The Annals of Probability
2002, Vol. 30, No. 3, 1039–1043

ON UNIQUENESS OF SOLUTIONS TO STOCHASTIC EQUATIONS: A COUNTER-EXAMPLE[1]

BY H. J. ENGELBERT

University of Jena

We consider the one-dimensional stochastic equation

$$X_t = X_0 + \int_0^t b(X_s)\, dM_s$$

where M is a continuous local martingale and b a measurable real function. Suppose that b^{-2} is locally integrable. D. N. Hoover asserted that, on a saturated probability space, there exists a solution X of the above equation with $X_0 = 0$ having no occupation time in the zeros of b and, moreover, the pair (X, M) is unique in law for all such X. We will give an example which shows that the uniqueness assertion fails, in general.

Let $(\Omega, \mathcal{F}, P, \mathbb{F})$ be an arbitrary filtered probability space. We consider a continuous local martingale $(M, \mathbb{F})$. The square variation process of M is denoted by $\langle M \rangle$. This is the unique continuous increasing process such that $\langle M \rangle_0 = 0$ and $(M^2 - \langle M \rangle, \mathbb{F})$ is a local martingale. To exclude the trivial case, we assume $P(\{\langle M \rangle_\infty > 0\}) > 0$ where $\langle M \rangle_\infty = \sup_{t \geq 0} \langle M \rangle_t$.

In this note, we deal with the stochastic equation

$$X_t = X_0 + \int_0^t b\,(X_s)\, dM_s, \qquad t \geq 0, \tag{1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a measurable function and $\mathbb{R}$ denotes the real line. Let us introduce the notation

$$E_b = \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} b^{-2}(y)\, dy = +\infty,\ \forall\, \varepsilon > 0 \right\}$$

and

$$N_b = \{x \in \mathbb{R} : b(x) = 0\}.$$

In the case if M is a Brownian motion, it is known (cf. [1], [2]) that equation (1) possesses, for all initial values $X_0 = x_0 \in \mathbb{R}$, a weak solution $(X, \mathbb{F})$ satisfying

$$\int_0^\infty 1_{N_b \cap E_b^c}(X_s)\, d\langle M \rangle_s = 0, \qquad P\text{-a.s.} \tag{2}$$

Received December 1998; revised August 2001.

[1]Supported by Grant INTAS 97-30204.

AMS 2000 subject classifications. Primary 60H10; secondary 60G44.

Key words and phrases. Stochastic equations, continuous local martingales, uniqueness in law.

if and only if the condition

$$E_b \subseteq N_b \tag{3}$$

holds. Moreover, the solution $(X, \mathbb{F})$ of equation (1) satisfying (2) is unique in law. If b^{-2} is locally integrable then $E_b = \varnothing$ and (3) trivially holds. Hence, for every initial condition $X_0 = x_0 \in \mathbb{R}$, there is a unique solution $(X, \mathbb{F})$ of equation (1) satisfying

$$\int_0^\infty 1_{N_b}(X_s)\, d\langle M\rangle_s = 0, \qquad P\text{-a.s.} \tag{4}$$

This solution is nontrivial in the sense that

$$P(\{X_t = X_0,\ \forall t \geq 0\}) < 1$$

(cf. [1]). (In fact, this probability is equal to zero.) If, additionally, $b(x) \neq 0$ for all $x \in \mathbb{R}$ then, for all initial conditions $X_0 = x_0 \in \mathbb{R}$, equation (1) has a unique solution $(X, \mathbb{F})$ which is, moreover, nontrivial. Conversely, the local integrability of b^{-2} is also necessary for the last statement being true.

Hoover ([3], Theorem 3.3) has considered the stochastic equation (1) for general continuous local martingales $(M, \mathbb{F})$. His principal existence result is only stated for $X_0 = 0$ but it is certainly true for arbitrary initial values. The first part of Theorem 3.3 of [3] states the following: If b^{-2} is locally integrable then for every given continuous local martingale $(M, \mathbb{F})$ defined on a *saturated* filtered probability space (see [3]) there exists a solution $(X, \mathbb{F})$ of equation (1) satisfying (4). This result is remarkable because it is more than the usual concept of weak solutions requires: For a weak solution $(X, \mathbb{F})$ we only have to find a filtered probability space and a continuous local martingale $(M, \mathbb{F})$ on it with *prescribed* distribution law (e.g., that of Brownian motion) such that equation (1) holds. But this stronger result relies on the fact that saturated filtered probability spaces together with their filtrations are "very large."

The second part of Theorem 3.3 in [3], however, asserts that the law of the pair (X, M) is uniquely determined if $(M, \mathbb{F})$ is a given continuous local martingale on a saturated filtered probability space and $(X, \mathbb{F})$ is a solution of equation (1) satisfying (4). By giving a counter-example we will show that this is not true. As we shall see, even the law of X is not unique, in general.

