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HOLOMORPHIC DIFFUSIONS AND BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

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We study a family of differential operators $\{L^{\alpha},\ \alpha \geq 0\}$ in the unit ball D of C^n with $n \geq 2$ that generalize the classical Laplacian, $\alpha = 0$, and the conformal Laplacian, $\alpha = 1/2$ (that is, the Laplace–Beltrami operator for Bergman metric in D). Using the diffusion processes associated with these (degenerate) differential operators, the boundary behavior of L^{α}-harmonic functions is studied in a unified way for $0 \leq \alpha \leq 1/2$. More specifically, we show that a bounded L^{α}-harmonic function in D has boundary limits in approaching regions at almost every boundary point and the boundary approaching region increases from the Stolz cone to the Korányi admissible region as α runs from 0 to 1/2. A local version for this Fatou-type result is also established.

1. Introduction. Let D be the unit ball in R^m. It is well known that every bounded harmonic function u in D has a nontangential limit at almost every boundary point. To state this precisely, define for $\xi \in \partial D$ and $\beta > 0$ the Stolz cone with opening β and vertex ξ:

$$\Gamma_{\beta}(\xi) = \{z \in D\colon |z - \xi| < \beta(1 - |z|)\}.$$

The classical result asserts that for every bounded harmonic function u in the unit ball D, $\lim_{\Gamma_{\beta}(\xi) \ni z \to \xi} u(z)$ exists for a.e. $\xi \in \partial D$. This can also be proved probabilistically by running a Brownian motion in R^{2n} (see, e.g., [3] for the case of D being the upper half space of R^m). This nontangential convergence result in fact holds for any domain in R^m with Lipschitz smooth boundary (see [1] for a probabilistic proof of this result).

When D is the unit ball in C^n with $n \geq 2$ and f is a bounded holomorphic function in D, more is true. Let $w \cdot z = \sum_{i=1}^{n} w_i \bar{z}_i$ denote the inner product in C^n between two vectors $w, z \in \mathrm{C}^n$, and define for $\xi \in \partial D$ and $\beta \in (0, 1)$ the admissible region with opening β and vertex ξ:

$$\mathscr{A}_{\beta}(\xi) = \left\{z \in D\colon |\xi \cdot (z - \xi)| < \beta(1 - |z|) \text{ and } |z - \xi| < \beta(1 - |z|)^{1/2}\right\},$$

which is the intersection of a wedge $|\xi \cdot (z - \xi)| < \beta(1 - |z|)$ with a paraboloid $|z - \xi| < \beta(1 - |z|)^{1/2}$. Korányi [9] showed that for any bounded holomorphic function f in D, $\lim_{\mathscr{A}_{\beta}(\xi) \ni z \to \xi} f(z)$ exists for a.e. $\xi \in \partial D$.

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The key to Korányi's analysis is that f is harmonic for the invariant Laplacian,

$$\bar{\Delta} = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \partial_j \bar{\partial}_k, \tag{1.1}$$

where δ_{jk} is Kronecker's delta with $\delta_{kk} = 1$ and $\delta_{jk} = 0$ if $j \neq k$, and

$$\partial_j = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right), \qquad \bar{\partial}_k = \frac{1}{2}\left(\frac{\partial}{\partial x_{2k-1}} + i\frac{\partial}{\partial x_{2k}}\right) \tag{1.2}$$

when $z = (z_1, \dots, z_n) = (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n})$. The name invariant comes from the fact that $\bar{\Delta}$ is invariant under the automorphisms of the unit ball (see [7] or Section 2.2 of [11]). Another special property of $\bar{\Delta}$ is that it is the Laplace–Beltrami operator corresponding to the Bergman metric in D, under which D has constant negative curvature.

Holomorphic functions are harmonic for the ordinary Laplacian Δ as well, but identifying C^n with R^{2n} and using the results quoted in the first paragraph of this section lead only to nontangential convergence, a weaker result than Koranyi's admissible limits. Given this, it is natural, if somewhat optimistic, to ask if we can find another operator which will allow us to get better results. To explain what we have in mind, note that a holomorphic function f has $Lf = 0$ whenever

$$L = \sum_{i,j=1}^{n} a_{ij}(z) \partial_j \bar{\partial}_k, \tag{1.3}$$

since by definition we have $\bar{\partial}_k f = 0$ for $1 \le k \le n$. A diffusion process Z_t with a generator of this form is called a holomorphic diffusion, since if τ is the exit time from D and f is a holomorphic function f, then $f(Z_t)$, $t < \tau$, is a local martingale and, furthermore, a time change of a complex Brownian motion. In this connection, we would like to mention that Fukushima and Okada [4] used suitably chosen holomorphic diffusions (and the associated Dirichlet forms) to prove five properties of plurisubharmonic functions.

To motivate the class of operators we will consider, note that if Z is the diffusion process corresponding to $(1-|z|^2)^{-2}\bar{\Delta}/2$, which we will call the (time-changed) invariant Brownian motion in D, then up to the exit time τ from the ball (which turns out to be $\tau < \infty$) we have $dZ_t = \sigma(Z_t)\,dB_t$ where B is a $2n$-dimensional Brownian motion in $\mathrm{R}^{2n} = \mathrm{C}^n$ and σ is a $2n \times 2n$ real matrix given by

$$\sigma(z) = |z|^{-1}(P, Q)\begin{pmatrix} I_2 & 0 \\ 0 & (1-|z|^2)^{-1/2} I_{2n-2} \end{pmatrix} |z|^{-1} \begin{pmatrix} P' \\ Q' \end{pmatrix}, \tag{1.4}$$

where I_k denotes the $k \times k$ real identity matrix. Here the superscript prime denotes the matrix transpose and we make the identification

$$z = (x_1, x_2, \dots, x_{2n-1}, x_{2n})' \quad \text{and} \quad iz = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1})', \tag{1.5}$$

P is the $2n\times 2$ real matrix (z, iz) and Q is a $2n\times(2n-2)$ real matrix such that $|z|^{-1}(P, Q)$ is a $2n \times 2n$ orthonormal matrix. We set $\sigma(0) = I_{2n}$, the $2n \times 2n$ real identity matrix. At a point z in $D \setminus \{0\}$, the process Z oscillates with magnitude 1 along the complex direction $\mathbb{C}z$ and along the space of complex dimension $n-1$ which is perpendicular to $\mathbb{C}z$; it oscillates with magnitude $(1-|z|^2)^{-1/2}$. Note that if we multiply the oscillations by $1-|z|^2$, then we get the dimensions of the admissible region $\mathscr{A}_\beta(\xi)$ (with $1-|z|$ replaced by $1-|z|^2$).

Consider now the family of diffusion processes Z^α in D for $\alpha \geq 0$, given by $dZ_t^\alpha = \sigma_\alpha(Z_t^\alpha)\,dB_t$, where

$$\sigma_\alpha(z) = |z|^{-1}(P, Q)\begin{pmatrix} I_2 & 0 \\ 0 & (1-|z|^2)^{-\alpha} I_{2n-2}\end{pmatrix}|z|^{-1}\begin{pmatrix} P' \\ Q'\end{pmatrix}. \tag{1.6}$$

Since each entry of $\sigma_\alpha(z)$ is a C^∞ function in D, the diffusion process Z^α exists up to the exit time τ from D. It is easy to see that $Z_\tau^\alpha = \lim_{t\uparrow\tau} Z_t^\alpha$ exists almost surely and takes value in ∂D. The infinitesimal generator of Z^α is

$$L^\alpha = \frac{2}{(1-|z|^2)^{2\alpha}}\sum_{j,k=1}^{n}\left(\delta_{jk} - \frac{1-(1-|z|^2)^{2\alpha}}{|z|^2} z_j \bar{z}_k\right)\partial_j\bar{\partial}_k \tag{1.7}$$

and therefore every pluriharmonic function in D (that is, the real part of some holomorphic function in D) is an L^α-harmonic function. However, the space of L^α-harmonic functions is larger than that of pluriharmonic functions. Forelli's theorem (see page 63 of [11]) asserts that if u defined in D satisfies $u \in C^\infty(\{0\})$ as well as the differential equation $\sum_{j,k=1}^n z_j\bar{z}_k\partial_j\bar{\partial}_k u(z) = 0$ for $z \in D$, then u is pluriharmonic in D. So by (1.7), if u is Δ-harmonic and L^α-harmonic for some $\alpha > 0$, then u is pluriharmonic. Generalizing the classical results for the Laplacian Δ and the invariant Laplacian $\bar{\Delta}$, we have the following theorem.

THEOREM 1.1. *For every bounded real-valued L^α-harmonic function u in the unit ball D in $\mathbb{C}^n$ $(n \geq 2)$ with $0 \leq \alpha \leq 1/2$ and for every $\beta > 0$, $\lim_{\mathscr{A}_\beta^\alpha(\xi)\ni z\to\xi} u(z) = \psi(\xi)$ exists for a.e. $\xi \in \partial D$, where*

$$\mathscr{A}_\beta^\alpha(\xi) = \left\{z \in D\colon |\xi\cdot(z-\xi)| < \beta(1-|z|) \text{ and } |z-\xi| < \beta(1-|z|)^{1-\alpha}\right\}.$$

Furthermore, u can be recovered from its boundary value ψ by $u(z) = E_z[\psi(Z_\tau^\alpha)]$.

Note that for each fixed opening $\beta > 0$, the approaching region $\mathscr{A}_\beta^\alpha(\xi)$ increases from the Stolz cone $\Gamma_\beta(\xi)$ to the admissible region $\mathscr{A}_\beta(\xi)$ as α runs from 0 to 1/2. We believe that when $\alpha > 1/2$, Theorem 1 holds with $1-\alpha$ replaced by 1/2. It is clear that $1-\alpha$ cannot be correct in that case since for $\alpha > 1/2$ the region $\mathscr{A}_\beta^\alpha(\xi)$ no longer fits inside the ball. To explain why we think the answer is 1/2 for $\alpha > 1/2$ and why we have difficulty proving this, consider the two-dimensional analogue

$$\sigma_\alpha(z) = |z|^{-1}\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (1-|z|^2)^{-\alpha} \end{pmatrix}|z|^{-1}\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}. \tag{1.8}$$

Using Itô's formula it is easy to see that the radial and angular parts are given by

$$dR_t = dB_t^1 + (1-R_t^2)^{-2\alpha}R_t^{-1}\,dt,$$
$$d\theta_t = (1-R_t^2)^{-\alpha}\,dB_t^2,$$

where the B^1 and B^2 are two independent Brownian motions. If we let $R_0 = 1 - r\varepsilon$, $\theta_0 = \theta\varepsilon^{1-\alpha}$ and we consider

$$X_t^\varepsilon = (1 - R_{t\varepsilon^2})/\varepsilon \quad \text{and} \quad Y_t^\varepsilon = \theta_{t\varepsilon^2}/\varepsilon^{1-\alpha},$$

then using $1 - R_{t\varepsilon^2} = 1 - \{1 - \varepsilon X_t^\varepsilon\}^2 \approx 2\varepsilon X_t^\varepsilon$ and Brownian scaling $\varepsilon^{-1}\,dB^i_{t\varepsilon^2} = d\bar{B}^i_t$ we have

$$\begin{aligned} dX_t^\varepsilon &\approx \varepsilon^{-1}\,dB^1_{t\varepsilon^2} + \varepsilon^{-1}(2\varepsilon X_t^\varepsilon)^{-2\alpha}(1+\varepsilon X_t^\varepsilon)^{-1}\varepsilon^2\,dt \\ &\approx dB_t^1 + \varepsilon^{1-2\alpha}(2X_t^\varepsilon)^{-2\alpha}\,dt, \\ dY_t^\varepsilon &\approx \varepsilon^{-(1-\alpha)}(2\varepsilon X_t^\varepsilon)^{-\alpha}\,dB^2_{t\varepsilon^2} \approx (2X_t^\varepsilon)^{-\alpha}\,d\bar{B}_t^2, \end{aligned}$$

where $\approx$ denotes "approximate equality." When $\alpha < 1/2$ the drift in dX_t^ε converges to 0, but when $\alpha > 1/2$ it explodes. In the latter case, to get a sensible limit we let $R_0 = 1 - r\varepsilon$, $\theta_0 = \theta\varepsilon^{1/2}$ and we consider

$$X_t^\varepsilon = (1 - R_{t\varepsilon^{1+2\alpha}})/\varepsilon \quad \text{and} \quad Y_t^\varepsilon = \theta_{t\varepsilon^{1+2\alpha}}/\varepsilon^{1/2}.$$

This time $\varepsilon^{-1}\,dB^1_{t\varepsilon^{2\alpha}} \Rightarrow 0$, so

$$\begin{aligned} dX_t^\varepsilon &\approx \varepsilon^{-1}dB^1_{t\varepsilon^{2\alpha}} + \varepsilon^{-1}(2\varepsilon X_t^\varepsilon)^{-2\alpha}(1+\varepsilon X_t^\varepsilon)^{-1}\varepsilon^{1+2\alpha}\,dt \approx (2X_t^\varepsilon)^{-2\alpha}\,dt, \\ dY_t^\varepsilon &\approx \varepsilon^{-1/2}(2\varepsilon X_t^\varepsilon)^{-\alpha}\,dB^2_{t\varepsilon^{1+2\alpha}} \approx (2X_t^\varepsilon)^{-\alpha}\,d\bar{B}_t^2 \end{aligned}$$

and in the limit the first component becomes deterministic. This degeneration kills our proof of the Harnack inequality in Section 4.

In spite of our failure to prove results for $\alpha > 1/2$, Theorem 1 gives one more indication why the invariant Laplacian is well suited for the study of holomorphic functions in the ball. It would be interesting to investigate other regions in $\mathbb{C}^n$ as well. Stein [12] showed that bounded holomorphic functions in a bounded C^2-smooth domain in $\mathbb{C}^n$ ($n \geq 2$) have admissible limits at almost every boundary point. However, this is only optimal for strongly pseudoconvex domains (see [6]). A simple example is provided by the "flat" upper half space $H = \{z\colon \operatorname{Re} z_1 > 0\}$, where bounded holomorphic functions have unrestricted limits at almost every boundary point. This indicates that the nature of boundary limits depends on the geometry of the boundary near the point in question. For a list of known results in this direction, see page 72 of [10]. It would be interesting to have a probabilistic approach to those results. However, we have not been able to overcome the problem of proving the existence of exit densities for the very singular diffusion processes involved.

