently negative to produce (6.7).

PROPOSITION 6.1. *Assume that $\delta \in (0, \varepsilon^4/90M^2]$, with ε being chosen sufficiently small. Let $\eta_\bullet$ be an exclusion process with $\eta_0 \in \Xi$, such that η_0 contains r heterogeneous intervals, with $r \ge 2$. Then, for large enough N (not depending on η_0),*

$$\hat{f}_N(\eta_0) \le -r\varepsilon^4 N^2/30. \tag{6.7}$$

PROOF. By Proposition 2.1,

$$\hat{f}_N(\eta_0) \le \hat{f}_N(\eta_0') + \frac{MN}{2} \sum_{i=1}^{m} \gamma_i. \tag{6.8}$$

Since η_0' is constant over each S_i, and $|S_i| = 2MN$, it follows from Proposition 2.2 that

$$\hat{f}_N(\eta_0') \le \sum_{i=1}^{r} \hat{f}_N(\eta_0^{V_i}) + \sum_{i=1}^{n} \hat{f}_N(\eta_0^{G_i}) + C(r + n + 1)e^{-cN} \tag{6.9}$$

for appropriate $c > 0$ and C, and large enough N not depending on η_0.

As remarked above (6.2), between any two homogeneous intervals there must be at least one heterogeneous interval. So $n \le 2r$, and the last term in (6.9) is at most $4Cre^{-cN}$. On the other hand, by Proposition 5.2, $\hat{f}_N(\eta_0^{G_i}) \le 0$ for each i and large N, not depending on η_0 or G_i. It follows from (6.8), (6.9) and these observations that

$$\hat{f}_N(\eta_0) \le \sum_{i=1}^{r} \hat{f}_N(\eta_0^{V_i}) + \frac{MN}{2} \sum_{j=1}^{m} \gamma_j + 4Cre^{-cN}. \tag{6.10}$$

It was also remarked above (6.2) that each boundary interval borders at least one heterogeneous interval. As before, label the boundary intervals bordering V_i by S_{j_i} and $S_{j_{i+1}}$. It follows immediately from (6.10) that

$$\hat{f}_N(\eta_0) \le \sum_{i=1}^{r} \left(\hat{f}_N(\eta_0^{V_i}) + \frac{MN}{2}(\gamma_{j_i} + \gamma_{j_{i+1}}) + 4Ce^{-cN} \right). \tag{6.11}$$

In order to bound the summands in (6.11), we consider three separate cases for V_i, where (a) V_i is long, (b) V_i is short and it contains a live ε-interface and (c) V_i is short and it contains no live ε-interface.

Suppose that a given V_i satisfies (a). Always, γ_{j_i} and $\gamma_{j_{i+1}}$ are each at most MN. Also, by Lemma 6.1, $\hat{f}_N(\eta_0^{V_i}) \le -3M^2N^2$ holds for large N, not depending on η_0 or V_i. So in this case,

$$\hat{f}_N(\eta_0^{V_i}) + \frac{MN}{2}(\gamma_{j_i} + \gamma_{j_{i+1}}) + 4Ce^{-cN} \le -M^2N^2. \tag{6.12}$$

Suppose that V_i satisfies (b). By (6.2), each of the boundary intervals bordering V_i is within $2\delta MN$ of unanimity, and so γ_{j_i} and $\gamma_{j_{i+1}}$ are each at most $2\delta MN$. By Proposition 4.1 (with $K = B_2$, $\varepsilon = \varepsilon$, $\delta = \varepsilon$, $n = 1$ and $\varepsilon_1 = \varepsilon^4/4$), $\hat{f}_N(\eta_0^{V_i}) \le -\varepsilon^4 N^2/4$ for large N, not depending on η_0 or V_i. Because $\delta \le \varepsilon^4/90M^2$, it follows that

$$\begin{aligned} &\hat{f}_N(\eta_0^{V_i}) + \frac{MN}{2}(\gamma_{j_i} + \gamma_{j_{i+1}}) + 4Ce^{-cN} \\ &\quad \le -\varepsilon^4 N^2/4 + 2\delta M^2 N^2 + 4Ce^{-cN} \\ &\quad \le -\varepsilon^4 N^2/5. \end{aligned} \tag{6.13}$$

