The Annals of Probability
1997, Vol. 25, No. 3, 1471–1501

ORDER OF MAGNITUDE BOUNDS FOR EXPECTATIONS OF Δ_2-FUNCTIONS OF NONNEGATIVE RANDOM BILINEAR FORMS AND GENERALIZED *U*-STATISTICS

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Let $X_1, Y_1, Y_2, \dots, X_n, Y_n$ be independent nonnegative rv's and let $\{b_{ij}\}_{1 \le i,\, j \le n}$ be an array of nonnegative constants. We present a method of obtaining the order of magnitude of

$$E\Phi\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i\, Y_j\right),$$

for any such $\{X_i\}$, $\{Y_j\}$ and $\{b_{ij}\}$ and any nondecreasing function Φ on $[0, \infty)$ with $\Phi(0) = 0$ and satisfying a Δ_2 growth condition. Furthermore, this technique is extended to provide the order of magnitude of

$$E\Phi\left(\sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j)\right),$$

where $\{f_{ij}(x, y)\}_{1 \le i,\, j \le n}$ is any array of nonnegative functions.

For arbitrary functions $\{g_{ij}(x, y)\}_{1 \le i \ne j \le n}$, the aforementioned approximation enables us to identify the order of magnitude of

$$E\Phi\left(\left|\sum_{1 \le i \ne j \le n} g_{ij}(X_i, X_j)\right|\right)$$

whenever decoupling results and Khintchine-type inequalities apply, such as Φ is convex, $L(g_{ij}(X_i, X_j)) = L(g_{ji}(X_j, X_i))$ and $Eg_{ij}(X_i, x) \equiv 0$ for all x in the range of X_j.

1. Introduction and summary. Let $X_1, Y_1, Y_2, \dots, X_n, Y_n$ be independent nonnegative random variables and let $\{b_{ij}\}_{1 \le i,\, j \le n}$ be nonnegative constants. Set

$$\Delta_2 \equiv \{\text{symmetric functions } \Phi, \text{ nondecreasing on } [0, \infty) \text{ with } \Phi(0) = 0 \text{ and such that for some } \alpha > 0,\ \Phi(cx) \le |c|^{\alpha}\Phi(x) \text{ for all } |c| \ge 2 \text{ and all } x\}.$$

Such a $\Phi \in \Delta_2$ is said to have parameter α (and hence it has parameter β for all $\beta \ge \alpha$).

Received November 1995; revised October 1996.

[1]Supported by NSF Grant DMS-93-10263.

[2]Supported by the Swedish Council for Research in the Humanities and the Social Sciences.

AMS 1991 *subject classifications*. Primary 60E15, 60F25, 60G50.

Key words and phrases. Expectations of functions, generalized *U*-statistics, nonnegative rv's, random bilinear forms.

We are interested in approximating

$$E\Phi(B(\mathbf{X},\mathbf{Y})) \equiv E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_j\Big) \tag{1.1}$$

for all $\{X_i\}$, $\{Y_j\}$, $\{b_{ij}\}$ and Φ as above. Define

$$\begin{aligned}\Phi(\mathbf{b},\mathbf{X},\mathbf{Y}) \equiv \max\Big\{&E\max_{1\le i,\, j\le n}\Phi\big(b_{ij}X_iY_j\big), E\max_{1\le i\le n}\Phi(v_{1i}X_i),\\ &E\max_{1\le j\le n}\Phi\big(v_{2j}Y_j\big), \Phi(v_{1*}), \Phi(v_{2*}), \Phi(w_*)\Big\},\end{aligned} \tag{1.2}$$

where

$$v_{1i} = \sup\Big\{v\ge 0\colon \sum_{j=1}^{n} E\big((b_{ij}Y_j)\wedge v\big)\ge v\Big\}, \tag{1.3}$$

$$v_{2j} = \sup\Big\{v\ge 0\colon \sum_{i=1}^{n} E\big((b_{ij}X_i)\wedge v\big)\ge v\Big\}, \tag{1.4}$$

$$v_{1*} = \sup\Big\{v\ge 0\colon \sum_{i=1}^{n} E\big((v_{1i}X_i)\wedge v\big)\ge v\Big\}, \tag{1.5}$$

$$v_{2*} = \sup\Big\{v\ge 0\colon \sum_{j=1}^{n} E\big((v_{2j}Y_j)\wedge v\big)\ge v\Big\}, \tag{1.6}$$

$$B_{1ij} = \big\{b_{ij}Y_j\le v_{1i}\big\}, \tag{1.7}$$

$$B_{2ij} = \big\{b_{ij}X_i\le v_{2j}\big\} \tag{1.8}$$

and

$$w_* = \sup\Big\{w\ge 0\colon \sum_{1\le i,\, j\le n} E\Big[\big((b_{ij}X_iY_j)\wedge w\big)I\big(B_{1ij}^cB_{2ij}^c\big)\Big]\ge w\Big\}. \tag{1.9}$$

We prove that

$$E\Phi(B(\mathbf{X},\mathbf{Y})) \approx_\alpha \Phi(\mathbf{b},\mathbf{X},\mathbf{Y}), \tag{1.10}$$

where $\approx_\alpha$ means that there are constants $0<\underline{c}_\alpha<\bar{c}_\alpha<\infty$ depending only on the parameter α of Φ such that

$$\underline{c}_\alpha\Phi(\mathbf{b},\mathbf{X},\mathbf{Y}) \le E\Phi(B(\mathbf{X},\mathbf{Y})) \le \bar{c}_\alpha\Phi(\mathbf{b},\mathbf{X},\mathbf{Y}).$$

Though the quantities which comprise $\Phi(\mathbf{b},\mathbf{X},\mathbf{Y})$ may seem bewildering at first sight, their presence actually makes good intuitive sense. Note first that for i.i.d. $Z_j\ge 0$, Lemma 2.3 of Klass and Zhang (1994) shows that whenever $q_n>0$ satisfies

$$E\sum_{j=1}^{n}\big(Z_j\wedge q_n\big) = q_n, \tag{1.11}$$

q_n can be considered to be a "typical value" of $S\equiv\sum_{j=1}^{n}Z_j$ in that $P(S\ge q_n/3)\ge 0.2$ and $P(S\le 3q_n)\ge 0.3$. The same qualitative fact holds for the case of nonidentically distributed variables.

Think of Z_j as $b_{ij}Y_j$ and note that $\sum_{j=1}^n Z_j$ is the coefficient of X_i in $B(\mathbf{X}, \mathbf{Y})$. Thus, v_{1i} represents a typical value of the coefficient of X_i. Substituting v_{1i} for its coefficient, we observe that v_{1*} is the typical value of $\sum_{i=1}^n v_{1i} X_i$. Now, in fact, for arbitrary independent $Z_j \ge 0$, $E\Phi(\sum_{j=1}^n Z_j)$ has order of magnitude $\Phi(q_n) + E \max_{1 \le j \le n} \Phi(Z_j)$ whenever $q_n \ge 0$ is the largest root of (1.11). Therefore,

$$E\Phi\left(\sum_{i=1}^n \left(\sum_{j=1}^n b_{ij} Y_j\right) X_i\right) \ge_\alpha E\Phi\left(\sum_{i=1}^n v_{1i} X_i\right)$$

$$\approx_\alpha \max\Big\{\Phi(v_{1*}), E \max_{1 \le i \le n} \Phi(v_{1i} X_i)\Big\},$$

where the one-sided bound $\ge_\alpha$ *or* $\le_\alpha$ has the obvious interpretation. Reversing X_i and Y_j,

$$E\Phi\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j\right) \ge_\alpha \max\Big\{\Phi(v_{2*}), E \max_{1 \le j \le n} \Phi(v_{2j} Y_j)\Big\}.$$

Large values of $\Phi(\sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j)$ might also be due to the coefficient of various X_i's being abnormally large. We associate this contingency with at least one $b_{ij}Y_j$ from among $b_{i1}Y_1, \dots, b_{in}Y_n$ exceeding v_{1i}. Simultaneously, it would seem that some $b_{ij}X_i$ from among $b_{1j}X_1, \dots, b_{nj}X_n$ should exceed v_{2j}. Thus, we are induced to analyze the random sum Q, where

$$Q \equiv \sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j I\big(B_{1ij}^c B_{2ij}^c\big).$$

Although Q is not merely a sum of independent nonnegative rv's, w_* still identifies the "center" of its distribution and $E\Phi(Q)$ is roughly

$$\Phi(w_*) + E \max_{1 \le i,\, j \le n} \Phi\big(b_{ij} X_i Y_j\big) I\big(B_{1ij}^c B_{2ij}^c\big).$$

Combining these heuristics and assertions with the trivial observation that

$$\max_{1 \le i,\, j \le n} \Phi\big(b_{ij} X_i Y_j\big) \le \Phi\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j\right)$$

accounts for the presence of each of the six quantities found in our approximation of $E\Phi(B(\mathbf{X}, \mathbf{Y}))$. Each of these quantities is needed; none can be dispensed with, as analysis following Theorem 3.5 shows.

What motivated our investigation of $E\Phi(B(\mathbf{X}, \mathbf{Y}))$ as above? The merging of many streams. Historically, the consideration of L^p norms of quadratic forms $\sum_{1 \le i,\, j \le n} a_{ij} \varepsilon_i \varepsilon_j$, where the $\{\varepsilon_i\}$ are i.i.d. ± 1's, dates back to Khintchine. The next major step was provided by McConnell and Taqqu (1986), who extended these results to independent symmetric ε_i (of otherwise arbitrary distribution), and reformulated the L^p approximation by introducing an independent copy $\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n$ of the $\varepsilon_1, \dots, \varepsilon_n$. Thereby, "Khintchine's inequalities" became "decoupling inequalities."

It was observed that such results had a variety of uses [e.g., see Cambanis, Rosiński and Woyczyński (1985), Bourgain and Tzafriri (1987) and Krakowiak and Szulga (1988)]. The conditions were gradually weakened, with de la Peña and Klass (1994) establishing that, for arbitrary independent mean zero random variables $H_1, \dots, H_n$ with independent copies $\tilde{H}_1, \dots, \tilde{H}_n$,

$$E\Psi\left(\left|\sum_{1\le i\ne j\le n} a_{ij}H_iH_j\right|\right) \approx_\alpha E\Psi\left(\sqrt{\sum_{1\le i\ne j\le n}\frac{(a_{ij}+a_{ji})^2}{4}(H_i)^2\left(\tilde{H}_j\right)^2}\right), \tag{1.12}$$

for any *convex* Δ_2 function Ψ of some parameter, say $2\alpha > 0$. Thus (for convex Δ_2 Ψ), (1.12) converts the problem of approximating

$$E\Psi\left(|\Sigma_{1\le i\ne j\le n}\, a_{ij}H_iH_j|\right)$$

into the problem of approximating the expected value of a function $\Phi(x) = \Psi(\sqrt{|x|})$ of a nonnegative random bilinear form of nonnegative independent random variables—the problem considered in this paper. Putting $b_{ij} = ((a_{ij} + a_{ji})^2/4)I(i \ne j)$, $X_i = H_i^2$ and $Y_j = \tilde{H}_j^2$, and applying our Theorem 3.5 to (1.12), we obtain

$$E\Psi\left(\left|\sum_{1\le i\ne j\le n} a_{ij}H_iH_j\right|\right) \approx_\alpha \Phi(\mathbf{b}, \mathbf{X}, \mathbf{Y}). \tag{1.13}$$

The approximation of the left-hand side of (1.13) by a semiequivalent version of its right-hand side was first obtained by de la Peña and Klass (1994) for convex Ψ with $\Psi(\sqrt{|x|})$ concave on $[0,\infty)$. The same authors also provided a method of identifying the order of magnitude of $E\Psi(|\Sigma_{1\le i\ne j\le n}\, a_{ij}H_iH_j|)$ whenever, for some integer $k \ge 1$, $\Psi(x^{2^{-k}})$ and $-\Psi(x^{2^{-k-1}})$ were both convex functions on $[0,\infty)$. The approximation in such cases included additional $2k$ quantities whose construction (1.13) demonstrates to be superfluous.