In the discussion above we have restricted our discussion to bounded functions to keep attention focussed on the shape of the approach region. Korányi [9] proved his results for $\bar{\Delta}$-harmonic functions that are in the Hardy space H^p with $1 \leq p < \infty$. By working harder we could prove the convergence in Theorem 1 in this generality or by using ideas from Section 5.3 of [3] we could extend the result to the Nevanlinna class (defined in Section 5.6 of [11]). See [2] for related work on the comparison of H^p space and martingale H^p space on the Hermitian hyperbolic space. To show that we can treat unbounded u, we will prove the following theorem.

THEOREM 1.2 (Local Fatou-type theorem). *Let* $0 \leq \alpha \leq 1/2$. *For an* L^α-*harmonic function* u *in* D, $\xi \in \partial D$ *and* $\beta > 0$, *define* $\tilde{\mathscr{A}}_\beta^\alpha(\xi) = \mathscr{A}_\beta^\alpha(\xi) \cap \{z \in D\colon \mathrm{Re}(z \cdot \xi) > 0\}$ *and let* $N_\beta^\alpha(\xi) = \sup_{\tilde{\mathscr{A}}_\beta^\alpha(\xi)} |u(z)|$. *Let* $F = \{\xi \in \partial D\colon N_\beta^\alpha(\xi) < \infty$ *for some* $\beta > 0\}$. *Then* $\lim_{\mathscr{A}_\beta^\alpha(\xi) \ni z \to \xi} u(z)$ *exists for every* $\beta > 0$ *and a.e.* $\xi \in F$.

As we will see from Theorem 4.1(i) and Corollary 4.2 below, if $N_\beta(\xi) < \infty$ for some $\beta > 0$, then $N_\beta(\xi) < \infty$ for all $\beta > 0$. Thus the set F in Theorem 1.2 can also be expressed as $\{\xi \in \partial D\colon N_\beta^\alpha(\xi) < \infty$ for all $\beta > 0\}$.

The rest of the paper is organized as follows. Holomorphic diffusion Z^α and some of its key estimates are studied in Section 2 for $\alpha \geq 0$. Unitary invariance of L^α and Z^α are discussed in Section 3 for $\alpha \geq 0$. Section 4 contains the Harnack inequality and uniform continuity for nonnegative L^α-harmonic functions near the boundary of D, where α is restricted to the closed interval [0, 1/2]. Estimates of the exit density functions and hitting probabilities of Z^α are given in Section 5, with the proof of the existence and minimality of the exit density function hidden away in an Appendix. Finally, Section 6 presents the proofs for Theorems 1.1 and 1.2.

Throughout the remainder of this paper, D denotes the unit ball in $\mathbb{C}^n$ with $n \geq 2$ and $1 = (1, 0, \ldots, 0, 0)' \in \mathbb{R}^{2n}$. In the sequel, when no confusion may be caused, we will suppress the super- and subscript α from L^α, Z^α, $\mathscr{A}_\beta^\alpha$, σ_α, and so forth. The notation "$\equiv$" stands for "be defined as."

2. Holomorphic diffusions.

In this section, $\alpha \in [0, \infty)$ and we will derive some key estimates for the holomorphic diffusions, which are important for our later development.

For $\alpha \geq 0$, let σ_α be the matrix defined by (1.6). Since each entry of σ_α is C^∞-smooth in D, for any given $2n$-dimensional real Brownian motion in $\mathbb{C}^n = \mathbb{R}^{2n}$, the stochastic differential equation

$$dZ_t^\alpha = \sigma(Z_t^\alpha)\, dB_t$$

has a unique strong solution. This process Z^α has continuous sample paths and its lifetime is denoted by τ^α. As we mentioned at the end of the last section, we will suppress α from Z^α, σ_α and τ^α in the sequel.

For $r > 0$, let D_r denote the ball in $\mathrm{C}^n = \mathrm{R}^{2n}$ of radius r centered at the origin. When $r = 1$, we simply denote D_1 by D. For $0 < r < 1$, define $\tau_r = \inf\{t > 0: |Z_t| = r\}$. Clearly the lifetime τ for process Z^α is given by $\lim_{r\uparrow 1} \tau_r$. Since each component of $\{Z_{\tau_r};\ 0 < r < 1\}$ is a bounded martingale, $\lim_{r\uparrow 1} Z_{\tau_r}$ exists almost surely and takes values in ∂D. The limit $\lim_{r\uparrow 1} Z_{\tau_r}$ will be denoted as Z_τ in the sequel.

The generator of Z is

$$L = \frac{1}{2}\sum_{i,\,j=1}^{2n} (\sigma\sigma')_{ij}(z)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

where $z = (z_1, \dots, z_n)' = (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n})'$. Since $|z|^{-1}(P, Q)$ is a $2n \times 2n$ orthonormal matrix,

$$\begin{aligned}
\sigma(z)\sigma(z)' &= |z|^{-1}(P, Q)\begin{pmatrix} I_2 & 0 \\ 0 & (1-|z|^2)^{-2\alpha} I_{2n-2}\end{pmatrix}|z|^{-1}\begin{pmatrix} P' \\ Q'\end{pmatrix} \\
&= |z|^{-2}\big(PP' + (1-|z|^2)^{-2\alpha} QQ'\big) \\
&= |z|^{-2}\big(PP' + (1-|z|^2)^{-2\alpha}\big(|z|^2 I_{2n} - PP'\big)\big) \\
&= |z|^{-2}\big((1-|z|^2)^{-2\alpha}|z|^2 I_{2n} + (1-(1-|z|^2)^{-2\alpha})PP'\big) \\
&= \frac{1}{(1-|z|^2)^{2\alpha}}\left(I_{2n} - \frac{1-(1-|z|^2)^{2\alpha}}{|z|^2}PP'\right).
\end{aligned} \tag{2.1}$$

If we use the notion ∂_j and $\bar\partial_k$ defined in (1.2), then the generator of Z is given by

$$L = \frac{2}{(1-|z|^2)^{2\alpha}}\sum_{j,\,k=1}^{n}\left(\delta_{jk} - \frac{1-(1-|z|^2)^{2\alpha}}{|z|^2} z_j \bar z_k\right)\partial_j\bar\partial_k. \tag{2.2}$$

We record the following observation as a lemma.

LEMMA 2.1. *For $z = (x_1, x_2, \dots, x_{2n})'$ in D,*

$$\sum_{i,\,j=1}^{2n} x_i x_j (\sigma\sigma')_{ij}(z) = |z|^2.$$

PROOF.

$$\begin{aligned}
\sum_{i,\,j}^{2n} x_i x_j(\sigma\sigma')_{ij}(z) &= (x_1, x_2, \dots, x_{2n})\sigma(z)\sigma(z)'(x_1, x_2, \dots, x_{2n})' \\
&= (|z|, 0, \dots, 0)\begin{pmatrix} I_2 & 0 \\ 0 & (1-|z|^2)^{-2\alpha} I_{2n-2}\end{pmatrix}\begin{pmatrix} |z| \\ 0 \\ \vdots \\ 0\end{pmatrix} \\
&= |z|^2. \qquad \square
\end{aligned}$$

THEOREM 2.2. *Let $R = |Z|$ be the radial process of Z. Then for $t < \tau$,*

$$R_t = R_0 + W_t + \int_0^t \frac{n-1}{R_s(1-R_s^2)^{2\alpha}}\,ds + \frac{1}{2}\int_0^t \frac{1}{R_s}\,ds, \tag{2.3}$$

$$R_t^2 = R_0^2 + 2\int_0^t R_s\,dW_s + 2\int_0^t \frac{n-1}{(1-R_s^2)^{2\alpha}}\,ds + 2t, \tag{2.4}$$

where W is a one-dimensional Brownian motion.

PROOF. Let $r(z) = |z| = (\sum_{i=1}^{2n} x_i^2)^{1/2}$. Then $\partial r/\partial x_i = x_i/r$ and $\partial^2 r/\partial x_i\,\partial x_j = \delta_{ij}/r - x_i x_j/r^3$. Let $\tau_r = \{t \geq 0\colon |Z_t| > r\}$ and denote the coordinate processes in R^{2n} of Z by $X^{(i)}$, $1 \leq i \leq 2n$. By Itô's formula, for $0 < r < 1$,

$$\begin{aligned} R_{t\wedge\tau_r} &= R_0 + \sum_{i=1}^{2n}\int_0^{t\wedge\tau_r} \frac{X_s^{(i)}}{R_s}\sum_{j=1}^{2n}\sigma_{ij}(Z_s)\,dB_s^j \\ &\qquad + \frac{1}{2}\int_0^{t\wedge\tau_r}\left(\sum_{i=1}^{2n}\frac{1}{R_s}(\sigma\sigma')_{ii}(Z_s) - \sum_{i,\,j=1}^{2n}\frac{X_s^{(i)}X_s^{(j)}}{R_s^3}(\sigma\sigma')_{ij}(Z_s)\right)ds \\ &\equiv R_0 + W_{t\wedge\tau_r} + A_{t\wedge\tau_r}. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} \langle W\rangle_{t\wedge\tau_r} &= \sum_{j=1}^{2n}\int_0^{t\wedge\tau_r}\left(\sum_{i=1}^{2n}\frac{X_s^{(i)}\sigma_{ij}(Z_s)}{R_s}\right)^2 ds \\ &= \sum_{j=1}^{2n}\int_0^{t\wedge\tau_r}\frac{1}{R_s^2}\sum_{i,\,k=1}^{2n}X_s^{(i)}X_s^{(j)}\sigma_{ij}(Z_s)\sigma_{kj}(Z_s)\,ds \\ &= \int_0^{t\wedge\tau_r}\frac{1}{R_s^2}\sum_{i,k=1}^{2n}X_s^{(i)}X_s^{(j)}(\sigma\sigma')_{ik}(Z_s)\,ds \\ &= t\wedge\tau_r. \end{aligned}$$

Thus W is a one-dimensional Brownian motion up to time $\tau = \lim_{r\uparrow 1}\tau_r$. For $1 \leq k \leq n$, let $Z^{(k)} = (X^{(2k-1)}, X^{(2k)})$. Then

$$\begin{aligned} A_{t\wedge\tau_r} &= \frac{1}{2}\int_0^{t\wedge\tau_r}\sum_{i=1}^{2n}\frac{1}{R_s}(\sigma\sigma')_{ii}(Z_s)\,ds - \frac{1}{2}\int_0^{t\wedge\tau_r}\sum_{i,\,j=1}^{2n}\frac{X_s^{(i)}X_s^{(j)}}{R_s^3}(\sigma\sigma')_{ij}(Z_s)\,ds \\ &= \int_0^{t\wedge\tau_r}\frac{1}{R_s(1-R_s^2)^{2\alpha}}\sum_{k=1}^{n}\left(1 - \frac{1-(1-R_s^2)^{2\alpha}}{R_s^2}|Z_s^{(k)}|^2\right)ds - \frac{1}{2}\int_0^{t\wedge\tau_r}\frac{1}{R_s}\,ds \\ &= \int_0^{t\wedge\tau_r}\frac{1}{R_s(1-R_s^2)^{2\alpha}}\left(n-1+(1-R_s^2)^{2\alpha}\right)ds - \frac{1}{2}\int_0^{t\wedge\tau_r}\frac{1}{R_s}\,ds \\ &= \int_0^{t\wedge\tau_r}\frac{n-1}{R_s(1-R_s^2)^{2\alpha}}\,ds + \frac{1}{2}\int_0^{t\wedge\tau_r}\frac{1}{R_s}\,ds. \end{aligned}$$

This proves (2.3) since $\tau = \lim_{r\uparrow 1} \tau_r$. Formula (2.4) follows from (2.3) and Itô's formula. □

COROLLARY 2.3. *For $z \in D$, we have*

$$E_z\left[\int_0^\tau R_s^{-1}(1-R_s^2)^{-2\alpha}\,ds\right] \le \frac{1-|z|^2}{2(n-1)} \quad \text{and} \quad E_z[\tau] \le \frac{1-|z|^2}{2n}. \tag{2.5}$$

PROOF. The proof follows directly from (2.3), by first evaluating at τ_r for $r \in (0,1)$ and then letting $r \uparrow 1$. □

LEMMA 2.4. *Let X be a one-dimensional diffusion on $I = (r_1, r_2)$ satisfying $dX_t = b(X_t)\,dt + a(X_t)\,dW_t$, where $b: I \to \mathrm{R}$ and $a: I \to \mathrm{R}$ are continuous functions with $a(x) > 0$ for $x \in I$, and W is Brownian motion on* R. *Suppose that g is a bounded continuous function on I. Then*

$$E_x\left[\int_0^\zeta g(X_s)\,ds\right] = \int_{r_1}^{r_2} g(y)G(x,y)\,dy, \qquad x \in I.$$

Here ζ is the exit time of X from I (that is, $\zeta = \lim_{s\downarrow r_1,\, r\uparrow r_2} \inf\{t > 0: X_t \notin (s,r)\}$) and

$$G(x,y) = \begin{cases} 2\,\dfrac{\phi(x)-\phi(r_1)}{\phi(r_2)-\phi(r_1)}(\phi(r_2)-\phi(y))\dfrac{1}{a^2(y)f(y)}, & \text{if } y > x,\\[2ex] 2\,\dfrac{\phi(r_2)-\phi(x)}{\phi(r_2)-\phi(r_1)}(\phi(y)-\phi(r_1))\dfrac{1}{a^2(y)f(y)}, & \text{if } y < x,\end{cases}$$

where

$$f(y) = \exp\left(-\int_x^y \frac{2b(t)}{a^2(t)}\,dt\right) \quad \text{and} \quad \phi(t) = \int_0^t f(s)\,ds.$$

PROOF. This is a classical result. See, for example, Exercise 5.39 on page 352 of [8]. □

For $Z = (X^{(1)}, X^{(2)}, \dots, X^{(2n)})'$, let $Y = X^{(2)}$ and $V = (X^{(3)}, X^{(4)}, \dots, X^{(2n)})'$.