Suppose that V_i satisfies (c). As in (b), γ_{j_i} and $\gamma_{j_{i+1}}$ are each at most $2\delta MN$. In this case, Proposition 4.1 no longer provides a negative upper bound. But, setting $n = 0$ and $\varepsilon_1 = \varepsilon^4/40$ in the proposition implies that $\hat{f}_N(\eta_0^{V_i}) \le \varepsilon^4 N^2/40$ for large N, not depending on η_0 or V_i. So, here one obtains

$$\begin{aligned} &\hat{f}_N(\eta_0^{V_i}) + \frac{MN}{2}(\gamma_{j_i} + \gamma_{j_{i+1}}) + 4Ce^{-cN} \\ &\quad \le \varepsilon^4 N^2/40 + 2\delta M^2 N^2 + 4Ce^{-cN} \\ &\quad \le \varepsilon^4 N^2/20. \end{aligned} \tag{6.14}$$

We claim that between two heterogeneous intervals each satisfying (c), there must be at least one heterogeneous interval satisfying either (a) or (b). Since an interval satisfying (c) is short and does not contain a live ε-interface, it must contain an inert ε-interface. By (6.1), there must be a live ε-interface between the inert ε-interfaces contained in the two intervals satisfying (c). But, by the comment after (6.3), this live ε-interface is either contained in an interval satisfying (a) or (b), or intersects a boundary interval between intervals satisfying (a), that also lie between the intervals satisfying (c).

It follows that of the r heterogeneous intervals, there are at least $[r/2]$ intervals satisfying either (a) or (b). So, for $r \geq 2$, the proportion of intervals satisfying either (a) or (b) is at least $1/3$. Together with the bounds in (6.11)–(6.14), this implies that

$$\hat{f}_N(\eta_0) \leq r\left(\frac{1}{3}\frac{(-\varepsilon^4 N^2)}{5} + \frac{2}{3}\frac{\varepsilon^4 N^2}{20}\right) = -r\varepsilon^4 N^2/30,$$

as desired. $\square$

In Proposition 6.2, we derive upper bounds on $\hat{f}_N(\eta_0)$ when η_0 contains a single heterogeneous interval V_1. Although η_0 need not contain a live ε-interface as it must when there are at least two heterogeneous intervals, the structure of η_0 is simpler than before. In (6.16), we derive negative upper bounds on $\hat{f}_N(\eta_0)$ when $V_1 \subset [-2MN, 2MN]^c$. The upper bound (6.15) holds in general. [When $V_1 \cap [-2MN, 2MN] \neq \phi$, we will use (6.15), together with negative upper bounds on $\hat{g}_N(\eta_0)$ in the next subsection, to obtain negative upper bounds on $\hat{h}_N(\eta_0)$.] As in the proof of Proposition 6.1, we will employ Propositions 2.1, 2.2, 4.1 and 5.2. Unlike in the proof of Proposition 6.1, we employ the full strength of Proposition 5.2 to show that $\sum_i \hat{f}_N(\eta_0^{G_i})$ is strictly negative when deriving (6.16). For both (6.15) and (6.16), we only employ Proposition 4.1, with $n = 0$, to show that $\hat{f}_N(\eta_0^{V_1})$ is not too positive.

