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INFINITE CLUSTERS IN DEPENDENT AUTOMORPHISM INVARIANT PERCOLATION ON TREES

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We study dependent bond percolation on the homogeneous tree T_n of order $n \geq 2$ under the assumption of automorphism invariance. Excluding a trivial case, we find that the number of infinite clusters a.s. is either 0 or ∞. Furthermore, each infinite cluster a.s. has either 1, 2 or infinitely many topological ends, and infinite clusters with infinitely many topological ends have a.s. a branching number greater than 1. We also show that if the marginal probability that a single edge is open is at least $2/(n+1)$, then the existence of infinite clusters has to have positive probability. Several concrete examples are considered.

1. Introduction. In standard bond percolation, one takes a large (usually infinite) graph G and, for some $p \in [0,1]$, removes each edge independently with probability $1-p$, thus keeping it with probability p. One is then interested in connectivity properties of the obtained random subgraph of G. When G is the infinite homogeneous tree of order $n \geq 2$, this model becomes rather easy to analyze, because it can be reduced to the study of Galton–Watson processes in which each individual has a binomially distributed (n, p) number of offspring. This is the most basic model of percolation on a tree. There are several interesting directions in which it can be generalized. One such direction, considered by Lyons [12, 13], is to look at general trees rather than just homogeneous trees. Another direction is to weaken the assumption that the edges are independent. In this paper, we will consider this second direction of generalization, replacing the i.i.d. assumption by the assumption of *automorphism invariance*, which is weak enough to cover several interesting examples and yet strong enough to have some striking consequences.

Write T_n for the homogeneous tree of order $n \geq 2$, that is, for the unique infinite connected graph having no circuits and in which there are exactly $n+1$ edges emanating from each vertex. Also write E_n (resp. V_n) for the edge set (resp. vertex set) of T_n. A subset of E_n will be identified with an element of $\{0,1\}^{\mathsf{E}_n}$, where a 1 indicates the presence of an edge and a 0 indicates its absence. Sometimes, we will use the words "open" and "closed" instead of "present" and "absent."

An automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$ is a measure which is invariant under graph automorphisms for T_n. A graph automorphism for T_n is a bijection π: $\mathsf{V}_n \to \mathsf{V}_n$ such that for $v, w \in \mathsf{V}_n$ there is an edge joining

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$\pi(v)$ and $\pi(w)$ if and only if there is an edge joining v and w, together with the induced mapping π': $\mathsf{E}_n \to \mathsf{E}_n$.

We are interested in the infinite connected components (clusters) of a configuration $\eta \in \{0,1\}^{\mathsf{E}_n}$ picked randomly according to an automorphism invariant measure μ. Possibly the most basic question is to ask for the number of infinite clusters, and our first result says that the number of infinite clusters a.s. is either 0 or ∞, except for the possibility of all edges in E_n being present (in which case we of course have exactly one infinite cluster). To avoid this triviality, we call a measure μ on $\{0,1\}^{\mathsf{E}_n}$ *nice* if $\mu(\bar{\eta}) = 0$, where $\bar{\eta}$ is the element of $\{0,1\}^{\mathsf{E}_n}$ which assigns value 1 to each $e \in \mathsf{E}_n$. Write $K(\eta)$ for the number of infinite clusters of η and write X for a random element of $\{0,1\}^{\mathsf{E}_n}$ picked according to μ.

THEOREM 1.1. *Suppose that μ is a nice automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$. Then*

$$\mu(K(X) = 0 \text{ or } K(X) = \infty) = 1.$$

Next, it is natural to ask for the number of *topological ends* of an infinite cluster C, which is defined as the number of different (but not necessarily disjoint) infinite self-avoiding paths in C leading out of a given vertex v in C. Note that this definition is independent of the choice of v. Say that an infinite cluster is of *type j* if it has exactly j topological ends. Any infinite cluster C has to be of type j for some $j \in \{1,2,\dots\} \cup \{\infty\}$; the existence of infinite clusters of type 0 is impossible since the degree of a vertex is finite (even bounded). For $\eta \in \{0,1\}^{\mathsf{E}_n}$, write $K_j(\eta)$ for the number of infinite clusters of type j in η. It is of course easy to construct, deterministically, infinite clusters contained in T_n of any type $j \in \{1,2,\dots\} \cup \{\infty\}$. Our next result, however, shows that only three of these types actually arise for automorphism invariant measures.

THEOREM 1.2. *Let μ be a nice automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$. Then, with probability 1, each infinite cluster of X is of type 1, 2 or ∞. Moreover, for $j = 1, 2, \infty$, we have*

$$\mu(K_j(X) = 0 \text{ or } K_j(X) = \infty) = 1.$$

In Section 3, examples will be given of measures giving rise to each of the three possible types of infinite clusters, and we will moreover present an example of a measure for which all three types coexist with positive probability.