THEOREM 2.5. *For $0 < \varepsilon < \nu < 1/2$, we have*

$$E_{(\sqrt{1-\varepsilon},\,0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \le \begin{cases} 2(n-1)\left(\dfrac{\nu^{1-2\alpha}-\varepsilon^{1-2\alpha}}{1-2\alpha}\,\varepsilon + \dfrac{\varepsilon^{2-2\alpha}}{2(1-\alpha)}\right), \\ \qquad\qquad \text{for } 0 \le \alpha < 1/2, \\[1ex] \varepsilon, \quad \text{for } \alpha \ge 1/2, \end{cases} \tag{2.6}$$

and

$$E_{(\sqrt{1-\varepsilon},\,0)}\big[|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \le \nu\varepsilon + \left(\frac{8n}{(n-1)^2}\left(\frac{4}{3}\right)^4 - \frac{1}{2}\right)\varepsilon^2 \quad \text{for } \alpha \ge 0. \tag{2.7}$$

PROOF. (i) By Itô's formula,

$$
\begin{aligned}
&E_{(\sqrt{1-\varepsilon},0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big]\\
&\quad = E_{(\sqrt{1-\varepsilon},0)}[|V_0|^2]\\
&\qquad + 2E_{(\sqrt{1-\varepsilon},0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}(1-R_s^2)^{-2\alpha}\bigg(n-1-\frac{1-(1-R_s^2)^{2\alpha}}{R_s^2}|V_s|^2\bigg)\,ds\bigg] \qquad (2.8)\\
&\quad \le 2(n-1)E_{(\sqrt{1-\varepsilon},0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}(1-R_s^2)^{-2\alpha}\,ds\bigg].
\end{aligned}
$$

By Theorem 2.2, R^2 is the solution for the following stochastic differential equation

$$
dX_t = 2\sqrt{X_t}\,dW_t + 2\bigg(\frac{n-1}{(1-X_t)^{2\alpha}}+1\bigg)\,dt, \qquad 0 \le t < \tau.
$$

Applying Lemma 2.4 to the radial process R^2 on the interval $(1-\nu, 1)$, we have from (2.8) that

$$
E_{(\sqrt{1-\varepsilon},0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \le \int_{1-\nu}^1 \frac{2(n-1)}{(1-t)^{2\alpha}}\,G(1-\varepsilon,\ t)\,dt, \qquad (2.9)
$$

where

$$
G(1-\varepsilon,t) = \begin{cases} 2\,\dfrac{\phi(1-\varepsilon)-\phi(1-\nu)}{\phi(1)-\phi(1-\nu)}\,(\phi(1)-\phi(t))\dfrac{1}{4tf(t)}, & \text{if } 1-\varepsilon < t < 1,\\[2ex] 2\,\dfrac{\phi(1)-\phi(1-\varepsilon)}{\phi(1)-\phi(1-\nu)}\,(\phi(t)-\phi(1-\nu))\dfrac{1}{4tf(t)}, & \text{if } 1-\nu < t < 1-\varepsilon, \end{cases}
$$

with $f(t) = \exp(-\int_{1-\varepsilon}^t((n-1)/(1-s)^{2\alpha}+1)s^{-1}\,ds)$. Note that $f(t)$ is a decreasing function in $t \in (1-\nu, 1)$. Thus for $1-\varepsilon < t < 1$,

$$
G(1-\varepsilon,\ t) \le 2\int_t^1 \frac{f(s)}{f(t)}\,ds\,\frac{1}{4t} \le \frac{1-t}{2(1-\varepsilon)}.
$$

When $1-\nu < t < 1-\varepsilon$,

$$
G(1-\varepsilon,\ t) \le 2\int_{1-\varepsilon}^1 \frac{f(s)}{f(t)}\,ds\,\frac{1}{4t} \le \frac{\varepsilon}{2t} \le \frac{\varepsilon}{2(1-\nu)}.
$$

In summary, when $0 < \varepsilon < \nu < 1/2$,

$$
G(1-\varepsilon,t) \le \begin{cases} 1-t, & \text{if } 1-\varepsilon < t < 1,\\ \varepsilon, & \text{if } 1-\nu < t < 1-\varepsilon. \end{cases} \qquad (2.10)
$$

For $0 < \alpha < 1/2$, it follows from (2.9) that

$$\begin{aligned}
&E_{(\sqrt{1-\varepsilon},\,0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \\
&\quad \le \int_{1-\nu}^{1-\varepsilon} \frac{2(n-1)}{(1-t)^{2\alpha}}\,\varepsilon\,dt + \int_{1-\varepsilon}^{1} \frac{2(n-1)}{(1-t)^{2\alpha}}\,(1-t)\,dt \\
&\quad = \frac{2(n-1)\,\varepsilon}{1-2\alpha}\big(\nu^{1-2\alpha} - \varepsilon^{1-2\alpha}\big) + \frac{2(n-1)}{2-2\alpha}\varepsilon^{2-2\alpha} \\
&\quad = 2(n-1)\bigg(\frac{\nu^{1-2\alpha} - \varepsilon^{1-2\alpha}}{1-2\alpha}\,\varepsilon + \frac{\varepsilon^{2-2\alpha}}{2(1-\alpha)}\bigg).
\end{aligned}$$

When $\alpha \ge 1/2$, by (2.8) and Corollary 2.3, we have

$$E_{(\sqrt{1-\varepsilon},\,0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \le 2(n-1)\,\frac{\varepsilon}{2(n-1)} = \varepsilon.$$

(ii) The proof of (2.7) is more demanding. By Itô's formula and (2.1),

$$\begin{aligned}
&E_{(\sqrt{1-\varepsilon},\,0)}\big[|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\big] \\
&\quad = E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} (\sigma\sigma')_{22}(Z_s)\,ds\bigg] \\
&\quad = E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} (1-R_s^2)^{-2\alpha} \\
&\qquad\qquad \times\bigg(1 - \frac{1-(1-R_s^2)^{2\alpha}}{R_s^2}\big((X_s^{(1)})^2 + (X_s^{(2)})^2\big)\bigg)\,ds\bigg] \qquad (2.11)\\
&\quad = E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} \bigg(1 + \frac{1-(1-R_s^2)^{2\alpha}}{R_s^2(1-R_s^2)^{2\alpha}}|V_s|^2\bigg)\,ds\bigg] \\
&\quad \le E_{(\sqrt{1-\varepsilon},\,0)}\big[\tau\wedge\tau_{\sqrt{1-\nu}}\big] + \frac{1}{1-\nu}E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} \frac{|V_s|^2}{(1-R_s^2)^{2\alpha}}\,ds\bigg].
\end{aligned}$$

By Lemma 2.4 and (2.10),

$$\begin{aligned}
E_{(\sqrt{1-\varepsilon},\,0)}\big[\tau\wedge\tau_{\sqrt{1-\nu}}\big] &= \int_{1-\nu}^{1} G(1-\varepsilon,\ t)\,dt \\
&\le \int_{1-\nu}^{1-\varepsilon} \varepsilon\,dt + \int_{1-\varepsilon}^{1}(1-t)\,dt \qquad (2.12)\\
&= (\nu-\varepsilon)\varepsilon + \frac{\varepsilon^2}{2} \\
&= \nu\,\varepsilon - \frac{\varepsilon^2}{2}.
\end{aligned}$$

Let

$$A = E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} \frac{|V_s|^2}{(1-R_s^2)^{2\alpha}}\,ds\bigg]$$

and

$$V^* = \sup_{0\le s\le \tau\wedge\tau_{\sqrt{1-\nu}}} |V_s|.$$

Then

$$\begin{aligned} A &\le E_{(\sqrt{1-\varepsilon},\,0)}\bigg[(V^*)^2\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}(1-R_s^2)^{-2\alpha}\,ds\bigg] \\ &\le \sqrt{E_{(\sqrt{1-\varepsilon},\,0)}[(V^*)^4]}\sqrt{E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\bigg(\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}(1-R_s^2)^{-2\alpha}\,ds\bigg)^2\bigg]}. \end{aligned} \tag{2.13}$$

By Doob's inequality,

$$E_{(\sqrt{1-\varepsilon},\,0)}[(V^*)^4] \le \left(\tfrac{4}{3}\right)^4 E_{(\sqrt{1-\varepsilon},\,0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^4\big],$$

while by Itô's formula,

$$\begin{aligned} &E_{(\sqrt{1-\varepsilon},\,0)}\big[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^4\big] \\ &\quad= E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}\bigg(|V_s|^2\sum_{i=3}^{2n}(\sigma\sigma')_{ii}(Z_s) \\ &\qquad\qquad + 2\sum_{i,j=3}^{2n} X_s^{(i)}X_t^{(j)}(\sigma\sigma')_{ij}(Z_s)\bigg)\,ds\bigg] \\ &\quad\le E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}\frac{2|V_s|^2}{(1-R_s^2)^{2\alpha}}\bigg(n-\frac{1-(1-R_s^2)^{2\alpha}}{R_s^2}|V_s|^2\bigg)\,ds\bigg] \\ &\quad\le E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}\frac{2n\,|V_s|^2}{(1-R_s^2)^{2\alpha}}\,ds\bigg]. \end{aligned} \tag{2.14}$$

In the first inequality above, we used the fact that by (2.1), for $z = (x_1, x_2, \dots, x_{2n})' \in D$,

$$\begin{aligned} \sum_{i,\,j=3}^{2n} x_i x_j(\sigma\sigma')(z) &= \frac{1}{1-|z|^{2\alpha}}\bigg(|w|^2 - \frac{1-(1-|z|^2)^{2\alpha}}{|z|^2}|w|^4\bigg) \\ &\le \frac{|v|^2}{1-|z|^{2\alpha}}, \end{aligned}$$

where $v = (x_3, x_4, \dots, x_{2n})'$. Thus

$$E_{(\sqrt{1-\varepsilon},\,0)}[(V^*)^4] \le \bigg(\frac{4}{3}\bigg)^4 E_{(\sqrt{1-\varepsilon},\,0)}\bigg[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}}\frac{2n\,|V_s|^2}{(1-R_s^2)^{2\alpha}}\,ds\bigg] = 2n\bigg(\frac{4}{3}\bigg)^4 A. \tag{2.15}$$

Let $\rho_s = R_s^2$ and $T = \tau\wedge\tau_{\sqrt{1-\nu}}$. By Theorem 2.2, ρ is a strong Markov process and T is a stopping time of ρ. By Corollary 2.3, for $0 \le r < 1$,

$$E_r\bigg[\int_0^T (1-\rho_s)^{-2\alpha}\,ds\bigg] \le \frac{1-r}{2(n-1)}.$$

Hence

$$
\begin{aligned}
&E_{(\sqrt{1-\varepsilon},\,0)}\biggl[\int_0^{\tau\wedge\tau_{\sqrt{1-\nu}}} \frac{1}{(1-R_s^2)^{2\alpha}}\,ds\biggr] \\
&\quad= E_{1-\varepsilon}\biggl[\biggl(\int_0^T (1-\rho_s)^{-2\alpha}\,ds\biggr)^2\biggr] \\
&\quad= 2E_{1-\varepsilon}\biggl[\int_0^T (1-\rho_s)^{-2\alpha}\biggl(\int_s^T (1-\rho_t)^{-2\alpha}\,dt\biggr)ds\biggr] \\
&\quad= 2E_{1-\varepsilon}\biggl[\int_0^T (1-\rho_s)^{-2\alpha}E_{\rho_s}\biggl[\int_0^T (1-\rho_u)^{-2\alpha}\,du\biggr]ds\biggr] \qquad (2.16)\\
&\quad\le 2E_{1-\varepsilon}\biggl[\int_0^T (1-\rho_s)^{-2\alpha}\frac{1-\rho_s}{2(n-1)}\,ds\biggr] \\
&\quad\le 2E_{1-\varepsilon}\biggl[\int_0^T (1-\rho_s)^{-2\alpha}\frac{\varepsilon}{(n-1)}\,ds\biggr] \\
&\quad\le \frac{2}{(n-1)^2}\,\varepsilon^2.
\end{aligned}
$$

Now by (2.13), (2.15) and (2.16), we have

$$
A \le \biggl(\frac{4}{3}\biggr)^2 \sqrt{2nA}\,\frac{\sqrt{2}\,\varepsilon}{n-1}
$$

and, therefore,

$$
A \le \biggl(\frac{4}{3}\biggr)^4 \frac{4n}{(n-1)^2}\,\varepsilon^2. \tag{2.17}
$$

By (2.11), (2.12) and (2.17), for $0<\varepsilon<\nu<1/2$,

$$
E_{(\sqrt{1-\varepsilon},\,0)}\bigl[|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2\bigr] \le \nu\,\varepsilon - \frac{\varepsilon^2}{2} + 2A = \nu\,\varepsilon + \biggl(\frac{8n}{(n-1)^2}\biggl(\frac{4}{3}\biggr)^4 - \frac{1}{2}\biggr)\varepsilon^2.
$$

This completes the proof of Theorem 2.5. □

Letting $\nu = 2\varepsilon$ in Theorem 2.5, we get the following corollary.

COROLLARY 2.6. *For $\alpha \ge 0$ and $0<\varepsilon<1/4$,*

$$
E_{(\sqrt{1-\varepsilon},\,0)}\bigl[|V_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}|^2\bigr] \le C_\alpha\,\varepsilon^{2\phi(\alpha)} \tag{2.18}
$$

and

$$
E_{(\sqrt{1-\varepsilon},\,0)}\bigl[|Y_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}|^2\bigr] \le C\varepsilon^2 \quad \text{for } \alpha \ge 0, \tag{2.19}
$$

where $\phi(\alpha) = \max\{1-\alpha, 1/2\}$,

$$C_\alpha = \begin{cases} 2(n-1)\left(\dfrac{2^{1-2\alpha}-1}{1-2\alpha} + \dfrac{1}{2(1-\alpha)}\right), & \text{for } 0 \le \alpha < 1/2, \\ 1, & \text{for } \alpha \ge 1/2, \end{cases}$$

and

$$C = \frac{3}{2} + \frac{8n}{(n-1)^2}\left(\frac{4}{3}\right)^4.$$

LEMMA 2.7. (i) *For* $0 < \varepsilon < \nu < 1$, $P_{(\sqrt{1-\varepsilon},0)}(\tau > \tau_{\sqrt{1-\nu}}) \le \varepsilon/\nu$. *In particular*, $P_{(\sqrt{1-\varepsilon},0)}(\tau > \tau_{\sqrt{1-2\varepsilon}}) \le 1/2$.

(ii) *For* $0 < \varepsilon < 1$, $P_{(\sqrt{1-\varepsilon},0)}(X_\tau^{(1)} \le 0) \le \varepsilon/(2(n-1)(1-\varepsilon))$.

