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ASYMPTOTICS FOR THE PRINCIPAL EIGENVALUE AND EIGENFUNCTION OF A NEARLY FIRST-ORDER OPERATOR WITH LARGE POTENTIAL

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The asymptotic behaviors of the principal eigenvalue and the corresponding normalized eigenfunction of the operator $G^\varepsilon f = (\varepsilon/2)\Delta f + g\nabla f + (l/\varepsilon)f$ for small ε are studied. Under some conditions, the first order expansions for them are obtained. Two applications to risk-sensitive control problems are also mentioned.

1. Introduction. The theory of diffusion processes with small noise by now is well developed and is still a good source for interesting problems. See [20]. In this paper, we will consider operators which arise by adding a potential function to the generators of such diffusion processes and study the asymptotic behavior of their principal eigenvalue and eigenfunction. More specifically, let the diffusion process in R^n satisfy the following stochastic differential equation

$$dx_t = g(x_t)\,dt + \varepsilon^{1/2}\,db_t, \tag{1.1}$$

with initial state $x_0 = x$, where $b_.$ is an n-dimensional Brownian motion. We assume that the noise intensity ε is small. Here g: $R^n \to R^n$ will be chosen so that the process x_t is ergodic. Some conditions will be given in Section 2. These conditions are motivated by questions in risk-sensitive stochastic control, considered in [12] and [14]. We will consider the following operator:

$$G^\varepsilon f = \frac{\varepsilon}{2}\Delta f + g \cdot \nabla f + \frac{l}{\varepsilon} f. \tag{1.2}$$

Here l: $R^n \to R$. Under some suitable conditions, this operator possesses the principal eigenvalue λ^ε with eigenfunction $\psi^\varepsilon(\cdot)$ in the sense that the spectrum of G^ε is contained in $\{\lambda \in C; \Re\lambda \le \lambda^\varepsilon\}$. We have $G^\varepsilon\psi^\varepsilon = \lambda^\varepsilon\psi^\varepsilon$, ψ^ε is an everywhere positive function and ψ^ε becomes unique when we require $\psi^\varepsilon(0) = 1$. In order to make this rigorous, we have to specify the space that G^ε acts on. This will be discussed in Section 2. In the particular case, which will be referred to as the gradient case in this paper, $g = -\nabla U$ for some smooth strictly convex function U satisfying some growth conditions; G^ε is self-adjoint

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as an operator in the space $L^2(R^n, d\mu^\varepsilon)$ with

$$d\mu^\varepsilon(x) = \exp\left(-\frac{2}{\varepsilon}U(x)\right)dx.$$

Therefore, ψ^ε is required to be in $L^2(R^n, d\mu^\varepsilon)$. In this case, λ^ε also has the following variational expression:

$$\lambda^\varepsilon = -\inf_{\int f^2(x)\,d\mu^\varepsilon(x)=1} \int \left(\frac{\varepsilon}{2}|\nabla f(x)|^2 - \frac{1}{\varepsilon}l(x)f^2(x)\right)d\mu^\varepsilon(x). \tag{1.3}$$

In the general case, we may establish the following:

$$\lambda^\varepsilon = \lim_{T\to\infty} \frac{1}{T}\log E_x\left[\exp\left(\frac{1}{\varepsilon}\int_0^T l(x_t)\,dt\right)\right]. \tag{1.4}$$

That is, λ^ε determines the rate of growth of the expectation in (1.4). Relation (1.4) holds when the process is in a compact state space and satisfies a strong kind of ergodicity condition. See [6, 7, 16, 41]. Donsker and Varadhan ([5]) deduce the following variational formula for λ^ε as a consequence of the large deviation principle:

$$\lambda^\varepsilon = \sup_\mu\left\{\int \frac{l}{\varepsilon}\,d\mu - I(\mu)\right\}, \tag{1.5}$$

where $I(\mu)$ is the large deviation rate function. Equation (1.5) reduces to (1.3) when $g = -\nabla U$. See [8]. On the other hand, denote by $\phi(T, x)$ the expectation in (1.4). The following result, which relates the exact asymptotics of $\phi(T, x)$ and λ^ε, ψ^ε, was mentioned in [16]:

$$\phi(T, x) \approx c\exp(\lambda^\varepsilon T)\psi^\varepsilon(x) \qquad \text{as } T\to\infty,$$

where c is a constant and ψ^ε is uniquely determined from this relation. The last relation also holds in our case.

The control interpretation of λ^ε and ψ^ε is useful and worth mention. The idea was first given by Holland [24] (see also [1, 2, 27, 37]). Write

$$\hat\lambda^\varepsilon = \varepsilon\lambda^\varepsilon, \qquad W^\varepsilon(x) = \varepsilon\log\psi^\varepsilon(x). \tag{1.6}$$

Then $\hat\lambda^\varepsilon, W^\varepsilon(x)$ satisfy formally the dynamic programming equation for a stochastic control problem with expected average cost per unit time criterion. This will be explained in Section 2. The scaling chosen in (1.6) will also become clear there. In [14], by considering the corresponding infinite horizon discounted cost control problem in the small discount factor limit, an a priori bound for its value function is found under suitable conditions. Then the existence of $\hat\lambda^\varepsilon, W^\varepsilon(x)$ as well as an a priori bound independent of ε can be deduced by letting the discount factor go to zero. This in turn implies the convergence of $\hat\lambda^\varepsilon, W^\varepsilon(x)$ to the limit $\hat\lambda^0, W^0(x)$ as $\varepsilon\to 0$. Then $\hat\lambda^0, W^0(x)$ satisfy the dynamic programming equation for a deterministic control problem with average cost per unit time criterion. This result will be generalized in Section 2 under slightly weaker conditions. Some study has been made in [14] where a conjecture also was mentioned.

In Sections 3 and 4, the uniqueness of the Lipschitz solution for (2.9) is proved under conditions (3.3) or (4.1), (4.6). This implies the convergence of $W^\varepsilon(\cdot)$. We remark that in [10] a similar result for a different model in discrete time is obtained. A PDE argument to prove the uniqueness of the solution for equations similar to (2.9) can be found in [25]. We next study more accurate asymptotics of $\hat\lambda^\varepsilon$, $W^\varepsilon(x)$. More precisely, the limit of $(\hat\lambda^\varepsilon - \hat\lambda^0)/\varepsilon$, $(W^\varepsilon(x) - W^0(x))/\varepsilon$ will be shown to exist under suitable conditions. Section 3 considers the gradient case. For the "general" case considered in Section 4, the strictly convex function $U(x)$ is replaced by a large deviations quasipotential function $I(x)$. Here we remark that in the gradient case, G^ε in $L^2(R^n, d\mu^\varepsilon)$ is equivalent to the operator H^ε in $L^2(R^n, dx)$ by the transformation:

$$\begin{aligned} &L^2(R^n, d\mu^\varepsilon) \to L^2(R^n, dx) \\ &f \to f \exp\left(-\frac{U}{\varepsilon}\right). \end{aligned} \tag{1.7}$$

Here

$$\begin{aligned} H^\varepsilon(f) &= \exp\left(-\frac{U}{\varepsilon}\right) G^\varepsilon\left(f \exp\left(\frac{U}{\varepsilon}\right)\right) \\ &= \frac{\varepsilon}{2}\Delta f + \frac{V^\varepsilon}{\varepsilon} f, \end{aligned}$$

with $V^\varepsilon = l - \frac{1}{2}(|\nabla U|^2 - \varepsilon\Delta U)$. This transformation enables us to apply the result in [39] and obtain the asymptotic expansion for λ^ε, $\psi^\varepsilon(\sqrt{\varepsilon}x)$. Here we shall rederive a weaker form of this result using a simpler argument. The asymptotic behavior of W^ε has also been studied in the literature and relates to the semiclassical rate of degeneracy of the lowest eigenvalues of the operator H^ε. See [26] and the references therein. The behavior appears to be very unstable with respect to perturbations of V. We refer to [14] for further examples. The conditions we pose are motivated by the study in [12], which reveals interesting connections between risk-sensitive control and H_∞ control. Our condition implies that $\hat\lambda^0 = 0$. Then a dissipation inequality, which is familiar in robust control, holds. The statements of the conditions include: 0 is the unique stable point for $\dot x = g(x)$ and l behaves quadratically near 0. See Sections 3 and 4 for a more precise statement. The argument for the gradient case is rather easy compared to the general case. It will be treated separately.