Thus, what specifically motivated this research effort was the desire to approximate $E\Psi(|\Sigma_{1\le i\ne j\le n}\, a_{ij}H_iH_j|)$ for more general Ψ, $\{H_i\}$. For mean zero H_i we have not eliminated the convexity condition on Ψ but have relaxed the growth condition to membership in Δ_2. However, for nonnegative b_{ij} and nonnegative H_i, the convexity condition is no longer required. A forthcoming paper is planned that will eliminate both the convexity condition on Ψ and any and all conditions on the $\{H_i\}$. The task seemed too ambitious for one paper. Fortuitously, it seems to divide quite naturally into two separate works.

Subsequent to the initiation of this endeavor, de la Peña and Montgomery-Smith (1995) obtained a stunning decoupling result for tail probabilities (no integrations necessary). They showed that for *any* $g_{ij}(x, y)$ and independent $H_1, \dots, H_n$ with independent copy $\tilde{H}_1, \dots, \tilde{H}_n$ such that $L(g_{ij}(H_i, H_j)) = L(g_{ji}(H_j, H_i))$,

$$P\left(\left|\sum_{1\le i\ne j\le n} g_{ij}(H_i, H_j)\right| \ge t\right) \approx P\left(\left|\sum_{1\le i\ne j\le n} g_{ij}\left(H_i, \tilde{H}_j\right)\right| \ge ct\right).$$

We therefore became interested in determining whether we could extend Theorem 3.5 to include nonnegative generalized U-statistics and proved the following result, to be found in Section 4:

$$
\begin{aligned}
&E\Phi\Big(\sum_{1\le i,\,j\le n} f_{ij}(X_i, Y_j)\Big)\\
&\quad\approx_\alpha \max\Big\{E\max_{1\le i,\,j\le n}\Phi\big(f_{ij}(X_i,Y_j)\big),\, E\max_{1\le i\le n}\Phi(v_{1i}(X_i)),\\
&\qquad E\max_{1\le j\le n}\Phi\big(v_{2j}(Y_j)\big), \Phi(v_{1*}), \Phi(v_{2*}), \Phi(w_*)\Big\},
\end{aligned}
\tag{1.14}
$$

where $X_1, Y_1, \dots, X_n, Y_n$ are independent rv's, $\Phi \in \Delta_2$ has parameter $\alpha > 0$, $\{f_{ij}(x, y)\}_{1\le i,\, j\le n}$ is any array of nonnegative functions,

$$v_{1i}(x) = \sup\Big\{v \ge 0,\ \sum_{j=1}^{n} E\big(f_{ij}(x, Y_j) \wedge v\big) \ge v\Big\}, \tag{1.15}$$

$$v_{2j}(y) = \sup\Big\{v \ge 0,\ \sum_{i=1}^{n} E\big(f_{ij}(X_i, y) \wedge v\big) \ge v\Big\}, \tag{1.16}$$

$$v_{1*} = \sup\Big\{v \ge 0,\ \sum_{i=1}^{n} E\big(v_{1i}(X_i) \wedge v\big) \ge v\Big\}, \tag{1.17}$$

$$v_{2*} = \sup\Big\{v \ge 0,\ \sum_{j=1}^{n} E\big(v_{2j}(Y_j) \wedge v\big) \ge v\Big\}, \tag{1.18}$$

and

$$
\begin{aligned}
w_* = \sup\Big\{w \ge 0:\ &\sum_{1\le i,\,j\le n} E\Big(\big(f_{ij}(X_i, Y_j) \wedge w\big)\\
&\times I\Big(f_{ij}(X_i, Y_j) > \big(v_{1i}(X_i) \vee v_{2j}(Y_j)\big)\Big)\Big) \ge w\Big\}.
\end{aligned}
\tag{1.19}
$$

Results in this direction were previously obtained by Giné and Zinn (1992). Specifically, they showed that for any independent rv's $X_1, Y_1, \dots, X_n, Y_n$ such that $L(X_i) = L(Y_i)$ for $1 \le i \le n$ and any function $f(x, y)$ satisfying $f(x, y) = f(y, x)$ for all x, y with the further property that $Ef(X_i, y) = 0$ for all y and i,

$$
\begin{aligned}
E\Big|\sum_{1\le i,\,j\le n} f(X_i, Y_j)\Big|^p \le_p\ & E\Big[\max_{1\le i\le n}\Big|\sum_{j=1}^{n} f(X_i, Y_j)\Big|\Big]^p\\
&+\Big[E\Big|\sum_{1\le i,\,j\le n} f(X_i, Y_j)\Big|\Big]^p \quad \text{for } p \ge 1.
\end{aligned}
\tag{1.20}
$$

In their paper they also gave a proof of (1.20) based on Hoffmann-Jørgensen's inequality due to M. Arcones. That proof would extend to nonnegative functions $f_{ij}(x, y)$ and arbitrary independent rv's X_i and Y_j in the following form:

$$
\begin{aligned}
E\left|\sum_{1 \le i, j \le n} f_{ij}(X_i, Y_j)\right|^p \le_p\ & E\left[\max_{1\le i\le n}\left|\sum_{j=1}^n f_{ij}(X_i, Y_j)\right|\right]^p \\
&+ E\left[\max_{1\le j\le n}\left|\sum_{i=1}^n f_{ij}(X_i, Y_j)\right|\right]^p \qquad (1.21)\\
&+ \left[E\left|\sum_{1\le i, j\le n} f_{ij}(X_i, Y_j)\right|\right]^p \quad \text{for } p \ge 1.
\end{aligned}
$$

Since $p \ge 1$, the reverse inequalities also hold (by Jensen's inequality).

2. Preliminaries. In the sequel, Φ will denote an arbitrary but fixed function in Δ_2 with some parameter $\alpha > 0$. The parameter α indicates that an adjustment of the argument x of $\Phi(x)$ by a factor of $|c| \ge 1$ can affect the magnitude of $\Phi(x)$ by a factor of as much as $(|c| \vee 2)^\alpha$. We will require the following properties of Φ.

LEMMA 2.1.

(i) Φ *also has parameter* β *for every* $\beta \ge \alpha$;
(ii) $\Phi(cx) \le (2^\alpha \vee |c|^\alpha)\Phi(x) \le (2^\alpha + |c|^\alpha)\Phi(x)$ *for all* c, x;
(iii) $(|c|^\alpha \wedge 2^{-\alpha})\Phi(x) \le \Phi(cx)$ *for all* c, x;
(iv) *For two nonnegative rv's* X *and* Y,

$$E\Phi(X+Y) \approx_\alpha \max\{E\Phi(X), E\Phi(Y)\}.$$

PROOF. Properties (i)–(iii) are straightforward. So is (iv), since

$$
\begin{aligned}
\max\{\Phi(X), \Phi(Y)\} &\le \Phi(X+Y) \le \Phi(2X) + \Phi(2Y)\\
&\le 2^\alpha\Phi(X) + 2^\alpha\Phi(Y). \qquad \square
\end{aligned}
$$

The following lemma shows how various L^p approximations of a random variable $|Y|$ can enable us to obtain *two-sided approximations* of expectations of Δ_2-functions of $|Y|$. This technique is central to our approach. The lower bound is based on a probability inequality for the event that a nonnegative rv is at least a half of its expectation. This inequality may date back to Paley and Zygmund (1932) and Marcinkiewicz and Zygmund (1937) [cf. also Kahane (1985)]. The upper bound makes direct use of the definition of the parameter of a Δ_2-function.

LEMMA 2.2. *Let Y be a nonnegative valued rv, $\Phi \in \Delta_2$, with parameter α, and $v = E(Y) > 0$. Then*

$$2^{-\alpha-2}\Phi(v)\frac{v^2}{E(Y^2)} \le E\Phi(Y) \le \Phi(v)(2^\alpha + v^{-\alpha}EY^\alpha). \tag{2.1}$$

Moreover, if $E(Y^\alpha) \le c_\alpha q^\alpha$, then

$$E\Phi(Y) \le \Phi(v \vee q)(2^\alpha + c_\alpha). \tag{2.2}$$

PROOF. Bounding below in (2.1),

$$\begin{aligned} E\Phi(Y) &\ge E\left[\Phi(Y) I\left(Y \ge \frac{v}{2}\right)\right] \ge \Phi\left(\frac{v}{2}\right) P\left(Y \ge \frac{v}{2}\right) \\ &\ge 2^{-\alpha}\Phi(v) P\left(Y \ge \frac{v}{2}\right). \end{aligned}$$

Since Y is a nonnegative rv with $E(Y) = v > 0$,

$$P\left(Y \ge \frac{v}{2}\right) \ge \left(1 - \frac{1}{2}\right)^2 \frac{v^2}{E(Y^2)},$$

which entails the left-hand side of (2.1).

To obtain the upper bounds in (2.1) and (2.2) write

$$\begin{aligned} \Phi(Y) &= \Phi\left(\frac{Y}{w} w\right) \le \Phi(2w) + \Phi\left(\frac{Y}{w} w\right) I(Y \ge 2w) \\ &\le \Phi(w)(2^\alpha + w^{-\alpha} Y^\alpha). \end{aligned}$$

Putting $w = v$ and taking expectations gives the right-hand side of (2.1), while putting $w = v \vee q$ implies the right-hand side of (2.2). □

To approximate moments of a random variable $|Y|$, as required by Lemma 2.2, we recall Hoffmann-Jørgensen's inequality for positive variables [Hoffmann-Jørgensen (1974)]. A proof can be found, for example, in Ledoux and Talagrand (1991), inequality (6.8), Proposition 6.8 (given that $\max_{k \le N}|S_k| = S_N$ for nonnegative variables).

LEMMA 2.3. *Let $\{Y_j\}_{j=1}^n$ be a sequence of independent, nonnegative rv's. Then, for any $\beta \ge 1$,*

$$E\left(\sum_{j=1}^n Y_j\right)^\beta \approx_\beta \left(E \max_{1 \le j \le n} Y_j^\beta + \left(\sum_{j=1}^n EY_j\right)^\beta\right).$$

Combining Lemmas 2.2 and 2.3, we immediately obtain the following corollary.

COROLLARY 2.4. *Let* $\{Y_j\}_{j=1}^n$ *be a sequence of independent, nonnegative rv's. Let* Φ *be any* Δ_2*-function with parameter* α *and suppose that* $Y_j \le w_n$, *for each* $j = 1, \dots, n$, *and that* $E\sum_{j=1}^n Y_j \le w_n$. *Then*

$$E\Phi\left(\sum_{j=1}^n Y_j\right) \le_\alpha \Phi(w_n). \tag{2.3}$$

Moreover, if $E\sum_{j=1}^n Y_j = \lambda_* w_n$ *for some* $0 < c \le \lambda_* \le 1$, *then*

$$E\Phi\left(\sum_{j=1}^n Y_j\right) \approx_{\alpha,c} \Phi(w_n). \tag{2.4}$$

Inequalities (2.3) and (2.4) of Corollary 2.4 convert the problem of upper-bounding or obtaining the actual order of magnitude of the n-dimensional integral $E\Phi(\sum_{j=1}^n Y_j)$ (for independent nonnegative suitably bounded Y_j) to that of applying Φ to a sum of n one-dimensional integrals. Lemma 2.5 generalizes this idea to a sum of n^2 nonnegative random quantities which depend in turn on only $2n$ independent variates.