The set of possible types of infinite clusters is even smaller if we impose the finite energy condition, which is of great significance in percolation theory; see, for example, [16, 4]. For $\eta \in \{0,1\}^{\mathsf{E}_n}$ and $\Lambda \subseteq \mathsf{E}_n$, let $\eta(\Lambda)$ denote the configuration η restricted to Λ. A probability measure μ on $\{0,1\}^{\mathsf{E}_n}$ is said to have *finite energy* if for any $e \in \mathsf{E}_n$ and μ-a.e. $\eta \in \{0,1\}^{\mathsf{E}_n}$ we have that the conditional probability that the edge e gets value 1, given the configuration $\eta(\mathsf{E}_n \setminus \{e\})$ off e, is strictly between 0 and 1. It is not hard to check that this is equivalent to the (at first sight stronger looking) condition that for each finite $\Lambda \subset \mathsf{E}_n$, each $\eta' \in \{0,1\}^{\Lambda}$ and μ-a.e. $\eta \in \{0,1\}^{\mathsf{E}_n \setminus \Lambda}$ we have $\mu(X(\Lambda) = \eta' \mid X(\mathsf{E}_n \setminus \Lambda) = \eta) > 0$.

THEOREM 1.3. *Let μ be an automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$ and suppose furthermore that μ satisfies the finite energy condition. Then, with probability 1, each infinite cluster of X is of type ∞.*

Further information can be given about infinite clusters of type ∞. Call a topological end *isolated* if the corresponding infinite self-avoiding open path starting at a given vertex $v \in \mathsf{V}_n$ eventually does not intersect any other infinite self-avoiding open path starting at v, and note that this definition is independent of the choice of v.

PROPOSITION 1.4. *Suppose μ is an automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$. Then μ assigns zero probability to the existence of isolated topological ends in infinite clusters of type ∞.*

The next result concerns the *branching number*, to be defined in Section 2, of an infinite cluster of type ∞.

THEOREM 1.5. *Let μ be an automorphism invariant measure on $\{0,1\}^{\mathsf{E}_n}$. Then μ-a.s. every infinite cluster of type ∞ in X has branching number strictly greater than 1.*

An immediate consequence of this is the exponential growth rate of infinite clusters of type ∞. These clusters also have uncountably many topological ends, as can be deduced either from Proposition 1.4 or from Theorem 1.5.

The final result in this section is a sufficient condition for infinite clusters to occur, in terms of the marginal probability p that a given edge e is open. With a slight abuse of language, we call p the *edge density* of μ.

THEOREM 1.6. *Suppose μ is an automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$ with edge density $p \geq 2/(n+1)$. Then μ assigns positive probability to the existence of infinite clusters. This bound is sharp in the sense that for any $p < 2/(n+1)$ there exists an automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$ which has edge density p and which assigns probability* 0 *to the existence of infinite clusters.*

The existence of such a threshold strictly less than 1 follows from a result of Adams and Lyons [1]. It is interesting to note that Theorem 1.6 reflects a qualitative difference between T_n and Z^d; for Z^d the corresponding threshold is trivial ($p = 1$).

Our interest in studying the number of topological ends in this general context was triggered by the results in [9], where two specific probability measures on $\{0,1\}^{\mathsf{E}_n}$ (not directly related to each other) both turn out to be concentrated on the event that every connnected component is infinite and of type 1; see Examples 3.1 and 3.2.

Perhaps the most important specific example of dependent percolation on Z^d is the random-cluster model, due to its relation to (and usefulness in studying) the Ising and Potts models (see, e.g., [7]). It has recently [8] been generalized to a tree setting. Random-cluster measures satisfy the finite energy condition, so Theorem 1.3 applies to the automorphism invariant random-cluster measures studied in [8]. Further motivation for studying automorphism invariant measures on trees is given in [19], where they are analyzed from a different point of view (mixing properties). The number of topological ends of trees embedded in Z^d has been studied in various particular cases; see [18, 2, 15].

If attention is focused on the connected component containing a given vertex $v \in V_n$, then the models considered here can be thought of as a class of branching processes allowing dependencies. On the other hand, our setup does not incorporate a natural time direction, and for this reason we think that other approaches (e.g., those of Klebaner [11] and Olofsson [17]) to the problem of interaction in branching structures may be more natural than ours from the perspective of population dynamics.

The organization of the remainder of the paper is as follows. Section 2 is devoted to the proof of Theorem 1.5. In Section 3, we give a number of illustrative examples and in Section 4 we prove Theorem 1.6. In order to keep down the length of this paper, we omit the proofs of Theorems 1.1–1.3 and Proposition 1.4. These can be found in an unpublished supplement [10] (available from the author) and are all based on density arguments in the spirit of Burton and Keane [4, 5].