In Section 5 two applications are given to robust and risk-sensitive control. The first of these provides asymptotics for λ^ε and $W^\varepsilon(x)$ considered in [14]. These results can be interpreted as more precise statements about the stochastic risk-sensitive approximation to the deterministic robust control limit, as the noise intensity parameter ε tends to 0. The second application is an extension of results in [12] to a particular class of controlled dynamical systems. In this case two small parameters μ and ε are considered, where $\mu^{-1} = \gamma^2$ and γ is a H_∞-norm bound parameter. The asymptotic results are as ε tends to 0 and $\mu = \beta\varepsilon$ with β a constant. A different model closely related to ours is also studied in [25]. We finally remark that from the viewpoint of H^∞-control and risk-sensitive control, it is important to allow l to be of quadratic growth, which, however, is excluded in this paper. We mention that McEneaney [33] has some new results in this case.

2. Associated control problem and preliminary results. In this section, we shall give conditions and specify the space that G^ε acts on. We show that on this space there exists the principal eigenvalue λ^ε and eigenfunction $\psi^\varepsilon(\cdot)$ in the sense that $G^\varepsilon\psi^\varepsilon = \lambda^\varepsilon\psi^\varepsilon$; $\lambda - G^\varepsilon$ has an inverse on this space if $\Re\lambda > \lambda^\varepsilon$. The eigenfunction ψ^ε is everywhere positive and is the unique one satisfying $\psi^\varepsilon(0) = 1$. The existence of ψ^ε will be proved through $\psi^\varepsilon = \exp(W^\varepsilon/\varepsilon)$ with W^ε defined using a control problem formulation. Some estimates for W^ε can also be derived. Then the uniqueness of ψ^ε follows easily. This approach is taken from [12, 13, 14].

Another goal of this section is to discuss the convergence of $\hat{\lambda}^\varepsilon$ and W^ε as ε tends to 0. This problem has been studied in [14] where a conjecture of our main result was also made and a proof for the one-dimensional case was given. The basic approach is similar to that used in that paper. We will regard $\hat{\lambda}^\varepsilon$, W^ε as objects satisfying the dynamic programming PDE for the cost per unit time control problem. By first considering the corresponding infinite horizon discounted cost control problem instead and deriving an a priori bound for its cost function, then letting the discount factor go to zero, an a priori bound for ∇W^ε can be obtained. Convergence of $\hat{\lambda}^\varepsilon$ and W^ε can be achieved by the compactness argument and the uniqueness of viscosity solution for the limiting PDE. We first introduce the conditions used in this section.

Assume that $l \in C^2(R^n)$, $g \in C^2(R^n)$. Let g_x denote the matrix of partial derivatives of g. We assume the following conditions.

CONDITION 2.1.

(a) The first and second partial derivatives of g are bounded;
(b) There is $c_0 > 0$ such that

$$z \cdot g_x(x)z \leq -c_0|z|^2$$

for x outside a bounded set, say $\{x; |x| \geq R_0\}$, and all z;
(c) l and its partial derivatives up to second order are bounded.

A condition such as (c) can be weakened. We leave this for the interested reader. We first describe the spaces that G^ε defines.

For any nonnegative c, define

$$B_c = \{f \in C(R^n); |f(x)| \exp(-c|x|) \text{ is bounded}\}.$$

Then B_c is a Banach space with the norm defined by

$$\|f\|_{B_c} = \sup_x |f(x)| \exp(-c|x|).$$

The term B_0 is just the space of bounded continuous functions defined on R^n. Under Condition 2.1, G^ε generates a semigroup T_t^ε on B_c. Some basic properties for T_t^ε will be given in Appendix 4. Let $\psi^\varepsilon = \exp(W^\varepsilon/\varepsilon)$ with W^ε, λ^ε

defined in the folllowing. By Theorem 2.2, ψ^ε is in B_{c_ε}, $c_\varepsilon = c/\varepsilon$. Here c is the constant appearing in the theorem. In fact, we will show in Appendix 4 that ψ^ε is of polynomial growth; therefore, it is in B_1. Since B_0 is a natural domain for G^ε, it is an interesting question if ψ^ε is in B_0. However, if we allow l to be of linear growth, then ψ^ε in general is not in B_0. This can be seen, for example, by taking $g(\cdot), l(\cdot)$ to be linear.

We now assume that ψ^ε is a positive function satisfying

$$G^\varepsilon \psi^\varepsilon = \lambda^\varepsilon \psi^\varepsilon.$$

Let $\hat{\lambda}^\varepsilon$, W^ε be defined by (1.6). In the next paragraph we briefly sketch a formal procedure to derive a control problem associated with $\hat{\lambda}^\varepsilon$ and W^ε. See [15, 17] for the details.

Now, $\hat{\lambda}^\varepsilon$ and W^ε satisfy the equation

$$\hat{\lambda}^\varepsilon = \varepsilon \exp(-\varepsilon^{-1} W^\varepsilon) G^\varepsilon(\exp(\varepsilon^{-1} W^\varepsilon). \tag{2.2}$$

Then the generators G^v and function $k(x, v)$ can be chosen so that (2.2) can be rewritten as the dynamic programming equation

$$\hat{\lambda}^\varepsilon = \sup_v \{G^v W^\varepsilon(x) + k(x, v)\}, \tag{2.3}$$

where

$$G^v f = \frac{\varepsilon}{2}\Delta f + (g + v) \cdot \nabla f,$$
$$k(x, v) = l(x) - \tfrac{1}{2}|v|^2.$$

The associated stochastic control problem can be described as follows. The admissible control class $\mathscr{V}$ consists of processes which are bounded and progressively measurable with respect to some reference probability system [17]. For each $v_\cdot$ in $\mathscr{V}$, the controlled Markov process ξ is governed by the stochastic differential equation

$$d\xi_t = (g(\xi_t) + v_t)\, dt + \varepsilon^{1/2}\, db_t. \tag{2.4}$$

The criterion is given by an expected average cost per unit time

$$J(v) = \limsup_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T k(\xi_t, v_t)\, dt\right]. \tag{2.5}$$

The goal is to find a $v_\cdot$ in $\mathscr{V}$ which maximizes J and we have

$$\hat{\lambda}^\varepsilon = \sup_{v\in\mathscr{V}} J(v).$$

The corresponding infinite horizon discounted cost control problem with discount factor $\rho > 0$ uses the following as the criterion instead:

$$J_\rho(x, v) = E_x\left[\int_0^\infty e^{-\rho t} k(\xi_t, v_t)\, dt\right]. \tag{2.6}$$

The value function is denoted by

$$W_\rho(x) = \sup_{v \in \mathscr{V}} J_\rho(x, v). \tag{2.7}$$

Then W_ρ is in $C^2(R^n)$ and is a solution of

$$\rho W_\rho(x) = \frac{\varepsilon}{2}\Delta W_\rho(x) + g(x) \cdot \nabla W_\rho(x) + \frac{1}{2}|\nabla W_\rho(x)|^2 + l(x). \tag{2.8}$$

As in [14], we will obtain an a priori bound for ∇W_ρ independent of ρ. Then we can conclude, as there, that $\rho W_\rho(x)$ and $W_\rho(x) - W_\rho(0)$ converge to $\hat{\lambda}^\varepsilon$ and $W^\varepsilon(x)$ as $\rho \to 0$. Note that $W^\varepsilon(0) = 0$. We now state our main results.

LEMMA 2.1. *For every $\rho > 0$, $x \in R^n$,*

$$\rho W_\rho \le ||l|| \quad \text{and} \quad |\nabla W_\rho(x)| \le c$$

for some constant c independent of ρ and ε. The function $-\Delta W_\rho$ is bounded above, independent of ρ, ε.