LEMMA 2.5. *Let* Φ *be any* Δ_2*-function with parameter* α *and let* $\{X_i\}_{i=1}^n$, $\{Y_j\}_{j=1}^n$ *be two independent sequences of independent rv's. Let* $\{Z_{ij}\}_{1\le i,\, j\le n}$ *be nonnegative rv's such that* Z_{ij} *depends only on* X_i *and* Y_j. *Assume further the existence of* z_* *such that we have the following*:

(i) $\operatorname{ess\,sup}_{1\le i,\, j\le n} Z_{ij} \le z_*$;
(ii) $\operatorname{ess\,sup}_{1\le j\le n} \sum_{i=1}^n E(Z_{ij}|Y_j) \le z_*$;
(iii) $\operatorname{ess\,sup}_{1\le i\le n} \sum_{j=1}^n E(Z_{ij}|X_i) \le z_*$;
(iv) $\sum_{1\le i,\, j\le n} EZ_{ij} \le z_*$.

Then

$$E\Phi\left(\sum_{1\le i,\, j\le n} Z_{ij}\right) \le_\alpha \Phi(z_*). \tag{2.5}$$

Furthermore, if for some $0 < c \le \lambda_* \le 1$,

$$\sum_{1\le i,\, j\le n} EZ_{ij} = \lambda_* z_*, \tag{2.6}$$

then

$$E\Phi\left(\sum_{1\le i,\, j\le n} Z_{ij}\right) \ge_{\alpha,c} \Phi(z_*). \tag{2.7}$$

Observe that dependency on c *in* (2.7) *as well as in* (2.4) *can be eliminated if* c *is known to be bounded away from* 0 *by an explicit constant.*

PROOF. Without loss of generality, we assume that $\alpha \ge 1$. We show that

$$E\left(\sum_{1\le i,\, j\le n} Z_{ij}\right)^\alpha \le_\alpha z_*^\alpha. \tag{2.8}$$

We have

$$
\begin{aligned}
E\Bigl(\sum_{1\le i,\,j\le n} Z_{ij}\Bigr)^{\alpha} &= E\Biggl[E\Biggl(\sum_{j=1}^{n}\Bigl(\sum_{i=1}^{n} Z_{ij}\Bigr)\Biggr)^{\alpha}\Bigg|\{X_k\}\Biggr]\\
&\le_{\alpha} E\Biggl[E\max_{1\le j\le n}\Bigl(\sum_{i=1}^{n} Z_{ij}\Bigr)^{\alpha}\Bigg|\{X_k\}\Biggr]\\
&\quad + E\Biggl[\sum_{i=1}^{n}\Bigl(\sum_{j=1}^{n} E(Z_{ij}\mid X_i)\Bigr)\Biggr]^{\alpha} \quad \text{(by Lemma 2.3)}\\
&\le_{\alpha} E\max_{1\le j\le n}\Bigl(\sum_{i=1}^{n} Z_{ij}\Bigr)^{\alpha} + E\max_{1\le i\le n}\Biggl[\sum_{j=1}^{n} E(Z_{ij}\mid X_i)\Biggr]^{\alpha}\\
&\quad + \Biggl[\sum_{1\le i,\,j\le n} EZ_{ij}\Biggr]^{\alpha} \quad \text{(by Lemma 2.3 again)}\\
&\le_{\alpha} \sum_{j=1}^{n} E\Biggl(E\Bigl(\sum_{i=1}^{n} Z_{ij}\Bigr)^{\alpha}\Bigg| Y_j\Biggr) + z_*^{\alpha}\\
&\hspace{6cm}\text{[by assumptions (iii) and (iv)]}\\
&\le_{\alpha} \sum_{j=1}^{n} E\Bigl[E\Bigl(\max_{1\le i\le n} Z_{ij}^{\alpha}\mid Y_j\Bigr)\Bigr] + \sum_{j=1}^{n} E\Biggl(\sum_{i=1}^{n} E(Z_{ij}\mid Y_j)\Biggr)^{\alpha} + z_*^{\alpha}\\
&\hspace{6cm}\text{(by Lemma 2.3)}\\
&\le_{\alpha} z_*^{\alpha-1}\sum_{j=1}^{n} E\max_{1\le i\le n} Z_{ij} + z_*^{\alpha-1}\sum_{j=1}^{n} E\Biggl(\sum_{i=1}^{n} E(Z_{ij}\mid Y_j)\Biggr) + z_*^{\alpha}\\
&\hspace{6cm}\text{[by (i) and (ii)]}\\
&\le_{\alpha} z_*^{\alpha} \quad \text{[by (iv)]},
\end{aligned}
$$

which proves (2.8). Combining (2.2) of Lemma 2.2 with (2.8), (2.5) holds. Given the left-hand side of (2.1) of Lemma 2.2, together with (2.8) for $\alpha = 2$, yields (2.7). □

To enable first-moment type considerations as discussed above to apply, we separate off all potentially abnormally (and uncontrollably) large individual summands or potentially abnormally and uncontrollably large individual factors of various groups of summands. The "rare event" cases that have thereby been isolated are treated by the method of Lemma 2.6. This produces a simplification in approximating expectations involving sums by noticing that, though formally consisting of many summands, the actual number of nonzero terms (or nonzero major factors) is a random variable having exponentially decaying tail probability.

LEMMA 2.6. *For* $1 \le j \le n$, *let the ordered pair* (B_j, Z_j) *be an event and a nonnegative random variable, respectively. Suppose there is a* σ-*field* F (*which could be trivial*) *such that*

$$\sum_{j=1}^{n} P(B_j|F) \le 1 \quad a.s. \tag{2.9}$$

and such that for each $1 \le j \le n$, $Z_j I(B_j)$ *is conditionally independent of* $N_j = \sum_{i=1,\, i \ne j}^{n} I(B_i)$ *given* F *and that the* $\{B_j\}$ *are mutually independent given* F. *Then, for* $\Phi \in \Delta_2$ *with parameter* α,

$$E\Phi\left(\sum_{j=1}^{n} Z_j I(B_j)\right) \approx_\alpha \sum_{j=1}^{n} E\Phi(Z_j) I(B_j) \approx_\alpha E \max_{1 \le j \le n} \Phi(Z_j) I(B_j). \tag{2.10}$$

PROOF. Since $E(Y) = E(E(Y|F))$ for all Y, rather than conditioning on F and then making our computations, we may as well assume that $Z_j I(B_j)$ and N_j are independent to begin with, as are $I(B_1), \dots, I(B_n)$:

$$\begin{aligned} E \max_{1 \le j \le n} \Phi(Z_j) I(B_j) &\le E\Phi\left(\sum_{j=1}^{n} Z_j I(B_j)\right) \le E \sum_{j=1}^{n} \Phi\big(Z_j I(B_j)(1 + N_j)\big) \\ &\le E \sum_{j=1}^{n} (1 + N_j)^\alpha \Phi\big(Z_j I(B_j)\big) \\ &= \sum_{j=1}^{n} E(1 + N_j)^\alpha E\Phi\big(Z_j I(B_j)\big) \\ &\le E(1 + P_1)^{\alpha \vee 1} \sum_{j=1}^{n} E\Phi(Z_j) I(B_j) \\ &\qquad \text{[by Lemma 1.1 of Klass (1981)]} \end{aligned}$$

where $P_1 \sim$ Poisson with parameter 1. To complete the cycle of inequalities, we lower-bound $E \max_{1 \le j \le n} \Phi(Z_j) I(B_j)$ in terms of $\sum_{j=1}^{n} E\Phi(Z_j) I(B_j)$. Since

$$\max_{1 \le j \le n} \Phi(Z_j) I(B_j) \ge \tfrac{1}{2} \sum_{j=1}^{n} \Phi(Z_j) I(B_j) I(N_j \le 1)$$

and

$$P(N_j \le 1) = 1 - P(N_j \ge 2) \ge 1 - \tfrac{1}{2} E(N_j) \ge \tfrac{1}{2},$$

we have

$$\begin{aligned} E \max_{1 \le j \le n} \Phi(Z_j) I(B_j) &\ge \tfrac{1}{2} \sum_{j=1}^{n} E\Phi(Z_j) I(B_j) EI(N_j \le 1) \\ &\ge \tfrac{1}{4} \sum_{j=1}^{n} E\Phi(Z_j) I(B_j). \qquad \square \end{aligned}$$

Synthesizing Lemma 2.2, 2.3 and Corollary 2.4 with Lemma 2.6 we have the following.

COROLLARY 2.7. *Let $X_1, \dots, X_n$ be independent, nonnegative random variables. Put*

$$v_n = \sup\left\{v: \sum_{j=1}^{n} E(X_j \wedge v) \ge v\right\}. \tag{2.11}$$

Then, for $\Phi \in \Delta_2$,

$$E\Phi\left(\sum_{j=1}^{n} X_j\right) \approx_\alpha \max\left\{\Phi(v_n), E \max_{1\le j\le n} \Phi(X_j)\right\}. \tag{2.12}$$

PROOF. Clearly,

$$\begin{aligned} E\Phi\left(\sum_{j=1}^{n} X_j\right) &\approx_\alpha \max\left\{E\Phi\left(\sum_{j=1}^{n}(X_j \wedge v_n)\right), E\Phi\left(\sum_{j=1}^{n} X_j I(X_j > v_n)\right)\right\} \\ &\approx_\alpha \max\left\{\Phi(v_n), E \max_{1\le j\le n} \Phi(X_j) I(X_j > v_n)\right\} \\ &\qquad\qquad [\text{by (2.4) and Lemma 2.6}] \\ &\approx_\alpha \max\left\{\Phi(v_n), E \max_{1\le j\le n} \Phi(X_j)\right\}. \qquad \square \end{aligned}$$

REMARK 2.8. (2.12) can be also inferred from Klass (1981).

3. Uniform two-sided bounds for the 2-linear random sum with nonnegative terms. In this section we obtain the order of magnitude of

$$E\Phi(B(\mathbf{X}, \mathbf{Y})), \tag{3.1}$$

where $B(\mathbf{X}, \mathbf{Y}) = \sum_{1\le i,\ j\le n} b_{ij} X_i Y_j$, $b_{ij} \ge 0$, $\{X_i\}$ and $\{Y_j\}$ are independent nonnegative rv's and $\Phi \in \Delta_2$ has some parameter $\alpha > 0$.