All results in this paper have direct analogues for site percolation on T_n, and the proofs go through essentially unchanged. The threshold in Theorem 1.6 becomes $(n+1)/2n$ in the case of site percolation. Our choice to primarily study bond rather than site models is due to the motivating examples in [8] and [9], which are bond models.

2. Proof of Theorem 1.5.

In this section, we prove Theorem 1.5. We first need to define branching number. Given any infinite locally finite tree Γ with vertex set V_Γ rooted at $\rho \in V_\Gamma$, we call a finite set $\Pi \subset V_\Gamma$ a *cutset* for Γ if every infinite self-avoiding path starting at ρ has to intersect Π and no proper subset of Π has this property. Informally, Π is a cutset if it is a minimal set which cuts off ρ from infinity. For $v \in V_\Gamma$, let $|v|$ denote the distance between ρ and v. For a sequence of cutsets $\Pi_1, \Pi_2, \ldots$, write $\Pi \to \infty$ if $\min\{|v|: v \in \Pi\} \to \infty$.

DEFINITION 2.1. The *branching number* of Γ, denoted $\mathrm{br}(\Gamma)$, is defined

$$\mathrm{br}(\Gamma) = \inf\left\{\lambda > 0\colon \liminf_{\Pi \to \infty} \sum_{v \in \Pi} \lambda^{-|v|} = 0\right\}.$$

Note that $\mathrm{br}(\Gamma)$ is independent of the choice of root $\rho \in V_\Gamma$. It appears that the branching number is of relevance for almost any interesting probability model on Γ; see, for example, [12, 14, 3]. In many references, $\log(\mathrm{br}(\Gamma))$ is called the Hausdorff dimension of Γ.

The proof of Theorem 1.5 will be based on simple random walk on the infinite cluster containing a given vertex $v \in \mathsf{V}_n$. For this we need some more definitions. Simple random walk on a tree Γ started at $\rho \in V_\Gamma$ is the Markov chain $\{Y_\rho(i)\}_{i=0}^\infty$ taking values in V_Γ with $Y_\rho(0) = \rho$ and transition matrix $(M_{v_1 v_2})_{v_1, v_2 \in V_\Gamma}$ given by

$$M_{v_1 v_2} = \begin{cases} \dfrac{1}{d(v_1)}, & \text{if } v_1 \text{ and } v_2 \text{ are nearest neighbors,} \\ 0, & \text{otherwise,} \end{cases}$$

where $d(v_1)$ is the degree of v_1. Write $|Y_\rho(i)|$ for the distance between ρ and $Y_\rho(i)$.

For $v \in \mathsf{V}_n$ and a measure μ on $\{0,1\}^{\mathsf{E}_n}$, let $C(v)$ denote the (random) connected component containing v. We define $\{Y_{v,\mu}(i)\}_{i=0}^\infty$ to be simple random walk on $C(v)$ started at v [this causes a slight problem if $C(v)$ happens to consist of v only, but we are not interested in this case so we may define it arbitrarily then, e.g. $Y_{v,\mu}(i) = v$ for all i]. Write $|Y_{v,\mu}(i)|$ for the distance between v and $Y_{v,\mu}(i)$.

Most of the work needed in proving Theorem 1.5 is contained in the following two lemmas, the second of which is due to Yuval Peres.

LEMMA 2.2. *Let μ be an automorphism invariant probability measure on $\{0,1\}^{\mathsf{E}_n}$ which assigns positive probability to the existence of infinite clusters of type ∞ without "dead ends" (i.e., in which each edge belongs to some bi-infinite self-avoiding open path). Conditional on the event that a given vertex $v \in \mathsf{V}_n$ is in such an infinite cluster, we a.s. have that*

$$\liminf_{i\to\infty} i^{-1}|Y_{v,\mu}(i)| > 0. \tag{1}$$

LEMMA 2.3. *Suppose we run simple random walk $\{Y_\rho(i)\}_{i=0}^\infty$ on an infinite locally finite tree Γ with root $\rho \in V_\Gamma$. If, for $s \in (0,1)$, the event that*

$$\liminf_{i\to\infty} i^{-1}|Y_\rho(i)| > s$$

has positive probability, then

$$\mathrm{br}(\Gamma) \geq e^{I(s)},$$

where

$$I(s) = \frac{(1+s)\log(1+s) + (1-s)\log(1-s)}{2}.$$

PROOF OF THEOREM 1.5 FROM LEMMAS 2.2 AND 2.3. Pick $X \in \{0,1\}^{\mathsf{E}_n}$ according to μ, and obtain $X' \in \{0,1\}^{\mathsf{E}_n}$ from X by deleting each edge which does not belong to some bi-infinite open path. Clearly, the distribution of X' is automorphism invariant, so by first applying Lemma 2.2 and then Lemma

