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LIAPOUNOV EXPONENTS OF STOCHASTIC FLOWS

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We obtain a formula for Liapounov exponents of stochastic flows generated by stochastic differential equations on compact manifolds. As an application, the exponents of a class of stochastic flows on spheres are determined explicitly.

1. Introduction. Consider an SDE (stochastic differential equation) on a compact d-dimensional manifold M,

$$dx_t = \sum_{i=1}^{k} X_i(x_t) \circ dw_t^i + X_0(x_t)\, dt, \tag{1}$$

where $X_0, X_1, \dots, X_k$ are smooth vector fields on M, $w_t = (w_t^1, \dots, w_t^k)$ is Brownian motion on R^k and $\circ\, d$ denotes the Stratonovich stochastic differential. We will use P to denote the probability measure associated with Brownian motion w_t.

Let ϕ_t denote the stochastic flow generated by (1). The reader is referred to Arnold [1] or Elworthy [7] for the general theory of such stochastic flows. We will assume that the SDE (1) is nondegenerate enough so that it has a unique stationary measure ρ on M. This assumption is satisfied if the Lie algebra generated by $X_0, X_1, \dots, X_k$ spans the tangent space at every point of M. Recall that a stationary measure ρ is a probability measure on M satisfying $\rho(dx) = \int_M \rho(dy) P(\phi_t(y) \in dx)$ for any $t \geq 0$.

Equip M with a Riemannian metric. Let $D\phi_t$ denote the differential map of ϕ_t. By a version of Oseledec's multiplicative ergodic theorem (see [5]), for $\rho \times P$-almost all (x, ω), there is a filtration of the tangent space $T_x M$: $T_x M = V_1 \supset V_2(\omega) \supset \cdots \supset V_r(\omega) \supset V_{r+1} = \{0\}$, where $V_i(\omega)$ are subspaces, such that $\forall\ v \in V_i - V_{i+1}$, $1 \leq i \leq r$, the limit $\mu_i = \lim_{t \to \infty} (1/t) \log \| D\phi_t(v) \|$ exists, where the norm $\|\cdot\|$ is given by the Riemannian metric. We may assume $\mu_1 > \mu_2 > \cdots > \mu_r$. These numbers are called the Liapounov exponents of the stochastic flow ϕ_t, which are nonrandom and independent of the Riemannian metric on M. The number $d_i = \dim(V_i) - \dim(V_{i+1})$ is called the multiplicity of the exponent μ_i. Sometimes it is convenient to list the Liapounov exponents as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, where an exponent of multiplicity k repeats k times in this list.

Extending a formula of Khas'minskii for linear stochastic differential equations, Carverhill [6] obtained a formula for the top Liapounov exponent

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μ_1. This formula is extended by Baxendale [4] to express the sums of the exponents. More recently, Arnold and Imkeller [2] derived a formula for the complete set of Liapounov exponents for linear SDE's using anticipative calculus. In this paper, we will obtain a formula for all the Liapounov exponents of SDE's on compact manifolds based on a different idea. Our method uses results from linear algebra and Itô's formula, and seems simpler than the approach in [2]. Arnold and Imkeller also obtained the Oseledec spaces, which are not discussed here.

As an application, we will consider a class of stochastic flows on the $(n-1)$-dimensional sphere S^{n-1} constructed as follows. Think of S^{n-1} as the unit sphere embedded in R^n. Let $x_1, x_2, \dots, x_n$ be Cartesian coordinates on R^n, let Y_j be the vector field on S^{n-1} obtained by orthogonal projection of the coordinate vector field $\partial/\partial x_j$ on R^n and let $Y_{j_1 j_2 \cdots j_m} = x_{j_1} x_{j_2} \cdots x_{j_{m-1}} Y_{j_m}$. Consider an SDE on S^{n-1} of the following form:

$$dx_t = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} Y_{j_1 \cdots j_m}(x_t) \circ dw_t^{j_1 \cdots j_m}, \tag{2}$$

where $w_t = (w_t^{j_1 \cdots j_m}\}$ is an n^m-dimensional Brownian motion. We will see that the Liapounov exponents of the stochastic flow ϕ_t generated by (2) can be expressed in terms of n and m in surprisingly simple formulas. The one-point motion of ϕ_t is Brownian motion on S^{n-1}. We will also see that ϕ_t becomes unstable in the sense that its top exponent is positive when n and m are sufficiently large.

We note that, when $m = 1$ or 2, ϕ_t is finite dimensional in the sense that it is contained in a finite-dimensional transformation group on S^{n-1} (one can see this by checking the dimension of the Lie algebra generated by the vector fields involved). Its Liapounov exponents can be determined using the group structure, as in [3] and [9]. However, for $m \geq 3$, ϕ_t is infinite dimensional.

2. A matrix-valued process. A frame $u = (u_1, u_2, \dots, u_d)$ at $x \in M$ is an ordered set of d linearly independent vectors in $T_x M$, which can be identified with the linear map: $R^d \to T_x M$ by sending $\xi \in R^d$ into $u\xi = \sum_j u_j \xi_j$. Let $O(M)$ be the bundle of orthonormal frames on M and let $\pi\colon O(M) \to M$ be the natural projection.

Let $G = GL(d, R)_+$ be the group of $d \times d$ real matrices of positive determinant and let S be the subgroup of upper triangular matrices with positive diagonal elements. Fix $x \in M$ and $u \in O(M)$ with $\pi(u) = x$. The frame $D\phi_t(u) = (D\phi_t(u_1), \dots, D\phi_t(u_d))$ at $x_t = \phi_t(x)$ in general is not orthonormal, but, by performing a Gram–Schmidt orthogonalization procedure to the ordered set of vectors in $D\phi_t(u)$, we obtain an $O(M)$-valued process u_t with $u_0 = u$ such that

$$D\phi_t(u) = u_t s_t, \tag{3}$$

where s_t is an S-valued process with $s_0 = I_d$, the $d \times d$ identity matrix. We note that u_t is a diffusion process on $O(M)$.

Let $K = SO(d)$, the group of $d \times d$ orthogonal matrices of determinant 1. Any $g \in G$ has a Cartan decomposition $g = ha^+k$, where $h, k \in K$ and a^+ is a diagonal matrix with positive and descending diagonal elements. Although the choices for h and k are not quite unique, a^+ is uniquely determined by g. Let $s_t = h_t a_t^+ k_t$ be the Cartan decomposition of the process s_t. The diagonal elements $\alpha_t^1 \geq \alpha_t^2 \geq \cdots \geq \alpha_t^d$ of a_t^+ are the factors by which the length of a tangent vector is changed under $D\phi_t$. It is natural to expect that the Liapounov exponents λ_j are the limiting exponential rates of α_t^j as $t \to \infty$ in the sense that $\lambda_j = \lim_{t\to\infty}(1/t)\log \alpha_t^j$. This fact, whose proof is nontrivial can be considered as a part of Oseledec's multiplicative theorem; see [10] (the condition there is verified in [5]). Let

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \log a_t^+. \tag{4}$$

Then $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$.