We begin by decomposing the sum $\sum_{1\le i,\ j\le n} b_{ij} X_i Y_j$ into six quantities (Lemma 3.1) of four essentially different types. We approximate each part separately (Lemmas 3.2–3.4) via the results developed in Section 2. The grand approximation is then obtained by taking the maximum of these bounds. Note that definitions (1.3)–(1.6) and (1.9) entail

$$\sum_{i=1}^{n} P(b_{ij} X_i > v_{2j}) \le 1, \tag{3.2a}$$

$$\sum_{j=1}^{n} P(b_{ij} Y_j > v_{1i}) \le 1, \tag{3.2b}$$

$$\sum_{i=1}^{n} P\left(X_i > \frac{v_{1*}}{v_{1i}}\right) \le 1, \tag{3.3a}$$

$$\sum_{j=1}^{n} P\left(Y_j > \frac{v_{2*}}{v_{2j}}\right) \le 1 \tag{3.3b}$$

and

$$\sum_{1\le i,\, j\le n} P\Big(b_{ij}X_iY_jI\big(B_{1ij}^cB_{2ij}^c\big) > w_*\Big) \le 1. \tag{3.4}$$

LEMMA 3.1 (Key splitting lemma).

$$\begin{aligned} &E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_j\Big)\\ &\quad\approx_\alpha E\Phi\Big(\sum_{1\le i,\, j\le n}\big((b_{ij}Y_j)\wedge v_{1i}\big)X_i\Big) + E\Phi\Big(\sum_{1\le i,\, j\le n}\big((b_{ij}X_i)\wedge v_{2j}\big)Y_j\Big)\\ &\qquad + E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i})\Big). \end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned} &E\Phi\Big(\sum_{1\le i,\, j\le n}\big((b_{ij}Y_j)\wedge v_{1i}\big)X_i\Big)\\ &\quad\approx_\alpha E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}Y_j)\wedge v_{1i}\big)\Big(X_i\wedge\frac{v_{1*}}{v_{1i}}\Big)\bigg)\\ &\qquad + E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}Y_j)\wedge v_{1i}\big)X_iI\Big(X_i > \frac{v_{1*}}{v_{1i}}\Big)\bigg), \end{aligned} \tag{3.6a}$$

$$\begin{aligned} &E\Phi\Big(\sum_{1\le i,\, j\le n}\big((b_{ij}X_i)\wedge v_{2j}\big)Y_j\Big)\\ &\quad\approx_\alpha E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}X_i)\wedge v_{2j}\big)\Big(Y_j\wedge\frac{v_{2*}}{v_{2j}}\Big)\bigg)\\ &\qquad + E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}X_i)\wedge v_{2j}\big)Y_jI\Big(Y_j > \frac{v_{2*}}{v_{2j}}\Big)\bigg), \end{aligned} \tag{3.6b}$$

and

$$\begin{aligned} &E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i})\Big)\\ &\quad\approx_\alpha E\Phi\Big(\sum_{1\le i,\, j\le n}\big((b_{ij}X_iY_j)\wedge w_*\big)I(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i})\Big)\\ &\qquad + E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})\\ &\qquad\qquad \times I(b_{ij}Y_j > v_{1i})I(b_{ij}X_iY_j > w_*)\Big). \end{aligned} \tag{3.7}$$

PROOF. Observe that

$$\begin{aligned} Z_{ij} \equiv b_{ij} X_i Y_j = \max\Big\{\big((b_{ij}Y_j) \wedge v_{1i}\big) X_i, \big((b_{ij}X_i) \wedge v_{2j}\big) Y_j, \\ b_{ij} X_i Y_j I\big(b_{ij} X_i > v_{2j}, b_{ij} Y_j > v_{1i}\big)\Big\} \\ \equiv \max\{Z_{ij,1}, Z_{ij,2}, Z_{ij,3}\}. \end{aligned}$$

Put $Z = \sum_{1 \le i,\ j \le n} Z_{ij}$ and $Z_m = \sum_{1 \le i,\ j \le n} Z_{ij,m}$ for $m = 1, 2, 3$. Since $Z_1 + Z_2 + Z_3 \ge Z$ and $Z \ge \max(Z_1, Z_2, Z_3)$ we obtain that

$$\begin{aligned} \tfrac{1}{3}\big[E\Phi(Z_1) + E\Phi(Z_2) + E\Phi(Z_3)\big] &\le E\Phi(Z) \\ &\le E\Phi(3Z_1) + E\Phi(3Z_2) + E\Phi(3Z_3) \\ &\le 3^{\alpha}\big[E\Phi(Z_1) + E\Phi(Z_2) + E\Phi(Z_3)\big], \end{aligned}$$

which establishes (3.5). The approximations (3.6a), (3.6b) and (3.7) can be proved by analogous arguments. □

We now direct an effort toward extracting the order of magnitude of each of the six quantities given in the right-hand side of (3.6a), (3.6b) and (3.7).

LEMMA 3.2. *Let*

$$V_{1ij} = \big((b_{ij}Y_j) \wedge v_{1i}\big)\left(X_i \wedge \frac{v_{1*}}{v_{1i}}\right),$$

$$V_{2ij} = \big((b_{ij}X_i) \wedge v_{2j}\big)\left(Y_j \wedge \frac{v_{2*}}{v_{2j}}\right)$$

and

$$W_{ij} = \big(b_{ij} X_i Y_j\big) I\big(b_{ij} X_i > v_{2j}\big) I\big(b_{ij} Y_j > v_{1i}\big).$$

Then

$$E\Phi\Bigg(\sum_{1 \le i,\ j \le n} V_{1ij}\Bigg) \approx_{\alpha} \Phi(v_{1*}), \tag{3.8a}$$

$$E\Phi\Bigg(\sum_{1 \le i,\ j \le n} V_{2ij}\Bigg) \approx_{\alpha} \Phi(v_{2*}) \tag{3.8b}$$

and

$$E\Phi\Bigg(\sum_{1 \le i,\ j \le n} \big(W_{ij} \wedge w_*\big)\Bigg) \approx_{\alpha} \Phi(w_*). \tag{3.9}$$

PROOF. Note that the conditions required in Lemma 2.5 hold for V_{1ij}:

(i) $\operatorname{ess\,sup}_{1 \le i,\ j \le n} V_{1ij} \le v_{1*}$;
(ii) $\operatorname{ess\,sup}_{1 \le j \le n} \sum_{i=1}^{n} E(V_{1ij} | Y_j) \le \sum_{i=1}^{n} E((v_{1i} X_i) \wedge v_{1*}) \le v_{1*}$;
(iii) $\operatorname{ess\,sup}_{1 \le i \le n} \sum_{j=1}^{n} E(V_{1ij} | X_i) \le \operatorname{ess\,sup}_{1 \le i \le n} ((v_{1i} X_i) \wedge v_{1*}) \le v_{1*}$;
(iv) $\sum_{1 \le i,\ j \le n} EV_{1ij} = v_{1*}$.

Hence Lemma 2.5 validates (3.8a). Moreover, the same argument proves (3.8b). Equation (3.9) is proved in similar fashion, employing bounds such as

$$\operatorname*{ess\,sup}_{1\le j\le n} \sum_{i=1}^{n} E\big((W_{ij}\wedge w_*)|Y_j\big) \le \sup_{1\le j\le n} \sum_{i=1}^{n} Ew_* I(b_{ij}X_i > v_{2j}) \le w_*$$

and

$$\sum_{1\le i,\, j\le n} E(W_{ij}\wedge w_*) = w_*. \qquad \square$$

LEMMA 3.3.

$$\begin{aligned} &E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}X_i)\wedge v_{2j}\big)Y_j I\bigg(Y_j > \frac{v_{2*}}{v_{2j}}\bigg)\bigg)\\ \text{(3.10a)}\qquad &\approx_\alpha \sum_{j=1}^{n} E\Phi\bigg(v_{2j}Y_j I\bigg(Y_j > \frac{v_{2*}}{v_{2j}}\bigg)\bigg)\\ &\approx_\alpha E\max_{1\le j\le n}\Phi(v_{2j}Y_j)I(v_{2j}Y_j > v_{2*}) \end{aligned}$$

and

$$\begin{aligned} &E\Phi\bigg(\sum_{1\le i,\, j\le n}\big((b_{ij}Y_j)\wedge v_{1i}\big)X_i I\bigg(X_i > \frac{v_{1*}}{v_{1i}}\bigg)\bigg)\\ \text{(3.10b)}\qquad &\approx_\alpha \sum_{i=1}^{n} E\Phi\bigg(v_{1i}X_i I\bigg(X_i > \frac{v_{1*}}{v_{1i}}\bigg)\bigg)\\ &\approx_\alpha E\max_{1\le i\le n}\Phi(v_{1i}X_i)I(v_{1i}X_i > v_{1*}). \end{aligned}$$

PROOF. Set

$$Z_{ij} = \big((b_{ij}X_i)\wedge v_{2j}\big)Y_j I\bigg(Y_j > \frac{v_{2*}}{v_{2j}}\bigg).$$

By virtue of (3.3b) we can apply Lemma 2.6 conditionally on the set of $X_1, \dots, X_n$ to obtain

$$E\Phi\bigg(\sum_{1\le i,\, j\le n} Z_{ij}\bigg) = EE\bigg(\Phi\bigg(\sum_{j=1}^{n}\bigg(\sum_{i=1}^{n} Z_{ij}\bigg)\bigg)\bigg|\{X_i\}\bigg) \approx_\alpha \sum_{j=1}^{n} E\Phi\bigg(\sum_{i=1}^{n} Z_{ij}\bigg).$$

Since

$$E\bigg(\sum_{i=1}^{n} Z_{ij}|Y_j\bigg) = v_{2j}Y_j I\bigg(Y_j > \frac{v_{2*}}{v_{2j}}\bigg)$$

and

$$Z_{ij} \le v_{2j}Y_j I\bigg(Y_j > \frac{v_{2*}}{v_{2j}}\bigg),$$

Corollary 2.4 entails

$$E\bigg(\Phi\bigg(\sum_{i=1}^{n} Z_{ij}\bigg)\bigg|\{Y_j\}\bigg) \approx_\alpha \Phi(v_{2j}Y_j)I(v_{2j}Y_j > v_{2*}).$$

Sum this equivalence on j and invoke Lemma 2.6 to conclude (3.10a). Similarly, (3.10b) holds. □

LEMMA 3.4.

$$E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij}X_iY_jI\big(b_{ij}X_i > v_{2j},\, b_{ij}Y_j > v_{1i},\, b_{ij}X_iY_j > w_*\big)\Big)$$
$$\approx_\alpha \sum_{1\le i,\, j\le n} E\Phi\big(b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i})I(b_{ij}X_iY_j > w_*)\big)$$
$$\approx_\alpha E\max_{1\le i,\, j\le n}\Phi\big(b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i})I(b_{ij}X_iY_j > w_*)\big).$$

PROOF. Set

$$W_{ij} = b_{ij}X_iY_jI(b_{ij}X_i > v_{2j})I(b_{ij}Y_j > v_{1i}),$$
$$W'_{ij} = W_{ij}I(W_{ij} > w_*),$$
$$N'_{ij} = \sum_{1\le i',\, j'\le n:\, i'\ne i \text{ and } j'\ne j} I(W'_{i'j'} \ne 0),$$
$$N'_{i\cdot}(j) = \sum_{j'=1,\, j'\ne j}^{n} I(b_{ij'}Y_{j'} > v_{1i}),$$
$$N'_{\cdot j}(i) = \sum_{i'=1,\, i'\ne i}^{n} I(b_{i'j}X_{i'} > v_{2j}).$$

Clearly,

$$\begin{aligned}
E\max_{1\le i,\, j\le n}\Phi(W'_{ij}) &\le E\Phi\Big(\sum_{1\le i,\, j\le n} W'_{ij}\Big)\\
&\le E\sum_{1\le i,\, j\le n}\Phi\big(W'_{ij}(1 + N'_{ij} + N'_{i\cdot}(j) + N'_{\cdot j}(i))\big)\\
&\le E\sum_{1\le i,\, j\le n}\big(1 + N'_{ij} + N'_{i\cdot}(j) + N'_{\cdot j}(i)\big)^\alpha\Phi(W'_{ij})\\
&\le E\sum_{1\le i,\, j\le n} 3^\alpha\Big((1 + N'_{ij})^\alpha + (N'_{i\cdot}(j))^\alpha + (N'_{\cdot j}(i))^\alpha\Big)\Phi(W'_{ij})\\
&= 3^\alpha\sum_{1\le i,\, j\le n} E(1 + N'_{ij})^\alpha E\Phi(W'_{ij})\\
&\quad + 3^\alpha\sum_{1\le i,\, j\le n} E(N'_{i\cdot}(j))^\alpha E\Phi(W'_{ij})\\
&\quad + 3^\alpha\sum_{1\le i,\, j\le n} E(N'_{\cdot j}(i))^\alpha E\Phi(W'_{ij})
\end{aligned}$$

(by linearity and independence).