The following results are direct consequence of this lemma as mentioned earlier.

THEOREM 2.2. *We have the estimates*

$$\hat{\lambda}^\varepsilon \le ||l|| \quad \textit{and} \quad |\nabla W^\varepsilon(x)| \le c.$$

The function $-\Delta W^\varepsilon$ is bounded above independent of ε.

Moreover, possible limits of $\hat{\lambda}^\varepsilon$ and $W^\varepsilon(x)$ will be shown to satisfy the dynamic programming PDE of a deterministic control problem with averge cost per unit time criterion. To describe this, we need some notation.

Let the control class $\mathscr{V}^0$ be the set of bounded Lebesgue measurable R^n valued functions on $[0, \infty)$. For each $v \in \mathscr{V}^0$, the state dynamics is

$$\dot{\xi}_t^0 = g(\xi_t^0) + v_t$$

with intial state x . The criterion is

$$J^0(v) = \limsup_{T \to \infty} \frac{1}{T}\int_0^T k(\xi_t^0, v_t)\, dt.$$

THEOREM 2.3. *Let $\hat{\lambda}^0$ and $W^0(x)$ be the limit of a subsequence of $\hat{\lambda}^\varepsilon$ and $W^\varepsilon(x)$, as ε tends to 0. Then $W^0(\cdot)$ is a Lipschitz continuous viscosity solution of the following first order* PDE*:*

$$\hat{\lambda}^0 = \sup_{v \in R^n}\{(g(x) + v) \cdot \nabla W^0(x) + k(x, v)\}. \tag{2.9}$$

See (4.8) for $\hat{\lambda}^\varepsilon$ and W^ε. Equation (2.9) is the dynamic programming equation for the control problem mentioned above. Moreover, $\hat{\lambda}^0$ is unique and is given by the following

$$\hat{\lambda}^0 = \sup_{v \in \mathscr{V}^0} J^0(v).$$

For every $T < \infty$, $W^0(\cdot)$ satisfies the following relation,

$$W^0(x) = \sup_{v \in \mathscr{V}^0} \left[\int_0^T k(\xi_t^0, v_t)\, dt + W^0(\xi_T^0) \right] - \hat{\lambda}^0 T. \tag{2.10}$$

The argument which leads to Theorem 2.2 and Theorem 2.3 by using Lemma 2.1 can be found in [14] and is omitted here. We only remark that the uniqueness of the solution of (2.9) is not known in general. Therefore, we cannot conclude the convergence of W^ε as in [14]. In Section 3 and Section 4 we will see some cases where (2.9) has a unique solution.

In the rest, we shall prove Lemma 2.1. The following notation will be used. For a function f on R^n, $f_i \equiv \partial f/\partial x_i$; $f_{ij} \equiv \partial^2 f/\partial x_i \partial x_j$. The sup norm of f is $\|f\|$.

PROOF OF LEMMA 2.1. The first inequality is immediate. We shall show that $|\nabla W_\rho(x)| \le c$ with c independent of ε, ρ. Since the boundedness of the function $|\nabla W_\rho(x)|$, with a bound which may depend on ε, ρ, is needed in several places in the following argument, and we remark that this can be proved without too much difficulty, a proof for this fact will be sketched at the end. Therefore, it shall be assumed for what follows. The regularity of solutions of (2.8) can be found in [22]; see Chapter 17 therein in particular. In the rest of the proof we omit ρ as index.

We differentiate (2.8) with respect to x_i:

$$\begin{aligned} \rho W_i(x) = {} & \frac{\varepsilon}{2} \Delta W_i(x) + (g(x) + \nabla W(x)) \cdot \nabla W_i(x) \\ & + g_i(x) \nabla W(x) + l_i(x). \end{aligned} \tag{2.11}$$

Let x_t^* be the process generated by the following stochastic differential equation:

$$dx_t^* = (g + \nabla W)(x_t^*)\, dt + \varepsilon^{1/2}\, db_t.$$

This process does not explode, since $|\nabla W|$ is bounded as mentioned earlier. Then by Itô's differential rule, after a simple calculation using (2.11),

$$\begin{aligned} & d|\nabla W(x_t^*)|^2 \\ & \quad = 2\left[-\nabla W g_x \nabla W - \nabla l \cdot \nabla W + \frac{\varepsilon}{2} W_{ij}^2 + \rho |\nabla W|^2 \right](x_t^*)\, dt + dM_t, \end{aligned} \tag{2.12}$$

where M_t is a local martingale. By considering $|\nabla W(x_t^*)|^2 \exp(-2c_0 t)$ and using (2.1) with a stopping time argument, we obtain

$$|\nabla W(x)|^2 + E_x\left[\int_0^\infty \exp(-2c_0 t)\varepsilon W_{ij}^2(x_t^*)\,dt\right] \le cE_x\left[\int_0^\infty \exp(-2c_0 t)(|\nabla W(x_t^*)|^2 \chi_{\{|x_t^*|\le R_0\}} + \|\nabla l\|)\,dt\right], \tag{2.13}$$

where R_0 is given in Condition (2.1)(b), χ is the indicator function and c is a positive constant.

From (2.13),

$$\|\nabla W\|^2 \le c\left(\|\nabla l\| + \sup_{|x|\le R_0} |\nabla W(x)|^2\right). \tag{2.14}$$

Therefore, it is enough to obtain a bound for $|\nabla W(x)|$ in $\{x; |x| \le R_0\}$. By (2.8), this will follow if we can show $-\Delta W$ to be bounded above. The idea of proving this interesting result, which will be presented in the following, has been used in [32]. Related to this, we mention also the method of Krylov in [30] which gives a lower bound for $W_{x_i x_i}$ for the cost functions of some finite time horizon control problems. See also Section 4.9 in [17]. The method can also be applied here under stronger conditions.

Write $h = \Delta W$. We differentiate the equation (2.8) with respect to x_i twice and sum over i:

$$\frac{\varepsilon}{2}\Delta h + (g + \nabla W)\cdot \nabla h = -2g_i \cdot \nabla W_i - g_{ii}\cdot \nabla W - \sum |\nabla W_i|^2 - \Delta l + \rho h. \tag{2.15}$$

We use F to denote the right-hand side of (2.15). Let α be a positive constant. Then by Itô's differential rule,

$$-h(x) = E_x\left[\int_0^\infty e^{-\alpha t}(F - \alpha h)(x_t^*)\,dt\right]. \tag{2.16}$$

Here we use the property that

$$E_x[h(x_t^*)]e^{-\alpha t} \to 0 \quad \text{as } t \to \infty. \tag{2.17}$$

The last result can be proved as follows. Equation (2.8) and the boundedness of $|\nabla W|$, which is mentioned earlier, implies that

$$|h(x)| \le c(|x| + 1)$$

for some constant c, which possibly depends on ε and ρ. By a routine argument using the boundedness of $|\nabla W|$ and (2.1)(b), we can prove the existence of c such that

$$E_x[|x_t^*|] \le c$$

for all $t > 0$. These imply (2.17).

We now estimate the integral on the right-hand side of (2.16). The following are observed: $(\Delta W)^2 \le n \sum |\nabla W_i|^2$; for $|x| \le R_0$, there is $c > 0$ such that

$$|\nabla W(x)|^2 \le \frac{\varepsilon}{2} n^{1/2} \Big(\sum |\nabla W_i(x)|^2 \Big)^{1/2} + c$$

by (2.8). From these together with (2.13),(2.16) we can easily show that $-h$ has an upper bound, independent of ρ and ε.

In the rest, we will sketch a proof for the result that $|\nabla W|$ is a bounded function for any ε and ρ.

We may assume $\varepsilon = 1$. As in [14] Lemma 4.1, we need to estimate

$$\int_0^\infty e^{-\rho t} E[l(\xi_t^x) - l(\xi_t^y)]\, dt$$

where ξ^x is the solution of (2.4) with initial state x, and similarly for ξ^y. This can be done as follows.