PROOF. (i) By Theorem 2.2,

$$\begin{aligned} 1-\varepsilon &= E_{(\sqrt{1-\varepsilon},0)}[R_0^2] \\ &\le E_{(\sqrt{1-\varepsilon},0)}\big[R^2_{\tau\wedge\tau_{\sqrt{1-\nu}}}\big] \\ &= P_{(\sqrt{1-\varepsilon},0)}\big(\tau < \tau_{\sqrt{1-\nu}}\big) + (1-\nu)P_{(\sqrt{1-\varepsilon},0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big) \\ &= 1 - P_{(\sqrt{1-\varepsilon},0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big) + (1-\nu)P_{(\sqrt{1-\varepsilon},0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big). \end{aligned}$$

Thus $P_{(\sqrt{1-\varepsilon},0)}(\tau > \tau_{\sqrt{1-\nu}}) \le \varepsilon/\nu$.

(ii) For $0 < \varepsilon < 1$,

$$\begin{aligned} &P_{(\sqrt{1-\varepsilon},0)}\big(X_\tau^{(1)} \le 0\big) \\ &\quad = P_{(\sqrt{1-\varepsilon},0)}\left(\sum_{j=1}^{2n}\int_0^\tau \sigma_{ij}(Z_s)\,dB_s^j \le -\sqrt{1-\varepsilon}\right) \\ &\quad \le \frac{1}{1-\varepsilon}E_{(\sqrt{1-\varepsilon},0)}\left[\left(\sum_{j=1}^{2n}\int_0^\tau \sigma_{ij}(Z_s)\,dB_s^j\right)^2\right] \\ &\quad = \frac{1}{1-\varepsilon}E_{(\sqrt{1-\varepsilon},0)}\left[\sum_{j=1}^{2n}\int_0^\tau (\sigma\sigma')_{11}(Z_s)\,ds\right] \\ &\quad = \frac{1}{1-\varepsilon}E_{(\sqrt{1-\varepsilon},0)}\left[\int_0^\tau \frac{1}{(1-R_s^2)^{2\alpha}}\right. \\ &\qquad\qquad \left.\times\left(1 - \frac{1-(1-R_s^2)^{2\alpha}}{R_s^2}\big((X^{(1)})^2 + (X^{(2)})^2\big)\right)ds\right] \\ &\quad \le \frac{1}{1-\varepsilon}E_{(\sqrt{1-\varepsilon},0)}\left[\int_0^\tau \frac{1}{(1-R_s^2)^{2\alpha}}\,ds\right] \\ &\quad \le \frac{\varepsilon}{2(n-1)(1-\varepsilon)}. \end{aligned}$$

The last inequality comes from Corollary 2.3. □

THEOREM 2.8. *For $\alpha \ge 0$, there is a constant $B = B(\alpha) > 1$ such that*

$$P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} > 0,\ |Y_\tau| < B\varepsilon \text{ and } |V_\tau| < B\varepsilon^{\phi(\alpha)}\big) > \tfrac{1}{6}$$

for any $0 < \varepsilon < 1/4$, where $\phi(\alpha) = \max\{1-\alpha, 1/2\}$.

PROOF. Note that

$$\begin{aligned}
&P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} > 0,\ |Y_\tau| < B\varepsilon \text{ and } |V_\tau| < B\varepsilon^{\phi(\alpha)}\big)\\
&\quad\ge P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_\tau| < B\varepsilon \text{ and } |V_\tau| < B\varepsilon^{\phi(\alpha)}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}| < B\varepsilon \text{ and } |V_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}| < B\varepsilon^{\phi(\alpha)}\big)\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-2\varepsilon}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge 1 - P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}| \ge B\varepsilon\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(|V_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}| \ge B\varepsilon^{\phi(\alpha)}\big)\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-2\varepsilon}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge 1 - \frac{E_{(\sqrt{1-\varepsilon},\,0)}[|Y_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}|^2]}{B^2\,\varepsilon^2} - \frac{E_{(\sqrt{1-\varepsilon},\,0)}[|V_{\tau\wedge\tau_{\sqrt{1-2\varepsilon}}}|^2]}{B^2\,\varepsilon^{2\phi(\alpha)}}\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-2\varepsilon}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big),
\end{aligned}$$

which by Corollary 2.6, Lemma 2.7 and the assumption of $0 < \varepsilon < 1/4$,

$$\begin{aligned}
&\ge 1 - \frac{C + C_\alpha}{B^2} - \frac{\varepsilon}{2(n-1)(1-\varepsilon)} - \frac{1}{2}\\
&\ge \frac{1}{2} - \frac{C_\alpha + C}{B^2} - \frac{1}{6(n-1)}.
\end{aligned}$$

Therefore if we choose $B = B(\alpha) = \sqrt{6(C_\alpha + C)}$, where constants C_α and C are as in Corollary 2.6, then

$$P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} > 0,\ |Y_\tau| < B\varepsilon \text{ and } |V_\tau| < B\varepsilon^{\phi(\alpha)}\big) \ge \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{6} = \tfrac{1}{6}.$$

More specifically, $B(\alpha)$ can be taken as

$$B(\alpha) = \begin{cases} \sqrt{8}\sqrt{2(n-1)\left(\dfrac{2^{2-2\alpha}-1}{1-2\alpha} + \dfrac{1}{2(1-\alpha)}\right) + \dfrac{3}{2} + \dfrac{8n}{(n-1)^2}\left(\dfrac{4}{3}\right)^4}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \le \alpha < \dfrac{1}{2}, \\ \sqrt{8}\sqrt{\dfrac{5}{2} + \dfrac{8n}{(n-1)^2}\left(\dfrac{4}{3}\right)^4}, \quad \text{for } \alpha \ge \dfrac{1}{2}. & \end{cases}$$

□

THEOREM 2.9. *Suppose that $\alpha \ge 0$. For $0 < \gamma < 1$ and $0 < \delta < \phi(\alpha)$, where $\phi(\alpha) = \max\{1-\alpha, 1/2\}$, we have*

$$\lim_{\varepsilon\downarrow 0} P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} > 0,\ |Y_\tau| < \varepsilon^\gamma \text{ and } |V_\tau| < \varepsilon^\delta\big) = 1.$$

PROOF. Let $0 < \varepsilon < \nu < 1/2$. Then

$$
\begin{aligned}
&P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} > 0,\ |Y_\tau| < \varepsilon^\gamma \text{ and } |V_\tau| < \varepsilon^\delta\big)\\
&\quad\ge P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_\tau| < \varepsilon^\gamma \text{ and } |V_\tau| < \varepsilon^\delta\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}| < \varepsilon^\gamma \text{ and } |V_{\tau\wedge\tau_{\sqrt{1-\nu}}}| < \varepsilon^\delta\big)\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge 1 - P_{(\sqrt{1-\varepsilon},\,0)}\big(|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}| \ge \varepsilon^\gamma\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}| \ge \varepsilon^\delta\big)\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big)\\
&\quad\ge 1 - \frac{E_{(\sqrt{1-\varepsilon},\,0)}[|Y_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2]}{\varepsilon^{2\gamma}} - \frac{E_{(\sqrt{1-\varepsilon},\,0)}[|V_{\tau\wedge\tau_{\sqrt{1-\nu}}}|^2]}{\varepsilon^{2\delta}}\\
&\qquad - P_{(\sqrt{1-\varepsilon},\,0)}\big(\tau > \tau_{\sqrt{1-\nu}}\big) - P_{(\sqrt{1-\varepsilon},\,0)}\big(X_\tau^{(1)} \le 0\big).
\end{aligned}
\tag{2.20}
$$

Now let $\nu = \varepsilon^\beta$ with $0 < 1-\beta < \min\{1-\gamma,\ \phi(\alpha)-\delta\}$. When $0 \le \alpha < 1/2$, by Theorem 2.5 and Lemma 2.7, (2.20) is bounded below by

$$
\begin{aligned}
1 &- \left(\varepsilon^{1+\beta-2\gamma} + \left(\frac{8n}{(n-1)^2}\left(\frac{4}{3}\right)^4 - \frac{1}{2}\right)\varepsilon^{2-2\gamma}\right)\\
&- 2(n-1)\left(\frac{\varepsilon^{1+\beta(1-2\alpha)-2\delta} - \varepsilon^{2-2\alpha-2\delta}}{1-2\alpha} + \frac{\varepsilon^{2-2\alpha-2\delta}}{2(1-\alpha)}\right)\\
&- \varepsilon^{1-\beta} - \frac{\varepsilon}{2(n-1)(1-\varepsilon)}.
\end{aligned}
\tag{2.21}
$$

When $\alpha \ge 1/2$, by Theorem 2.5 and Lemma 2.7 again, (2.20) is bounded below by

$$
\begin{aligned}
1 &- \left(\varepsilon^{1+\beta-2\gamma} + \left(\frac{8n}{(n-1)^2}\left(\frac{4}{3}\right)^4 - \frac{1}{2}\right)\varepsilon^{2-2\gamma}\right)\\
&- \varepsilon^{1-2\delta} - \varepsilon^{1-\beta} - \frac{\varepsilon}{2(n-1)(1-\varepsilon)}.
\end{aligned}
\tag{2.22}
$$

Theorem 2.9 now follows from (2.20)–(2.22) by passing $\varepsilon \downarrow 0$. □

3. Unitary invariance.

In this section, $\alpha \in [0, \infty)$. Let $C = (c_{kj})_{n\times n}$ be an unitary complex matrix in $\mathbb{C}^n$ whose entries are $c_{kj} = \alpha_{kj} + i\beta_{kj}$, $1 \le k, j \le n$, where α_{kj} and β_{kj} are real numbers. Then with $z_j = x_{2j-1} + ix_{2j}$, $j = 1, 2, \dots, n$,

$$
\begin{aligned}
\sum_{j=1}^n c_{kj}z_j &= \sum_{j=1}^n (\alpha_{kj} + i\beta_{kj})(x_{2j-1} + ix_{2j})\\
&= \sum_{j=1}^n (\alpha_{kj}x_{2j-1} - \beta_{kj}x_{2j}) + i\sum_{j=1}^n (\beta_{kj}x_{2j-1} + \alpha_{kj}x_{2j}).
\end{aligned}
$$

Thus, under the identification of C^n with R^{2n} through (1.5), the unitary matrix C in C^n is identified with an orthogonal matrix

$$\Phi = \begin{pmatrix} \alpha_{11} & -\beta_{11} & \alpha_{12} & -\beta_{12} & \cdots & \alpha_{1n} & -\beta_{1n} \\ \beta_{11} & \alpha_{11} & \beta_{12} & \alpha_{12} & \cdots & \beta_{1n} & \alpha_{1n} \\ \alpha_{21} & -\beta_{21} & \alpha_{22} & -\beta_{22} & \cdots & \alpha_{2n} & -\beta_{2n} \\ \beta_{21} & \alpha_{21} & \beta_{22} & \alpha_{22} & \cdots & \beta_{2n} & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1} & -\beta_{n1} & \alpha_{n2} & -\beta_{n2} & \cdots & \alpha_{nn} & -\beta_{nn} \\ \beta_{n1} & \alpha_{n1} & \beta_{n2} & \alpha_{n2} & \cdots & \beta_{nn} & \alpha_{nn} \end{pmatrix} \tag{3.1}$$

in R^{2n}. That is, under the identification (1.5),

$$C\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \Phi\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix}.$$

Clearly, if

$$\Phi\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2n-1} \\ y_{2n} \end{pmatrix}, \tag{3.2}$$

then

$$\Phi\begin{pmatrix} -x_2 \\ x_1 \\ \vdots \\ -x_{2n} \\ x_{2n-1} \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \\ \vdots \\ -y_{2n} \\ y_{2n-1} \end{pmatrix}. \tag{3.3}$$

Hence the operator L is invariant under the mapping Φ; that is, for any smooth function f in D,

$$L(f \circ \Phi) = (Lf) \circ \Phi.$$

In fact, if let $\widetilde{Z} = \Phi(Z)$, then

$$d\widetilde{Z}_t = \sigma(\widetilde{Z}_t)\, d\widetilde{B}_t, \qquad t < \tau, \tag{3.4}$$

where $\widetilde{B} = \Phi \circ B$, which is a Brownian motion in R^{2n}. Therefore, for $0 < r \leq 1$ and $A \subset \partial D_r$,

$$P_z(Z_{\tau_r} \in A) = P_{\Phi(z)}(Z_{\tau_r} \in \Phi(A)), \qquad z \in D. \tag{3.5}$$

In particular, this implies the following proposition.

PROPOSITION 3.1. *Under P_0, the distribution of Z_{τ_r} is a uniform distribution on ∂D_r for $0 < r \leq 1$; the distribution of Z_{τ_r} is $(\sigma_r(dx))/(\sigma_r(\partial D_r))$, where σ_r is the Lebesgue surface measure on ∂D_r.*

4. Harnack inequality and uniform continuity. In this section, we assume that $0 \leq \alpha \leq 1/2$. For $\gamma > 0$ and $\varepsilon > 0$, define

$$\begin{aligned} I_\gamma(\varepsilon) = \{ z = (x_1, x_2, \dots, x_{2n})' \in \partial D \colon x_1 > 0, \\ |x_2| < \gamma\varepsilon,\ |(x_3, \dots, x_{2n})| < \gamma\varepsilon^{1-\alpha} \}, \end{aligned} \tag{4.1}$$

and for $0 < r < 1$, define

$$rI_\gamma(\varepsilon) = \left\{ z \in D \colon |z| = r,\ \frac{z}{r} \in I_\gamma(\varepsilon) \right\}.$$

THEOREM 4.1. (i) *For any given constant $\gamma > 0$, there are constants $\lambda = \lambda(\alpha, \gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(\alpha, \gamma) \in (0, \min\{1/4, 1/(4\gamma^2)\})$ such that for $0 < \varepsilon < \varepsilon_0$*

$$\sup_{z \in \sqrt{1-\varepsilon} I_\gamma(\varepsilon)} u(z) \leq \lambda \inf_{z \in \sqrt{1-\varepsilon} I_\gamma(\varepsilon)} u(z),$$

for any L-harmonic function $u \geq 0$ in D.