We will let A be the subgroup of diagonal matrices with positive diagonal elements and let N be the subgroup of upper triangular matrices with all the diagonal elements equal to 1. Then $S = AN$. We have the Iwasawa decomposition $G = KAN$ in the sense that any $g \in G$ can be uniquely decomposed as $g = zan$ with $z \in K$, $a \in A$ and $n \in N$. Since s_t is in S and $S = AN$, its Iwasawa decomposition $s_t = a_t n_t$ does not have a K-component. We note that s_t depends on the Brownian path w and the initial frame u. We may write $s_t(u, w)$ to indicate this dependence.

Because $O(M)$ is compact, the diffusion process u_t has at least a stationary measure. We will assume that it has a smooth stationary measure $\bar{\rho}$ on $O(M)$. This means that $\bar{\rho}$ has a smooth density under local coordinates on $O(M)$. By the following lemma, it is easy to see that, for $\bar{\rho} \times P$-almost all (u, w),

$$\lim_{t\to\infty} \frac{1}{t} \log a_t(u, w) = \lim_{t\to\infty} \frac{1}{t} \log a_t^+(u, w). \tag{5}$$

LEMMA 1. *There is a subset I of K of measure 0 such that if g_j is a sequence in G with Cartan decomposition $g_j = h_j a_j^+ k_j$ and Iwasawa decomposition $g_j = z_j a_j n_j$ satisfying*:

(i) $\lim_{j\to\infty}(1/j)\log a_j^+$ *exists, and*
(ii) *the sequence k_j has a limiting point not contained in I,*

then $\lim_{j\to\infty}(1/j)\log a_j^+ = \lim_{j\to\infty}(1/j)\log a_j$.

The lemma is a consequence of a more general result for semisimple Lie groups; see Corollary (2.4) in [8]. Here we present an elementary proof. By assumption, the diagonal elements of a_j^+ have limiting exponential rates $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ as $i \to \infty$ in the sense that $a_j^+ = \operatorname{diag}\{\exp(\mu_{j1}), \exp(\mu_{j2}), \ldots, \exp(\mu_{jd})\}$ with $\lim_{j\to\infty} \mu_{ji}/j = \lambda_i$. Without loss of generality, we may assume $k_\infty = \lim_{j\to\infty} k_j$ exists. Let $b_j = a_j^+ k_j$. By performing a

Gram–Schmidt orthogonalization procedure on the column vectors of the matrix b_j, one sees that there is an upper triangular matrix t_j such that $b_j t_j \in K$. Hence $a_j n_j = t_j^{-1}$. It suffices to show that the diagonal elements of t_j have limiting exponential rates $-\lambda_1, \dots, -\lambda_d$, unless k_∞ is contained in some subset of K of measure 0. To prove this, we will construct t_j as a product of upper triangular matrices following the Gram–Schmidt orthogonalization procedure.

Note that the element of b_j at place (p, q) is $(b_j)_{pq} = \exp(\mu_{jp})(k_j)_{pq}$, which has a limiting exponential rate λ_p unless $(k_\infty)_{pq} = 0$. If k_∞ is not contained in $\{k \in K;\ k_{11} = 0\}$, a subset of K of measure 0, then, by performing a column reduction on b_j, we may reduce its first column to a vector of unit length and change the rest of the first row to 0. This can be achieved by multiplying b_j on the right by the following upper triangular matrix $t_j^{(1)}$. Let c_{1j} be the norm of the first column vector of b_j, let $\{c_{1j}^{-1}, -(k_j)_{12}/(k_j)_{11}, \dots, -(k_j)_{1d}/(k_j)_{11}\}$ be the first row of $t_j^{(1)}$, let $\{c_{1j}^{-1}, 1, \dots, 1\}$ be the diagonal of $t_j^{(1)}$ and let all the other elements of $t_j^{(1)}$ be 0. We note that c_{1j} has a limiting exponential rate λ_1.

For $p, q \geq 2$,

$$\left(b_j t_j^{(1)}\right)_{pq} = -\exp(\mu_{jp})\left[(k_j)_{p1}(k_j)_{1q}/(k_j)_{11}\right] + \exp(\mu_{jp})(k_j)_{pq},$$

which has a limiting exponential rate λ_p unless $k_\infty \in I_{pq} = \{k \in K;\ -(k_{p1}k_{1q}/k_{11}) + k_{pq} = 0\}$, a subset of K of measure 0.

If k_∞ is not contained in I_{22}, we can now perform column reduction on the matrix $b_j t_j^{(1)}$ to get a new matrix whose first column is the same as that of $b_j t_j^{(1)}$, whose second column is orthogonal to the first one and has unit length and whose elements in the second row to the right of the diagonal are all 0. This amounts to multiplying the matrix $b_j t_j^{(1)}$ by an upper triangular matrix $t_j^{(2)}$ whose diagonal is $\mathrm{diag}\{1, c_{2j}^{-1}, 1, \dots, 1\}$, where c_{2j} has a limiting exponential rate λ_2 and whose elements off the diagonal, second row and second column are 0. Unless k_∞ belongs to a set of measure 0, $(b_j t_j^{(1)} t_j^{(2)})_{pq}$ has a limiting exponential rate λ_p for $\alpha, \beta \geq 3$. We can continue in this way to obtain upper triangular matrices $t_j^{(3)}, \dots, t_j^{(d)}$ such that $b_j t_j^{(1)} \cdots t_j^{(d)} \in K$ and $t_j = t_j^{(1)} \cdots t_j^{(d)}$ has the desired property.

We may take the subset I of K in Lemma 1 as the union of all the subsets of K of measure 0 appearing in the above construction. Note that I is independent of the sequence g_j.

3. A general formula. As before, let ϕ_t be the stochastic flow on M generated by the SDE (1). For a tangent vector e at $x \in M$, $D\phi_t(e)$ is a diffusion process in the tangent bundle TM and satisfies the following SDE:

$$d\,D\phi_t(e) = \sum_{i=1}^{k} \delta X_i\left(D\phi_t(e)\right) \circ dw_t^i + \delta X_0\left(D\phi_t(e)\right) dt, \tag{6}$$

where δX is the natural lift to TM of the vector field X on M. We note that, for $e \in TM$, $\delta X(e) \in T_e TM$ is the tangent vector to the curve $s \mapsto D\psi_s(e)$ at $s = 0$, where ψ_s is the flow of the vector field X on M.