It is not difficult to see that we can choose vector fields g_0, g_1 such that $g(x) = g_0(x) + g_1(x)$ for all x, g_1 has compact support and (2.1)(b) holds for g_0 for all x. Consider the processes η_t^x and η_t^y which solve the equation

$$d\eta_t = (g_0(\eta_t) + v_t)\, dt + db_t$$

with initial states x and y, respectively. We then use the Girsanov theorem to get the following expression:

$$\begin{aligned} E[l(\xi_t^x) - l(\xi_t^y)] &= E\bigg[(l(\eta_t^x) - l(\eta_t^y)) \exp\bigg(\int_0^t g_1(\eta_s^x)\, db_s - \tfrac{1}{2} \int_0^t |g_1(\eta_s^x)|^2\, ds \bigg) \bigg] \\ &\quad + E\bigg[l(\eta_t^y) \exp\bigg(\int_0^t g_1(\eta_s^y)\, db_s - \tfrac{1}{2} \int_0^t |g_1(\eta_s^y)|^2\, ds \bigg) \\ &\qquad \times \bigg(\exp\bigg(\int_0^t (g_1(\eta_s^x) - g_1(\eta_s^y) \bigg)\, db_s \\ &\qquad\qquad - \tfrac{1}{2} \int_0^t (|g_1(\eta_s^x)|^2 - |g_1(\eta_s^y)|^2)\, ds) - 1 \bigg) \bigg]. \end{aligned}$$

Using $|\eta_t^x - \eta_t^y| \le \exp(-c_0 t)|x - y|$ for all t, Hölder's inequality and tedious calculation, we can get the following estimate

$$\begin{aligned} E[l(\xi_t^x) - l(\xi_t^y)] &\le \exp(-c_0 t)\|\nabla l\||x - y| + c\|l\| \exp\bigg(\frac{1}{2} \|g_1\|^2 (q-1)t \bigg) \\ &\qquad \times \exp(cp|x - y|) \bigg(1 + t + \frac{\exp(c|x - y|) - 1}{|x - y|} \bigg) |x - y| \end{aligned}$$

for any $q, p > 1$ with $1/p + 1/q = 1$. We then choose q such that $\frac{1}{2}\|g_1\|^2(q-1) \le \frac{1}{2}\rho$. With these, a uniform bound for $\int_0^\infty e^{-\rho t} E[l(\xi_t^x) - l(\xi_t^y)]\, dt$ can be derived which implies the boundedness of $|\nabla W|$.

3. Finer asymptotics: the gradient case. In the last section we have proved that $\{\hat{\lambda}^\varepsilon, W^\varepsilon\}$ forms a compact family with limit satisfying (2.9). In

this and later sections, further conditions will be assumed such that (2.9) has a unique solution. This implies the convergence of $\{\hat{\lambda}^\varepsilon, W^\varepsilon\}$ as $\varepsilon \to 0$. Finer asymptotics for $\hat{\lambda}^\varepsilon, W^\varepsilon$ will also be studied in this and the next section. In this section we consider only the gradient case, that is, $g = -\nabla U$. Although this can be regarded as a special case, the treatment is less technical and is interesting on its own. Let us first give a remark. Recall G^ε given in (1.2). After the transformation (1.7), the variational formula (1.3) for λ^ε can be rewritten as

$$-\lambda^\varepsilon = \inf_{\int f^2\,dx=1} \int \left(\frac{\varepsilon}{2} |\nabla f|^2 + \frac{1}{\varepsilon} V^\varepsilon f^2 \right) dx, \tag{3.1}$$

where

$$\begin{aligned} V^\varepsilon &= V - \frac{\varepsilon}{2} \Delta U, \\ V &= \tfrac{1}{2} |\nabla U|^2 - l. \end{aligned} \tag{3.2}$$

Note that $\hat{\lambda}^\varepsilon \equiv \varepsilon \lambda^\varepsilon$. Then we have the following.

LEMMA 3.1. *Assume that $V(x)$ and $V(x) - \Delta U(x)$ tend to ∞ as $|x| \to \infty$. Then*

$$\begin{aligned} \lim_{\varepsilon \to 0} \hat{\lambda}^\varepsilon &= \hat{\lambda}^0 \\ &= -\min_x V(x). \end{aligned}$$

This follows easily from (3.1) by using suitable test functions. We omit the proof (see also [44]). The conditions ensure that G^ε has discrete eigenvalues. See [35]. In the rest we will assume the following. The functions U and l are smooth.

CONDITION 3.3.

(a) $U(x) \geq 0$, $l(0) = U(0) = 0$, $\nabla l(0) = 0$.
(b) U is convex and there are $c_1, c_2 > 0$ such that $c_1 \leq ((\partial^2 U)/\partial x_i \partial x_j) \leq c_2$, U has bounded third-order derivatives.
(c) Up to third order, l and its derivatives are bounded.
(d) There is $c_3 > 0$ such that $V(x) \equiv \frac{1}{2} |\nabla U(x)|^2 - l(x) \geq c_3 |x|^2$.

The inequalities in (b) are meant for nonnegative matrices. These conditions imply (2.1). Therefore, $\hat{\lambda}^\varepsilon$ converge to $\hat{\lambda}^0$ by Theorem 2.3. According to Lemma 3.1, $\hat{\lambda}^0 = 0$. Moreover, we will show that the Lipschitz solution of (2.9) is unique. Therefore, W^ε also converges to W^0. This result will be stated in the following theorem. Some additional properties of W^0 will also be mentioned. We denote

$$A = \left(\frac{\partial^2 U(0)}{\partial x_i \partial x_j} \right), \qquad B = \left(\frac{\partial^2 l(0)}{\partial x_i \partial x_j} \right), \qquad D = \left(\frac{\partial^2 V(0)}{\partial x_i \partial x_j} \right), \tag{3.4}$$

where A, D are positive definite and $D = A^2 - B$.

THEOREM 3.2. *As ε tends to 0, W^ε converges to $W^0(x)$, which satisfies the following equation:*

$$\tfrac{1}{2}|\nabla W^0(x)|^2 - \nabla U(x)\cdot\nabla W^0(x) + l(x) = 0, \tag{3.5}$$

in viscosity sense and almost everywhere. It has the following expression:

$$\begin{aligned} W^0(x) &= \sup_{\substack{\varphi_0=x,\\ \varphi_\infty=0}} \int_0^\infty \big(l(\varphi_t) - \tfrac{1}{2}|\dot\varphi_t + \nabla U(\varphi_t))|^2\big)\,dt \\ &= U(x) - \inf_{\substack{\varphi_0=x,\\ \varphi_\infty=0}} \int_0^\infty \big(\tfrac{1}{2}|\dot\varphi_t|^2 + V(\varphi_t)\big)\,dt. \end{aligned} \tag{3.6}$$

We have the estimate: there is $c>0$ such that

$$W^0(x) - U(x) \le -c|x|^2. \tag{3.7}$$

The smooth region for W^0 is connected, of full Lebesgue measure and contains a neighborhood of 0. Moreover,

$$\Delta W^0(0) = \mathrm{Tr}\ A - \mathrm{Tr}\ D^{1/2} \equiv 2\kappa. \tag{3.8}$$

PROOF. The first statement follows from [14], Section 5 and $\hat\lambda^0 = 0$. Equation (3.6) will be proved in Appendix 3. To obtain (3.7), we use (3.6) and replace $V(\varphi_t)$ by $c_3|\varphi_t|^2$ [see (3.3)(d)]. Then (3.7) is well known. We prove (3.8) in Appendix 1.

We now state our main results.

THEOREM 3.3. *Asssume (3.3). Then*

$$\lim_{\varepsilon\to 0} \lambda^\varepsilon = \kappa,$$

as κ is defined in (3.8). Moreover, there is $c>0$ such that

$$|\lambda^\varepsilon - \kappa| \le c\varepsilon^{1/2}.$$

THEOREM 3.4. *Assume (3.3). Let W^0 be smooth at x. Then*

$$\lim_{\varepsilon\to 0} \frac{W^\varepsilon(x) - W^0(x)}{\varepsilon} = \int_0^\infty \left(\kappa - \frac{1}{2}\Delta W^0\right)(\varphi_t^*)\,dt,$$

where φ_t^* is the unique minimizing curve of the variational problem in (3.6). The limit exists uniformly for x in compact subsets of the smooth region of W^0.