2.3, we have that each infinite cluster C' of type ∞ in X' a.s. has $\mathrm{br}(C') > 1$. A moment's thought reveals that each infinite cluster of type ∞ in X contains some infinite cluster of type ∞ in X', and since adding branches to a tree obviously cannot decrease its branching number, we a.s. have that each infinite cluster C of X has $\mathrm{br}(C) > 1$. □

PROOF OF LEMMA 2.2. Instead of simple random walk $\{Y_{v,\mu}(i)\}_{i=0}^{\infty}$, we consider the delayed random walk $\{Y^*_{v,\mu}(i)\}_{i=0}^{\infty}$ on $C(v)$ started at v, with transition matrix $(M_{v_1 v_2})_{v_1, v_2 \in V_\Gamma}$ given by

$$M_{v_1 v_2} = \begin{cases} \dfrac{1}{n+1}, & \text{if } v_1 \text{ and } v_2 \text{ are nearest neighbors,} \\ 1 - \dfrac{d(v_1)}{n+1}, & \text{if } v_1 = v_2, \\ 0, & \text{otherwise,} \end{cases}$$

where $d(v_1)$ is the degree of v_1 in $C(v)$. The point of considering $\{Y^*_{v,\mu}(i)\}_{i=0}^{\infty}$ instead of $\{Y_{v,\mu}(i)\}_{i=0}^{\infty}$ is that the former is "stationary" in a sense to be explained. Since $\{Y^*_{v,\mu}(i)\}_{i=0}^{\infty}$ is just a slowed down version of $\{Y_{v,\mu}(i)\}_{i=0}^{\infty}$, it clearly suffices to show

$$\liminf_{i\to\infty} i^{-1}|Y^*_{v,\mu}(i)| > 0$$

in order to show (1). Our approach will in fact even show the stronger result that the limit

$$\lim_{i\to\infty} i^{-1}|Y^*_{v,\mu}(i)| > 0 \tag{2}$$

exists.

Instead of the random walk taking a step from v to w, we can think of the process "as viewed from the walker," thus letting the walker stand still at v and moving the entire configuration X according to some graph automorphism which maps w on v. If we fix such a graph automorphism ψ_w for each nearest neighbor w of v, then this defines a Markov chain $\{X^*_{\mu,v}(i)\}_{i=0}^{\infty}$ with state space $\{0,1\}^{\mathsf{E}_n}$. Now let $\mathscr{F}_v$ be the σ-algebra consisting of all measurable subsets of $\{0,1\}^{\mathsf{E}_n}$ that are invariant under graph automorphisms that fix v; an example of an event in $\mathscr{F}_v$ is the event {there are exactly k open edges incident to v}. Writing μ_i for the distribution of $X^*_{\mu,v}(i)$ (so that in particular $\mu_0 = \mu$), we now claim that

$$\mu_i(F) = \mu_0(F) \tag{3}$$

for all $F \in \mathscr{F}_v$ and all i. This follows using induction and a reversibility argument (see [10] for details).

Now let $Z(i)$ denote the number of open edges incident to v in $X^*_{v,\mu}(i)$ and note that $\{Z(i)\}_{i=0}^{\infty}$ is a stationary sequence due to (3). If $C(v)$ contains no dead ends, then $Z(i) \geq 2$ for all i, and if $C(v)$ is of type ∞, then a.s. $Z(i) \geq 3$ for some i [to see the last statement, note that the event $\{Z(i) = 2$ for all $i\}$

would imply that the walker stays on a single bi-infinite open path, and by the fact that simple random walk on the integers a.s. visits every integer we have that this is only possible if $C(v)$ is of type 2]. Hence

$$\lim_{m\to\infty} m^{-1}\sum_{i=0}^{m-1} 1_{\{Z(i)\geq 3\}} > 0 \quad \text{a.s.} \tag{4}$$

(existence of the limit follows from the pointwise ergodic theorem). Writing E_P for expectation under P, we have

$$E_P\Big[|Y^*_{v,\mu}(j+1)| - |Y^*_{v,\mu}(j)| \,\Big|\, \{X^*_{\mu,v}(i)\}_{i=0}^{j}\Big] = \frac{Z(i)-2}{n+1},$$

with the inconsequential exception of when $Y^*_{v,\mu}(j) = v$, in which case the right-hand side becomes $(Z(i))/(n+1)$. Hence, using (4) and the law of large numbers for bounded martingale increments, we obtain (2), so the proof is complete. □

Proof of Lemma 2.3. As before, write $|v|$ for the distance between $v \in V_\Gamma$ and ρ. For $s \in (0,1)$ and some large integer L, pick a random element $X \in \{0,1\}^{V_\Gamma}$ as follows. First run simple random walk $\{Y_\rho(i)\}_{i=1}^\infty$ on V_Γ, and then, for each $v \in V_\Gamma$, let

$$X(v) = \begin{cases} 1, & \text{if either } |v| < L \text{ or } \inf_i\{Y_\rho(i) = v\} < |v|/s,\\ 0, & \text{otherwise}\end{cases}$$