Note that

$$E(1+N'_{ij})^{\alpha} \le E\Bigl(1+\sum_{1\le i,\, j\le n} I(W'_{ij}\ne 0)\Bigr)^{\alpha} \le_{\alpha} 1$$

[by applying inequalities (3.2a), (3.2b) and (3.4) to Lemma 2.5],

$$E(N'_{i\cdot}(j))^{\alpha} \le E\Bigl(\sum_{j=1}^{n} I(b_{ij}Y_j > v_{1i})\Bigr)^{\alpha} \le_{\alpha} 1$$

[by (3.2b) applied to Corollary 2.4]

and, similarly,

$$E(N'_{\cdot j}(i))^{\alpha} \le E\Bigl(\sum_{j=1}^{n} I(b_{ij}X_i > v_{2j})\Bigr)^{\alpha} \le_{\alpha} 1.$$

Hence,

$$E\max_{1\le i,\, j\le n}\Phi(W'_{ij}) \le E\Phi\Bigl(\sum_{1\le i,\, j\le n} W'_{ij}\Bigr) \le_{\alpha} \sum_{1\le i,\, j\le n} E\Phi(W'_{ij}).$$

Finally,

$$E\max_{1\le i,\, j\le n}\Phi(W'_{ij}) \ge \tfrac{1}{10}E\sum_{1\le i,\, j\le n}\Phi(W'_{ij})I(N'_{ij}\le 3,\ N'_{i\cdot}(j)\le 3,\ N'_{\cdot j}(i)\le 3)$$

$$\Bigl[\text{since for all } (i,j) \sum_{1\le i',\, j'\le n} I(W'_{ij}\ne 0) \le 10 \text{ on } \{W'_{ij}\ne 0,\ N'_{ij}\vee N'_{i\cdot}(j)\vee N'_{\cdot j}(i)\le 3\}\Bigr]$$

$$\ge \tfrac{1}{10}\sum_{1\le i,\, j\le n} E\Phi(W'_{ij})\bigl(1 - I(N'_{ij}\ge 4) - I(N'_{i\cdot}(j)\ge 4) - I(N'_{\cdot j}(i)\ge 4)\bigr)$$

$$= \tfrac{1}{10}\sum_{1\le i,\, j\le n} E\Phi(W'_{ij})\bigl(1 - P(N'_{ij}\ge 4) - P(N'_{i\cdot}(j)\ge 4) - P(N'_{\cdot j}(i)\ge 4)\bigr)$$

(by linearity and independence)

$$\ge \tfrac{1}{40}\sum_{1\le i,\, j\le n} E\Phi(W'_{ij}),$$

since

$$P(N'_{ij}\ge 4) \le \frac{1}{4}E\sum_{1\le i',\, j'\le n} I(W_{i'j'} > w_*) \le \frac{1}{4}\sum_{1\le i',\, j'\le n} E\Bigl(\frac{W_{i'j'}}{w_*}\wedge 1\Bigr) = \frac{1}{4},$$

$$P\big(N'_{i\cdot}(j) \ge 4\big) \le \frac{1}{4}\sum_{j=1}^{n} P\big(b_{ij}Y_j > v_{1i}\big) \le \frac{1}{4}\sum_{j=1}^{n} E\left(\frac{b_{ij}Y_j}{v_{1i}} \wedge 1\right) = \frac{1}{4}$$

and, similarly,

$$P\big(N'_{\cdot j}(i) \ge 4\big) \le \tfrac{1}{4}. \qquad \square$$

THEOREM 3.5. *Let Φ be any Δ_2-function with parameter $\alpha > 0$, $\{b_{ij}\}_{1\le i,\ j\le n}$ be nonnegative constants, $\{X_i\}$ and $\{Y_j\}$ be two independent sequences of independent, nonnegative rv's. Define v_{1i}, v_{2j}, v_{1*}, v_{2*} and w_* as in* (1.3)–(1.6), (1.9), *respectively. Then*,

$$\begin{aligned} E\Phi\Bigg(\sum_{1\le i,\ j\le n} b_{ij}X_iY_j\Bigg) &\approx_\alpha \Phi(\mathbf{b},\mathbf{X},\mathbf{Y}) \\ &\equiv \max\Big\{\Phi(v_{1*}),\ E\max_{1\le i\le n}\Phi(v_{1i}X_i),\ \Phi(v_{2*}),\\ &\qquad E\max_{1\le j\le n}\Phi\big(v_{2j}Y_j\big),\ \Phi(w_*),\ E\max_{1\le i,\ j\le n}\Phi\big(b_{ij}X_iY_j\big)\Big\}. \end{aligned} \tag{3.11}$$

Moreover, if Φ is convex on $[0,\infty)$, the approximation can be simplified to read

$$\begin{aligned} E\Phi\Bigg(\sum_{1\le i,\ j\le n} b_{ij}X_iY_j\Bigg) &\approx_\alpha \overline{\Phi}(\mathbf{b},\mathbf{X},\mathbf{Y}) \\ &\equiv \max\Bigg\{\Phi\Bigg(\sum_{1\le i,\ j\le n} b_{ij}EX_iEY_j\Bigg),\\ &\qquad E\max_{1\le i\le n}\Phi\Bigg(X_i\sum_{j=1}^{n} b_{ij}EY_j\Bigg),\ E\max_{1\le j\le n}\Phi\Bigg(Y_j\sum_{i=1}^{n} b_{ij}EX_i\Bigg),\\ &\qquad E\max_{1\le i,\ j\le n}\Phi\big(b_{ij}X_iY_j\big)\Bigg\}. \end{aligned} \tag{3.12}$$

PROOF. Combining Lemmas 3.1–3.4,

$$\begin{aligned} E\Phi\Bigg(\sum_{1\le i,\ j\le n} b_{ij}X_iY_j\Bigg) &\approx_\alpha \max\Big\{\Phi(v_{1*}),\ E\max_{1\le i\le n}\Phi(v_{1i}X_i)I(v_{1i}X_i > v_{1*}),\ \Phi(v_{2*})\\ &\qquad E\max_{1\le j\le n}\Phi\big(v_{2j}Y_j\big)I\big(v_{2j}Y_j > v_{2*}\big),\ \Phi(w_*),\ E\max_{1\le i,\ j\le n}\Phi\big(W'_{ij}\big)\Big\}, \end{aligned}$$

where W'_{ij} is defined as in the proof of Lemma 3.4. Due to the presence of $\Phi(v_{1*})$ above, $\Phi(v_{1i}X_i)$ may be ignored when $v_{1i}X_i \le v_{1*}$. Hence,

$$\max\Big\{\Phi(v_{1*}),\, E\max_{1\le i\le n}\Phi(v_{1i}X_i)I(v_{1i}X_i > v_{1*})\Big\}$$

may be replaced by

$$\max\Big\{\Phi(v_{1*}),\, E\max_{1\le i\le n}\Phi(v_{1i}X_i)\Big\}.$$

Similarly, we may substitute

$$\max\Big\{\Phi(v_{2*}),\, E\max_{1\le j\le n}\Phi\big(v_{2j}Y_j\big)\Big\}$$

for

$$\max\Big\{\Phi(v_{2*}),\, E\max_{1\le j\le n}\Phi\big(v_{2j}Y_j\big)I\big(v_{2j}Y_j > v_{2*}\big)\Big\}.$$

Finally, since

$$\max_{1\le i,\,j\le n}\Phi\big(W'_{ij}\big) \le \max_{1\le i,\,j\le n}\Phi\big(b_{ij}X_iY_j\big) \le \Phi\bigg(\sum_{1\le i,\,j\le n} b_{ij}X_iY_j\bigg)$$

we may drop $E\max_{1\le i,\,j\le n}\Phi(W'_{ij})$ in favor of $E\max_{1\le i,\,j\le n}\Phi(b_{ij}X_iY_j)$.

Suppose now that Φ is convex. By Jensen's inequality,

$$E\Phi\bigg(\sum_{1\le i,\,j\le n} b_{ij}X_iY_j\bigg) \ge \Phi\bigg(\sum_{1\le i,\,j\le n} b_{ij}E\big(X_iY_j\big)\bigg).$$

Let $m_{1i} = \sum_{j=1}^n b_{ij}EY_j$ and $m_{2j} = \sum_{i=1}^n b_{ij}\,EX_i$. Conditioning on $\{X_i\}$ and using Jensen's inequality again,

$$E\Phi\bigg(\sum_{1\le i,\,j\le n} b_{ij}X_iY_j\bigg) \ge \Phi\bigg(\sum_{i=1}^n m_{1i}EX_i\bigg) \ge E\max_{1\le i\le n}\Phi(m_{1i}X_i).$$

Similarly

$$E\Phi\bigg(\sum_{1\le i,\,j\le n} b_{ij}X_iY_j\bigg) \ge E\max_{1\le j\le n}\Phi\big(m_{2j}Y_j\big).$$

Hence the right-hand side in (3.12) is a lower bound for $E\Phi(\sum_{1\le i,\,j\le n} b_{ij}X_iY_j)$. However, since $v_{1i}\le m_{1i}$,

$$v_{1*} \le E\sum_{i=1}^n v_{1i}X_i \le \sum_{i=1}^n m_{1i}EX_i = \sum_{1\le i,\,j\le n} b_{ij}EX_iEY_j \equiv m_*.$$

Similarly, $v_{2j}\le m_{2j}$, $v_{2*}\le m_*$ and $w_*\le m_*$. Thus the right-hand side of (3.12) dominates the right-hand side of (3.11). Hence, by (3.11), (3.12) holds. $\square$

Despite the intuitive content of the six quantities which comprise $\Phi(\mathbf{b},\mathbf{X},\mathbf{Y})$, their sheer number should probably motivate a search for some kind of reformulated simplification.

For convex Φ one might further hope that two out of the four quantities which comprise $\overline{\Phi}(\mathbf{b}, \mathbf{X}, \mathbf{Y})$ could be omitted, namely

$$E \max_{1 \le i \le n} \Phi\left(X_i \textstyle\sum_{j=1}^n b_{ij} E Y_j \right)$$

and its counterpart. The following example illustrates the necessity of incorporating all four quantities.

EXAMPLE 3.6. Suppose that $b_{ij} \equiv 1 \equiv Y_j$, and $P(X_i = 1) = p_n = 1 - P(X_i = 0)$. Let $\Phi_\alpha(x) = x^\alpha$. Then for all $\alpha > 1$ and $p_n > 0$ such that $np_n \to 0$,

$$E\Phi_\alpha\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j \right) = E\left(n \sum_{i=1}^n X_i \right)^\alpha \sim n^{\alpha+1} p_n$$

and

$$E \max_{1 \le i \le n} \Phi_\alpha\left(X_i \sum_{j=1}^n b_{ij} Y_j \right) = E \max_{1 \le i \le n} (nX_i)^\alpha \sim n^{\alpha+1} p_n,$$

whereas the other three quantities

$$E \max_{1 \le i,\, j \le n} \Phi_\alpha\left(b_{ij} X_i Y_j \right),$$

$$E \max_{1 \le j \le n} \Phi_\alpha\left(Y_j \sum_{i=1}^n b_{ij} X_i \right)$$

and

$$\Phi_\alpha\left(\sum_{1 \le i,\, j \le n} b_{ij} E X_i E Y_j \right)$$

are of lower order. Hence the quantity $E \max_{1 \le i \le n} \Phi(X_i \sum_{j=1}^n b_{ij} Y_j)$ [and therefore $E \max_{1 \le j \le n} \Phi(Y_j \sum_{i=1}^n b_{ij} X_i)$ as well] cannot in general be eliminated from $\overline{\Phi}(\mathbf{b}, \mathbf{X}, \mathbf{Y})$ if we are to maintain the validity of (3.12). The necessity of incorporating the other two quantities into $\overline{\Phi}(\mathbf{b}, \mathbf{X}, \mathbf{Y})$ is obvious.

Can we dispense with any of the six quantities in $\Phi(\mathbf{b}, \mathbf{X}, \mathbf{Y})$? Clearly, $\Phi(v_{1*})$ and so $\Phi(v_{2*})$ are (separately) needed, as is $E \max_{1 \le i,\, j \le n} \Phi(b_{ij} X_i Y_j)$. The following example illustrates the necessity of the other three quantities.