REMARK. Here φ_t^* satisfies

$$\dot\varphi_t^* = -\nabla U(\varphi_t^*) + \nabla W^0(\varphi_t^*). \tag{3.9}$$

This is also a solution of the following characteristic system for a Hamitonian–Jacobi equation

$$\dot{\varphi} = H_p(\varphi, p),$$
$$\dot{p} = -H_x(\varphi, p).$$

Here $H(x, p) = \frac{1}{2}|p|^2 - V(x)$.

We begin the proof of these results.

PROOF OF THEOREM 3.3. In (3.1) we use the scaling $x \to \varepsilon^{1/2}x$. Then we can get the following expression:

$$\lambda^\varepsilon = -\inf_{\int f^2\,dx=1} \int \left(\tfrac{1}{2}|\nabla f|^2 + \hat{V}^\varepsilon f^2\right) dx, \tag{3.10}$$

where

$$\hat{V}^\varepsilon(x) = \frac{1}{\varepsilon} V(\varepsilon^{1/2}x) - \frac{1}{2}\Delta U(\varepsilon^{1/2}x). \tag{3.11}$$

Therefore, λ^ε formally converges to the right-hand side of (3.10) with $\hat{V}^\varepsilon$ replaced by $\hat{V}^0 \equiv \frac{1}{2} < Dx, x > -\frac{1}{2}\Delta U(0)$. The last number is exactly equal to κ. For a proof we proceed as follows. First, we get a lower bound. Let

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n/2} (\det D)^{1/4} \exp\left(-\frac{1}{2}\langle\sqrt{D}x, x\rangle\right).$$

Then

$$\begin{aligned} \lambda^\varepsilon &\geq -\int \left(\tfrac{1}{2}|\nabla f|^2 + \hat{V}^\varepsilon f^2\right) dx \\ &= \kappa - \int (\hat{V}^\varepsilon(x) - \hat{V}^0(x)) f^2(x)\,dx \\ &\geq \kappa - c\sqrt{\varepsilon}. \end{aligned} \tag{3.12}$$

The last step is by the expansion of $V(\varepsilon^{1/2}x)$ and $\Delta U(\varepsilon^{1/2}x)$ at $\varepsilon = 0$ using Taylor's formula and bounded third-order derivatives of U and l.

To get an upper bound, let inf in (3.10) be attained at f^ε. Then f^ε satisfies

$$\tfrac{1}{2}\Delta f^\varepsilon - \hat{V}^\varepsilon f^\varepsilon = \lambda^\varepsilon f^\varepsilon \tag{3.13}$$

and

$$\lambda^\varepsilon = -\int \left(\tfrac{1}{2}|\nabla f^\varepsilon|^2 + \hat{V}^\varepsilon (f^\varepsilon)^2\right) dx. \tag{3.14}$$

An easy argument (see Appendix 2) shows

$$f^\varepsilon(x) \leq c_1 \exp(-c_2|x|^2) \tag{3.15}$$

for some $c_1, c_2 > 0$. Then

$$\begin{aligned}\lambda^\varepsilon &= -\int (\tfrac{1}{2}|\nabla f^\varepsilon|^2 + V^0 (f^\varepsilon)^2)\,dx + \int (V^\varepsilon - V^0)(f^\varepsilon)^2\,dx \\ &\le \kappa - \int (V^\varepsilon - V^0)(f^\varepsilon)^2\,dx \\ &\le \kappa + c\varepsilon^{1/2}.\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 3.4. The argument sketched as follows is taken from [38]. Remark that we may also use the arguments in [18] to prove the result.

We consider the function

$$R^\varepsilon = \exp\biggl(\frac{1}{\varepsilon}(W^\varepsilon - W^0)\biggr).$$

It satisfies

$$\frac{\varepsilon}{2}\Delta R^\varepsilon + (-\nabla U + \nabla W^\varepsilon)\cdot \nabla R^\varepsilon = \biggl(\lambda^\varepsilon - \frac{1}{2}\Delta W^0\biggr) R^\varepsilon \tag{3.16}$$

in the set where W^0 is smooth. Moreover, $R^\varepsilon(0) = 1$.

The following fact will be needed. See the argument in Appendix 1 and [4], Theorem 6. Let W^0 be smooth at x. Then φ_t^*, the solution of (3.9), has the properties: φ_t^* tends to the origin as $t \to \infty$; there is $\delta > 0$ such that W^0 is smooth in a δ neighborhood of $\{\varphi_t^*\colon 0 \le t < \infty\}$ denoted by G.

Let ζ be the diffusion solving

$$d\zeta_t = (-\nabla U + \nabla W^0)(\zeta_t)\,dt + \varepsilon^{1/2}\,db_t$$

with initial state x until ζ_t hits the boundary of G. Fix $R > 0$. Define

$$\tau^\varepsilon = \inf\{t; |\zeta_t| = \varepsilon^{1/2} R \text{ or } \zeta_t \in \partial G\}.$$

Then

$$\begin{aligned} R^\varepsilon(x) &= E_x\biggl[R^\varepsilon(\zeta_{\tau^\varepsilon}) \exp\biggl\{\int_0^{\tau^\varepsilon} (\tfrac{1}{2}\Delta W^0 - \lambda^\varepsilon)\,dt\biggr\}\biggr] \\ &= E_x\biggl[R^\varepsilon(\zeta_{\tau^\varepsilon}) \exp\biggl\{\int_0^{\tau^\varepsilon} (\tfrac{1}{2}\Delta W^0 - \kappa)\,dt\biggr\} \exp\{-(\lambda^\varepsilon - \kappa)\tau^\varepsilon\}\biggr]\end{aligned} \tag{3.17}$$

As in [38], we can prove the following.

PROPERTY 3.18.

(a) For any positive $\delta_1 > 0$, $R^\varepsilon(x) \le \exp(\delta_1/\varepsilon)$ uniformly on compact sets;
(b) There is $\delta_2 > 0$ such that $P_x\{\zeta_{\tau^\varepsilon} \in \partial G\} \le \exp(-\delta_2/\varepsilon)$;
(c) There is $c > 0$ such that for any $r > 0$,

$$E_x\biggl[\exp\biggl(r\int_0^{\tau^\varepsilon} |\zeta_t|\,dt\biggr)\biggr] \le \exp(cr(U(x) - W^0(x))^{1/2}).$$

For example, (c), as in [38], Lemma 3.3, can be proved by applying Itô's differential rule to $\exp\{cr(U - W^0)^{1/2}(\zeta_t)\}$ and by choosing c large enough. Further, (b) is a consequence of large deviation properties of ζ_t and (a) follows from the convergence of W^ε to W^0 uniformly on compact sets as $\varepsilon \to 0$.

Since $|\frac{1}{2}\Delta W^0(x) - \kappa| \le c|x|$ holds in a neighborhood of origin, by using W^0 being C^3 near origin, $\frac{1}{2}\Delta W^0(0) = \kappa$, and $|\lambda^\varepsilon - \kappa| \le c\varepsilon^{1/2}$ and by (3.17), (3.18), we can prove the convergence of $R^\varepsilon(x)$ if we can show that $R^\varepsilon(\varepsilon^{1/2}x)$ converges uniformly on compact sets. Recall that $f^\varepsilon > 0$ is the unique function attaining the inf in (3.10). We observe

$$R^\varepsilon(\varepsilon^{1/2}x) = \frac{1}{f^\varepsilon(0)} f^\varepsilon(x) \exp\left(\frac{1}{\varepsilon}(U - W^0)(\varepsilon^{1/2}x)\right).$$

Since f^ε satisfies a uniformly elliptic equation (3.13), the estimate in (3.15) implies $\{f^\varepsilon\}$ is a compact family of functions. Therefore by a routine argument, they converge to f^0 as $\varepsilon \to 0$. In fact, f^0 is Gaussian and is given by

$$f^0(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n/2} (\det D)^{1/4} \exp\left(-\frac{1}{2}\langle \sqrt{D}x, x\rangle\right).$$

Remark that $\exp((1/\varepsilon)(U - W^0)(\varepsilon^{1/2}x))$ converges to $\exp(\frac{1}{2}\langle\sqrt{D}x, x\rangle)$. This in turn implies the uniform convergence of $R^\varepsilon(\varepsilon^{1/2}x)$ to 1 on compact sets.