A tangent vector $V \in T_e TM$ is called vertical if it is tangent to the curve $s \mapsto e + sY$ in $T_x M \subset TM$ at $s = 0$ for some $Y \in T_x M$. For simplicity, we may identify $Y \in T_x M$ with $V \in T_e TM$ as above. Using the Riemannian connection on M, any tangent vector W on TM can be uniquely decomposed as $W = W^h + W^v$, where W^h is horizontal and W^v is vertical. We can show that $(\delta X)^v(e) = \nabla_e X = \nabla X(e)$ for any vector field X on M, where the covariant differentiation ∇ is given by the Riemannian connection.

Let $u = (u_1, \dots, u_d)$ be an orthonormal frame at $x \in M$. We may write $(\delta X)^v(u) = \nabla X(u) = (\nabla X(u_1), \dots, \nabla X(u_d))$. Recall $D\phi_t(u) = u_t s_t$. We have $d\, D\phi_t(u) = (\circ\, du_t)s_t + u_t \circ ds_t$. We may substitute u for e in (6) to get the following SDE on $L(M)$, the bundle of linear frames on M:

$$du_t + u_t(\circ\, ds_t)s_t^{-1} = \sum_i \delta X_i(u_t) \circ dw_t^i + \delta X_0(u_t)\, dt. \tag{7}$$

For $u \in O(M)$ and a vector field X on M, let $\tilde{X}(u)$ be the $d \times d$ matrix defined by

$$\tilde{X}(u)_{\alpha\beta} = \langle u_\alpha, \nabla X(u_\beta) \rangle, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian inner product on $T_{\pi(u)}M$. We may write $\tilde{X}(u) = u^{-1}\nabla X(u)$ in the sense that $\forall\, \xi \in R^d$, $\tilde{X}(u)\xi = u^{-1}\nabla X(u\xi)$. We note that, for $k \in K$, $\tilde{X}(uk) = k^{-1}\tilde{X}(u)k$.

Let H: $TM \to TO(M)$ be the horizontal lift. We note that, for $u \in O(M)$ with $\pi(u) = x$ and $Y \in T_x M$, $H(Y)(u)$ is the tangent vector to the curve $s \mapsto u(s)$ in $O(M)$ at $s = 0$, where $u(s)$ is the parallel displacement of u along a curve z_s in M with $(d/ds)z_s|_{s=0} = Y$.

Let G, K, A, N and S be respectively the Lie algebras of G, K, A, N and S. We have $G = K \oplus S = K \oplus A \oplus N$. For $Y \in G$, let $Y = Y_K + Y_S = Y_K + Y_A + Y_N$ be such decompositions. From (7), by separating the components in the tangent spaces of $L(M)$ and using $\delta X^v(u) = \nabla X(u) = u\tilde{X}(u)$ and $\delta X^h(u) = H(X(\pi(u)))(u)$, we may write down the SDE's satisfied by u_t and s_t as follows. Recall that $x_t = \phi_t(x)$. Then

$$\begin{aligned} du_t = {} & \sum_{i=1}^{k} \Big\{ H(X_i(x_t))(u_t) + u_t\big[\tilde{X}_i(u_t)\big]_K \Big\} \circ dw_t^i \\ & + \Big\{ H(X_0(x_t))(u_t) + u_t\big[\tilde{X}_0(u_t)\big]_K \Big\}\, dt, \end{aligned} \tag{9}$$

$$ds_t = \sum_{i=1}^{k} \big[\tilde{X}_i(u_t)\big]_S\, s_t \circ dw_t^i + \big[\tilde{X}_0(u_t)\big]_S\, s_t\, dt. \tag{10}$$

Since $s_t = a_t n_t$, we have $ds_t = (\circ\, da_t)n_t + a_t \circ dn_t$ and $(\circ\, ds_t)s_t^{-1} = (\circ\, da_t)a_t^{-1} + \mathrm{Ad}(a_t)[(\circ\, dn_t)n_t^{-1}]$, where $\mathrm{Ad}(g)Y = gYg^{-1}$ for $g \in G$ and $Y \in G$. Since $\mathrm{Ad}(a)Y \in N$ for $a \in A$ and $Y \in N$, it follows from (10) and the uniqueness of the Iwasawa decomposition of $(\circ\, ds_t)s_t^{-1}$ that

$$dn_t = \sum_{i=1}^{k} \{\mathrm{Ad}(a_t^{-1})[\tilde{X}_i(u_t)]_N\} n_t \circ dw_t^i + \{\mathrm{Ad}(a_t^{-1})[\tilde{X}_0(u_t)]_N\} n_t\, dt, \tag{11}$$

$$da_t = \sum_{i=1}^{k} [\tilde{X}_i(u_t)]_A\, a_t \circ dw_t^i + [\tilde{X}_0(u_t)]_A\, a_t\, dt \tag{12}$$

and

$$d \log a_t = \sum_{i=1}^{k} \left[\tilde{X}_i(u_t)\right]_A \circ dw_t^i + \left[\tilde{X}_0(u_t)\right]_A dt. \tag{13}$$

Given a vector field X on M, let $Z = [\tilde{X}(u)]_K$. We may regard uZ as the tangent vector to the curve $s \mapsto ue^{sZ}$ in $O(M)$ at $s = 0$. As a vector field on $O(M)$, uZ acts on $\tilde{X}(u)$ as follows:

$$(uZ)\tilde{X}(u) = \frac{d}{ds}\tilde{X}(ue^{sZ})\bigg|_{s=0} = \frac{d}{ds} e^{-sZ}\tilde{X}(u)e^{sZ}\bigg|_{s=0} = -\left[Z, \tilde{X}(u)\right],$$

where $[X, Y] = XY - YX$ for $X, Y \in G$. It follows from (9) and (13) that

$$\log a_t = m_t + \int_0^t \bigg\{ \tfrac{1}{2} \sum_i H(X_i(x_s))\tilde{X}_i(u_s) - \tfrac{1}{2}\sum_i \Big[\big[\tilde{X}_i(u_s)\big]_K, \tilde{X}_i(u_s)\Big] + \tilde{X}_0(u_s) \bigg\}_A ds,$$

where $m_t = \int_0^t \sum_i [\tilde{X}_i(u_s)]_A\, dw_s^i$ (the Itô integral). Recall that Λ defined by (4) is a diagonal matrix with the Liapounov exponents arranged in descending order along the diagonal. By (5) and the ergodic theory, we have proved the following result.