PROPOSITION 6.2. *Assume that $0 < \delta < \varepsilon$, with ε being chosen sufficiently small. Let $\eta_\bullet$ be an exclusion process with $\eta_0 \in \Xi$, such that η_0 contains exactly one heterogeneous interval V_1. For large N,*

$$\hat{f}_N(\eta_0) \leq 4\delta M^2 N^2. \tag{6.15}$$

If $V_1 \subset [-2MN, 2MN]^c$, then, for large N,

$$\hat{f}_N(\eta_0) \leq -\mu M N^2/4. \tag{6.16}$$

(N *does not depend on η_0 in either case.*)

PROOF. As was observed above (6.2), each boundary interval always borders at least one heterogeneous interval. Therefore, since η_0 has a unique heterogeneous

interval, the partition of $\mathbb{Z}$ into heterogeneous, boundary and homogeneous intervals takes the form G_1, S_1, V_1, S_2, G_2, where the intervals are ordered according to their coordinates. All $I_i \subset G_1$ must have low density and all $I_i \subset G_2$ must have high density. By (6.2) and (6.3), S_1 has at most $2\delta MN$ particles and S_2 has at most $2\delta MN$ empty sites. In particular,

$$\begin{aligned}&\text{all } I_i \text{ to the left of } V_1 \text{ have low density and}\\&\text{all } I_i \text{ to the right of } V_1 \text{ have high density.}\end{aligned} \tag{6.17}$$

We define η_0' as in (6.5), where $q(1) = 0$ and $q(2) = 1$; $\eta_\bullet'$ is the corresponding exclusion process. We define $\eta_0^{V_1}$, $\eta_0^{G_1}$ and $\eta_0^{G_2}$ as in (6.6), with $\eta_\bullet^{V_1}$, $\eta_\bullet^{G_1}$ and $\eta_\bullet^{G_2}$ denoting the corresponding exclusion processes. Since $\eta_0'(x) \neq \eta_0(x)$ at most at $2\delta MN$ sites in each of S_1 and S_2, it follows from Proposition 2.1 that

$$\hat{f}_N(\eta_0) \le \hat{f}_N(\eta_0') + 2\delta M^2 N^2. \tag{6.18}$$

Application of Proposition 2.2 to $\eta_\bullet'$ implies that

$$\hat{f}_N(\eta_0') \le \hat{f}_N(\eta_0^{V_1}) + \hat{f}_N(\eta_0^{G_1}) + \hat{f}_N(\eta_0^{G_2}) + 4Ce^{-cN} \tag{6.19}$$

for appropriate $c > 0$ and C, and large enough N not depending on η_0. Consequently, by (6.18) and (6.19),

$$\hat{f}_N(\eta_0) \le \hat{f}_N(\eta_0^{V_1}) + \hat{f}_N(\eta_0^{G_1}) + \hat{f}_N(\eta_0^{G_2}) + 2\delta M^2 N^2 + 4Ce^{-cN}. \tag{6.20}$$

We first show (6.15). By Proposition 5.2, $\hat{f}_N(\eta_0^{G_i}) \le 0$ for $i = 1, 2$ and large N, not depending on η_0. By Proposition 4.1 (with $n = 0$ and $\varepsilon_1 = \delta$), $\hat{f}_N(\eta_0^{V_1}) \le \delta N^2$ for large N not depending on η_0. Together with (6.20), these inequalities imply that

$$\hat{f}_N(\eta_0) \le \delta N^2 + 2\delta M^2 N^2 + 4Ce^{-cN} \le 4\delta M^2 N^2 \tag{6.21}$$

for large N, which gives (6.15).

We now show (6.16). By symmetry, we may assume that $V_1 \subset (2MN, \infty)$. Hence, (a) $S_1 \subset (0, \infty)$ and (b) $(-\infty, 0] \subset G_1$. By (6.17) and (a), the number of empty sites in $(0, \infty)$ is at least $2(1-\varepsilon)MN$. Since $\eta_0 \in \Xi$, the number of particles in $(-\infty, 0)$ is also at least $2(1-\varepsilon)MN$. Together with Proposition 5.2 and (b), this bound implies that