(ii) *For any given $\delta > 0$, there is $\gamma_0 = \gamma_0(\alpha, \delta) > 0$ such that for $0 < \varepsilon < \varepsilon_0(\alpha, \gamma)$ and $0 < \gamma < \gamma_0$,*

$$\mathrm{osc}_{\sqrt{1-\varepsilon} I_\gamma(\varepsilon)}\, u \leq \delta \|u\|_\infty,$$

where $\mathrm{osc}_{\sqrt{1-\varepsilon} I_\gamma(\varepsilon)}\, u = \sup_{z, w \in \sqrt{1-\varepsilon} I_\gamma(\varepsilon)} |u(z) - u(w)|$ and $\|u\|_\infty = \sup_{z \in D} |u(z)|$.

PROOF. For $\gamma > 0$, let

$$\begin{aligned} U = U_\gamma(\varepsilon) = \Bigg\{ z = (x_1, x_2, \dots, x_{2n})' \in D \colon x_1 > 0,\ 1 - \frac{3\varepsilon}{2} < |z|^2 < 1 - \frac{\varepsilon}{2}, \\ |x_2| < 2\gamma\varepsilon,\ \text{and } |(x_3, \dots, x_{2n})| < 2\gamma\varepsilon^{1-\alpha} \Bigg\} \end{aligned}$$

and

$$\begin{aligned} K = K_\gamma(\varepsilon) = \Bigg\{ z = (x_1, x_2, \dots, x_{2n})' \in D \colon x_1 > 0,\ 1 - \frac{5\varepsilon}{4} \leq |z|^2 \leq 1 - \frac{3\varepsilon}{4}, \\ |x_2| \leq \gamma\varepsilon,\ \text{and } |(x_3, \dots, x_{2n})| \leq \gamma\varepsilon^{1-\alpha} \Bigg\}. \end{aligned}$$

We observe that $\sqrt{1-\varepsilon}I_\gamma(\varepsilon) \subset K \subset U$ and $U_\gamma(\varepsilon) \subset D$ when $\varepsilon < \min\{1/4, (2\gamma)^{-2}\}$. On $U = U_\gamma(\varepsilon)$, we introduce the following transformation $T = T_\varepsilon$:

$$(u_1, u_2, u_3, \dots, u_{2n})' = T_\varepsilon(x_1, x_2, x_3, \dots, x_{2n})' = \left(\frac{1-|z|^2}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon^{1-\alpha}}, \dots, \frac{x_{2n}}{\varepsilon^{1-\alpha}}\right)'.$$

Then

$$\frac{\partial(u_1, u_2, u_3, \dots, u_{2n})}{\partial(x_1, x_2, x_3, \dots, x_{2n})} = \begin{pmatrix} -\frac{2x_1}{\varepsilon} & -\frac{2x_2}{\varepsilon} & -\frac{2x_3}{\varepsilon} & \cdots & -\frac{2x_{2n}}{\varepsilon} \\ 0 & \frac{1}{\varepsilon} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\varepsilon^{1-\alpha}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\varepsilon^{1-\alpha}} \end{pmatrix}.$$

Notice that $T_\varepsilon(U) = \{(u_1, u_2, u_3, \dots, u_{2n}): |u_1| < 1/2, |u_2| < 2\gamma$ and $|(u_3, \dots, u_{2n})| < 2\gamma\}$ and $T_\varepsilon(K) = \{(u_1, u_2, u_3, \dots, u_{2n}): |u_1| \le 1/4, |u_2| \le \gamma$ and $|(u_3, \dots, u_{2n})| \le \gamma\}$. Let

$$a(z) = (a_{ij}(z))_{2n\times 2n} = ((\sigma\sigma')_{ij}(z))_{2n\times 2n}.$$

Under the new coordinates $(u_1, u_2, u_3, \dots, u_{2n})$,

$$\begin{aligned} L(z) &= \sum_{i,j=1}^{2n} a_{ij}(z)\frac{\partial^2}{\partial x_i\,\partial x_j} \\ &= \sum_{k,l=1}^{2n}\left(\sum_{i,j=1}^{2n} a_{ij}(z)\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right)\frac{\partial^2}{\partial u_k \partial u_l} + \frac{1}{\varepsilon}\sum_{k=1}^{2n}\left(\sum_{i,j=1}^{2n}\frac{\partial^2 u_k}{\partial x_i\,\partial x_j}\right)\frac{\partial}{\partial u_k} \\ &= \sum_{k,l=1}^{2n}\left(\sum_{i,j=1}^{2n} a_{ij}(z)\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right)\frac{\partial^2}{\partial u_k\,\partial u_l} - \frac{1}{\varepsilon}\left(\sum_{i,j=1}^{2n}\frac{\partial^2|z|^2}{\partial x_i\,\partial x_j}\right)\frac{\partial}{\partial u_1} \\ &= \sum_{k,l=1}^{2n}\left(\sum_{i,j=1}^{2n} a_{ij}(z)\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right)\frac{\partial^2}{\partial u_k\,\partial u_l} - \frac{2}{\varepsilon}\left(\sum_{i=1}^{2n} a_{ii}(z)\right)\frac{\partial}{\partial u_1}. \end{aligned} \tag{4.2}$$

Thus

$$\begin{aligned} \varepsilon^2 L(z) &= \sum_{k,l=1}^{2n}\left(\varepsilon^2\sum_{i,j=1}^{2n} a_{ij}(z)\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right)\frac{\partial^2}{\partial u_k\,\partial u_l} - 2\varepsilon\left(\sum_{i=1}^{2n} a_{ii}(z)\right)\frac{\partial}{\partial u_1} \\ &\equiv \sum_{k,l=1}^{2n}\alpha_{kl}(T_\varepsilon z)\frac{\partial^2}{\partial u_k\,\partial u_l} - \beta_1(T_\varepsilon z)\frac{\partial}{\partial u_1} \end{aligned} \tag{4.3}$$

Note that on U,

$$\beta_1(T_\varepsilon z) = 2\varepsilon\left(\sum_{i=1}^{2n} a_{ii}(z)\right) \tag{4.4}$$
$$= 2\varepsilon\left(\frac{2(n-1)}{(1-|z|^2)^{2\alpha}} + 2\right) \le 2^{2+2\alpha}(n-1)\varepsilon^{1-2\alpha} + 4\varepsilon,$$

which is uniformly bounded for $\varepsilon \in (0, 1/4)$. We now show that $(\alpha_{kj})_{2n\times 2n}$ is uniformly bounded and elliptic on U. For any unit vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \ldots, \xi_{2n})'$ in R^{2n} and $z \in U$, let

$$\boldsymbol{\eta} = \frac{\partial(u_1, u_2, u_3, \ldots, u_{2n})}{\partial(x_1, x_2, x_3, \ldots, x_{2n})}\boldsymbol{\xi};$$

that is, $\eta_1 = -2x_1\xi_1/\varepsilon$, $\eta_2 = -2x_2\xi_1/\varepsilon + \xi_2/\varepsilon$, $\eta_3 = -2x_3\xi_1/\varepsilon + \xi_3/\varepsilon^{1-\alpha}$, ... and $\eta_{2n} = -2x_{2n}\xi_1/\varepsilon + \xi_{2n}/\varepsilon^{1-\alpha}$. Let $P_z\boldsymbol{\eta} = (\eta\cdot z)z/|z|^2$ be the projection of $\boldsymbol{\eta}$ to the complex line $\mathrm{C}z$:

$$|P_z\boldsymbol{\eta}|^2 = \frac{1}{|z|^2}\left[\left(-\frac{2|z|^2\xi_1}{\varepsilon} + \frac{x_2\xi_2}{\varepsilon} + \sum_{j=3}^{2n}\frac{x_j\xi_j}{\varepsilon^{1-\alpha}}\right)^2\right. \tag{4.5}$$
$$\left. + \left(\frac{x_1\xi_2}{\varepsilon} + \sum_{k=2}^{n}\frac{(x_{2k-1}\xi_{2k} - x_{2k}\xi_{2k-1})}{\varepsilon^{1-\alpha}}\right)^2\right]$$
$$= \frac{4|z|^2\xi_1^2}{\varepsilon^2} + \frac{x_1^2\xi_2^2}{|z|^2\varepsilon^2} + A(\varepsilon, z, \boldsymbol{\xi}),$$

where

$$A(\varepsilon, z, \boldsymbol{\xi}) = \frac{1}{|z|^2}\left(\frac{x_2\xi_2}{\varepsilon} + \sum_{j=3}^{2n}\frac{x_j\xi_j}{\varepsilon^{1-\alpha}}\right)^2 + \frac{1}{|z|^2}\left(\sum_{k=2}^{n}\frac{(x_{2k-1}\xi_{2k} - x_{2k}\xi_{2k-1})}{\varepsilon^{1-\alpha}}\right)^2 \tag{4.6}$$
$$- \frac{4\xi_1}{\varepsilon}\left(\frac{x_2\xi_2}{\varepsilon} + \sum_{j=3}^{2n}\frac{x_j\xi_j}{\varepsilon^{1-\alpha}}\right) + \frac{2x_2\xi_2}{\varepsilon}\sum_{k=2}^{n}\frac{x_{2k-1}\xi_{2k} - x_{2k}\xi_{2k-1}}{\varepsilon^{1-\alpha}}.$$

It follows from the definition of U,

$$|A(\varepsilon, z, \boldsymbol{\xi})| \le 5\gamma^2 + \frac{10\gamma}{\varepsilon} \tag{4.7}$$

for $\varepsilon > 0$, $z \in U$ and unit vector $\boldsymbol{\xi} \in \mathrm{R}^{2n}$. On the other hand,

$$|\boldsymbol{\eta}|^2 = \sum_{j=1}^{2n}\eta_j^2$$
$$= \frac{4|z|^2\xi_1^2}{\varepsilon^2} + \frac{\xi_2^2}{\varepsilon^2} + \sum_{j=3}^{2n}\frac{\xi_j^2}{\varepsilon^{2(1-\alpha)}} - \left(\frac{4x_2\xi_2}{\varepsilon^2} + \sum_{k=3}^{2n}\frac{4x_k\xi_k}{\varepsilon^{2-\alpha}}\right)\xi_1.$$

Thus

$$
\begin{aligned}
|\boldsymbol{\eta} - P_z\boldsymbol{\eta}|^2 &= |\boldsymbol{\eta}|^2 - |P_z\boldsymbol{\eta}|^2 \\
&= \sum_{j=3}^{2n} \frac{\xi_j^2}{\varepsilon^{2(1-\alpha)}} + \frac{\xi_2^2}{\varepsilon^2}\frac{|z|^2 - x_1^2}{|z|^2} \\
&\quad - \left(\frac{4x_2\xi_2}{\varepsilon^2} + \sum_{k=3}^{2n}\frac{4x_k\xi_k}{\varepsilon^{2-\alpha}}\right)\xi_1 - A(\varepsilon, z, \boldsymbol{\xi}) \\
&= \sum_{j=3}^{2n} \frac{\xi_j^2}{\varepsilon^{2(1-\alpha)}} + \frac{\xi_2^2}{\varepsilon^2}\frac{|z|^2 - x_1^2}{|z|^2} + B(\varepsilon, z, \boldsymbol{\xi}),
\end{aligned} \tag{4.8}
$$

where

$$
\begin{aligned}
B(\varepsilon, z, \boldsymbol{\xi}) = -\frac{1}{|z|^2}\Bigg(&\left(\frac{x_2\xi_2}{\varepsilon} + \sum_{j=3}^{2n}\frac{x_j\xi_j}{\varepsilon^{1-\alpha}}\right)^2 \\
&+ \left(\sum_{k=2}^{n}\frac{(x_{2k-1}\xi_{2k} - x_{2k}\xi_{2k-1})}{\varepsilon^{1-\alpha}}\right)^2\Bigg) \\
&- \frac{2x_2\xi_2}{\varepsilon}\sum_{k=2}^{n}\frac{x_{2k-1}\xi_{2k} - x_{2k}\xi_{2k-1}}{\varepsilon^{1-\alpha}}.
\end{aligned} \tag{4.9}
$$

Thus for $0 < \varepsilon < \frac{1}{4}$, $z \in U$ and unit vector $\boldsymbol{\xi} \in \mathrm{R}^{2n}$,

$$
|B(\varepsilon, z, \boldsymbol{\xi})| \le 22\gamma^2. \tag{4.10}
$$

Since

$$
\sum_{k,l=1}^{2n} \alpha_{kl}(T_\varepsilon z)\,\xi_k\xi_l = \varepsilon^2 \sum_{i,j=1}^{2n} a_{ij}(z)\,\eta_i\eta_j = \varepsilon^2|P_z\boldsymbol{\eta}|^2 + \varepsilon^2\frac{|z - P_z\boldsymbol{\eta}|^2}{(1-|z|^2)^{2\alpha}}
$$

and for $z \in U$, $\varepsilon/2 < 1 - |z|^2 < 3\varepsilon/2$, it follows from (4.5)–(4.10) that for $0 < \varepsilon < 1/4$,

$$
\begin{aligned}
1 - \frac{33\gamma^2}{3}\varepsilon^2 - \left(10\gamma + \frac{102\gamma^2}{5}\right)\varepsilon &< \sum_{k,l=1}^{2n}\alpha_{kl}(T_\varepsilon z)\,\xi_k\xi_k \\
&< 4 + 5\gamma^2\varepsilon^2 + (10\gamma + 44\gamma^2)\,\varepsilon.
\end{aligned} \tag{4.11}
$$

Thus there is a constant $\varepsilon_0 = \varepsilon_0(\alpha, \gamma) \in (0, \min\{1/4, 1/(4\gamma^2)\})$, depending on γ only, such that for $\varepsilon \in (0, \varepsilon_0)$,

$$
\tfrac{1}{2} < \sum_{k,l=1}^{2n}\alpha_{kl}(T_\varepsilon z)\,\xi_k\xi_k < 5 \qquad \forall \boldsymbol{\xi} \in \mathrm{R}^{2n} \text{ with } |\boldsymbol{\xi}| = 1
$$

for any $z \in U = U_\gamma(\varepsilon)$. That is, $(\alpha_{kl})_{2n\times 2n}$ is uniformly elliptic and bounded on $T_\varepsilon(U)$. Thus by (4.3) and (4.4), $\{\varepsilon^2 L, 0 < \varepsilon < \varepsilon_0(\alpha, \gamma)\}$, under the coordinates

$(u_1, u_2, \dots, u_{2n})$ in $T_\varepsilon(U)$, is a family of uniformly elliptic and bounded differential operators. It follows from Corollary 9.25 of [5] that any nonnegative L-harmonic function on D has Harnack inequality

$$\sup_{z\in K_\gamma(\varepsilon)} u(z) \le \lambda \inf_{z\in K_\gamma(\varepsilon)} u(z)$$

with constant $\lambda = \lambda(\alpha, \gamma) > 0$ independent of u and $\varepsilon \in (0, \varepsilon_0)$. For $0 < \gamma < 1/2$, let

$$F_\gamma(\varepsilon) = T_\varepsilon^{-1}\big(\{(u_1, u_2, u_3, \dots, u_{2n}) \colon |u|_1 < \gamma,\ |u_2| < \gamma, \text{ and } |(u_3, \dots, u_{2n})| < \gamma\}\big);$$

that is,

$$F_\gamma(\varepsilon) = \{z = (x_1, x_2, \dots, x_{2n})' \in D \colon x_1 > 0, \ 1 - \varepsilon - \gamma\varepsilon \le |z|^2 \le 1 - \varepsilon + \gamma\varepsilon,\ |x_2| \le \gamma\varepsilon, \text{ and } |(x_3, \dots, x_{2n})| \le \gamma\varepsilon^{1-\alpha}\}.$$

Then by Corollary 9.24 of [5], the following Hölder estimate holds for any nonnegative L-harmonic function u on D:

$$\operatorname{osc}_{F_\gamma(\varepsilon)} u \le \lambda_1 (2\gamma)^r \operatorname{osc}_{F_{1/2}(\varepsilon)} u, \qquad 0 < \gamma < \tfrac{1}{2},$$

where λ_1 and $r > 0$ are constants independent of ε, γ and u. The theorem is thus proved by noting that $\sqrt{1-\varepsilon} I_\gamma(\varepsilon) \subset K_\gamma(\varepsilon) \bigcap F_\gamma(\varepsilon)$. □

REMARK 1. When $\alpha > 1/2$, the above argument in the proof of Theorem 4.1 breaks down, since $I_\gamma(\varepsilon)$ is no longer a relatively compact subset of D and $\varepsilon^2 L$ is not uniformly elliptic and bounded in $U(\varepsilon)$. However, we conjecture that Theorem 4.1 still holds for $\alpha > 1/2$ with $1-\alpha$ in the definition (4.1) of $I_\gamma(\varepsilon)$ be replaced by 1/2.