THEOREM 1. *Let $\bar{\rho}$ be a smooth stationary measure of u_t. Then*

$$\Lambda = \int_{O(M)} \bigg\{ \tfrac{1}{2}\sum_{i=1}^{k} H(X_i \circ \pi)\tilde{X}_i - \tfrac{1}{2}\sum_{i=1}^{k} \Big[\big[\tilde{X}_i\big]_K, \tilde{X}_i\Big] + \tilde{X}_0 \bigg\}_A \bar{\rho}(du). \tag{14}$$

REMARK 1. A Liapounov exponent is called simple if its multiplicity is equal to 1. If all the exponents are simple, then Λ has strictly descending diagonal elements. In this case, we can show that, for $\bar{\rho} \times P$-almost all (u, w), $\lim_{t\to\infty} n_t(u, w)$ exists. To show this, we need only to observe that in (11), $\mathrm{Ad}(a_t^{-1})[\tilde{X}_i(u_t)]_N$ has a negative limiting exponential rate as $t \to \infty$. Hence an argument similar to the one used in [11] shows the convergence of n_t. Moreover, with a proper choice of the component k_t in the Cartan decomposition $s_t = h_t a_t^+ k_t$, $\lim_{t\to\infty} k_t(u, w)$ exists for $\bar{\rho} \times P$-almost all (u, w). This follows from the convergence of n_t and Corollary 2 in [8].

4. A special case. We may write $\bar{\rho}(du) = \int_M \bar{\rho}_x(du)\rho(dx)$, where $\bar{\rho}_x$ is a smooth probability measure on the fiber $\pi^{-1}(x)$. Given an orthonormal frame u at x, this fiber can be identified with $K = SO(d)$ via the map $k \mapsto uk$. A measure on $\pi^{-1}(x)$ will be called a Haar measure if it is a Haar measure on K via the above identification. We will now simplify the formula (14) under the assumption that $\bar{\rho}_x$ is the normalized Haar measure for any $x \in M$.

We define an inner product on G by $\langle X, Y \rangle = \mathrm{Trace}(XY^*)$ for $X, Y \in G$, where Y^* is the transpose of the matrix Y. This is just the standard Euclidean inner product when X and Y are viewed as d^2-dimensional vectors. Let $\|X\| = \sqrt{\langle X, X \rangle}$ be the norm of X. Let P be the space of symmetric matrices of trace 0 and let R be the one-dimensional space spanned by the identity matrix I_n. Then $G = R \oplus K \oplus P$. For $Y \in G$, we will use Y_P to denote its P-component under this decomposition. We note that Y_K has been defined to be the K-component of Y under the Iwasawa decomposition, which is, in general, different from the K-component of Y under the above decomposition. Let H_d be the diagonal matrix defined by

$$H_d = \mathrm{diag}\{d-1, d-3, \dots, -(d-3), -(d-1)\}, \tag{15}$$

let dk be the normalized Haar measure on K and let $p = \dim(P)$.

LEMMA 2. *For any $X \in G$,*

$$\int_K [[\mathrm{Ad}(k)X]_K, \mathrm{Ad}(k)X]_A \, dk = -\frac{1}{p}\|X_P\|^2 H_d.$$

A more general result is proved in Section 4 of [9]. We provide a proof of Lemma 2 here for the reader's convenience.

We may assume that the R-component of X is 0 in the decomposition $G = F \oplus K \oplus P$ because it will not affect the value of $[[\mathrm{Ad}(k)X]_K, \mathrm{Ad}(k)X]_A$. There are an orthonormal basis $\{Z_1, Z_2, \dots, Z_q\}$ of K and an orthonormal basis $\{Y_1, Y_2, \dots, Y_q, Y_{q+1}, \dots, Y_p\}$ of P such that $\{Y_{q+1}, \dots, Y_p\}$ forms a basis of A, $[Y_j]_K = -Z_j$ for $1 \le j \le q$ and $[Z_i, Y_j]_A = \delta_{ij} H_i'$ for $1 \le i, j \le q$, where $H_i' \in A$ with $H_d = \sum_{i=1}^q H_i'$. To see the existence of such bases, let Y_{ij} be the symmetric matrix which has 1 at places (i,j) and (j,i), and 0 elsewhere; let Z_{ij} be the skew-symmetric matrix which has 1 at (i,j), -1 at (j,i) and 0 elsewhere; and let H_{ij} be the diagonal matrix which as 1 at (i,i), -1 at (j,j) and 0 elsewhere. Then we may let $\{Z_1, \dots, Z_q\}$ be $\{Z_{ij}/\sqrt{2};\ i<j\}$, let $\{Y_1, \dots, Y_q\}$ be $\{Y_{ij}/\sqrt{2};\ i<j\}$ and let $\{Y_{q+1}, \dots, Y_p\}$ be $\{H_{12}/\sqrt{2},\ H_{13}/\sqrt{2}, \dots, H_{1d}/\sqrt{2}\}$. In this case, $\{H_i';\ 1 \le i \le q\}$ is given by $\{H_{ij};\ i<j\}$. Because $[H_{ij}, Z_{ij}] = 2Y_{ij}$, if we set $Z_j = H_j' = 0$ for $q+1 \le j \le p$, the relations $[Y_j]_K = -Z_j$ and $[Z_i\ Y_j]_A = \delta_{ij} H_i'$ hold for all $1 \le i, j \le p$.

Let $X = Y + Z$ with $Y \in P$ and $Z \in K$. We now show that

$$[[\mathrm{Ad}(k)X]_K, \mathrm{Ad}(k)X]_A = [\mathrm{Ad}(k)[Z, Y]]_A - \sum_{j=1}^q \left\langle \mathrm{Ad}(k)Y, Y_j \right\rangle^2 H_j'. \tag{16}$$

Since $[K, K] \subset K$,

$$\begin{aligned}
&[[\mathrm{Ad}(k)X]_K, \mathrm{Ad}(k)X]_A \\
&\quad = [[\mathrm{Ad}(k)X]_K, \mathrm{Ad}(k)Y]_A \\
&\quad = [\mathrm{Ad}(k)Z, \mathrm{Ad}(k)Y]_A + [[\mathrm{Ad}(k)Y]_K, \mathrm{Ad}(k)Y]_A \\
&\quad = [\mathrm{Ad}(k)[Z, Y]]_A + [[\mathrm{Ad}(k)Y]_K, \mathrm{Ad}(k)Y]_A.
\end{aligned} \tag{17}$$