$$\hat{f}_N(\eta_0^{G_1}) \le -(1-\varepsilon)\mu MN^2/2 \quad \text{and} \quad \hat{f}_N(\eta_0^{G_2}) \le 0 \tag{6.22}$$

for small enough $\varepsilon > 0$ and large N not depending on η_0. But, by Proposition 4.1 (with $n = 0$ and $\varepsilon_1 = \mu/16$), $\hat{f}_N(\eta_0^{V_1}) \le \mu N^2/16$. Together with (6.20) and (6.22), this bound implies that

$$\hat{f}_N(\eta_0) \le \mu N^2/16 - (1-\varepsilon)\mu MN^2/2 + 2\delta M^2 N^2 + 4Ce^{-cN} \le -\mu MN^2/4$$

for large enough N, not depending on η_0. This implies (6.16). $\square$

Behavior of $E[g(\eta_N)] - g(\eta_0)$ and conclusion. In Propositions 6.1 and 6.2, we obtained upper bounds on $\hat{f}_N(\eta_0)$. These bounds are negative except when η_0 contains exactly one heterogeneous interval V_1 and $V_1 \cap [-2MN, 2MN] \neq \phi$; in this case, we only have the positive bound given in (6.15) for $\hat{f}_N(\eta_0)$. In Proposition 6.3, we derive a negative upper bound for $\hat{g}_N(\eta_0)$ in this case, and in Lemma 6.2, we derive a simple upper bound on $\hat{g}_N(\eta_0)$ which includes the other cases. Together, these bounds on $\hat{f}_N(\eta_0)$ and $\hat{g}_N(\eta_0)$ will imply that $\hat{h}_N(\eta_0)$ is negative and bounded away from 0 except on a finite set of $\eta_0 \in \Xi$. As in (1.8), for $\eta \in \Xi$,

$$g(\eta) = \sigma N\big((L(\eta) + \beta N)^- + (R(\eta) - \beta N)^+\big),$$

where $\sigma > 0$ is a small number and β is a large number which will be specified later.

LEMMA 6.2. *Let $\eta_\bullet$ be an exclusion process, with $\eta_0 \in \Xi$. Then, for large N (not depending on η_0),*

$$\hat{g}_N(\eta_0) \leq 2\sigma MN^2. \tag{6.23}$$

PROOF. By the bound on the expectation after (2.3),

$$E[(L(\eta_N) - L(\eta_0))^-] \leq MN \quad \text{and} \quad E[(R(\eta_N) - R(\eta_0))^+] \leq MN$$

for large N. Since $(c_2 - a)^+ - (c_1 - a)^+ \leq (c_2 - c_1)^+$ for any $a, c_1, c_2 \in \mathbb{R}$, it follows that

$$\hat{g}_N(\eta_0) \leq \sigma N\Big(E\big[(L(\eta_N) - L(\eta_0))^-\big] + E\big[(R(\eta_N) - R(\eta_0))^+\big]\Big) \leq 2\sigma MN^2. \quad \square$$

Proposition 6.3 states that when the unique heterogeneous interval V_1 intersects $[-2MN, 2MN]$, and either $L(\eta_0) \leq -(B_1 + 5M)N$ or $R(\eta_0) \geq (B_1 + 5M)N$, then $\hat{g}_N(\eta_0)$ is negative. The corollary to Proposition 5.1, together with our definition of g, is used here. The main idea is that, under these assumptions, the density of particles (empty sites) close to $L(\eta_0)$ ($R(\eta_0)$) will be low, which will induce a drift of $L(\eta_t)$ ($R(\eta_t)$) toward 0 and hence decrease $\hat{g}_t(\eta_0)$ over $[0, N]$. From now on, we fix the constant β in the definition of g, setting $\beta = B_1 + 4M$ (where $B_1 = 70M^3/\varepsilon^2\delta^2$). We also set

$$G = \{\eta : L(\eta) \leq -(B_1 + 5M)N \text{ or } R(\eta) \geq (B_1 + 5M)N\}. \tag{6.24}$$

(We will specify σ before Proposition 6.4.)