(the infimum of an empty set is ∞ by convention). This should be thought of as a (highly nonhomogeneous) site percolation process on Γ. Write P_L for the underlying probability measure of this process. The assumption of the lemma that

$$P_L\Big[\liminf_{i\to\infty} i^{-1}|Y_\rho(i)| > s\Big] > 0$$

implies that for L sufficiently large and for some $\varepsilon > 0$ we have

$$P_L[\rho \leftrightarrow \infty] > \varepsilon,$$

where $\rho \leftrightarrow \infty$ denotes the event that there is an infinite self-avoiding path starting at ρ, all of whose vertices v satisfy $X(v) = 1$. Fixing such an L, it follows that

$$P_L(X(v) = 0 \text{ for all } v \in \Pi) < 1 - \varepsilon \tag{5}$$

for all cutsets $\Pi \subset V_\Gamma$. We now claim that

$$P(X(v) = 1) < e^{-|v|I(s)} \tag{6}$$

for all v with $|v|$ sufficiently large. This follows from a standard large deviations result for simple random walk on the integers (see, e.g., [6]) if Γ consists solely of a naked branch from ρ to v, and adding other branches clearly delays (in the sense of stochastic order) the first visit of the random walk at v. By (6) we have, for all cutsets Π with $\min\{|v|: v \in \Pi\}$ sufficiently large, that

$$P_L(X(v) = 0 \text{ for all } v \in \Pi) \geq 1 - \sum_{v \in \Pi} P_L(X(v) = 1)$$
$$> 1 - \sum_{v \in \Pi} e^{-|v| I(s)}.$$

Together with (5), this implies $\sum_{v \in \Pi} e^{-|v| I(s)} > \varepsilon$, so $\mathrm{br}(\Gamma) \geq e^{I(s)}$ by the definition of branching number. □

3. Some examples. In this section, we shall give some examples of automorphism invariant measures with the three different types of infinite clusters that are not ruled out by Theorem 1.2. The simplest example is of course when the edges are i.i.d. The finite energy condition is then satisfied, so when $p \in (1/n, 1)$ we a.s. have infinitely many infinite clusters all of which are of type ∞.

Next we mention the two examples considered in [9].

EXAMPLE 3.1. *The minimal essential spanning forest*. Let μ be the measure on $\{0, 1\}^{\mathsf{E}_n}$ corresponding to picking $X \in \{0, 1\}^{\mathsf{E}_n}$ as follows. First assign i.i.d. random variables $\{Y_e\}_{e \in \mathsf{E}_n}$, uniformly distributed on $[0, 1]$, to the edges of T_n. Then, for each edge $e \in \mathsf{E}_n$ with end vertices $v, v' \in \mathsf{V}_n$, let $X(e) = 0$ if both v and v' are the starting points of some infinite self-avoiding paths $\{e_1, e_2, \ldots\}$ and $\{e'_1, e'_2, \ldots\}$ such that $Y_{e_i}, Y_{e'_i} < Y_e$ for each i, and let $X(e) = 1$ otherwise. (An equivalent formulation is to pick $\{Y_e\}_{e \in \mathsf{E}_n}$ as above and then run "invasion percolation" from each $v \in \mathsf{V}_n$ and let X be given by the union of the invasion clusters; see [9].) It is obvious from the construction that μ is automorphism invariant and furthermore it turns out that μ is concentrated on the event that every connected component is infinite and of type 1.

EXAMPLE 3.2. *The uniform essential spanning forest*. This example is a T_n-analogue of Pemantle's [18] uniform essential spanning forest for Z^d. Fix $v \in \mathsf{V}_n$ and for $i = 1, 2, \ldots$, let Λ_i be the subgraph of T_n consisting of all vertices and edges within distance i from v. Writing E_i for the edge set of Λ_i, we let μ_i be the measure on $\{0, 1\}^{E_i}$ which puts uniform distribution on the set of configurations in which each vertex in the "interior" of Λ_i has a unique open path to the "boundary." This is equivalent to picking a spanning tree uniformly for the graph Λ_i^* obtained by contracting all vertices in the boundary of Λ_i into a single vertex. It turns out that the μ_i's converge to a limiting measure μ on $\{0, 1\}^{\mathsf{E}_n}$, and that this limiting measure is automorphism invariant and assigns probability 1 to the event that all connencted components are infinite and of type 1. One may ask whether this measure is actually the same as the measure in Example 3.1, but the answer is no: the two measures are different.