Define $a_{ij} = a_{ij}(k)$ by $\mathrm{Ad}(k)Y_i = \sum_j a_{ij} Y_j$. Let $Y = \sum_j y_j Y_j$. We have

$$\begin{aligned}
[[\mathrm{Ad}(k)Y]_K, \mathrm{Ad}(k)Y]_A &= -\sum_{i,j,v,w} y_v y_w a_{vi} a_{wj} [Z_i, Y_j]_A \\
&= -\sum_{v,w,j} y_v y_w a_{vj} a_{wj} H_j'.
\end{aligned}$$

Since

$$\langle \mathrm{Ad}(k)Y, Y_j \rangle = \sum_{v,i} y_v a_{vi} \langle Y_i, Y_j \rangle = \sum_v y_v a_{vj},$$

we see that

$$[[\mathrm{Ad}(k)Y]_K, \mathrm{Ad}(k)Y]_A = -\sum_j \langle \mathrm{Ad}(k)Y, Y_j \rangle^2 H_j'.$$

This combined with (17) implies (16).

We note that, for any $Y \in P$, $\int_K \mathrm{Ad}(k)Y\,dk = 0$. To show this, we may assume Y is a traceless diagonal matrix because $\mathrm{Ad}(k)Y = kYk^{-1}$ is such a matrix for some $k \in K$, and the Haar measure dk is translation invariant. Let $Y = \mathrm{diag}\{y_1, y_2, \dots, y_d\}$. We have

$$\int_K \mathrm{Ad}(k)Y\,dk = \int_K kYk^{-1}\,dk = \sum_{i,\alpha,\beta} y_i \int k_{\alpha i} k_{\beta i}\,dk = \frac{1}{d}\sum_i y_i = 0,$$

because $\int_K k_{\alpha i} k_{\beta i}\,dk = (1/d)\delta_{\alpha\beta}$. This proves our claim.

Now Lemma 2 follows from (16) and Lemma 3 below. Before stating Lemma 3, we note that, for any nonzero $Y \in P$, the linear span of $O_Y = \{\mathrm{Ad}(k)Y;\ k \in K\}$ is P. To see this, we observe that O_Y is the set of symmetric matrices with the same eigenvalues as those of Y. We may assume $Y \in A$. It suffices to show that, for any nonzero $Y \in A$, the set of diagonal matrices obtained by permuting the diagonal entries of Y spans A. This reduces our claim to the elementary fact that, given any nonzero vector lying in the hyperplane $x_1 + x_2 + \cdots + x_d = 0$, by permuting its components, we will get enough vectors to span the whole hyperplane.

LEMMA 3. *For any* $Y, Y' \in P$ *with* $\|Y\| = \|Y'\| = 1$, $\int_K \langle \mathrm{Ad}(k)Y, Y' \rangle^2\,dk = 1/p$, *where* $p = \dim(P)$.

To prove Lemma 3, let S be the unit sphere in P and fix $Y \in S$. Consider the function $\psi(W) = \int_K \langle \mathrm{Ad}(k)Y, W \rangle^2\,dk$ defined on S. We will show that ψ is a constant on S. If not, let a and b be respectively its minimal and maximal values. Choose W_1 such that $\psi(W_1) = a$. Let S_1 be the intersection of S and the orthogonal complement of W_1 in P. Any $W \in S$ can be expressed as $W = xW_1 + yW_2$ for some $W_2 \in S_1$ and $x^2 + y^2 = 1$. We have

$$0 = \frac{d}{dt}\psi\left(\sqrt{1-t^2}\,W_1 + tW_2\right)\Big|_{t=0}$$

$$= 2\int_K \langle \mathrm{Ad}(k)Y, W_1\rangle\langle \mathrm{Ad}(k)Y, W_2\rangle\, dk.$$

It follows that

$$\psi(xW_1 + yW_2) = x^2\psi(W_1) + y^2\psi(W_2).$$

This is less than the maximal value b if $x \neq 0$. Therefore, b can only be obtained on S_1. If $\psi(W_2) = b$, then, by the invariance of the Haar measure dk, $\psi = b$ along the orbit of W_2 under $\mathrm{Ad}(K)$. Hence this orbit is orthogonal to W_1. Since the linear span of $\{\mathrm{Ad}(k)W_2;\ k \in K\}$ is equal to P, this is impossible. Therefore, ψ must be a constant.

Let $W_1, \dots, W_p$ be an orthonormal basis of P with respect to the inner product $\langle\cdot,\cdot\rangle$. Since $\mathrm{Ad}(k)$ is an isometry on P, $\{\langle \mathrm{Ad}(k)W_i, W_j\rangle\}$ is an orthogonal matrix. We have

$$p\psi = \sum_i \int_K \langle \mathrm{Ad}(k)W_i, W_1\rangle^2\, dk$$

$$= \int_K \sum_i \langle \mathrm{Ad}(k)W_i, W_1\rangle^2\, dk = 1.$$

This proves Lemma 3 and hence also Lemma 2.

Let X be a vector field on M. Fix $u_0 \in \pi^{-1}(x)$. Then, for any $u \in \pi^{-1}(x)$, $u = u_0 k$ for some $k \in K$. By Lemma 2, we have

$$\begin{aligned}
&\int_{\pi^{-1}(x)} \left[\left[\tilde{X}(u)\right]_K, \tilde{X}(u)\right]_A \bar{\rho}_x(du)\\
&= \int_K \left[\left[\mathrm{Ad}(k^{-1})\tilde{X}(u_0)\right]_K, \mathrm{Ad}(k^{-1})\tilde{X}(u_0)\right]_A dk \qquad (18)\\
&= -\frac{1}{p}\left\|\left[\tilde{X}(u_0)\right]_P\right\|^2 H_d.
\end{aligned}$$

In general, $TO(M)$ does not have a smooth section. However, it is possible to have a smooth section defined on an open subset of $O(M)$ whose complement has zero measure. Let $x \mapsto u(x)$ be such a smooth section. Given a vector field X on M, let z_s be the solution of the differential equation $(d/ds)z_s = X(z_s)$ with $z_0 = x$ and let $u(x, s)$ be the parallel displacement of $u(x)$ along the curve $s \mapsto z_s$. We have

$$\begin{aligned}
H(X(x))\tilde{X}(u(x)k) &= \frac{d}{ds}\tilde{X}(u(x,s)k)\Big|_{s=0}\\
&= \frac{d}{ds}\tilde{X}(u(z_s)k_s k)\Big|_{s=0}
\end{aligned}$$

for some $k_s \in K$. Hence

$$\int H(X(x))\,\tilde{X}(u)\,\bar{\rho}_x(du) = \frac{d}{ds}\int_K \tilde{X}(u(z_s)\,k_s k)\,dk\Big|_{s=0}$$
$$= \frac{d}{ds}\int_K \tilde{X}(u(z_s)\,k)\,dk\Big|_{s=0}$$
$$= \int_K X\left[\tilde{X}(u(x)\,k)\right]dk.$$