PROPOSITION 6.3. *Assume that $0 < \delta < \varepsilon$, where ε is sufficiently small. Let $\eta_\bullet$ be an exclusion process with $\eta_0 \in \Xi \cap G$; assume that η_0 contains exactly one heterogeneous interval V_1 and that $V_1 \cap [-2MN, 2MN] \neq \phi$. Then,*

$$\hat{g}_N(\eta_0) \leq -\sigma\mu N^2/6 \tag{6.25}$$

for large enough N (not depending on η_0).

PROOF. By symmetry, we may assume that $L(\eta_0) \le -(B_1 + 5M)N$. However, $|V_1| \le B_1 N$ and $V_1 \cap [-2MN, 2MN] \ne \phi$, and so $V_1 \subset [-(B_1 + 2M)N, (B_1 + 2M)N]$. It follows that $[L(\eta_0), L(\eta_0) + MN]$ lies to the left of V_1. Since V_1 is the unique heterogeneous interval in η_0, for the same reason as in the proof of Proposition 6.2, all of the intervals I_i to the left of V_1 have low density. Consequently, under η_0, there are at most $2\varepsilon MN$ particles in $[L(\eta_0), L(\eta_0) + MN]$.

By assumption, $L(\eta_0) + \beta N \le -MN$. It follows from this, the conclusion of the previous paragraph, and the corollary of Proposition 5.1, that

$$E\big[\big(L(\eta_N) + \beta N\big)^-\big] - \big(L(\eta_0) + \beta N\big)^- \le -\mu N/5 \tag{6.26}$$

for small $\varepsilon > 0$ and large enough N (not depending on η_0).

We also need to examine the behavior of $R(\eta_N)$. We consider two cases, depending on whether or not $R(\eta_0) \le (\beta - M)N$. Suppose the inequality holds. Then, translation of the process in (2.3) implies

$$E\big[\big(R(\eta_N) - \beta N\big)^+\big] \le 1 \tag{6.27}$$

for large N. On the other hand, if the inequality fails, then the distance from $R(\eta_0)$ to V_1 is greater than MN. Since all I_i to the right of V_1 have high density, there are at most $2\varepsilon MN$ empty sites in $[R(\eta_0) - MN, R(\eta_0)]$. So, by the corollary to Proposition 5.1,

$$E\big[\big(R(\eta_N) - \beta N\big)^+\big] - \big(R(\eta_0) - \beta N\big)^+ \le 1 \tag{6.28}$$

for large N. The inequalities (6.26), (6.27) and (6.28) imply that

$$\hat{g}_N(\eta_0) \le -\sigma N(\mu N/5 - 1) \le -\sigma \mu N^2/6$$

for large N, which is independent of η_0. This implies (6.25). $\square$

By applying the bounds on $\hat{f}_N(\eta_0)$ and $\hat{g}_N(\eta_0)$ in Propositions 6.1, 6.2 and 6.3, and in Lemma 6.2, it is a simple matter to demonstrate Theorem 1.3. We state the more explicit version Proposition 6.4, below. We will assume that $\varepsilon > 0$, $\delta > 0$ and $\sigma > 0$ satisfy

$$\sigma \le \varepsilon^4/60M \quad \text{and} \quad \delta \le \sigma\mu/48M^2, \tag{6.29}$$

as well as that εN, $1/\varepsilon$ and $1/\delta$ are all integers.

PROPOSITION 6.4. *Assume that $\eta_\bullet$ is an exclusion process and that ε, δ and σ satisfy* (6.29), *with ε being chosen sufficiently small. Then, $E[h_N(\eta_0)] < \infty$ for all $\eta_0 \in \Xi$ and N. If, in addition, $\eta_0 \in G$, then*

$$\hat{h}_N(\eta_0) \le -\sigma\mu N^2/12 \tag{6.30}$$

for large enough N (not depending on η_0).