It remains to give an example of an automorphism invariant measure which yields infinite clusters of type 2. We first need some more terminology. Designate an arbitrary vertex $\rho \in \mathsf{V}_n$ to be the root of T_n. We say that the edges incident to ρ are of first generation, and more generally we say that an edge e belongs to the ith generation if its two endvertices are at distance $i-1$ and i from ρ. For two edges e and e', call e a parent of e' (and e' a child of e) if they share an endvertex and the generation of e' is 1 plus the generation of e.

EXAMPLE 3.3. Let μ be the the distribution of the random element $X \in \{0,1\}^{\mathsf{E}_n}$ obtained as follows. Pick two out of the $n+1$ edges of the first generation at random [i.e., uniformly over all $\binom{n+1}{2}$ possibilities] and declare these to be open and the remaining $n-1$ edges to be closed. Then continue inductively in the following way. If the value $X(e)$ of an edge has been determined, pick $2-X(e)$ of its children at random and let them be open, leaving the other $n-2+X(e)$ children closed. Do this with the obvious independence structure (as in a Galton–Watson process). It is clear that μ will be concentrated on the event that all connected components are infinite and of type 2, and it is easy to check that μ is automorphism invariant.

Now that we have seen that each of the types 1, 2 and ∞ of infinite clusters can occur, it is natural to ask whether they can also coexist. The next example (which unfortunately is a bit involved) shows that, indeed, they can. We will use the words "parent" and "child" about vertices of T_n in the obvious way analogous to the edge terminology.

EXAMPLE 3.4. The construction in this example uses a kind of tree-indexed Markov chain, somewhat along the lines of the construction of automorphism invariant random-cluster measures in [8]. We will assign symbols $\xi(v)$ from the set $\{V_1, V_2, V_3, V_{1a}, V_{1b}, V_{2a}, V_{2b}\}$ to the vertices v of T_n. The value V_1 (resp., V_2) signifies that a vertex is in an infinite cluster of type 1 (resp., 2), while V_3 signifies that it is in either a finite cluster or an infinite cluster of type ∞. Values V_{1a} and V_{1b} carry the same information as V_1, plus some extra information soon to be explained, and similarly for V_{2a} and V_{2b}. Pick a constant $c \in (1/n, 1)$ and assign the root ρ value V_1, V_2 or V_3 according to the probability vector

$$\left(\frac{(1-c)(n+1)}{2(1-c)(n+1)+n-1}, \frac{(1-c)(n+1)}{2(1-c)(n+1)+n-1}, \frac{n-1}{2(1-c)(n+1)+n-1}\right)$$

(these numbers will pop out of a calculation below). If $\xi(\rho) = V_1$, then pick one of the edges from the first generation at random, declare it to be open and assign value V_{1a} to the corresponding child of ρ. Then let the rest of the edges of the first generation, independently of each other, be open with probability $1/n$ and let the children of ρ corresponding to these open edges have value V_{1b}. If $\xi(\rho) = V_2$, then pick two of the edges from the first generation at random, declare them to be open and give the corresponding children of ρ value V_{2b}. If $\xi(\rho) = V_3$, then let each edge of the first generation be independently open with probability c and assign value V_3 to the corresponding children of ρ. In all three cases, declare the rest of the edges of the first generation to be closed and assign independent values equidistributed on $\{V_{1a}, V_{2a}, V_3\}$ to each of the corresponding children of ρ.

Now we continue inductively in the following way, again (as in Example 3.3) with the natural independence structure. If a vertex v ($\neq \rho$) has value V_{1a}, then pick one of its children at random to have value V_{1a} and let the corresponding edge be open. Let each of the remaining edges independently be open with probability $1/n$ and assign value V_{1b} to the corresponding children of v. If $\xi(v) = V_{1b}$, then let each of its children independently get value V_{1b} with probability $1/n$ and open the corresponding edges. If v has value V_{2a} (resp. V_{2b}), then pick two (resp. one) of its children at random to have value V_{2b} and open the corresponding edges. Finally, if $\xi(v) = V_3$, then let each child independently get value V_3 with probability c, letting their respective edges be open. In all five cases, let the remaining edges leading outward from v (as seen from ρ) be closed and assign independent values equidistributed on $\{V_{1a}, V_{2a}, V_3\}$ to each of the corresponding children.

Note that on each connected component, the labels of the vertices are either all in $\{V_1, V_{1a}, V_{1b}\}$ or all in $\{V_2, V_{2a}, V_{2b}\}$ or all V_3. It is easy to verify that the arising connected components a.s. are of the right types (as indicated by their ξ-values). Indeed, each cluster with ξ-values in $\{V_1, V_{1a}, V_{1b}\}$ has a single infinite path with V_{1a}-vertices, while all the V_{1b}-branches are finite because they perform critical Galton–Watson branching. Clusters with ξ-values in $\{V_2, V_{2a}, V_{2b}\}$ are just bi-infinite strings as in Example 3.3, while clusters with ξ-values V_3 perform (due to the choice of c) supercritical Galton–Watson branching, so they must be finite or of type ∞. The interpretation of V_{1a} (resp. V_{1b}) is that the unique open path to infinity lead away from (resp. toward) the root, while the interpretation of V_{2a} (resp. V_{2b}) is that both (resp. only one of the) infinite open paths lead away from the root.