4. Finer asymptotics: general case. In this section, we consider the general case when g is not a gradient. We will see that the "gradient part" of g plays a crucial role. That is, g has an orthogonal decomposition: $g = -\nabla I + (g + \nabla I)$ for a function I, where I is the Wentzell–Freidlin quasipotential function. See [20]. Some interesting properties of this function are given in [4]. The function I also relates to the asymptotic behavior of the invariant density of the process in (1.1). See [3, 20, 34, 38]. Results concerning this will be reported in the following lemma. We assume the following condition:

(4.1) The function g is smooth with bounded first-order and second-order derivatives; $g(0) = 0$. There is $c_0 > 0$ such that $z \cdot g_x(x)z \le -c_0|z|^2$ for all x, z.

LEMMA 4.1. *Assume condition (4.1). Then the process in (1.1) has a unique invariant density p^ε. The following limit exists*

$$\lim_{\varepsilon\to 0} -\varepsilon \log p^\varepsilon(x) = 2I(x). \tag{4.2}$$

Here I has the following properties. It satisfies the following equation in the viscosity sense and also almost everywhere

$$|\nabla I(x)|^2 + g(x) \cdot \nabla I(x) = 0. \tag{4.3}$$

It is a classical solution in the smooth region of I, which is an open set containing the origin with complement of Lebesgue measure 0. The following expression *holds for I:*

$$2I(x) = \inf_{\substack{\phi_0=x,\\ \phi_\infty=0}} \tfrac{1}{2}\int_0^\infty |\dot{\phi}_t + g(\phi_t)|^2\,dt. \tag{4.4}$$

Moreover, we have the estimate: there are positive constants c_1, c_2 such that

$$\begin{gathered} c_1|x|^2 \le I(x) \le c_2|x|^2,\\ c_1|x| \le |\nabla I(x)| \le c_2|x|. \end{gathered} \tag{4.5}$$

The last inequalities hold when I is differentiable at x.

We remark that (4.3) is the the dynamic programming equation for the calculus of variations problem in (4.4). Equation (4.2) can be found in [3, 34, 38]. A related result can be found in [20], Chapter 4. A proof of (4.5) will be given in Appendix 3. After this preliminary result we now state assumptions for l.

ASSUMPTION 4.6.

(a) The function l is smooth with bounded derivatives of all orders and $l(0) = 0$, $\nabla l(0) = 0$.

(b) There is $c > 0$, $V(x) = \frac{1}{2}|\nabla I(x)|^2 - l(x) \ge c|x|^2$.

Here (b) holds at x where I is differentiable.

Recall that $\psi^\varepsilon, \lambda^\varepsilon$ are the first eigenfunction and eigenvalue of G^ε normalized by $\psi^\varepsilon(0) = 1$,

$$G^\varepsilon\psi^\varepsilon = \lambda^\varepsilon\psi^\varepsilon, \tag{4.7}$$

with

$$G^\varepsilon f = \frac{\varepsilon}{2}\Delta f + g\cdot\nabla f + \frac{l}{\varepsilon}f.$$

Let $\hat{\lambda}^\varepsilon = \varepsilon\lambda^\varepsilon$, $W^\varepsilon = \varepsilon\log\psi^\varepsilon$. Then $W^\varepsilon(0) = 0$ and

$$\frac{\varepsilon}{2}\Delta W^\varepsilon + g\cdot\nabla W^\varepsilon + \frac{1}{2}|\nabla W^\varepsilon|^2 + l = \hat{\lambda}^\varepsilon. \tag{4.8}$$

Some properties of W^ε, λ^ε will be stated in the following. Some of these have already been mentioned in Section 2. See also [14].

THEOREM 4.2. *We have the estimate*

$$\begin{gathered} |\hat{\lambda}^\varepsilon| \le \|l\|,\\ \|\nabla W^\varepsilon\| \le c_0^{-1}\|\nabla l\| \end{gathered} \tag{4.9}$$

with c_0 as in (4.1). Then $W^\varepsilon, \hat{\lambda}^\varepsilon$ converge to W^0, $\hat{\lambda}^0$ as $\varepsilon \to 0$, $\hat{\lambda}^0 = 0$ and W^0 satisfies

$$g\cdot\nabla W^0 + \tfrac{1}{2}|\nabla W^0|^2 + l = 0, \tag{4.10}$$

in the viscosity sense and almost everywhere and has the expression

$$W^0(x) = \sup_{\substack{\phi_0 = x,\\ \phi_\infty = 0}} \int_0^\infty \left(l(\phi_t) - \tfrac{1}{2}|\dot\phi_t - g(\phi_t)|^2\right) dt. \tag{4.11}$$

The smooth region for W^0 *is connected, of full Lebesgue measure and contains a neighborhood of the origin. Moreover, we have the estimate*

$$\begin{aligned} \|\nabla W^0\| &\le c_0^{-1}\|\nabla l\|, \\ I(x) - W^0(x) &\ge c|x|^2. \end{aligned} \tag{4.12}$$

Here c *is some positive constant.*

These results will be proved in Appendix 3. The following are statements corresponding to Theorems 3.3 and 3.4.

THEOREM 4.3. *Assume (4.1), (4.6). Then*

$$\lim_{\varepsilon \to 0} \lambda^\varepsilon = \kappa$$

exists and is finite and κ *can be defined explicitly; see (4.20). There is a positive number* c *such that*

$$|\lambda^\varepsilon - \kappa| \le c\varepsilon^{1/2}.$$

THEOREM 4.4. *Assume (4.1), (4.6). Let* W^0 *be smooth at* x. *Then*

$$\lim_{\varepsilon \to 0} \frac{W^\varepsilon(x) - W^0(x)}{\varepsilon} = \int_0^\infty \left(\kappa - \frac{1}{2}\Delta W^0\right)(\varphi_t^*)\, dt,$$

where φ_t^* *is the unique minimizing curve of the variations problem in (4.11). The limit exists uniformly on compact subsets of a smooth region of* W^0.

In the rest, whenever no confusion will arise, we will drop ε as an index. We denote

$$\begin{aligned} \hat g^\varepsilon(x) &= \frac{1}{\sqrt{\varepsilon}} g(\varepsilon^{1/2}x), \\ \hat l^\varepsilon(x) &= \frac{1}{\varepsilon} l(\varepsilon^{1/2}x), \\ \hat W^\varepsilon(x) &= \frac{1}{\varepsilon} W^\varepsilon(\varepsilon^{1/2}x), \\ \hat\psi^\varepsilon(x) &= \psi^\varepsilon(\varepsilon^{1/2}x), \\ \hat G^\varepsilon f &= \tfrac{1}{2}\Delta f + \hat g^\varepsilon \nabla f + \hat l^\varepsilon. \end{aligned}$$

Let $x_t^{\varepsilon *}$ be the diffusion generated by the stochastic differential equation

$$dx_t^{\varepsilon *} = (g + \nabla W^\varepsilon)(x_t^{\varepsilon *})\, dt + \varepsilon^{1/2}\, db_t.$$

The unique invariant density of this process is denoted by $p^{\varepsilon *}$.

Reexamine the proof of Theorem 3.3. The following play a role: variational formula (1.3) , using a suitable test function and properties of $p^{\varepsilon *}$. Here the situation is more complicated: $p^{\varepsilon *}$, depending on W^{ε}, cannot be directly analyzed. On the other hand, (1.5) should be used instead of (1.3), which, however, is not rigorously established. The following are some technical results we need to solve these difficulties.