EXAMPLE 3.7. First, using Example 3.6 with $p_n > 0$, $np_n \to 0$ and $0 < \alpha < 1$, we have $v_{1i} = n$, $v_{2j} = v_{2*} = v_{1*} = w_* = 0 = E \max_{1 \le j \le n} \Phi_\alpha(v_{2j} Y_j)$ and so (by default)

$$E\Phi_\alpha\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i Y_j \right) \approx_\alpha E \max_{1 \le i \le n} \Phi_\alpha(v_{1i} X_i).$$

Hence $E \max_{1 \le i \le n} \Phi(v_{1i} X_i)$ and $E \max_{1 \le j \le n} \Phi(v_{2j} Y_j)$ are vital members of $\Phi(\mathbf{b}, \mathbf{X}, \mathbf{Y})$.

Finally, to show that w_* also cannot be excluded, take any $\Phi \in \Delta_2$ of some parameter $\alpha > 0$, $b_{ii} \equiv 1$ and $b_{ij} = 0$ for $i \neq j$. Then (for each $1 \leq i \leq n$) take any (independent) nonnegative, nonconstant random variables X_i and Y_i such that

$$E \max_{1 \leq i \leq n} \Phi(X_i Y_i) \ll E\Phi\left(\sum_{i=1}^{n} X_i Y_i\right).$$

Then (since Y_i is nonconstant) $v_{1i} = 0$ and so $v_{1*} = 0$. Similarly $v_{2j} = 0 = v_{2*}$ and so $E \max_{1 \leq i \leq n} \Phi(v_{1i} X_i) = 0 = E \max_{1 \leq j \leq n} \Phi(v_{2j} Y_j)$. Since we have taken X_i and Y_j to satisfy

$$\max_{1 \leq i,\, j \leq n} \Phi\big(b_{ij} X_i Y_j\big) = E \max_{1 \leq i \leq n} \Phi(X_i Y_i) \ll E\Phi\left(\sum_{i=1}^{n} X_i Y_i\right)$$

and

$$E\Phi\left(\sum_{i=1}^{n} X_i Y_i\right) = E\Phi\left(\sum_{1 \leq i,\, j \leq n} b_{ij} X_i Y_j\right)$$

it follows from (3.11) (or Corollary 2.7) that

$$E\Phi\left(\sum_{1 \leq i,\, j \leq n} b_{ij} X_i Y_j\right) \approx_{\alpha} \Phi(w_*).$$

[Note that $w_* = \sum_{i=1}^{n} E((X_i Y_i) \wedge w_*) > 0$.]

It would be possible to replace the three quantities v_{1*}, v_{2*} and w_* by a single quantity q^* which acts as a rough approximation to their maximum, where

$$q^* = \sup\left\{q\colon \sum_{i=1}^{n} E\big((v_{1i} X_i) \wedge q\big) + \sum_{j=1}^{n} E\big((v_{2j} Y_j) \wedge q\big) + \sum_{1 \leq i,\, j \leq n} E\big(W_{ij} \wedge q\big) \geq q\right\} \tag{3.13}$$

and W_{ij} is defined as in Lemma 3.2.

However, it is not possible to substitute v^* for $\max\{v_{1*}, v_{2*}, w_*\}$ where

$$v^* = \sup\left\{v\colon \sum_{1 \leq i,\, j \leq n} E\big((b_{ij} X_i Y_j) \wedge v\big) \geq v\right\}, \tag{3.14}$$

as the following example shows.

Example 3.8. Fix $0 < \alpha < \beta < 1$. Suppose $\{X_i\}$ and $\{Y_j\}$ have a common distribution determined by

$$P(X > y) = y^{-\beta} \wedge 1, \qquad y > 0.$$

Then

$$E\left|\sum_{1\le i,\, j\le n} X_i Y_j\right|^{\alpha} = E\left|\sum_{i=1}^{n} X_i\right|^{\alpha}\left|\sum_{j=1}^{n} Y_j\right|^{\alpha} = \left(E\left|\sum_{i=1}^{n} X_i\right|^{\alpha}\right)^2 \approx n^{2\alpha/\beta}.$$

However, since

$$P(X_1 X_2 > y) \sim \frac{\beta \log y}{y^{\beta}} + y^{-\beta}, \qquad y > 1,$$

direct calculation yields

$$v^* \sim \left(\frac{2\beta n^2 \log n}{1-\beta}\right)^{1/\beta}$$

whence $(v^*)^{\alpha} \gg E|\sum_{1\le i,\, j\le n} X_i Y_j|^{\alpha}$.

One might wonder how the order of magnitude of $E\Phi(\sum_{1\le i,\, j\le n} b_{ij} X_i Y_j)$ is affected as the terms inside Φ are made increasingly more independent of one another [de la Peña raised such a question in a paper on martingales (1990)]. Perhaps more importantly, how does this additional independence affect and alter the method of approximation?

Happily, the structure of Theorem 3.5 is broad enough to encompass the situations which introduce these questions. We begin by showing how $E\Phi(\sum_{1\le i,\, j\le n} b_{ij} X_i Y_j)$ and $E\Phi(\sum_{1\le i,\, j\le n} b_{ij} X_i^{(j)} Y_j^{(i)})$, where $X_i^{(1)}, \dots, X_i^{(n)}$ are i.i.d. copies of X_i and $Y_j^{(1)}, \dots, Y_j^{(n)}$ are i.i.d copies of Y_j, are related to each other.

First, note that for any convex (concave) h and any i.i.d. rv's X, X_1 and X_2,

$$g(\lambda) \equiv Eh(\lambda X_1 + (1-\lambda) X_2)$$

is convex (concave). Therefore, if h is convex,

$$\sup_{0\le\lambda\le 1} g(\lambda) = \max\{g(0), g(1)\} = Eh(X),$$

and so

$$Eh(X) \ge Eh(\lambda X_1 + (1-\lambda) X_2) \tag{3.15}$$

for all $0 \le \lambda \le 1$. Similarly, if h is concave,

$$Eh(X) \le Eh(\lambda X_1 + (1-\lambda) X_2) \tag{3.16}$$

for all $0 \le \lambda \le 1$. If everything in sight is independent, then introducing $X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(n)}$ and $Y_j^{(1)}, Y_j^{(2)}, \dots, Y_j^{(n)}$ one by one (while suitably conditioning on the others) and using (3.15) repeatedly, it is easy to see that

$$\begin{aligned} E\Phi\left(\sum_{1\le i,\, j\le n} b_{ij} X_i Y_j\right) &\ge E\Phi\left(\sum_{1\le i,\, j\le n} b_{ij} X_i^{(j)} Y_j\right) \\ &\ge E\Phi\left(\sum_{1\le i,\, j\le n} b_{ij} X_i^{(j)} Y_j^{(i)}\right), \end{aligned} \tag{3.17}$$

for convex Φ, with the inequalities reversing for concave Φ.

Next, we present two examples to further illustrate how one obtains the actual order of magnitude of such quantities for general Δ_2 Φ.

EXAMPLE 3.9 (Making all the terms independent). Given $\Phi \in \Delta_2$ with parameter α, constants $b_{ij} \geq 0$ and independent nonnegative random variables $\{X_i, Y_j\}$, let $\{X_{ij}, Y_{ij}: 1 \leq i, j \leq n\}$ be independent random variables such that for all $1 \leq i, j \leq n$, $L(X_{ij}) = L(X_i)$ and $L(Y_{ij}) = L(Y_j)$. How does one approximate

$$E\Phi\left(\sum_{1 \leq i,\, j \leq n} b_{ij} X_{ij} Y_{ij}\right)? \tag{3.18}$$

We could approximate (3.18) directly from Corollary 2.7. This would be the most natural approach. Set

$$v^* = \sup\left\{v: \sum_{1 \leq i,\, j \leq n} E\big((b_{ij} X_i Y_j) \wedge v\big) \geq v\right\}. \tag{3.19}$$

By Corollary 2.7,

$$E\Phi\left(\sum_{1 \leq i,\, j \leq n} b_{ij} X_{ij} Y_{ij}\right) \approx_\alpha \max\left\{\Phi(v^*), E \max_{1 \leq i,\, j \leq n} \Phi\big(b_{ij} X_{ij} Y_{ij}\big)\right\}. \tag{3.20}$$

However, we could also put the problem in the framework of Theorem 3.5 and invoke its approximation. To do this, let

$$\begin{aligned} \tilde{b}_{(i-1)n+j,\,k} &= b_{ij} I(k = (i-1)n + j), \\ \tilde{X}_{(i-1)n+j} &= X_{ij}, \\ \tilde{Y}_{(i-1)n+j} &= Y_{ij}. \end{aligned}$$

As i and j vary from 1 to n, $(i-1)n + j$ varies from 1 to n^2.

Defining $\tilde{v}_{1i}$, $\tilde{v}_{2j}$, $\tilde{v}_{1*}$, $\tilde{v}_{2*}$ *and* $\tilde{w}_*$ in the obvious way we obtain

$$\tilde{v}_{1i} = \begin{cases} \tilde{b}_{ii} \tilde{Y}_i, & \text{if } \tilde{Y}_j \text{ is constant}, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly,

$$\tilde{v}_{2j} = \begin{cases} \tilde{b}_{jj} \tilde{X}_j, & \text{if } \tilde{X}_j \text{ is constant}, \\ 0, & \text{otherwise}. \end{cases}$$

For simplicity, let us assume $\tilde{v}_{1i} \equiv 0 \equiv \tilde{v}_{2j}$. Then $\tilde{v}_{1*} = 0 = \tilde{v}_{2*}$ and $\tilde{w}_* = v^*$ [of (3.19)]. Thus the two approximations turn out to be identical.

EXAMPLE 3.10 (Making all terms independent in the **X** variables). With Φ, b_{ij}, X_i, Y_j as in Example 3.9, introduce independent random variables $X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(n)}$ for $1 \le i \le n$ such that $L(X_i^{(j)}) = L(X_i)$. How do we approximate

$$E\Phi\left(\sum_{1 \le i,\, j \le n} b_{ij} X_i^{(j)} Y_j\right)? \tag{3.21}$$

While Corollary 2.7 does not apply, the second method we employed to handle Example 3.9 does. Let, for $1 \le i,\ j \le n$, $1 \le k \le n^2$,

$$\begin{aligned} \tilde{\tilde{b}}_{(i-1)n+j,\,k} &= b_{ij} I(k = j), \\ \tilde{\tilde{X}}_{(i-1)n+j} &= X_i^{(j)}, \\ \tilde{\tilde{Y}}_k &= Y_k I(k \le n). \end{aligned}$$

By construction

$$\sum_{1 \le i,\, j \le n} b_{ij} X_i^{(j)} Y_j = \sum_{1 \le i,\, j \le n^2} \tilde{\tilde{b}}_{ij} \tilde{\tilde{X}}_i \tilde{\tilde{Y}}_j. \tag{3.22}$$

Therefore (for $1 \le i,\ j \le n$)

$$\begin{aligned} \tilde{\tilde{v}}_{1,(i-1)n+j} &= \sum_{k=1}^{n^2} E\left(\left(\tilde{\tilde{b}}_{(i-1)n+j,\,k} \tilde{\tilde{Y}}_k\right) \wedge \tilde{\tilde{v}}_{1,(i-1)n+j}\right) \\ &= E\left(\left(b_{ij} Y_j\right) \wedge v_{1,(i-1)n+j}\right) I(k = j) \\ &= \begin{cases} 0, & \text{if } b_{ij} Y_j \text{ is nonconstant}, \\ b_{ij} Y_j, & \text{if } b_{ij} Y_j \text{ is positive and constant}. \end{cases} \end{aligned}$$