Now let the measure μ on $\{0,1\}^{\mathsf{E}_n}$ be given by the edge-marginal of this process. Obviously, μ assigns probability 1 to the existence of infinite clusters of all three types. We need to show that μ is automorphism invariant.

If we look at the ξ-process along a single self-avoiding path from ρ, then we see a $\{V_{1a}, V_{1b}, V_{2a}, V_{2b}, V_3\}$-valued Markov chain with transition matrix M and stationary distribution π given by

$$M = \begin{bmatrix} \dfrac{1}{n} + \dfrac{(n-1)^2}{3n^2} & \dfrac{n-1}{n^2} & \dfrac{(n-1)^2}{3n^2} & 0 & \dfrac{(n-1)^2}{3n^2} \\ \dfrac{n-1}{3n} & \dfrac{1}{n} & \dfrac{n-1}{3n} & 0 & \dfrac{n-1}{3n} \\ \dfrac{n-2}{3n} & 0 & \dfrac{n-2}{3n} & \dfrac{2}{n} & \dfrac{n-2}{3n} \\ \dfrac{n-1}{3n} & 0 & \dfrac{n-1}{3n} & \dfrac{1}{n} & \dfrac{n-1}{3n} \\ \dfrac{1-c}{3} & 0 & \dfrac{1-c}{3} & 0 & c + \dfrac{1-c}{3} \end{bmatrix}$$

and

$$\pi = \begin{bmatrix} \frac{n(1-c)}{2(1-c)(n+1)+n-1} \\ \frac{1-c}{2(1-c)(n+1)+n-1} \\ \frac{(n-1)(1-c)}{2(1-c)(n+1)+n-1} \\ \frac{2(1-c)}{2(1-c)(n+1)+n-1} \\ \frac{n-1}{2(1-c)(n+1)+n-1} \end{bmatrix}.$$

In order for this to make sense for the first transition of the Markov chain, V_1 and V_2 have to be interpreted as the appropriate convex combinations of V_{1a} and V_{1b} (resp. V_{2a} and V_{2b}) [more precisely, V_1 is V_{1a} with probability $n/(n+1)$ and V_{1b} with probability $1/(n+1)$, while V_2 is V_{2a} with probability $(n-1)/(n+1)$ and V_{2b} with probability $2/(n+1)$]. It is easy to see that the ξ-variables a.s. are identifiable from the edge configuration, whence the Markov chain observed along a single self-avoiding path has to be stationary if we are to have any hope for automorphism invariance of μ. This is where the probability vector for $\xi(\rho)$ comes from.

The symmetries of the construction of the tree-indexed Markov chain make it obvious that μ is invariant under graph automorphisms which map ρ to itself. Since the composition of two graph automorphisms is again a graph automorphism, it suffices to show that μ is invariant under some graph automorphism which maps v on ρ, where v is a nearest neighbor of ρ. The spatial homogeneity (except at ρ) of the tree-indexed Markov chain implies that it suffices to check certain identities for probabilities of events concerning ρ, v and their nearest neighbors. For instance, we need to check that the events $\{\xi(\rho) = V_3 \cap \xi(v) = V_{2a}\}$ and $\{\xi(\rho) = V_2 \cap \xi(v) = V_3\}$ have the same probability and that conditional on $\{\xi(\rho) = V_1 \cap \xi(v) = (V_{1a} \text{ or } V_{1b})\}$ it is equally probable that ρ is on the unique infinite open path starting at v and vice versa. We omit the details.

Further examples can readily be constructed. For instance, to see that infinite clusters of types 1 and 2 can coexist without the presence of infinite clusters of type ∞, one can let μ be the distribution corresponding to picking $X \in \{0,1\}^{\mathsf{E}_n}$ as in Example 3.4 and then deleting each edge which belongs to an infinite cluster of type ∞. To see that infinite clusters of type 2 need not consist solely of naked bi-infinite strings (as in Examples 3.3 and 3.4), one can modify $X \in \{0,1\}^{\mathsf{E}_n}$ chosen as in Example 3.4 by letting each finite cluster which has one or more neighboring infinite cluster of type 2 attach to one of these, chosen uniformly at random. It is clear that both of these last examples are automorphism invariant.

4. The percolation threshold. In this section, we prove Theorem 1.6. Intuitively, the reason why we can find a nontrivial threshold like the one in Theorem 1.6 for T_n but not for Z^d is the following. In order for a connected component to be finite, it has to be surrounded by closed edges. On T_n, this "surface set" of closed edges has to have a cardinality of the same order of magnitude as the "volume set" which it surrounds, while on Z^d the surface/volume ratio can be made arbitrarily small by, for example, letting the volume sets be large enough cubes.