Note that in the above we have used the left invariance of the Haar measure dk. Since $\tilde{X}(uk) = k^{-1}\tilde{X}(u)k$ and $\int_K k_{\beta\alpha}k_{\gamma\alpha}\,dk = (1/d)\delta_{\gamma\beta}$, we have

$$\int_K X\left[\tilde{X}(u(x)\,k)\right]_{\alpha\alpha} dk = \int_K \sum_{\beta,\gamma} X\left[\tilde{X}(u(x))\right]_{\beta\gamma} k_{\beta\alpha}k_{\gamma\alpha}\,dk$$
$$= \frac{1}{d}\sum_{\beta} X\left[\tilde{X}(u(x))\right]_{\beta\beta}.$$

Note that since $\sum_\beta[\tilde{X}(u(x))]_{\beta\beta} = \operatorname{div} X(x)$, we have

$$\int_{O(M)}\left[H(X(\pi(u)))\,\tilde{X}(u)\right]_{\mathbf{A}}\bar{\rho}(du) = \left\{\frac{1}{d}\int_M X(\operatorname{div} X)(x)\,\rho(dx)\right\}I_d. \tag{19}$$

Because

$$\int\left[\tilde{X}_0\right]_{\mathbf{A}}\bar{\rho}_x(du) = \int_K\left[\operatorname{Ad}(k^{-1})\,\tilde{X}_0(u_0)\right]_{\mathbf{A}} dk$$
$$= \int_K\left[\operatorname{Ad}(k^{-1})\left[\tilde{X}_0(u_0)\right]_{\mathbf{P}}\right]_{\mathbf{A}} dk = 0,$$

by Theorem 1, (18) and (19), we obtain the following result.

THEOREM 2. *Under the hypothesis of Theorem* 1, *assume* $\bar{\rho}(du) = \int_M \bar{\rho}_x(du)\rho(dx)$, *where* ρ *is a stationary measure of SDE* (1) *and* $\bar{\rho}_x$ *is the normalized Haar measure on* $\pi^{-1}(x)$ *for all* $x \in M$. *Then*

$$\Lambda = C_1 I_d + C_2 H_d, \tag{20}$$

where

$$C_1 = (1/2d)\sum_{i=1}^{k}\int_M X_i(\operatorname{div} X_i)(x)\,\rho(dx),$$
$$C_2 = (1/2p)\sum_{i=1}^{k}\int_M \left\|\left[\tilde{X}_i(u(x))\right]_{\mathbf{P}}\right\|^2\rho(dx),$$

$p = \dim(\mathbf{P})$ *and* $u(x)$ *is a smooth section on the bundle* $O(M)$ *defined on an open subset of* M *whose complement has zero measure. We note that we may choose a different section* $u(x)$ *in computing each integral in* C_2.

5. Stochastic flows on spheres. We now consider the stochastic flow ϕ_t on the $(n-1)$-dimensional sphere S^{n-1} generated by the SDE (2) in Section

1, which involves the vector fields $Y_{j_1 \cdots j_m}$. We will assume $n \geq 3$. Let u_t be the process on $O(S^{n-1})$ defined in Section 1. As before, let $\pi: O(S^{n-1}) \to S^{n-1}$ be the natural projection. For $x \in S^{n-1}$, let $\bar{\rho}_x$ be the normalized Haar measure on the fiber $\pi^{-1}(x) \equiv SO(n-1)$ and define $\bar{\rho}(du) = \int_{S^{n-1}} \bar{\rho}_x(du)\rho(dx)$, where ρ is the uniform distribution on S^{n-1}.

THEOREM 3. (i) *The measure $\bar{\rho}$ defined above is a stationary measure for the process u_t.*

(ii) *$x_t = \phi_t(x)$ for $x \in S^{n-1}$ is a Brownian motion on S^{n-1}.*

(iii) *The Liapounov exponents of ϕ_t are given by*

$$\Lambda = -\frac{n+m-2}{2} I_{n-1} + \frac{m-1}{2} H_{n-1}.$$

REMARK 2. Before proving the above theorem, we make the following remarks.

(a) When $m = 1$, the stochastic flow ϕ_t has a single exponent $-(n-1)/2$ of multiplicity $n-1$. In this case, ϕ_t is known as the gradient flow and has been studied in more detail in [3] and [7].

(b) When $m \geq 2$, ϕ_t has $n-1$ simple exponents:

$$\lambda_i = \frac{(n-1)(m-2)}{2} - (m-1)i, \qquad i = 1, 2, \dots, (n-1).$$

The case when $m = 2$ is considered in [9]. We note that when $n = 3$ the two simple exponents are -1 and $-m$.

(c) When $m = 1, 2$ or when $n = 3$, the stochastic flow ϕ_t is asymptotically stable in the sense that all its exponents are negative. However, ϕ_t becomes unstable for larger n and m. For example, its top exponent is equal to 0 when $n = m = 4$, and is positive when $n = 4$ and $m > 4$.

The rest of this paper is devoted to the proof of Theorem 3.

To prove (i), note that Y_i, as the orthogonal projection of $\partial/\partial x_i$ to S^{n-1}, is given by $(\partial/\partial x_i) - x_i \sum_h x_h (\partial/\partial x_h)$. Equation (2) can be written under the Cartesian coordinates $x_1, \dots, x_n$ as

$$dx_i = \sum_{j_1, \dots, j_m} \left[x_{j_1} \cdots x_{j_{m-1}} \delta_{i j_m} - x_{j_1} \cdots x_{j_m} x_i \right] \circ dw_t^{j_1 \cdots j_m}.$$

From this, one can show that (2) is invariant under orthogonal transformations in the sense that if x_t is a solution of (2), then, for any $a \in SO(n)$, x'_t defined by $x'_i(t) = \sum_j a_{ij} x_j(t)$ is also a solution. Fix a point $x \in S^{n-1}$ and let u_t be given the initial distribution $\bar{\rho}$. If we consider all the rotations which fix x, it is not hard to see that, given $\pi(u_t) = x$, u_t is equally likely to be any orthonormal frame at x. Hence $\bar{\rho}$ must be a stationary measure for u_t.