To construct the counter-example, we consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. For keeping the frame put by Hoover, we assume that $(\Omega, \mathcal{F}, P, \mathbb{F})$ is saturated. However, this is not important for our construction and the only thing we use from saturated filtered probability spaces is that they are large enough to carry a Brownian motion (cf. [3]). So, let $(W, \mathbb{F})$ be a Brownian motion on $(\Omega, \mathcal{F}, P, \mathbb{F})$. We now define

$$X_t = \begin{cases} W_t, & \text{if } t \leq 1 \text{ or if } t > 1 \text{ and } W_1 \geq 0, \\ W_1 + \sqrt{2}(W_t - W_1), & \text{if } t > 1 \text{ and } W_1 < 0, \end{cases}$$

for all $t \geq 0$. Then $(X, \mathbb{F})$ is a continuous martingale such that

$$\langle X \rangle_t = t + (t-1)^+ \cdot 1_{(-\infty,0)}(X_1), \qquad t \geq 0.$$

Furthermore, we set

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0, \end{cases}$$

for all $x \in \mathbb{R}$ and introduce the continuous martingale $(M, \mathbb{F})$ by

$$M_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s, \qquad t \geq 0,$$

which, obviously, has the same square variation process as X, that is, $\langle M \rangle = \langle X \rangle$. Integrating $\operatorname{sgn}(X)$ by M yields

$$X_t = \int_0^t \operatorname{sgn}(X_s)\, dM_s, \qquad t \geq 0. \tag{5}$$

This means that $(X, \mathbb{F})$ is a solution of equation (1) for $b = \operatorname{sgn}$ with respect to the driving continuous martingale $(M, \mathbb{F})$.

Next we show that $(\tilde{X}, \mathbb{F})$ with $\tilde{X} = -X$ is also a solution of (5). Indeed, this immediately follows from $-\operatorname{sgn}(x) = \operatorname{sgn}(-x)$, $x \neq 0$, and the property

$$\int_0^t 1_{\{0\}}(X_s)\, dM_s = 0, \qquad P\text{-a.s.}$$

for all $t \geq 0$: Using the definition of M, the isometry of the stochastic integral and the occupation time formula we get

$$\begin{aligned} E\left(\int_0^t 1_{\{0\}}(X_s)\, dM_s\right)^2 &= E\left(\int_0^t 1_{\{0\}}(X_s)\operatorname{sgn}(X_s)\, dX_s\right)^2 \\ &= E\left(\int_0^t 1_{\{0\}}(X_s)\, d\langle X \rangle_s\right) \\ &= E\left(\int_{\mathbb{R}} 1_{\{0\}}(a) L^X(t,a)\, da\right) \\ &= 0 \end{aligned}$$

where $L^X(t,a)$ denotes the local time of X in a up to time t.

Obviously, the laws of the continuous martingales X and $\tilde{X}$ are different. Finally, relation (4) for both X and $\tilde{X}$ is satisfied because sgn is defined such that $\operatorname{sgn}(x) \neq 0$ everywhere. Thus the counter-example is complete.

For better understanding the situation let us add a few remarks. We define the $\mathbb{F}$-time change T as the right inverse of the increasing process $A := \langle X \rangle = \langle M \rangle$,

$$T_t = \inf\{s \geq 0 : A_s > t\}, \qquad t \geq 0,$$

and put

$$Y_t = X_{T_t}, \quad B_t = M_{T_t}, \quad \mathcal{G}_t = \mathcal{F}_{T_t}, \qquad t \geq 0.$$

Then $(Y, \mathbb{G})$ and $(B, \mathbb{G})$ with filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ are continuous local martingales such that $\langle Y\rangle_t = \langle B\rangle_t = t$ for all $t \geq 0$ and, using the martingale characterization of Brownian motion by P. Lévy, we conclude that both processes $(Y, \mathbb{G})$ and $(B, \mathbb{G})$ are Brownian motions. Now, by time change in the stochastic integral, from (5) we get

$$Y_t = \int_0^t \operatorname{sgn}(Y_s)\, dB_s, \qquad t \geq 0. \tag{6}$$

The solution to equation (6) exists and is always a Brownian motion, hence it is unique in law. But it is well-known that the solution to equation (6) is neither pathwise unique (with Y is $-Y$ a second solution) nor there exists a strong solution of it. This observation is due to H. Tanaka and depends on the fact that $\mathcal{F}_t^B \subsetneq \mathcal{F}_t^Y$ for all $t > 0$. Here $\mathbb{F}^Z = (\mathcal{F}_t^Z)$ denotes the filtration generated by the process Z and completed in $\mathcal{F}$. More precisely, it is well known (cf., e.g., Revuz and Yor [5], Corollary 2.2) that

$$\mathcal{F}_t^B = \mathcal{F}_t^{|Y|} \subsetneq \mathcal{F}_t^Y, \qquad t > 0,$$

for any solution $(Y, \mathbb{F})$ of equation (6) and any driving Brownian motion $(B, \mathbb{F})$. It is also an easy exercise to verify that, for any fixed $u \geq 0$, $\operatorname{sgn}(Y_u)$ and the process $|Y|$, and hence $\operatorname{sgn}(Y_u)$ and the Brownian motion B, are independent.