COROLLARY 4.2. *By the unitary invariance of L, Theorem 4.1 holds on $\Phi(\sqrt{1-\varepsilon}\, I_\gamma(\varepsilon))$ for any orthogonal transformation of the form (3.1).* □

5. Exit density and hitting probabilities. In this section, $\alpha \in [0, \infty)$ unless otherwise specified, as in Theorems 5.5–5.7 and Corollaries 5.6–5.8, where $\alpha \in [0, 1/2]$. Let $h(z, \xi)$, $z \in D$, $\xi \in \partial D$, denote the exit density function of the holomorphic diffusion Z_t with respect to the Lebesgue surface measure σ on ∂D (or equivalently, the Poisson kernel for L in D); that is, h is such a function that for any continuous function ϕ on ∂D,

$$E_z[\phi(Z_\tau)] = \int_{\partial D} \phi(\xi) h(z, \xi)\, d\sigma(\xi).$$

For each fixed $\xi \in \partial D$, $h_\xi(z) = h(z, \xi)$ is a minimal C^∞-smooth L-harmonic function in $z \in D$ and $h \in C(D \times \partial D)$. The existence of $h(z, \xi)$ as well as its aforementioned properties are shown in the Appendix. By the unitary invariance (3.5), we have

$$h(z,\xi)=h(\Phi(z),\,\Phi(\xi)) \tag{5.1}$$

for any orthogonal transformation Φ of (3.1). From Proposition 3.1, we see that

$$h(0,\xi)=\frac{1}{\sigma(\partial D)},\qquad \xi\in\partial D. \tag{5.2}$$

LEMMA 5.1. *Let τ_r be the first exit time of Z from D_r. For $z\in D$, $0<r<1$ and $A\in\mathscr{F}_{\tau_r}$, let*

$$P_z^{\xi}(A)=E_z[h(Z_{\tau_r},\xi);\,A]/h(z,\xi). \tag{5.3}$$

Then (5.3) uniquely defines a probability measure P_z^{ξ} on $\mathscr{F}_\tau\equiv\sigma(\bigcup_{0<r<1}\mathscr{F}_{\tau_r})$, the σ-algebra generated by $\bigcup_{0<r<1}\mathscr{F}_{\tau_r}$.

PROOF. For $0<s<r<1$, since $h(z,\xi)$ is a bounded L-harmonic for $z\in D_{r+\varepsilon}$, where $0<\varepsilon<1-r$, then by the optional stopping theorem, we have

$$E_z[h(Z_{\tau_r},\xi);\,A]/h(z,\xi)=E_z[h(Z_{\tau_s},\xi);\,A]/h(z,\xi),\qquad A\in\mathscr{F}_{\tau_s}.$$

That is, the definitions of P_z^{ξ} in (5.3) on $\mathscr{F}_{\tau_r}$, $0<r<1$, are consistent and therefore P_z^{ξ} is σ-additive on $\bigcup_{0<r<1}\mathscr{F}_{\tau_r}$. Then by the Carathéodory extension theorem, P_z^{ξ} in (5.3) uniquely defines a probability measure on $\sigma(\bigcup_{0<r<1}\mathscr{F}_{\tau_r})$. $\square$

REMARK 2. We will show in Corollary 5.8 below that under P_z^{ξ}, $\{Z_t,\ 0\le t<\tau\}$ is the process conditioned to exit D at ξ.

LEMMA 5.2. *Let $A\in\mathscr{F}_\tau$ and $z\in D$. Let $g(\xi)=P_z^{\xi}(A)$. Then $g(Z_\tau)=P_z(A|Z_\tau)$.*

PROOF. The proof is the same as that for (4) in Section 3.2 of [3]. $\square$

LEMMA 5.3. *If u is a bounded L-harmonic function in D, then $P_z^{\xi}(\lim_{r\uparrow 1}u(Z_{\tau_r})$ exists$)=1$ for each $z\in D$ and for a.e. $\xi\in\partial D$.*

PROOF. Let $A=\{\lim_{r\uparrow 1}u(Z_{\tau_r})$ exists$\}$ and $g(\xi)=P_z^{\xi}(A)$. Since $A\in\mathscr{F}_\tau$, it follows from Lemma 5.2 that $g(Z_\tau)=P_z(A|Z_\tau)$. Hence

$$E_z[g(Z_\tau)]=E_z[P_z(A|Z_\tau)]=P_z(A)=1,$$

where the last equality holds since $\{Z_{\tau_r},\mathscr{F}_{\tau_r}\}_{0\le r<1}$ is a bounded P_z-martingale. Combining

$$\int_{\partial D}g(\xi)h(z,\xi)\,\sigma(d\xi)=E_zg(Z_\tau)=1$$

with the fact that $g(\xi) \leq 1$ and $\int_{\partial D} h(z,\xi)\,\sigma(d\xi) = 1$ yields $g(\xi) = 1$ a.e. on ∂D. □

DEFINITION 5.1. An event $A \in \mathscr{F}_\tau$ is said to be shift invariant if for any stopping time $T < \tau$, $\theta_T^{-1}(A) = A$ P_z-a.s for all $z \in D$, where θ is the shifting operator on the sample space such that $Z_s \circ \theta_t = Z_{s+t}$ for $0 \leq t < \tau(\omega) - s$.

LEMMA 5.4 (0-1 law). *Let $A \in \mathscr{F}_\tau$ be a shift invariant event. Then $z \to P_z^\xi(A)$ is a constant function which is either* 0 *or* 1*.*

PROOF. With the fact that the exit density functions $\{h(z,\xi),\ \xi \in \partial D\}$ are minimal L-harmonic functions in $z \in D$ and with the establishment of representation of a nonnegative L-harmonic $u \leq h(\cdot,\xi)$ in terms of $h(\cdot,\xi)$ in Theorems A.4 and A.5 in the Appendix, the rest of the proof is the same as that in Section 3.5 of [3]. □

THEOREM 5.5. *For $0 \leq \alpha \leq 1/2$ and $\beta > 0$, the exit density function $h(z,\xi)$ of $Z = Z^\alpha$ from D has the estimate*

$$h(z,1) \geq \lambda_1 (1 - |z|^2)^{-1-2(n-1)(1-\alpha)}, \qquad z \in \mathscr{A}_\beta(1) \quad \text{with } |z|^2 > 1 - \varepsilon_0(\alpha,\beta),$$

where $\varepsilon_0(\alpha,\beta)$ is the constant in Theorem 4.1(1) and $\lambda_1 = \lambda_1(\alpha,\beta) > 0$ is a constant independent of z.

PROOF. Without loss of generality, we may assume that $\beta > B$, where B is the constant in Theorem 2.8. For $z \in \mathscr{A}_\beta(1)$ with $|z|^2 > 1 - \varepsilon_0(\alpha,\beta)$, let $\varepsilon = 1 - |z|^2$. Note that $0 < \varepsilon < 1/4$ and $\beta^2\varepsilon < 1/4$. By Theorem 2.8 and the mean value theorem,

$$\tfrac{1}{6} < p \equiv \int_{I_B(\varepsilon)} h\big(\sqrt{1-\varepsilon}\,1,\ \xi\big)\sigma(d\xi) = h\big(\sqrt{1-\varepsilon}\,1,\ \eta\big)\sigma(I_B(\varepsilon)) \tag{5.4}$$

for some $\eta \in I_B(\varepsilon)$. Let Ψ be the reflection transformation with respect to the complex line Cw with $w = (\eta+1)/|\eta+1|$; that is,

$$\Psi(z) = (z\cdot w)w - (z - (z\cdot w)w) = 2(z\cdot w)w - z, \qquad z \in \mathrm{C}^n. \tag{5.5}$$

The transformation Ψ is a unitary transformation of C^n such that $\Psi(\eta) = 1$ and $\Psi(1) = \eta$. Observe that

$$\sigma(I_\gamma(\varepsilon)) = c_n \gamma^{2n-1} \varepsilon^{1+2(n-1)(1-\alpha)} \quad \text{for } \gamma\varepsilon^{1-\alpha} < 1, \tag{5.6}$$

where c_n is a constant independent of γ. Note that $\mathscr{A}_\beta(1) \cap \{w \in D\colon |w|^2 = 1-\varepsilon\} \subset \sqrt{1-\varepsilon}\, I_\beta(\varepsilon)$. Thus by Theorem 4.1(1), the unitary invariance property

(5.1) and (5.4), we have that for $z \in \mathscr{A}_\beta(1)$ with $|z|^2 > 1 - \varepsilon_0(\alpha, \beta)$,

$$\begin{aligned} h(z, 1) &\geq \lambda^{-1} h(\sqrt{1-\varepsilon}\,\eta,\ 1) \\ &= \lambda^{-1}\, h(\sqrt{1-\varepsilon}\,\Psi(\eta),\ \Psi(1)) \\ &= \lambda^{-1}\, h(\sqrt{1-\varepsilon}\,1,\ \eta) \\ &> \frac{1}{6\lambda}\,\sigma(I_B(\varepsilon))^{-1} \\ &= (6c_n\lambda)^{-1} B^{-(2n-1)} \varepsilon^{-1-2(n-1)(1-\alpha)} \\ &= \lambda_1 (1 - |z|^2)^{-1-2(n-1)(1-\alpha)}, \end{aligned}$$

where $\lambda = \lambda(\alpha, \beta)$ is the constant in Theorem 4.1(1) and $\lambda_1 = (6c_n\lambda)^{-1} B^{-(2n-1)} > 0$ which is independent of z. □

For $\xi \in \partial D$, let Φ be an orthogonal transformation of (3.1) such that $\Phi(1) = \xi$. Then $\mathscr{A}_\beta(\xi) = \Phi\mathscr{A}_\beta(1)$ for $\beta > 0$. By the unitary invariance (5.1), we have

COROLLARY 5.6. *For $0 \leq \alpha \leq 1/2$ and $\xi \in \partial D$, the exit density function $h(z, \xi)$ of $Z = Z^\alpha$ has the estimate*

$$h(z, \xi) \geq \lambda_1 (1 - |z|^2)^{-1-2(n-1)(1-\alpha)}, \qquad z \in \mathscr{A}_\beta(\xi) \quad \text{with } |z|^2 > 1 - \varepsilon_0(\alpha, \beta),$$

where λ_1 is the constant in Theorem 5.5. □

Recall that $I_\gamma(\varepsilon)$ is an open subset of ∂D defined by (4.1).