For simplicity, let us assume $\tilde{\tilde{v}}_{1,k} \equiv 0$ for $1 \le k \le n^2$. Then $\tilde{\tilde{v}}_{1*} = 0$ as well. For $1 \le k \le n$,

$$\tilde{\tilde{v}}_{2,k} = \sum_{1 \le i,\, j \le n} E\left(\left(\tilde{\tilde{b}}_{(i-1)n+j,\,k} \tilde{\tilde{X}}_{(i-1)n+j}\right) \wedge \tilde{\tilde{v}}_{2,k}\right) = \sum_{i=1}^{n} E\left(b_{ik} X_i \wedge \tilde{\tilde{v}}_{2,k}\right).$$

Hence, for $1 \le k \le n^2$,

$$\tilde{\tilde{v}}_{2,k} = v_{2k} I(k \le n) \tag{3.23}$$

and so

$$\tilde{\tilde{v}}_{2*} = v_{2*}, \tag{3.24}$$

$$\begin{aligned} \tilde{\tilde{w}}_* = &\sum_{1 \le i,\, j,\, i',\, j' \le n} E\left(\left(\tilde{\tilde{b}}_{(i-1)n+j,(i'-1)n+j'} \tilde{\tilde{X}}_{(i-1)n+j} \tilde{\tilde{Y}}_{(i'-1)n+j'}\right) \wedge \tilde{\tilde{w}}_*\right) \\ &\times I\left(i' = 1,\ j' = j,\ b_{ij} \tilde{\tilde{X}}_{(i-1)n+j} > v_{2,j}\right) \\ = &\sum_{1 \le i,\, j \le n} E\left(b_{ij} X_i Y_j \wedge \tilde{\tilde{w}}_*\right) I\left(b_{ij} X_i > v_{2,j}\right). \end{aligned} \tag{3.25}$$

(Therefore, $w_* \le \tilde{\tilde{w}}_*$.) Consequently,

$$E\Phi\Big(\sum_{1\le i,\, j\le n} b_{ij} X_i^{(j)} Y_j\Big) \tag{3.26}$$
$$\approx_\alpha \max\Big\{\Phi(v_{2*}), \Phi(\tilde{\tilde{w}}_*), E\max_{1\le j\le n}\Phi(v_{2j}Y_j), E\max_{1\le i,\, j\le n}\Phi\big(b_{ij}X_i^{(j)}Y_j\big)\Big\}.$$

4. Two-sided uniform bounds for nonnegative generalized *U*-statistics. The method we have used to identify the order of magnitude of $E\Phi(\sum_{1\le i,\, j\le n} b_{ij}X_iY_j)$ can be abstracted, enabling us to approximate

$$E\Phi\Big(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\Big)$$

whenever $\{f_{ij}(x, y)\}_{1\le i,\, j\le n}$ is any array of nonnegative functions, $\Phi \in \Delta_2$ with parameter α, and $\{X_i\}_{i=1}^n$ and $\{Y_j\}_{j=1}^n$ are two independent sequences of independent rv's. Define $v_{1i}(x)$, $v_{2j}(y)$, v_{1*}, v_{2*} and w_* as in (1.15)–(1.19), and note that

$$\sum_{j=1}^n P\big(f_{ij}(x, Y_j) > v_{1i}(x)\big) \le 1 \quad \text{for all } x, \tag{4.1a}$$

$$\sum_{i=1}^n P\big(f_{ij}(X_i, y) > v_{2j}(y)\big) \le 1 \quad \text{for all } y, \tag{4.1b}$$

$$\sum_{i=1}^n P\big(v_{1i}(X_i) > v_{1*}\big) \le 1, \tag{4.2a}$$

$$\sum_{j=1}^n P\big(v_{2j}(Y_j) > v_{2*}\big) \le 1 \tag{4.2b}$$

and

$$\sum_{1\le i,\, j\le n} P\Big(f_{ij}(X_i, Y_j) > \big(v_{1i}(X_i) \vee v_{2j}(Y_j) \vee w_*\big)\Big) \le 1. \tag{4.3}$$

Next, define the following sets of events:

$$A_{1ij} = \big\{f_{ij}(X_i, Y_j) > v_{1i}(X_i)\big\}, \tag{4.4}$$

$$A_{2ij} = \big\{f_{ij}(X_i, Y_j) > v_{2j}(Y_j)\big\}, \tag{4.5}$$

$$B_{ij} = \big\{f_{ij}(X_i, Y_j) \le w_*\big\}, \tag{4.6}$$

$$C_{1i} = \big\{v_{1i}(X_i) \le v_{1*}\big\}, \tag{4.7}$$

$$C_{2j} = \big\{v_{2j}(Y_j) \le v_{2*}\big\} \tag{4.8}$$

and

$$D_{ij} = \{v_{1i}(X_i) \vee v_{2j}(Y_j) \le v_{1*} \vee v_{2*}\}. \tag{4.9}$$

We now record some results that will be necessary for our approximation of $E\Phi(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j))$.

LEMMA 4.1.

$$E\Phi\Bigl(\sum_{1\le i,\, j\le n}\bigl(f_{ij}(X_i,Y_j)\wedge w_*\bigr)I(A_{1ij}A_{2ij})\Bigr) \approx_\alpha \Phi(w_*), \tag{4.10}$$

$$\begin{aligned} &E\Phi\Bigl(\sum_{1\le i,\, j\le n}\bigl(f_{ij}(X_i,Y_j)\wedge \bigl(v_{1i}(X_i)\vee v_{2j}(Y_j)\bigr)I(D_{ij})\bigr)\Bigr)\\ &\quad\le_\alpha \Phi(v_{1*}\vee v_{2*}) \end{aligned} \tag{4.11}$$

with the reverse inequality holding if

$$\begin{aligned} &\max\Bigl\{\sum_{i=1}^n P\bigl(v_{1i}(X_i) > v_{1*}\vee v_{2*}\bigr),\ \sum_{j=1}^n P\bigl(v_{2j}(Y_j) > v_{1*}\vee v_{2*}\bigr)\Bigr\}\\ &\quad\le \tfrac{1}{2}. \end{aligned} \tag{4.12}$$

PROOF. Put

$$W_{ij} = \bigl(f_{ij}(X_i,Y_j)\wedge w_*\bigr)I(A_{1ij}A_{2ij})$$

and

$$V_{ij} = f_{ij}(X_i,Y_j)\wedge\bigl(v_{1i}(X_i)\vee v_{2j}(Y_j)\bigr)I(D_{ij}).$$

We intend to employ Lemma 2.5. Observe that $W_{ij}\le w_*$,

$$\begin{aligned} &\sum_{j=1}^n E\bigl(W_{ij}|X_i\bigr) \le \sum_{j=1}^n w_* P(A_{1ij}\,|\,X_i) \le w_* \quad [\text{by (4.1a)}],\\ &\sum_{i=1}^n E\bigl(W_{ij}|Y_j\bigr) \le w_* \quad [\text{as above but incorporating (4.1b)}] \end{aligned}$$

and

$$\sum_{1\le i,\, j\le n} EW_{ij} = w_* \quad [\text{by (1.19)}].$$

Now (4.10) follows from Lemma 2.5.

As for V_{ij}, w.l.o.g. assume that $v_{1*} \geq v_{2*}$. Observe that $V_{ij} \leq v_{1*}$,

$$\begin{aligned}\sum_{j=1}^{n} E\big(V_{ij}|X_i\big) &\leq \sum_{j=1}^{n} E\Big(\big(f_{ij}(X_i, Y_j) \wedge v_{1*}\big) I\big(v_{1i}(X_i) \leq v_{1*}\big)|X_i\Big)\\ &\leq v_{1*} I\big(v_{1i}(X_i) \leq v_{1*}\big) \quad [\text{by } (1.15)]\\ &\leq v_{1*},\end{aligned}$$

$$\sum_{i=1}^{n} E\big(V_{ij}|Y_j\big) \leq \sum_{i=1}^{n} E\Big(\big(f_{ij}(X_i, Y_j) \wedge v_{1*}\big) I\big(v_{2j}(Y_j) \leq v_{1*}\big)|Y_j\Big) \leq v_{1*}$$

$$[\text{as above but using } (1.16)]$$

and

$$\begin{aligned}\sum_{1 \leq i,\, j \leq n} EV_{ij} &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} E\big(f_{ij}(X_i, Y_j) \wedge v_{1i}(X_i)\big) I\big(v_{1i}(X_i) \leq v_{1*}\big)\\ &\quad + \sum_{j=1}^{n}\sum_{i=1}^{n} E\big(f_{ij}(X_i, Y_j) \wedge v_{2j}(Y_j)\big) I\big(v_{2j}(Y_j) \leq v_{1*}\big)\\ &\leq \sum_{i=1}^{n} Ev_{1i}(X_i) I(C_{1i}) + \sum_{j=1}^{n} Ev_{2j}(Y_j) I\big(v_{2j}(Y_j) \leq v_{1*}\big)\\ &\leq \sum_{i=1}^{n} E\big(v_{1i}(X_i) \wedge v_{1*}\big) + \sum_{j=1}^{n} E\big(v_{2j}(Y_j) \wedge v_{1*}\big) \leq 2v_{1*}\\ &\qquad (\text{since } v_{2*} \leq v_{1*}).\end{aligned}$$

Hence (4.11) holds by (2.5) of Lemma 2.5.

For the reverse bound, observe that

$$\begin{aligned}\sum_{1 \leq i,\, j \leq n} EV_{ij} &\geq \sum_{i=1}^{n}\sum_{j=1}^{n} Ef_{ij}(X_i, Y_j) \wedge v_{1i}(X_i)\big(I(C_{1i}) - I\big(C_{1i} D_{ij}^c\big)\big)\\ &\geq \sum_{i=1}^{n} Ev_{1i}(X_i) I(C_{1i}) - \sum_{i=1}^{n}\sum_{j=1}^{n} Ev_{1i}(X_i) I\big(C_{1i} D_{ij}^c\big)\\ &= \sum_{i=1}^{n} Ev_{1i}(X_i) I(C_{1i})\left(1 - \sum_{j=1}^{n} P\big(v_{2j}(Y_j) > v_{1*}\big)\right)\\ &\geq \tfrac{1}{2}\sum_{i=1}^{n} Ev_{1i}(X_i) I(C_{1i}) \quad [\text{by } (4.12)]\\ &= \tfrac{1}{2} v_{1*}\left(1 - \sum_{i=1}^{n} P(C_{1i}^c)\right) \geq \tfrac{1}{4} v_{1*} \quad [\text{by } (4.12)].\end{aligned}$$

Invoking (2.7) of Lemma 2.5 we obtain the reverse inequality. □

LEMMA 4.2.

$$E\Phi\Big(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j) I(A_{1ij} A_{2ij} B_{ij}^c)\Big) \le_\alpha E \max_{1\le i,\, j\le n} \Phi\big(f_{ij}(X_i, Y_j)\big). \tag{4.13}$$

PROOF. Proceed exactly as in the proof of Lemma 3.4, using $W_{ij} = f(X_i, Y_j) I(A_{1ij} A_{2ij})$ and $W'_{ij} = W_{ij} I(B_{1i}^c)$ with N'_{ij} and $N'_{\cdot j}(i)$ defined analogously. □

THEOREM 4.3. *Take any nonnegative functions $f_{ij}(x, y)$. Let $X_1, Y_1, \dots, X_n, Y_n$ be independent random variables and $\Phi \in \Delta_2$ having parameter $\alpha > 0$. Define $v_{1i}(\cdot)$, $v_{2j}(\cdot)$, v_{1*}, v_{2*} and w_* as in* (1.15)–(1.19). *Then*

$$\begin{aligned} E\Phi\Big(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\Big) &\approx_\alpha \max\Big\{E \max_{1\le i,\, j\le n} \Phi\big(f_{ij}(X_i, Y_j)\big), E \max_{1\le i\le n} \Phi(v_{1i}(X_i)), \\ &\qquad E \max_{1\le j\le n} \Phi\big(v_{2j}(Y_j)\big), \Phi(v_{1*}), \Phi(v_{2*}), \Phi(w_*)\Big\}. \end{aligned} \tag{4.14}$$