Now the square variation A can also be written as

$$A_t = t + \tfrac{1}{2}(t-1)^+[1 - \operatorname{sgn}(Y_1)], \qquad t \geq 0, \tag{7}$$

and, in view of the independence of $\operatorname{sgn}(Y_1)$ and B, A is an $\mathbb{F}^Y$-time change independent of B. Furthermore, we have

$$X_t = Y_{A_t}, \quad M_t = B_{A_t}, \qquad t \geq 0. \tag{8}$$

In particular, the continuous martingale $(M, \mathbb{F})$ results from the *randomized* time change A of the Brownian motion $(B, \mathbb{G})$: There is needed an additional experiment independent of B to decide whether $\operatorname{sgn}(Y_1) = +1$ or -1. This randomization, however, disturbs the representation property for the continuous martingale $(M, \mathbb{F}^M)$ (cf. Jacod [4]). Indeed, using relations (7) and (8) and the independence between A and B we can easily compute the distribution P_M of M on the space $C([0, +\infty))$ of continuous functions $x : [0, +\infty) \to \mathbb{R}$ as

$$P_M = \tfrac{1}{2}(P_1 + P_2) \tag{9}$$

where P_1 is the Wiener measure and P_2 is the distribution of a continuous Gaussian martingale with expectation zero and variance function v defined by $v(t) = t + (t-1)^+$, $t \geq 0$. This shows that P_M is not extremal in the convex set of martingale measures on $C([0,+\infty))$ and hence $(M,\mathbb{F}^M)$ does not satisfy the representation property (cf. Jacod [4]).

It can easily be seen that we have $\mathbb{F}^X = \mathbb{F}^W$. From this and the well-known fact that $(W,\mathbb{F}^W)$ possesses the representation property (cf. [4] or [5]), it is not difficult to derive that $(M,\mathbb{F}^X)$ possesses the representation property, too. Since $(M,\mathbb{F}^M)$ does not satisfy the representation property, we can conclude $\mathbb{F}^M \neq \mathbb{F}^X$. On the other side, $\mathbb{F}^M \subseteq \mathbb{F}^X$. This yields that X is not adapted to $\mathbb{F}^M$ and hence X is *not* a strong solution to equation (5).

It is also interesting to notice that in the definition of X we can replace the time 1 by any time $u > 0$. Let $X^{(u)}$ be the resulting process and introduce $M^{(u)}$ by

$$M_t^{(u)} = \int_0^t \operatorname{sgn}(X_s^{(u)})\, dX_s^{(u)}, \qquad t \geq 0.$$

Examining the above procedure we observe that the distribution of the continuous martingale $(M^{(u)},\mathbb{F})$ is given by (9), the same as of $M^{(1)} = M$, and hence does not depend on u. However, the distributions of $(X^{(u)},\mathbb{F})$ are pairwise different and every $(X^{(u)},\mathbb{F})$ represents a (weak) solution of equation (5). Thus we have found a lot of (weak) solutions of equation (5) all with different laws.

In conclusion, we state the following conjecture: Let $(M,\mathbb{F})$ be a continuous local martingale. Then the solution $(X,\mathbb{F})$ of equation (1) satisfying (2) is unique in law for every real function b such that $E_b \subseteq N_b$ holds if and only if $(M,\mathbb{F}^M)$ possesses the representation property for continuous local martingales.

REFERENCES

[1] ENGELBERT, H. J. and SCHMIDT, W. (1985). On solutions of one-dimensional stochastic differential equations without drift. *Z. Wahrsch. Verw. Gebiete* **68** 287–314.

[2] ENGELBERT, H. J. and SCHMIDT, W. (1991). Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations, part III. *Math. Nachr.* **151** 149–197.

[3] HOOVER, D. N. (1987). A characterization of adapted distribution. *Ann. Probab.* **15** 1600–1611.

[4] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Springer, Berlin.

[5] REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

INSTITUTE FOR STOCHASTICS
FRIEDRICH-SCHILLER UNIVERSITY OF JENA
ERNST-ABBE-PLATZ 1-4
D-07743 JENA
GERMANY
E-MAIL: engelbert@minet.uni-jena.de