THEOREM 5.7. *Suppose that $0 \leq \alpha \leq 1/2$. For a sequence of points $\{z_k;\ k \geq 1\}$ in $\mathscr{A}_\beta(\xi)$ with $z_k \to \xi \in \partial D$ as $k \to \infty$, let Φ_k be an orthogonal transformation of (3.1) such that $\Phi_k(1) = z_k/|z_k|$. For $\gamma > 0$, let $R_k \equiv R_k(\gamma) = |z_k|\Phi_k(I_\gamma(1 - |z_k|^2))$, for $k \geq 1$. Then*

$$P_z^\xi(Z_t,\ 0 \leq t < \tau,\ \text{hits infinitely many } R_k) = 1, \qquad z \in D.$$

PROOF. Let τ_k be the first hitting time of $\{z\colon |z| = |z_k|\}$ by Z. Then by the definition (5.3) of P_0^ξ, Proposition 3.1 and (5.2),

$$\begin{aligned} p_k &\equiv P_0^\xi(Z_t \text{ hits } R_k \text{ for } t < \tau) \\ &= E_0[h(Z_{\tau_k}, \xi);\ Z_{\tau_k} \in R_k]/h(0, 0) \\ &= \frac{1}{\sigma_k(\partial D_{|z_k|})\,\sigma(\partial D)} \int_{R_k} h(z, \xi)\,\sigma_k(dz), \end{aligned}$$

where σ_k is the Lebesgue surface measure on $\partial D_{|z_k|}$. A simple calculation shows that $R_k \subset \mathscr{A}_{\beta_1}(\xi)$, where $\beta_1 = \max\{\beta + 2\gamma,\ \sqrt{2}\gamma(\beta + 1 + \sqrt{4\gamma^2 + 1})\}$. Applying Corollaries 4.2 and 5.6 to the L-harmonic function $h(\cdot, \xi)$ on R_k, we obtain that for some constants $\lambda = \lambda(\alpha, \gamma) > 0$ and $\lambda_1 = \lambda_1(\alpha, \beta_1) > 0$,

$$\begin{aligned} p_k &\geq \frac{1}{\sigma_k(\partial D_{|z_k|})\,\sigma(\partial D)}\,\frac{1}{\lambda}\,h(z_k, \xi)\sigma_k(R_k) \\ &\geq \frac{1}{\lambda\,\sigma_k(\partial D_{|z_k|})\,\sigma(\partial D)}\lambda_1(1-|z_k|^2)^{-1-2(n-1)(1-\alpha)}\,c_n\gamma^{2n-1}(1-|z_k|^2)^{1+2(n-1)(1-\alpha)} \\ &\geq \frac{c_n\lambda_1\gamma^{2n-1}}{\lambda\,|z_k|^{2n-1}\,\sigma(\partial D)^2}, \end{aligned}$$

where constant c_n in the second inequality is the one in (5.6). It follows from Fatou's lemma and the 0-1 law of Lemma 5.4 that

$$P_z^{\xi}\Big(\bigcap_{m\geq 1}\bigcup_{k\geq m}\{Z_t \text{ hits } R_k\}\Big) = 1. \qquad \square$$

COROLLARY 5.8. *Suppose that* $0 \leq \alpha \leq 1/2$. *For each* $\xi \in \partial D$, *we have*

$$P_z^{\xi}\Big(\lim_{t\uparrow\tau} Z_t = \xi\Big) = 1, \qquad z \in D.$$

PROOF. Let $A = \{\lim_{t\uparrow\tau} Z_t$ exists and $\lim_{t\uparrow\tau} Z_t \in \partial D\}$, which is a measurable event in $\mathscr{F}_\tau$. Applying Lemma 5.3 to the holomorphic function $u(z) = z$, we have that for a.e. $\xi \in \partial D$, $P_z^{\xi}(A) = 1$ for all $z \in D$. It follows from the unitary invariance of Z [see, in particular, (3.4)] that $P_z^{\xi}(A) = P_{\Phi(z)}^{\Phi(\xi)}(A)$ for any orthogonal transformation Φ in R^{2n} of the form (3.1). Therefore for each $\xi \in \partial D$, $P_z^{\xi}(A) = 1$ for all $z \in D$ and Corollary 5.8 is proved in view of Theorem 5.7. $\square$

6. Proof of Theorems 1.1 and 1.2.

Now we are in the position to prove the main results of this paper. In this section, $\alpha \in [0, 1/2]$. Keep in mind that throughout this section the super- and subscript α are suppressed from the ones that appeared in the statement of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. From Lemma 5.3, $\lim_{r\uparrow 1} u(Z_{\tau_r})$ exists P_z^{ξ} almost surely for every $z \in D$ and a.e. $\xi \in \partial D$. Let $\psi(\xi) = \lim_{r\uparrow 1} u(Z_{\tau_r})$ under P_0^{ξ}. We show that $\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u(z) = \psi(\xi)$.

Let $b = \limsup_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u(z)$. By the uniform continuity property of Corollary 4.2, for any $\delta > 0$, there exists $0 < \gamma < 1/2$ and $\varepsilon_0 = \varepsilon_0(\alpha, \gamma)$ so that for all $0 < \varepsilon < \varepsilon_0$ and every orthogonal transformation Φ of form (3.1), $\operatorname{osc}_{\sqrt{1-\varepsilon}\Phi(I_\gamma(\varepsilon))} u < \delta$. Let $z_k \in \mathscr{A}_\beta(\xi)$ so that $z_k \to \xi$ as $k \to \infty$ and $u(z_k) > b - \delta$ for all k. Let Φ_k be an orthogonal transformation of the form (3.1) such that $\Phi_k(1) = z_k/|z_k|$. Then $u \geq b - 2\delta$ on $\sqrt{1-\varepsilon_k}\Phi_k(I_\gamma(\varepsilon))$ for all k such that $\varepsilon_k \equiv 1 - |z_k| < \varepsilon_0$. Since Z_t hits infinitely many $\sqrt{1-\varepsilon_k}\Phi_k(I_\gamma(\varepsilon))$ under P_z^{ξ} by Theorem 5.7, $\psi(\xi) = \lim_{r\uparrow 1} u(Z_{\tau_r}) \geq b - 2\delta$ P_z^{ξ}-a.s. After letting $\delta \downarrow 0$, we get

$$\psi(\xi) \geq b = \limsup_{\mathscr{A}_\beta(\xi) \ni z \to \xi} u(z), \qquad P_z^\xi\text{- a.s.}$$

Applying this to $-u$, we see that

$$\psi(\xi) \leq \liminf_{\mathscr{A}_\beta(\xi) \ni z \to \xi} u(z), \qquad P_z^\xi\text{-a.s.}$$

Consequently, $\psi(\xi)$ is not random and equals $\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u(z)$ for a.e. $\xi \in \partial D$.

To show the probabilistic representation of u in terms of its boundary data ψ, let $\tau_k = \tau_{|z_k|}$. Since u is a bounded L-harmonic function in D, by the optional stopping theorem we see that $\{Z_{\tau_k}, \mathscr{F}_{\tau_k}\}_{k\geq 1}$ is a bounded martingale under P_z and therefore $u(z) = E_z[u(Z_{\tau_k})]$ for $z \in D$. Let $A = \{\lim_{k\uparrow\infty} u(Z_{\tau_k}) = \psi(Z_\tau)\}$. Note that $P_z^\xi(A) = 1$ for a.e. $\xi \in \partial D$ and every $z \in D$, because $P_z^\xi(Z_\tau = \xi) = 1$ by Corollary 5.8. Thus it follows from Lemma 5.2 that $P_z(A) = \int_{\partial D} P_z^\xi(A) h(z,\xi)\,\sigma(d\xi) = 1$. That is, $\lim_{k\to\infty} u(Z_{\tau_k}) = \psi(Z_\tau)$, P_z-a.s. for all $z \in D$. Now passing $k\to\infty$ in $u(z) = E_z[u(Z_{\tau_k})]$ yields that $u(z) = E_z[\psi(Z_\tau)]$. □

We are now going to prove a local version of Theorem 1.1. We begin with two lemmas. For $\xi \in \partial D$, define $I_\gamma^\xi(\varepsilon) = \Phi I_\gamma(\varepsilon)$, where Φ is any orthogonal transformation in R^{2n} of the form (3.1) such that $\Phi(1) = \xi$. This definition is well defined since, as can be seen from (3.2) and (3.3), $\Psi I_\gamma(\varepsilon) = I_\gamma(\varepsilon)$ for any orthogonal transformation Ψ of (3.1) with $\Psi(1) = 1$. In fact, $I_\gamma^\xi(\varepsilon)$ can be explicitly written down:

$$I_\gamma^\xi(\varepsilon) = \bigl\{\eta \in \partial D\colon \mathrm{Re}(\eta\cdot\xi) > 0,\ |\mathrm{Im}(\eta\cdot\xi)| < \gamma\varepsilon, \text{ and } |\eta - (\eta\cdot\xi)\xi| < \gamma\varepsilon^{1-\alpha}\bigr\}. \tag{6.1}$$

LEMMA 6.1. *For $k > \sup_{\{|z|<1/2\}}|u(z)|$, let $F_\beta^k = \{\xi \in \partial D\colon N_\beta(\xi) \leq k\}$. Set $D_0 = D_{1/2} \cup \bigcup_{\xi\in F_\beta^k} \tilde{\mathscr{A}}_\beta(\xi)$. Then for $\beta > B\sqrt{4B^2+1}+1$ and $z \in D\setminus D_0$, we have $\partial D \setminus F_\beta^k \supset I_B^\xi(\varepsilon)$, where B is the constant in Theorem 2.8, $\xi = z/|z|$ and $\varepsilon = 1 - |z|^2$.*

PROOF. Note that $\eta \in \partial D$, $|\eta - \xi| \geq |z-\eta| - |z-\xi| \geq |z-\eta| - \varepsilon$ and $|(\xi-\eta)\xi| = |(\eta-\xi)\eta| \geq |(z-\eta)\eta| - |(z-\xi)\eta| \geq |(z-\eta)\eta| - \varepsilon$. Thus for $\eta \in F_\beta^k$, since $z \notin \tilde{\mathscr{A}}_\beta(\eta)$, either $\mathrm{Re}(\eta\xi) = |z|^{-1}\mathrm{Re}(z\xi) \leq 0$ or $|\eta - \xi| > (\beta-1)\varepsilon^{1-\alpha}$ or $|(\xi-\eta)\xi| > (\beta-1)\varepsilon$. However, an easy calculation shows that

$$\begin{aligned} I_B^\xi(\varepsilon) &\subset \bigl\{\eta\in\partial D\colon \mathrm{Re}(\eta\xi) > 0,\ |(\xi-\eta)\xi| < B\sqrt{4B^2+1}\,\varepsilon \\ &\qquad\qquad \text{and } |\eta-\xi| < 2B\varepsilon^{1-\alpha}\bigr\} \\ &\subset \bigl\{\eta\in\partial D\colon \mathrm{Re}(\eta\xi) > 0,\ |(\xi-\eta)\xi| < B\sqrt{4B^2+1}\,\varepsilon \\ &\qquad\qquad \text{and } |\eta-\xi| < B\sqrt{4B^2+1}\,\varepsilon^{1-\alpha}\bigr\}, \end{aligned} \tag{6.2}$$

since $B > 1$. Thus for $\beta > B\sqrt{4B^2+1}+1$, we have $F_\beta^k \cap I_B^\xi(\xi) = \varnothing$; that is, $\partial D\setminus F_\beta^k \supset I_B^\xi(\varepsilon)$. □

LEMMA 6.2. *Let k, F^k_β and D_0 be as in Lemma 6.1 with $\beta > B\sqrt{4B^2+1}+1$. Let $\tau_0 = \inf\{t > 0\colon Z_t \in D \setminus D_0\}$. Then for $z \in D \setminus D_0$, $P_z(Z_\tau \notin F^k_\beta) \geq \frac{1}{6} P_z(\tau_0 < \tau)$.*

PROOF. Note that D_0 is an open set since each $\tilde{\mathscr{A}}_\beta(\xi)$ is open. By the continuity of Z inside D, $Z_{\tau_0} \in \partial D_0 \cap D$ on $[\tau_0 < \tau]$. Thus by the strong Markov property of Z, Theorem 2.8 and unitary invariance property (3.5),

$$\begin{aligned} P_z(Z_\tau \notin F^k_\beta) &= P_z(Z_\tau \notin F^k_\beta;\ \tau_0 < \tau) \\ &= E_z\big[P_{Z_{\tau_0}}(Z_\tau \notin F^k_\beta;\ \tau_0 < \tau)\big] \\ &\geq \tfrac{1}{6} P_z(\tau_0 < \tau). \end{aligned}$$

□

PROOF OF THEOREM 1.2. Let k, F^k_β and D_0 be as in Lemma 6.1 with $\beta > B\sqrt{4B^2+1}+1$ and $\tau_0 = \inf\{t > 0\colon Z_t \in D \setminus D_0\}$. Note that by Theorem 4.1(1) and Corollary 4.2, $F = \{\xi \in \partial D\colon N^\alpha_\gamma(\xi) < \infty \text{ for all } \gamma > 0\}$ and therefore $F = \bigcup_{k=1}^\infty F^k_\beta$. Let $A = \{\lim_{r\uparrow 1} u(Z_{\tau_r}) \text{ exists}\}$, which is a shift invariant event in $\mathscr{F}_\tau$. On A, denote the limit of $u(Z_{\tau_r})$ as $r \uparrow 1$ by η. It follows from Theorem 5.7 that $|\eta| \leq k\ P^\xi_z$-a.e. on A for $\xi \in F^k_\beta$. Since the L-harmonic function u is bounded on D_0 by k, $\{u(Z_{\tau_0\wedge\tau_r}), \mathscr{F}_{\tau_0\wedge\tau_r},\ 0 \leq r < 1\}$ is a bounded martingale and therefore $\lim_{r\uparrow 1} u(Z_{\tau_0\wedge\tau_r})$ exists almost surely. Thus $[\tau_0 = \tau] \subset A\ P_z$-a.e and $\lim_{r\uparrow 1} u(Z_{\tau_0\wedge\tau_r}) = \eta\ P_z$-a.s. on $[\tau_0 = \tau]$ for $z \in D$, while on $[\tau_0 < \tau]$, $\lim_{r\uparrow 1} u(Z_{\tau_0\wedge\tau_r}) = u(Z_{\tau_0})$. Therefore for $z \in D_0$,

$$\begin{aligned} u(z) &= \lim_{r\uparrow 1} u(Z_{\tau_0\wedge\tau_r}) \\ &= E_z[u(Z_{\tau_0});\ \tau_0 < \tau] + E_z[\eta;\ \tau_0 = \tau] \\ &= E_z[u(Z_{\tau_0});\ \tau_0 < \tau] + E_z[\eta;\ Z_\tau \in F^k_\beta] - E_z[\eta;\ Z_\tau \in F^k_\beta,\ \tau_0 < \tau] \\ &\equiv u_1(z) + u_2(z) + u_3(z). \end{aligned} \tag{6.3}$$

Since η and $Z_\tau \in F^k_\beta$ are both $\mathscr{F}_\tau$-measurable and shift invariant, u_2 is a bounded L-harmonic function on D and therefore by Theorem 1.1, $\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u_2(z)$ exists for a.e. $\xi \in \partial D$. On the other hand, for $i = 1, 3$, $|u_i(z)| \leq k P_z(\tau_0 < \tau)$. From Lemma 6.2, we have

$$|u_i(z)| \leq 6k\, E_z\big[1_{\partial D\setminus F^k_\beta}(Z_\tau)\big].$$

For $z \in D$, let $v(z) = 6kE_z[u(1_{\partial D\setminus F^k_\beta}(Z_\tau))]$, which is a bounded L-harmonic function in D. Therefore by Theorem 1.1 there is a bounded function ψ defined on ∂D such that

$$\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} v(z) = \psi(\xi), \quad \text{a.e. } \xi \in \partial D,$$

whereas $\psi(\xi)$ is also the limit of $v(Z_{\tau_r})$ under P^ξ_z as $r \uparrow 1$ as is seen in the proof of Theorem 1.1. By the martingale convergence theorem,

$\lim_{r\uparrow 1} v(Z_{\tau_r}) = 6k\,1_{\partial D\setminus F_\beta^k}(Z_\tau)$ P_z-a.s. and therefore it follows from Lemma 5.3 that $\lim_{r\uparrow 1} v(Z_{\tau_r}) = 6k\,1_{\partial D\setminus F_\beta^k}(\xi)$ P_z^ξ-a.s. for a.e. $\xi \in \partial D$. This says that $\psi(\xi) = 6k\,1_{\partial D\setminus F_\beta^k}(\xi)$ for a.e. ξ. Hence we have for $i = 1, 3$,

$$\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u_i(z) = 0 \quad \text{for } \sigma\text{-a.e. } \xi \in F_\beta^k. \tag{6.4}$$

Therefore (6.3) implies that $\lim_{\mathscr{A}_\beta(\xi)\ni z\to\xi} u(z)$ exists for σ-a.e. $\xi \in F_\beta^k$ and hence for a.e. $\xi \in F = \bigcup_{k=1}^\infty F_\beta^k$. □

APPENDIX

In this Appendix, $\alpha \in [0, \infty)$, and L^α and Z^α are simply denoted as L and Z. We will show the existence of an exit density function $h_\xi(z) \equiv h(z, \xi)$ for the diffusion Z (or equivalently, the Poisson kernel for L) in D as well as its minimal and continuity properties.