PROOF. It follows without difficulty from (2.3) that for $\eta_0 \in \Xi$, $E[f(\eta_N)] < \infty$ and $E[g(\eta_N)] < \infty$, and consequently, $E[h(\eta_N)] < \infty$. In order to demonstrate (6.30), we consider the different cases covered in Propositions 6.1–6.3, where (a) η_0 contains at least two heterogeneous intervals, (b) η_0 contains exactly one heterogeneous interval V_1 and $V_1 \subset [-2MN, 2MN]^c$, and (c) $\eta_0 \in G$, such that η_0 contains exactly one heterogeneous interval V_1 and $V_1 \cap [-2MN, 2MN] \neq \phi$.

When η_0 satisfies (a), it follows from Proposition 6.1 and Lemma 6.2, that for large enough N,

$$\begin{aligned} \hat{h}_N(\eta_0) &= \hat{f}_N(\eta_0) + \hat{g}_N(\eta_0) \\ &\leq -\varepsilon^4 N^2/15 + 2\sigma M N^2. \end{aligned} \tag{6.31}$$

When η_0 satisfies (b), it follows from (6.16) of Proposition 6.2 and Lemma 6.2, that for large N,

$$\hat{h}_N(\eta_0) \leq -\mu M N^2/4 + 2\sigma M N^2. \tag{6.32}$$

Also, when η_0 satisfies (c), it follows from (6.15) of Proposition 6.2 and Proposition 6.3 that for large N,

$$\hat{h}_N(\eta_0) \leq 4\delta M^2 N^2 - \sigma \mu N^2/6. \tag{6.33}$$

In all three cases, N does not depend on η_0. For ε, δ and σ satisfying (6.29), one has in each case that $\hat{h}_N(\eta_0) \leq -\sigma \mu N^2/12$ for large enough N, which implies (6.30). $\square$

Acknowledgments. The authors thank Ellen Saada and the referee for carefully reading the manuscript.

REFERENCES

BRAMSON, M., LIGGETT, T. M. and MOUNTFORD, T. (2002). Characterization of stationary measures for one-dimensional exclusion processes. *Ann. Probab.* To appear.

EVANS, L.C. (1998). *Partial Differential Equations*. Amer. Math. Soc., Providence, RI.

FERRARI, P. A., LEBOWITZ, J. L. and SPEER, E. (2001). Blocking measures for asymmetric exclusion processes via coupling. *Bernoulli* **7** 935–950.

KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Particle Systems*. Springer, Berlin.

LIGGETT, T. M. (1975). Ergodic theorems for the asymmetric simple exclusion process. *Trans. Amer. Math. Soc.* **213** 237–261.

LIGGETT, T. M. (1976). Coupling the simple exclusion process. *Ann. Probab.* **4** 339–356.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, Berlin.

LIGGETT, T. M. (1999). *Stochastic Interacting Systems*: *Contact, Voter and Exclusion Processes*. Springer, Berlin.

MOUNTFORD, T. (2001). An extension of a result of Andjel. *Ann. Appl. Probab.* **11** 405–418.

REZAKHANLOU, F. (1991). Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. *Comm. Math. Phys.* **140** 417–448.

SMOLLER, J. (1983). *Shock Waves and Reaction–Diffusion Equations*. Springer, Berlin.
SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290.

UNIVERSITY OF MINNESOTA
TWIN CITIES CAMPUS
SCHOOL OF MATHEMATICS
INSTITUTE OF TECHNOLOGY
127 VINCENT HALL
206 CHURCH STREET S.E.
MINNEAPOLIS, MINNESOTA 55455
E-MAIL: bramson@math.umn.edu

DÉPARTEMENT DE MATHÉMATIQUES
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
ECUBLENS CH 1015
SWITZERLAND
E-MAIL: thomas.mountford@epfl.ch