To show (ii), it suffices to show that its generator L is $(1/2)\Delta'$, where Δ' is the Laplace–Beltrami operator on S^{n-1}. We will reserve Δ for the ordinary Laplace operator on R^n. By the invariance of (2) under orthogonal transformations, the generator L of x_t must be a constant multiple of the Laplace–Beltrami operator Δ' on S^{n-1}. We have

$$L = \frac{1}{2} \sum_{j_1, \dots, j_m} \left\{ \sum_i \left[x_{j_1} \cdots x_{j_{m-1}} \delta_{i j_m} - x_{j_1} \cdots x_{j_m} x_i \right] \frac{\partial}{\partial x_i} \right\}^2 .$$

By direct computation, its leading terms are given by

$$(1/2)\left[\Delta - \sum_{i,j} x_i x_j (\partial/\partial x_i)(\partial/\partial x_j)\right],$$

which coincide with those of $(1/2)\Delta'$. This proves (ii).

We will use spherical polar coordinates on S^{n-1}:

$$x_1 = \cos\theta_1, \qquad x_2 = \sin\theta_1 \cos\theta_2, \dots, x_{n-1} = \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1},$$
$$x_n = \sin\theta_1 \cdots \sin\theta_{n-1},$$

where $0 < \theta_i < \pi$ for $1 \le i \le n-2$ and $0 < \theta_{n-1} < 2\pi$. The Riemannian metric tensor is

$$g_{ij} = \left\langle \frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j} \right\rangle = \delta_{ij} \sin^2\theta_1 \cdots \sin^2\theta_{i-1},$$

with $g_{11} = 1$. The uniform distribution on S^{n-1} is given by

$$d\rho = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, d\theta_1\, d\theta_2 \cdots d\theta_{n-1}/\sigma_n,$$

where σ_n is the area of S^{n-1}. We can show

$$\int x_1^2\, d\rho = \frac{1}{n}, \qquad \int x_1^4\, d\rho = \frac{3}{n(n+2)}, \qquad \int x_1^2 x_2^2\, d\rho = \frac{1}{n(n+2)}. \tag{21}$$

The Christoffel symbols are given by

$$\Gamma_{jj}^i = -\sin^2\theta_i \sin^2\theta_{i+1} \cdots \sin^2\theta_{j-1} \cot\theta_i$$

and

$$\Gamma_{ij}^j = \Gamma_{ji}^j = \cot\theta_i \quad \text{for } i < j,$$

and all other $\Gamma_{jk}^i = 0$. As a smooth section on $O(S^{n-1})$, let

$$u = (u_1, u_2, \dots, u_{n-1}) = \left(\frac{\partial}{\partial\theta_1}, \csc\theta_1 \frac{\partial}{\partial\theta_2}, \dots, \csc\theta_1 \cdots \csc\theta_{n-2} \frac{\partial}{\partial\theta_{n-1}} \right).$$

Then $\nabla_{u_1} u_1 = 0$, $\nabla_{u_j} u_1 = (\cot\theta_1) u_j$ for $j > 1$.

To prove (iii), we need to show $C_1 = -(n+m-2)/2$ and $C_2 = (m-1)/2$, where C_1 and C_2 are constants in Theorem 2. Note that $Y_1 = -(\sin\theta_1)u_1$. Let

$$Y = Y_\alpha = Y_{j_1 \cdots j_{m-1} 1} = -(\sin\theta_1)\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} u_1,$$

where α_i are nonnegative integers and $\sum_{i=1}^n \alpha_i = m-1$. In the sequel, we may omit some elementary details in rather tedious computations.

Let $\beta_i = \sum_{j=i+1}^{n} \alpha_j$ for $1 \le i \le n-1$. Then

$$Y = -\sin\theta_1 \prod_{i=1}^{n-1} \sin^{\beta_i}\theta_i \cos^{\alpha_i}\theta_i \, u_1,$$

$$\begin{aligned}\operatorname{div} Y &= \sum_i \langle u_i, \nabla Y(u_i)\rangle \\ &= \Big[-(n+\beta_1-1)\sin^{\beta_1}\theta_1 \cos^{\alpha_1+1}\theta_1 \\ &\qquad + \alpha_1 \sin^{\beta_1+2}\theta_1 \cos^{\alpha_1-1}\theta_1\Big] \prod_{i=2}^{n-1} \sin^{\beta_i}\theta_i \cos^{\alpha_i}\theta_i .\end{aligned}$$

Let $X_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Using $\alpha_1 + \beta_1 = m-1$, we have

$$\begin{aligned}Y(\operatorname{div} Y) &= m(n+m-2)\, x_1^2 X_\alpha^2 - (n+m-2)\, X_\alpha^2 \\ &\qquad -(n+2m-4)\alpha_1 X_\alpha^2 + \alpha_1(\alpha_1-1)\, x_1^{-2} X_\alpha^2 .\end{aligned}$$

Let $C_\alpha^{m-1} = (m-1)!/[\alpha_1!\alpha_2! \cdots \alpha_n!]$. Then, by symmetry, we have

$$\begin{aligned}C_1 &= \frac{n}{2(n-1)} \sum_\alpha C_\alpha^{m-1} \int Y_\alpha(\operatorname{div} Y_\alpha)\, d\rho \\ &= \frac{n}{2(n-1)} \int \Big\{ m(n+m-2)\, x_1^2 \sum C_\alpha^{m-1} X_\alpha^2 \\ &\qquad -(n+m-2) \sum C_\alpha^{m-1} X_\alpha^2 \\ &\qquad -(n+2m-4) \sum C_\alpha^{m-1} \alpha_1 X_\alpha^2 \\ &\qquad + \sum C_\alpha^{m-1} \alpha_1(\alpha_1-1)\, x_1^{-2} X_\alpha^2 \Big\}\, d\rho .\end{aligned}$$

Using the following combinatoric identities,

$$\sum C_\alpha^{m-1} X_\alpha^2 = 1, \qquad \sum C_\alpha^{m-1} \alpha_1 X_\alpha^2 = (m-1)\, x_1^2,$$

$$\sum C_\alpha^{2} \alpha_1(\alpha_1-1)\, x_1^{-2} X_\alpha^2 = (m-1)(m-2)\, x_1^2$$

and $\int x_1^2\, d\rho = 1/n$, we have

$$\begin{aligned}C_1 &= \frac{n}{2(n-1)} \left[\frac{m(n+m-2)}{n} - (n+m-2) \right. \\ &\qquad \left. - \frac{(n+2m-4)(m-1)}{n} + \frac{(m-1)(m-2)}{n} \right] \\ &= -\frac{n+m-2}{2} .\end{aligned}$$