Let us denote ∂D by S. For $0 < r < 1$, we identify ∂D_r with rS. The diffusion matrix $\sigma\sigma'$, as can be seen from (2.1), has C^∞-smooth entries on $\overline{D}_r$ and its smallest eigenvalue is 1 and the largest eigenvalue on $\overline{D}_r$ is $(1 - r^2)^{-2\alpha}$. This warrants the existence of an exit density function $h_{r\xi}(z)$ (with respect to the Lebesgue surface measure σ_r on ∂D_r) of Z starting at $z \in D_r$ and exit ∂D_r at $r\xi$. Function $h_{r\xi}(z)$ is a nonnegative C^∞-smooth L-harmonic function in $z \in D_r$ for each $\xi \in S$ and is continuous on S as a function of ξ for each fixed z. Thus $h_{r\xi}(z)$ is a continuous function in $(z, \xi) \in D \times S$. We show below that the sequence of continuous functions $\{h_{r\xi}(z);\ r \in (r_1, 1)\}$ is equicontinuous in $D_{r_0} \times S$ for any $r_1 > r_0$.

LEMMA A.1. *The function $h_{r\xi}(z)$ is equicontinuous in (z, ξ) on $D_{r_0} \times S$ for $r \in (r_1, 1)$ where $r_1 > r_0$.*

PROOF. Let $z, w \in D_{r_0}$ and $\xi, \eta \in S$. By the unitary invariance of $h_{r\xi}(z)$ [see (3.5)], we may assume that ξ and η are in the complex plane spanned by z_1 and z_2 axes in C^n; that is, the coordinates of ξ and η in R^{2n} are $(\xi_1, \xi_2, \xi_3, \xi_4, 0, \dots, 0)'$ and $(\eta_1, \eta_2, \eta_3, \eta_4, 0, \dots, 0)'$. For such $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, 0, \dots, 0)' \in S$, let

$$\Phi_\xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & 0 & 0 & \cdots & 0 & 0 \\ -\xi_2 & \xi_1 & -\xi_4 & \xi_3 & 0 & 0 & \cdots & 0 & 0 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 & 0 & 0 & \cdots & 0 & 0 \\ -\xi_4 & -\xi_3 & \xi_2 & \xi_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{A.1}$$

which is an orthogonal transformation in R^{2n} of the form (3.1) and maps ξ to 1. By the unitary invariance (3.5) and the mean value theorem, we have

$$\begin{aligned}|h_{r\xi}(z)-h_{r\eta}(w)| &= |h_{r1}(\Phi_\xi(z))-h_{r1}(\Phi_\eta(w))| \\ &\le \Big(\sup_{z\in\bar{D}_{r_0}} |\nabla h_{r1}(z)|\Big)|\Phi_\xi(z)-\Phi_\eta(w)|.\end{aligned} \tag{A.2}$$

Since Φ_ξ and Φ_η are of the form (A.1),

$$\begin{aligned}|\Phi_\xi(z)-\Phi_\eta(w)| &\le |\Phi_\xi(z)-\Phi_\xi(w)|+|\Phi_\xi(w)-\Phi_\eta(w)| \\ &\le |z-w|+2|\xi-\eta|.\end{aligned} \tag{A.3}$$

For $0<r\le 1$ and a differentiable function u on D_r, define

$$[u]^*_{1,D_r} = \sup_{1\le k\le 2n}\sup_{z\in D_r}\left|(r-|z|)\frac{\partial u(z)}{\partial x_k}\right|,$$

$$[u]^*_{2,D_r} = \sup_{1\le k,l\le 2n}\sup_{z\in D_r}\left|(r-|z|)^2\frac{\partial^2 u(z)}{\partial x_k\,\partial x_l}\right|.$$

Then by (4.17)$''$, (6.82) and (6.14) of [5], there are constants $\lambda_2>0$ and $\lambda_3>0$ depending on r_1 but independent of $r\in(r_1,1)$ such that

$$\begin{aligned}(r_1-r_0)\sup_{z\in\bar{D}_{r_0}}|\nabla h_{r1}(z)| &\le [h_{r1}]^*_{1,D_{r_1}} \\ &\le \lambda_2\sup_{x\in D_{r_1}} h_{r1}(z)+[h_{r1}]^*_{2,D_{r_1}} \\ &\le \lambda_2\sup_{z\in D_{r_1}} h_{r1}(z)+\lambda_3\sup_{z\in D_{r_1}} h_{r1}(z).\end{aligned} \tag{A.4}$$

Note that by Proposition 3.1, $h_{r1}(0)=r^{-(2n-1)}\sigma(S)^{-1}$. Thus applying the Harnack inequality (cf. Corollary 9.25 of [5]) for the uniformly elliptic and bounded operator L on D_{r_1}, we have $\sup_{r_1<r<1}\sup_{z\in D_{r_1}} h_{r1}(z)<\infty$. This, combined with (A.2)–(A.4), completes the proof for this lemma. □

By the Arzela–Ascoli theorem and a diagonal selecting procedure, $\{h_{r1}(z)\colon 0<r<1\}$ has a subsequence $\{h_{r_k1}(z)\colon k\ge 1\}$ with $r_k\uparrow 1$ converging uniformly on each $D_{r_0}\times S$, $0<r_0<1$, as $k\to\infty$. Denote the limiting function as $h_\xi(z)$, which is nonnegative. Clearly $h_\xi(z)$ is continuous in $D_{r_0}\times S$, hence in $D\times S$. It follows from the following lemma that $h_\xi(z)$ is a C^∞-smooth L-harmonic function in $z\in D$ for each fixed $\xi\in S$.

LEMMA A.2. *If $\{u_k; k\ge 1\}$ is a sequence of L-harmonic functions in D_r, $0<r<1$, and is uniformly convergent to a function u on D_r, then u is C^∞-smooth and L-harmonic in D_r.*

PROOF. For any $r_1 \in (0, r)$ and $z \in D_{r_1}$,

$$u(z) = \lim_{k\to\infty} u_k(z) = \lim_{k\to\infty} E_z[u_k(Z_{\tau_{r_1}})] = E_z[u(Z_{\tau_{r_1}})] = \int_{r_1 S} u(\eta) h_\eta(z) \sigma_{r_1}(d\eta).$$

Thus u is C^∞-smooth and L-harmonic in D_{r_1} and hence in D_r. □

THEOREM A.3. *The function $h_\xi(z)$ obtained above is the exit density function of Z from D starting at $z \in D$.*

PROOF. By Dynkin's π-λ theorem, it suffices to show that for any open set U in S with $\sigma(\partial U) = 0$, $P_z(Z_\tau \in U) = \int_U h_\xi(z)\, \sigma(d\xi)$ holds for any $z \in D$, where ∂U is the boundary of U in S. Using the unitary invariance (3.5) for $r = 1$ and Theorem 2.9, we have $\lim_{r\uparrow 1} P_{r\xi}(Z_\tau \in U) = 1_U(\xi)$ for all $\xi \in S \setminus \partial U$. Thus by the strong Markov property of Z in D and the dominated convergence theorem,

$$\begin{aligned} P_z(Z_\tau \in U) &= \lim_{r\uparrow 1} E_z\big[P_{Z_{\tau_r}}(Z_\tau \in U)\big] \\ &= \lim_{r\uparrow 1} \int_S P_{r\xi}(Z_\tau \in U) h_{r\xi}(z)\, r^{2n-1}\, \sigma(d\xi) \\ &= \int_S 1_U(\xi) h_\xi(z)\, \sigma(d\xi). \end{aligned}$$

This proves the theorem. □

We now proceed to show the minimal L-harmonicity of $h_\xi(z)$ in D for each $\xi \in S$. The first step is to establish a representation theorem for nonnegative L-harmonic functions in terms of $\{h_\xi;\ \xi \in S\}$.

THEOREM A.4. *If $u \ge 0$ is L-harmonic in D, then there exists a nonnegative measure μ on S such that $u(z) = \int_S h_\xi(z)\, d\mu(\xi)$ for all $z \in D$.*

PROOF. The proof essentially imitates (2) in Section 3.5 of [3] for the classical harmonic functions and the Poisson kernel. For $0 < r < 1$, let μ_r be the measure on S that has the density function $u(r\xi)$ with respect to the Lebesgue surface measure σ on S. By Proposition 3.1, we have for $0 < r < 1$,

$$\begin{aligned} \mu_r(S) &= \int_S u(r\xi)\, \sigma(d\xi) = r^{-(2n-1)} \int_{rS} u(\eta)\, \sigma_r(d\eta) \\ &= \sigma(S)\, E_0[u(Z_{\tau_r})] = u(0)\, \sigma(S). \end{aligned}$$

Therefore we can use the Helly selection theorem to get a subsequence $\{\mu_{r_k};\ k \ge 1\}$ that converges weakly on the unit sphere S to a measure μ as $r_k \uparrow 1$. According to Lemma A.1, we may and do assume that $h_{r_k\xi}(z)$ converges uniformly in $(z, \xi) \in D_r \times S$ to $h_\xi(z)$ for each $0 < r < 1$.

For $z \in D$, let r_k be such that $z \in D_{r_k}$. Then

$$u(z) = E_z[u(Z_{\tau_{r_k}})] = \int_S u(r_k\xi)\, h_{r_k\xi}(z)\, \sigma(d\xi) = \int_S h_{r_k\xi}(z)\, d\mu_{r_k}.$$

Since

$$\begin{aligned}
&\lim_{k\to\infty}\left|\int_S h_{r_k\xi}(z)\,d\mu_{r_k} - \int_S h_\xi(z)\,d\mu\right| \\
&\quad\le \lim_{k\to\infty}\int_S |h_{r_k\xi}(z) - h_\xi(z)|\,d\mu_{r_k} + \lim_{k\to\infty}\left|\int_S h_\xi(z)\,d\mu_{r_k} - \int_S h_\xi(z)\,d\mu\right| \\
&\quad\le \sup_{\xi\in S}|h_{r_k\xi}(z) - h_\xi(z)| \lim_{k\to\infty}\mu_{r_k}(S) + 0 \\
&\quad= 0,
\end{aligned}$$

we have $u(z) = \int_S h_\xi(z)\,d\mu$. □

THEOREM A.5. *The exit density functions $\{h_\xi;\ \xi \in S\}$ constructed above are minimal L-harmonic functions in D in the sense that if u is L-harmonic in D and $0 \le u(z) \le h_{\xi_0}(z)$ for some $\xi_0 \in S$ and all $z \in D$, then there exists a constant c such that $u(z) = ch_{\xi_0}(z)$ for all $z \in D$.*

PROOF. Without loss of generality, we may assume that $\xi_0 = 1$. By Theorem A.4, there is a nonnegative measure μ on S such that $u(z) = \int_S h_0(z)\,d\mu(z)$ for $z \in D$. We show that μ concentrates at point 1. For this, let $\mu_{r_k}(d\xi) = u(r_k\xi)\sigma(d\xi)$ be the sequence of measures on S in the proof of Theorem A.4 that converges weakly to μ. For $0 < \varepsilon < 1$, let $J_\varepsilon = \{\eta \in S: |\eta - 1| \le \varepsilon\}$. Since $S \setminus J_\varepsilon$ is an open subset of S,

$$\begin{aligned}
\mu(S\setminus J_\varepsilon) &\le \liminf_{k\to\infty}\ \mu_{r_k}(S\setminus J_\varepsilon) \\
&= \liminf_{k\to\infty}\int_{S\setminus J_\varepsilon} u(r_k\xi)\,\sigma(d\xi) \qquad \text{(A.5)} \\
&\le \liminf_{k\to\infty}\int_{S\setminus J_\varepsilon} h_1(r_k\xi)\,\sigma(d\xi).
\end{aligned}$$

For $\xi \in S\setminus\{-1\}$, let Ψ_ξ be the reflection map of C^n with respect to the complex line $\mathrm{C}w$ as defined by (5.5), where $w = (\xi+1)/|\xi+1|$. That is, $\Psi_\xi(z) = 2(z\cdot w)w - z$ for $z \in \mathrm{C}^n$. If $\xi = -1$, define $\Psi_{-1}(z) = -z$. In summary, for $\xi \in S$, Ψ_ξ is a unitary transformation of C^n such that $\Psi_\xi(\xi) = 1$ and $\Psi_\xi(1) = \xi$. Thus by the unitary invariance (5.1), the last term in (A.5) equals

$$\begin{aligned}
\liminf_{k\to\infty}\int_{S\setminus J_\varepsilon} h_{\Psi_\xi(1)}(\Psi_\xi(r_k\xi))\,\sigma(d\xi) &= \liminf_{k\to\infty}\int_{S\setminus J_\varepsilon} h_\xi(r_k 1)\,\sigma(d\xi) \\
&= \liminf_{k\to\infty} P_{r_k 1}(Z_\tau \in S\setminus J_\varepsilon) \\
&= 0.
\end{aligned}$$

The last equality comes from Theorem 2.9. This shows that $\mu(S\setminus J_\varepsilon) = 0$ for any $\varepsilon \in (0,1)$ and therefore μ is a measure on S with mass concentrated at point 1. Hence $u(z) = \int_S h_\xi(z)\mu(d\xi) = \mu(\{1\})h_1(z)$. This proves the minimality of h_1 and therefore of h_ξ for any $\xi \in S$. □

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