Moreover, if Φ is convex on $[0, \infty)$,

$$\begin{aligned} E\Phi\Big(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\Big) &\approx_\alpha \max\Big\{E \max_{1\le i,\, j\le n} \Phi\big(f_{ij}(X_i, Y_j)\big), E \max_{1\le i\le n} \Phi(\bar v_{1i}(X_i)), \\ &\qquad E \max_{1\le j\le n} \Phi\big(\bar v_{2j}(Y_j)\big), \Phi(\bar v)\Big\}, \end{aligned} \tag{4.15}$$

where

$$\bar v_{1i}(x) = \sum_{j=1}^n E f_{ij}(x, Y_j), \tag{4.16}$$

$$\bar v_{2j}(y) = \sum_{i=1}^n E f_{ij}(X_i, y) \tag{4.17}$$

and

$$\bar v = \sum_{1\le i,\, j\le n} E f_{ij}(X_i, Y_j). \tag{4.18}$$

PROOF. Since $f_{ij}(X_i, Y_j) \geq 0$ for all i, j it is clear that

$$E\Phi\Bigg(\sum_{1\leq i,\, j\leq n} f_{ij}(X_i, Y_j)\Bigg) \geq \max\Bigg\{E\max_{1\leq i,\, j\leq n}\Phi\big(f_{ij}(X_i, Y_j)\big),\, E\max_{1\leq i\leq n}\Phi\Bigg(\sum_{j=1}^{n} f_{ij}(X_i, Y_j)\Bigg),\, E\max_{1\leq j\leq n}\Phi\Bigg(\sum_{i=1}^{n} f_{ij}(X_i, Y_j)\Bigg)\Bigg\}.$$

Let σ be the first index in $[1, n]$ satisfying $v_{1\sigma}(X_\sigma) = \max_{1\leq i\leq n} v_{1i}(X_i)$.

$$\begin{aligned}
E\max_{1\leq i\leq n}\Phi\Bigg(\sum_{j=1}^{n} f_{ij}(X_i, Y_j)\Bigg)
&\geq \sum_{i=1}^{n} E\Bigg(E\Bigg(\Phi\Bigg(\sum_{j=1}^{n}\big(f_{ij}(X_i, Y_j)\wedge v_{1i}(X_i)\big)\Bigg)I(\sigma = i)\,|\, X_1, \dots, X_n\Bigg)\Bigg)\\
&=_\alpha \sum_{i=1}^{n} E\Phi(v_{1i}(X_i))I(\sigma = i) \quad \text{[by (2.6) of Corollary 2.5]}\\
&= E\max_{1\leq i\leq n}\Phi(v_{1i}(X_i)).
\end{aligned}$$

Hence

$$E\Phi\Bigg(\sum_{1\leq i,\, j\leq n} f_{ij}(X_i, Y_j)\Bigg) \geq_\alpha E\max_{1\leq i\leq n}\Phi(v_{1i}(X_i))$$

and similar reasoning gives the third lower bound $E\max_{1\leq j\leq n}\Phi(v_{2j}(Y_j))$.

Suppose (4.12) fails. Then

$$\begin{aligned}
E\Phi\Bigg(\sum_{1\leq i,\, j\leq n} f_{ij}(X_i, Y_j)\Bigg)
&\geq_\alpha E\max_{1\leq i\leq n}\Phi(v_{1i}(X_i)) + E\max_{1\leq j\leq n}\Phi\big(v_{2j}(Y_j)\big)\\
&\geq \Phi(v_{1*}\vee v_{2*})\Bigg(P\Bigg(\bigcup_{i=1}^{n}\{v_{1i}(X_i) > v_{1*}\vee v_{2*}\}\Bigg) + P\Bigg(\bigcup_{j=1}^{n}\{v_{2j}(Y_j) > v_{1*}\vee v_{2*}\}\Bigg)\Bigg)\\
&\geq \Phi(v_{1*}\vee v_{2*})\inf\Bigg\{1 - \prod_{i=1}^{n}(1 - x_i) + 1 - \prod_{j=1}^{n}(1 - y_j) : 0 \leq x_i, y_i \leq 1,\\
&\qquad \max\Bigg\{\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_i\Bigg\} \geq \tfrac{1}{2}\Bigg\}\\
&= \tfrac{1}{2}\Phi(v_{1*}\vee v_{2*}).
\end{aligned}$$

On the other hand, if (4.12) holds, then by Lemma 4.1,

$$E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\right) \ge_\alpha \Phi(v_{1*} \vee v_{2*}).$$

Incorporating (4.10) we may conclude that the right-hand side of (4.14) is of no larger order than the left-hand side. Bounding above, we obtain

$$\begin{aligned}
&E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\right)\\
&\quad\le_\alpha E\Phi\left(\sum_{i=1}^n\left(\sum_{j=1}^n f_{ij}(X_i, Y_j)\right) I(C_{1i}^c)\right)\\
&\qquad + E\Phi\left(\sum_{i=1}^n\left(\sum_{j=1}^n f_{ij}(X_i, Y_j)\right) I(C_{2j}^c)\right)\\
&\qquad + E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j) I(A_{1ij}A_{2ij}B_{ij}^c)\right)\\
&\qquad + E\Phi\left(\sum_{1\le i,\, j\le n} \left(f_{ij}(X_i, Y_j) \wedge w_*\right) I(A_{1ij}A_{2ij})\right)\\
&\qquad + E\Phi\left(\sum_{1\le i,\, j\le n} \left(f_{ij}(X_i, Y_j) \wedge \left(v_{1i}(X_i) \vee v_{2j}(Y_j)\right)\right) I(D_{ij})\right)\\
&\quad\equiv T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}$$

By Lemma 4.1, $T_4 + T_5 \le_\alpha \Phi(v_{1*} \vee v_{2*} \vee w_*)$. Lemma 4.2 gives $T_3 \le E\max_{1\le i,\, j\le n} \Phi(f_{ij}(X_i, Y_j))$. Conditioning on the set of $\{Y_j\}$ and invoking Lemma 2.6 and unconditioning,

$$\begin{aligned}
T_1 &\approx_\alpha \sum_{i=1}^n E\Phi\left(\sum_{j=1}^n f_{ij}(X_i, Y_j)\right) I(C_{1i}^c)\\
&\approx_\alpha \sum_{i=1}^n E\Phi\left(\sum_{j=1}^n \left(f_{ij}(X_i, Y_j) \wedge v_{1i}(X_i)\right)\right) I(C_{1i}^c)\\
&\quad + \sum_{i=1}^n E\Phi\left(\sum_{j=1}^n f_{ij}(X_i, Y_j) I(A_{1ij})\right) I(C_{1i}^c)\\
&\le_\alpha \sum_{i=1}^n E\Phi(v_{1i}(X_i)) I(C_{1i}^c) + \sum_{i=1}^n E\max_{1\le j\le n} \Phi\left(f_{ij}(X_i, Y_j)\right) I(A_{1ij}C_{1i}^c)\\
&\qquad\qquad \text{(by Lemmas 2.5 and 2.6 and another conditioning argument)}\\
&\le_\alpha E\max_{1\le i\le n} \Phi(v_{1i}(X_i)) + E\max_{1\le i,\, j\le n} \Phi\left(f_{ij}(X_i, Y_j)\right)\\
&\qquad\qquad \text{(by Lemma 2.6 and conditioning as above).}
\end{aligned}$$

Similarly,

$$T_2 \le_\alpha E \max_{1 \le j \le n} \Phi\big(v_{2j}(Y_j)\big) + E \max_{1 \le i,\, j \le n} \Phi\big(f_{ij}(X_i, Y_j)\big).$$

Hence the left-hand side of (4.14) is no larger order than the right-hand side. □

REMARK 4.4. Let $\Phi_{*n}(f_{ij}, \{X_i\}, \{Y_j\})$ denote the right-hand side of (4.14). Fix any $\alpha > 0$ and let R_α denote the collection of all ratios of the form

$$\frac{E\Phi\big(\sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j)\big)}{\Phi_{*n}\big(f_{ij}, \{X_i\}, \{Y_j\}\big)},$$

which occur as we take all possible choices of $\Phi \in \Delta_2$ of parameter α, integers $n = 1, 2, \dots$ independent rv's $X_1, Y_1, \dots, X_n, Y_n$ and nonnegative functions $f_{ij}(x, y)$ such that $\Phi_{*n}(f_{ij}, \{X_i\}, \{Y_j\}) > 0$. Put

$$\bar{c}_\alpha = \sup R_\alpha, \tag{4.19}$$

$$\underline{c}_\alpha = \inf R_\alpha. \tag{4.20}$$

Since Φ has parameter α whenever it has parameter $0 < \beta \le \alpha$, it follows that $\bar{c}_\alpha$ is nondecreasing in α and $\underline{c}_\alpha$ is nonincreasing in α. Theorem 4.3 shows that

$$0 < \underline{c}_\alpha \le \bar{c}_\alpha < \infty.$$

Obviously, a similar story holds for Theorem 3.5.

REMARK 4.5. Due to a decoupling theorem of de la Peña and Montgomery-Smith (1995), Theorem 4.3 continues to apply if $Y_j = X_j$, provided $f_{jj}(x, y) = 0$ and $L(f_{ij}(X_i, Y_j)) = L(f_{ji}(Y_j, X_i))$.

REFERENCES

BOURGAIN, J. and TZAFRIRI, L. (1987). Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis. *Israel J. Math.* **57** 137–224.

CAMBANIS, S., ROSIŃSKI, J. and WOYCZYŃSKI, W. A. (1985). Convergence of quadratic forms in p-stable random variables and θ_p-radonifying operators. *Ann. Probab.* **13** 885–897.

DE LA PEÑA, V. H. (1990). Bounds on the expectation of functions of martingales and sums of positive R.V.'s in terms of norms of sums of independent random variables. *Proc. Amer. Math. Soc.* **108** 233–239.

DE LA PEÑA, V. H. and KLASS, M. J. (1994). Order-of-magnitude bounds for expectations involving quadratic forms. *Ann. Probab.* **22** 1044–1077.

DE LA PEÑA, V. H. and MONTGOMERY-SMITH, S. J. (1995). Decoupling inequalities for tail probabilities of multilinear forms of multivariate U-statistics. *Ann. Probab.* **23** 806–816.

GINÉ, E. and ZINN, J. (1992). On Hoffmann-Jørgensen's inequality for U-processes. In *Probability in Banach Spaces 8. Progress in Probability* **30** 80–91. Birkhäuser, Boston.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Applications*, 2nd ed. Academic Press, New York.

HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

KAHANE, J.-P. (1985). *Some Random Series of Functions*. Cambridge Univ. Press.

KLASS, M. J. (1981). A method of approximating expectations of functions of sum of independent random variables. *Ann. Probab.* **9** 413–428.

KLASS, M. J. and ZHANG, C.-H. (1994). On the almost sure minimal growth rate of partial sum maxima. *Ann. Probab.* **22** 1857–1878.

KRAKOWIAK, W. and SZULGA, J. (1988). On a p-stable multiple integral II. *Probab. Theory Related Fields* **78** 449–453.

LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*. Springer, Berlin.

MARCINKIEWICZ, J. and ZYGMUND, A. (1937). Sur les fonctions independantes. *Fund. Math.* **29** 60–90.

MCCONNELL, T. and TAQQU, M. (1986). Decoupling inequalities for multilinear forms in independent symmetric random variables. *Ann. Probab.* **14** 943–954.

PALEY, R. E. A. C. and ZYGMUND, A. (1932). A note on analytic functions on the unit circle. *Proc. Cambridge Phil. Soc.* **28** 266–272.

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