LEMMA 4.5. *There is* $c > 0$ *such that*

$$|\nabla \hat{W}(x)| \le c(1 + |x|) \quad \textit{for all } x.$$

LEMMA 4.6. *There is* $\delta > 0$ *and* $c_1, c_2 > 0$ *such that*

$$I(x) - W^{\varepsilon}(x) \ge c_1 |x|^2 - c_2 \varepsilon, \qquad |x| \le \delta.$$

LEMMA 4.7. *There are* r_0, $c > 0$ *such that*

$$\int \exp\left(c \frac{|x|^2}{\varepsilon}\right) p^{\varepsilon *}(x)\, dx \le \exp\left(\frac{r_0}{\varepsilon}\right).$$

LEMMA 4.8. *We denote* $p_t^{\varepsilon *}(x, y)$ *to be the transition density of* $x_t^{\varepsilon *}$. *Then*

$$\lim_{\varepsilon \to 0} \varepsilon \log p_t^{\varepsilon *}(x, y) = W^0(y) - W^0(x) - \inf_{\substack{\phi_0 = x \\ \phi_t = y}} \int_0^t \left(\tfrac{1}{2} |\dot{\phi}_s - g(\phi_s)|^2 - l(\phi_s)\right) ds. \tag{4.13}$$

Moreover,

$$\lim_{\varepsilon \to 0} \varepsilon \log p^{\varepsilon *}(x) = -I^*(x) \tag{4.14}$$

holds uniformly on compact sets with

$$I^*(x) = -W^0(x) + \inf_{\substack{\phi_0 = x \\ \phi_\infty = 0}} \int_0^\infty \left(\tfrac{1}{2} |\dot{\phi}_t + g(\phi_t)|^2 - l(\phi_t)\right) dt.$$

Let $P^{\varepsilon *}$ *be the invariant measure of* $x_t^{\varepsilon *}$. *Then*

$$\lim_{\varepsilon \to 0} \varepsilon \log P^{\varepsilon *}(D) = -\inf_{x \in D} I^*(x) \tag{4.15}$$

for any region D *with smooth boundary. In particular, for any positive* r *there is positive* $c(r)$ *such that*

$$P^{\varepsilon *}\{|x| \ge r\} \le \exp\left(-\frac{c(r)}{\varepsilon}\right). \tag{4.16}$$

LEMMA 4.9. *Let $\varpi(\cdot)$ be a smooth function such that*

$$0 \le \varpi(x) \le 1,$$

$$\varpi(x) = \begin{cases} 1, & \text{if } |x| \le \delta_1, \\ 0, & \text{if } |x| \ge \delta_2 \end{cases}$$

for small δ_1, δ_2 with $\delta_1 < \delta_2$. Then the integral

$$\int p^{\varepsilon *}(x) \frac{V(x)}{\varepsilon} \varpi(x) \exp\left(\frac{I(x) - W^{\varepsilon}(x)}{\varepsilon} \right) dx$$

is uniformly bounded.

Lemma 4.8 says that the process $x_t^{\varepsilon *}$ has quasipotential function I^*. In the proof, we also see that $x_t^{\varepsilon *}$ satisfies, in finite time intervals, the large deviation principle with rate function

$$I_T^*(\phi) = W^0(\phi_0) - W^0(\phi_T) + \int_0^T (\tfrac{1}{2}|\dot{\phi}_t - g(\phi_t)|^2 - l(\phi_t))\, dt.$$

We shall postpone the proof of these lemmas.

PROOF OF THEOREM 4.3. Let $\hat{g}^0$, $\hat{l}^0$ be the limit of $\hat{g}^{\varepsilon}$, $\hat{l}^{\varepsilon}$ as $\varepsilon \to 0$. Then

$$\hat{g}^0(x) = Ax, \quad \hat{l}^0 = \tfrac{1}{2}\langle Bx, x\rangle,$$
$$A = g_x(0), \quad B = \left(\frac{\partial^2 l(0)}{\partial x_i \partial x_j} \right).$$

Denote

$$\hat{G}^0 f = \tfrac{1}{2}\Delta f + \hat{g}^0 \cdot \nabla f + \hat{l}^0 f. \tag{4.17}$$

The first eigenvalue and corresponding eigenfunction of $\hat{G}^0$ are denoted by $\kappa, \hat{\psi}^0$. Then

$$\hat{\psi}^0(x) = \exp\left(\tfrac{1}{2}\langle Dx, x\rangle\right), \tag{4.18}$$

where D is a symmetric matrix and satisfies the following Riccati-type equation:

$$D^2 + A^T D + DA + B = 0, \tag{4.19}$$

such that $D + A$ is stable. Here A^T is the transpose of A:

$$\kappa = \tfrac{1}{2} \operatorname{Tr} D. \tag{4.20}$$

The equations (4.19), (4.20) follow by formally plugging (4.18) into

$$\hat{G}^0 \hat{\psi}^0 = \kappa \hat{\psi}^0. \tag{4.21}$$

The existence of D satisfying (4.19) follows from [45]. See also [9, 19, 23].

The convergence of $\hat{\psi}^\varepsilon$, λ^ε to $\hat{\psi}^0$, κ is expected since $\hat{\psi}^\varepsilon$, λ^ε satisfy

$$\hat{G}^\varepsilon \hat{\psi}^\varepsilon = \lambda^\varepsilon \hat{\psi}^\varepsilon, \tag{4.22}$$

with

$$\hat{G}^\varepsilon f = \tfrac{1}{2}\Delta f + \hat{g}^\varepsilon \cdot \nabla f + \hat{l}^\varepsilon f,$$

which has (4.21) as the limiting equation. A rigorous proof will go as follows: let y_t^ε be the diffusion with generator $\frac{1}{2}\Delta + (\hat{g}^\varepsilon + \nabla \hat{W}^\varepsilon)\cdot\nabla$ and let $\hat{p}^\varepsilon$ be the invariant density for y_t^ε. We first observe that

$$\int \frac{1}{u}\left(\frac{1}{2}\Delta u + \hat{g}\cdot\nabla u\right)\hat{p}\,dx \geq \int \frac{1}{\hat{\psi}}\left(\frac{1}{2}\Delta\hat{\psi} + \hat{g}\cdot\nabla\hat{\psi}\right)\hat{p}\,dx \tag{4.23}$$

for any smooth positive function u such that $\log u$ has bounded derivatives of any order. Equation (4.23) also holds for more general u by using approximation argument.

Equation (4.23) can be proved as follows. It is easy to see the following by differentiation:

$$\begin{aligned}&\frac{1}{e^v}\left(\frac{1}{2}\Delta e^v + (\hat{g} + \nabla\hat{W})\cdot\nabla e^v\right)\\ &\qquad \geq \frac{1}{2}\Delta v + (\hat{g} + \nabla\hat{W})\cdot\nabla v.\end{aligned}$$

We integrate the above inequality with respect to $\hat{p}$. Notice that the right-hand side is zero and

$$\begin{aligned}&\frac{1}{e^v}\left(\frac{1}{2}\Delta e^v + (\hat{g} + \nabla\hat{W})\cdot\nabla e^v\right)\\ &\quad = \frac{1}{e^v\hat{\psi}}\left(\frac{1}{2}\Delta(e^v\hat{\psi}) + \hat{g}\cdot\nabla(e^v\hat{\psi})\right) - \frac{1}{\hat{\psi}}\left(\frac{1}{2}\Delta\hat{\psi} + \hat{g}\cdot\nabla\hat{\psi}\right).\end{aligned}$$

We obtain (4.23) by taking $u = e^v\hat{\psi}$.