PROOF OF THEOREM 1.6. We begin with the first part of the theorem. For $v \in \mathsf{V}_n$ and a given configuration $\eta \in \{0,1\}^{\mathsf{E}_n}$, write $C(v)$ for the connected component containing v and write $|C(v)|$ for the number of edges in $C(v)$. It is easily seen by induction over $|C(v)|$ that the number of closed edges incident to $C(v)$ has to be $(n-1)|C(v)|+n+1$. Given $\eta \in \{0,1\}^{\mathsf{E}_n}$, define $\psi = \psi_\eta \in \mathsf{R}^{\mathsf{E}_n}$ as follows. For $e \in \mathsf{E}_n$ with end vertices v_1 and v_2, let

$$\psi(e) = \begin{cases} 1, & \text{if } \eta(e)=1 \text{ and } e \text{ is in an infinite cluster,} \\ 0, & \text{if } \eta(e)=1 \text{ and } e \text{ is in a finite cluster,} \\ 1+f(v_1)+f(v_2), & \text{if } \eta(e)=0, \end{cases} \tag{7}$$

where

$$f(v) = f_\eta(v) = \begin{cases} \dfrac{|C(v)|}{(n-1)|C(v)|+n+1}, & \text{if } v \text{ is in a finite cluster,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$0 \le \psi(e) < 1 + \frac{2}{n-1} \tag{8}$$

for all e. We interpret ψ as a distribution of mass over the edges and think of it as being obtained as follows. Originally every edge has mass 1. If $\eta(e)=1$ and e is in an infinite cluster, then nothing happens to the mass at e, while if $\eta(e)$ is in a finite cluster, then e gives away all its mass to the closed edges incident to this cluster, and it gives equal mass $((n-1)|C(v)|+n+1)^{-1}$ to each of these edges. If $\eta(e)=0$, then e just sits there and waits for the mass that other edges may care to give to it. For $e_1, e_2 \in \mathsf{E}_n$, write $\Delta\psi(e_1,e_2)$ for the flow of mass from e_1 to e_2 when ψ is given this interpretation. Clearly,

$$\psi(e) = 1 + \sum_{e' \in \mathsf{E}_n} \Delta\psi(e', e). \tag{9}$$

Moreover, if the random element $X \in \{0,1\}^{\mathsf{E}_n}$ is distributed according to the automorphism invariant measure μ, then automorphism invariance implies that for any $e_1, e_2 \in \mathsf{E}_n$,

$$\mathsf{E}_\mu(\Delta\psi_X(e_1, e_2)) = 0,$$

where E_μ denotes expectation under μ. The sum in (9) is absolutely summable and bounded, whence

$$\begin{aligned} \mathsf{E}_\mu(\psi_X(e)) &= 1 + \mathsf{E}_\mu\biggl(\sum_{e' \in \mathsf{E}_n} \Delta\psi_X(e', e)\biggr) \\ &= 1 + \sum_{e' \in \mathsf{E}_n} \mathsf{E}_\mu(\Delta\psi_X(e', e)) \\ &= 1. \end{aligned}$$

Combining this with (7) and (8), we obtain

$$\mu(X(e) = 1 \text{ and } e \text{ is in an infinite cluster}) > 1 - \left(\frac{n+1}{n-1}\right)\mu(X(e) = 0).$$

The probability that e is in an infinite cluster is thus strictly positive whenever $\mu(X(e) = 0) \le (n-1)/(n+1)$, that is, whenever $\mu(X(e) = 1) \ge 2/(n+1)$, so the first part of the theorem is proved.

To prove the second part of the theorem, we need to find, given any $\varepsilon > 0$, an automorphism invariant measure on $\{0,1\}^{\mathsf{E}_n}$ which has edge density $2/(n+1) - \varepsilon$ and which puts zero probability on the event that infinite clusters exist. Let μ' be the probability measure on $\{0,1\}^{\mathsf{E}_n}$ corresponding to picking $X' \in \{0,1\}^{\mathsf{E}_n}$ in the following way. First, pick $X \in \{0,1\}^{\mathsf{E}_n}$ at random according to the probability measure in Example 3.1 (Examples 3.2 or 3.3 would serve just as well). Then obtain X' from X by independently closing each open edge with probability $((n+1)\varepsilon)/2$. The edge density in Example 3.1 is easily seen to be $2/(n+1)$, so the edge density for μ' is $2/(n+1) - \varepsilon$. Moreover each $v \in \mathsf{V}_n$ a.s. has a unique self-avoiding open path to infinity in X, and this path is a.s. destroyed when going from X to X'. Hence there are a.s. no infinite clusters in X', and since μ' obviously is automorphism invariant, the proof is complete. □

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