We now calculate the matrix $\tilde{Y}$ for $Y = Y_\alpha$. We have $\tilde{Y}_{11} = -m x_1 X_\alpha + \alpha_1 x_1^{-1} X_\alpha$, $\tilde{Y}_{ii} = -x_1 X_\alpha$ and

$$\tilde{Y}_{1i} = (\sin\theta_1)\, X_\alpha x_i^{-1} \left[\alpha_i \sin^2\theta_i - \beta_i \cos^2\theta_i\right] \sin^{-1}\theta_i$$

for $2 \leq i \leq n-1$, and all other $\tilde{Y}_{ij} = 0$. Hence $(\tilde{Y})_P$ is given by

$$\begin{bmatrix} \frac{(n-2)[\alpha_1 x_1^{-1} - (m-1)x_1]}{n-1} X_\alpha & \frac{1}{2}\tilde{Y}_{12} & \cdots & \cdots & \frac{1}{2}\tilde{Y}_{1,n-1} \\ \frac{1}{2}\tilde{Y}_{12} & -\frac{\alpha_1 x_1^{-1} - (m-1)x_1}{n-1} X_\alpha & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}\tilde{Y}_{1,n-1} & 0 & \cdots & 0 & -\frac{\alpha_1 x_1^{-1} - (m-1)x_1}{n-1} X_\alpha \end{bmatrix}.$$

We have

$$\begin{aligned} \|(\tilde{Y})_P\|^2 &= \frac{n-2}{n-1}\left[\alpha_1 - (m-1)x_1^2\right]^2 x_1^{-2} X_\alpha^2 \\ &\quad + \frac{1}{2}\left(\sin^2\theta_1\right) X_\alpha^2 \sum_{i=2}^{n-1} x_i^{-2}\left[\alpha_i \sin^2\theta_i - \beta_i \cos^2\theta_i\right]^2 \sin^{-2}\theta_i. \end{aligned}$$

Using $\alpha_i + \beta_i = \beta_{i-1}$ and $x_{i+2}^{-2} \sin^2 \theta_{i+1} + x_i^{-2} \sin^{-2} \theta_i = x_i^{-2} + x_{i+1}^{-2}$ and simplifying, we obtain

$$\begin{aligned} &\sum_{i=2}^{n-1} x_i^{-2}\left[\alpha_i \sin^2\theta_i - \beta_i \cos^2\theta_i\right]^2 \sin^{-2}\theta_i \\ &\quad = \sum_{i=2}^{n-1} x_i^{-2}\left[\beta_{i-1} \sin^2\theta_i - \beta_i\right]^2 \sin^{-2}\theta_i \\ &\quad = \beta_1^2 x_2^{-2} \sin^2\theta_2 - 2\sum_{i=2}^{n-1} \beta_{i-1}\beta_i x_i^{-2} + \beta_{n-1}^2 x_{n-1}^{-2} \sin^{-2}\theta_{n-1} \\ &\qquad + \sum_{i=2}^{n-2} \beta_i^2\left[x_{i+1}^{-2} \sin^2\theta_{i+1} + x_i^{-2}\sin^{-2}\theta_i\right] \\ &\quad = \beta_1^2 x_2^{-2} \sin^2\theta_2 - 2\sum_{i=2}^{n-1} (\alpha_i + \beta_i)\beta_i x_i^{-2} \\ &\qquad + \alpha_n^2 x_{n-1}^{-2} \sin^{-2}\theta_{n-1} + \sum_{i=2}^{n-2} \beta_i^2\left[x_i^{-2} + x_{i+1}^{-2}\right] \\ &\quad = \cdots = -\beta_1^2 x_2^{-2} \cos^2\theta_2 + \sum_{i=2}^{n} \alpha_i^2 x_i^{-2}. \end{aligned}$$

Substituting the above in $(\tilde{Y})_P$ and using $\beta_1 = m - 1 - \alpha_1$, we obtain

$$\begin{aligned} \|(\tilde{Y}_\alpha)_P\|^2 &= \frac{n-2}{n-1}\alpha_1^2 x_1^{-2} X_\alpha^2 - \frac{(n-3)(m-1)}{n-1}\alpha_1 X_\alpha^2 \\ &\quad + \frac{(n-2)(m-1)^2}{n-1} x_1^2 X_\alpha^2 - \frac{(m-1)^2}{2} X_\alpha^2 \\ &\quad - \frac{1}{2}\alpha_1^2 X_\alpha^2 + \frac{1}{2}\left(1 - x_1^2\right)\sum_{i=2}^{n} \alpha_i^2 x_i^{-2} X_\alpha^2. \end{aligned}$$

The dimension of P is $p = (n+1)(n-2)/2$. Using symmetry, some combinatoric identities like those used before and (21), we have

$$\begin{aligned}
C_2 &= \frac{n}{(n+1)(n-2)} \sum_\alpha C_\alpha^{m-1} \int \left\| \left(\tilde{Y}_\alpha \right)_P \right\|^2 d\rho \\
&= \frac{n}{(n+1)(n-2)} \\
&\quad \times \int \left\{ \frac{n-2}{n-1} \left[(m-1)(m-2) x_1^2 + (m-1) \right] \right. \\
&\qquad - \frac{(n-3)(m-1)}{n-1} (m-1) x_1^2 \\
&\qquad + \frac{(n-2)(m-1)^2}{n-1} x_1^2 - \frac{(m-1)^2}{2} \\
&\qquad - \frac{1}{2} \left[(m-1)(m-2) x_1^4 + (m-1) x_1^2 \right] \\
&\qquad \left. + \frac{n-1}{2} \left(1 - x_1^2\right) \left[(m-1)(m-2) x_2^2 + (m-1) \right] \right\} d\rho \\
&= \frac{n}{(n+1)(n-2)} \left\{ + \frac{n-2}{n-1} \left[\frac{(m-1)(m-2)}{n} + (m-1) \right] \right. \\
&\qquad + \frac{(m-1)^2}{n(n-1)} - \frac{(m-1)^2}{2} \\
&\qquad - \frac{1}{2} \left[\frac{3(m-1)(m-2)}{n(n+2)} + \frac{m-1}{n} \right] \xi \\
&\qquad + \frac{n-1}{2} \left[\frac{(m-1)(m-2)}{n} + (m-1) \right] \\
&\qquad \left. - \frac{n-1}{2} \left[\frac{(m-1)(m-2)}{n(n+2)} + \frac{m-1}{n} \right] \right\} \\
&= \frac{m-1}{2}.
\end{aligned}$$

Theorem 3 is proved.

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