Now in (4.23) we take $u = e^v$ and use (4.22) and an integration by parts to obtain

$$\lambda^\varepsilon \leq \int \hat{p}\left(\tfrac{1}{2}|\nabla v|^2 + (\hat{g} - \tfrac{1}{2}\nabla\log\hat{p})\cdot\nabla v + \hat{l}\right)dx.$$

Let $\hat{I}(x) = (1/\varepsilon)I(\sqrt{\varepsilon}x)$. If we formally choose $v = \hat{I} + \frac{1}{2}\log\hat{p}$, after using the relation $\nabla\hat{I}(\hat{g} + \nabla\hat{I}) = 0$ and an integration by parts formula, the right-hand side in the last relation becomes

$$\int\left(\hat{l} - \tfrac{1}{2}|\nabla\hat{I}|^2 - \tfrac{1}{8}|\nabla\log\hat{p}|^2 - \tfrac{1}{2}\operatorname{div}\hat{g}\right)\hat{p}\,dx.$$

Therefore,

$$\lambda^\varepsilon + \int\left(\hat{V} + \tfrac{1}{8}|\nabla\log\hat{p}|^2 + \tfrac{1}{2}\operatorname{div}\hat{g}\right)\hat{p}\,dx \leq 0. \tag{4.24}$$

A rigorous proof of (4.24) can be done by an approximation argument. Indeed, let us choose $v = f + \frac{1}{2}\log\hat{p}$, where f is smooth and has compact support.

Then, after some simple calculations and using an integration by parts, the integral we consider is equal to

$$\int (\hat{l} - \tfrac{1}{8}|\nabla \log \hat{p}|^2 - \tfrac{1}{2}div(\hat{g}) + \tfrac{1}{2}|\nabla f|^2 + \hat{g}\cdot\nabla f)\hat{p}\,dx.$$

Now we can choose a sequence of f approximating $\hat{I}$ and we obtain (4.24).

Equation (4.24) implies, with $c = \frac{1}{2}\|\mathrm{div}(g)\|$,

$$\lambda^\varepsilon \le c.$$

Note also, by

$$\tfrac{1}{2}\Delta\hat{W}^\varepsilon + \hat{g}^\varepsilon\cdot\nabla\hat{W}^\varepsilon + \tfrac{1}{2}|\nabla\hat{W}^\varepsilon|^2 + \hat{l}^\varepsilon = \lambda^\varepsilon,$$

we have

$$\lambda^\varepsilon = \tfrac{1}{2}\Delta\hat{W}^\varepsilon(0) + \tfrac{1}{2}|\nabla\hat{W}^\varepsilon(0)|^2.$$

Here $\Delta\hat{W}^\varepsilon(0)$ has a lower bound, independent of ε, by an argument in Section 2. Therefore, λ^ε is bounded from below by a constant. From this, we can conclude the following: there is a constant c such that

$$\begin{aligned} |\lambda^\varepsilon| &\le c,\\ \int \hat{p}\hat{V}\,dx &\le c,\\ \int \hat{p}|\nabla\log\hat{p}|^2\,dx &\le c. \end{aligned} \tag{4.25}$$

We remark that (4.25) is enough to deduce the convergence of λ^ε to κ by a compactness argument. We need a further argument to claim that $\sqrt{\varepsilon}$ is an upper bound for the rate of convergence. Let $\hat{p}^0$ be the invariant density for the diffusion with generator $\frac{1}{2}\Delta + (Ax+Dx)\cdot\nabla$. An inequality similar to (4.23) holds with $\hat{p}^\varepsilon$, $\hat{g}^\varepsilon$ replaced by $\hat{p}^0$, $\hat{g}^0$. Recall (4.18). Then

$$\begin{aligned} \kappa &= \int \hat{p}^\varepsilon\left(\frac{1}{2}\Delta\hat{\psi}^0 + \hat{g}^0\cdot\nabla\hat{\psi}^0 + \hat{l}^0\hat{\psi}^0\right)\frac{1}{\hat{\psi}^0}\,dx\\ &= \int \hat{p}^\varepsilon\left(\frac{1}{2}\Delta\hat{\psi}^0 + \hat{g}^\varepsilon\cdot\nabla\hat{\psi}^0 + \hat{l}^\varepsilon\hat{\psi}^0\right)\frac{1}{\hat{\psi}^0}\,dx\\ &\quad + \int \hat{p}^\varepsilon[(\hat{g}^0 - \hat{g}^\varepsilon)\,Dx + (\hat{l}^0 - \hat{l}^\varepsilon)]\,dx\\ &\ge \lambda^\varepsilon + \int \hat{p}^\varepsilon[(\hat{g}^0 - \hat{g}^\varepsilon)\,Dx + (\hat{l}^0 - \hat{l}^\varepsilon)]\,dx. \end{aligned} \tag{4.26}$$

Here we use (4.23) with $u = \hat{\psi}^0$. After a simple calculation, we obtain

$$\begin{aligned} \kappa &\ge \lambda^\varepsilon - c\sqrt{\varepsilon}\int \hat{p}^\varepsilon|x|^3\,dx\\ &\ge \lambda^\varepsilon - c\sqrt{\varepsilon}. \end{aligned}$$

The last inequality follows by the fact that $\int \hat{p}^\varepsilon |x|^3\,dx$ is uniformly bounded. This is a consequence of Lemma 4.6 $\sim$ Lemma 4.9 for the following reason: first, notice that

$$\hat{p}^\varepsilon(x) = (\sqrt{\varepsilon})^n p^{\varepsilon *}(\sqrt{\varepsilon}x).$$

We consider the integral in two regions: $\{x\colon |x| \le \delta/\sqrt{\varepsilon}$ or $|x| \ge M/\sqrt{\varepsilon}\}$ and its complement. Here we choose a small δ and large M. From Lemma 4.6 and 4.9, there is a positive constant c_1 such that

$$\int_{|y|\le\delta} p^{\varepsilon *}(y)\frac{|y|^2}{\varepsilon}\exp\left(c_1\frac{|y|^2}{\varepsilon}\right)dy$$

is bounded if we choose $\delta \le \delta_1$. This implies the boundness of the integral

$$\int_{|x|\le\delta/\sqrt{\varepsilon}} \hat{p}^\varepsilon(x)|x|^3\,dx.$$

On the other hand, by Lemma 4.7, it is easy to deduce the boundedness of the integral

$$\int_{|x|\ge M/\sqrt{\varepsilon}} \hat{p}^\varepsilon(x)|x|^3\,dx.$$

Finally, using (4.16), we have the boundedness of the integral

$$\int_{\delta/\sqrt{\varepsilon}\le|x|\le M/\sqrt{\varepsilon}} \hat{p}^\varepsilon(x)|x|^3\,dx.$$

Similarly, we can prove $\lambda^\varepsilon \ge \kappa - c\sqrt{\varepsilon}$. Here we use Lemma 4.5. This completes the proof.

PROOF OF THEOREM 4.4. The proof is exactly the same as that of Theorem 3.3. We also consider the function

$$R^\varepsilon = \exp\left(\frac{1}{\varepsilon}(W^\varepsilon - W^0)\right),$$

which now satisfies the equation

$$\frac{\varepsilon}{2}\Delta R^\varepsilon + (g + \nabla W^0)\cdot\nabla R^\varepsilon = \left(\lambda^\varepsilon - \frac{1}{2}\Delta W^0\right)R^\varepsilon.$$

We then consider this function along the diffusion ζ_t defined by

$$d\zeta_t = (g + \nabla W^0)(\zeta_t)\,dt + \sqrt{\varepsilon}\,db_t.$$

This again gives the expression (3.17). Property 3.18 holds in our case except for a minor change. Estimation in Property 3.18(c) is replaced by the following: there is $c > 0$ such that for any $r > 0$ and if ε is small enough we have

$$E_x\left[\exp\left(r\int_0^{\tau^\varepsilon}|\zeta_t|\,dt\right)\right] \le \exp(cr(I(x) - W^0(x))^{1/2}).$$

This can be proved by applying Itô's differential rule to $\exp(cr(I - W^0)^{1/2}(\zeta_t))$.

In the rest, we will prove Lemmas 4.5–4.9.

PROOF OF LEMMA 4.5. By

$$\tfrac{1}{2}\Delta\hat{W}^\varepsilon + \hat{g}^\varepsilon\nabla\hat{W}^\varepsilon + \tfrac{1}{2}|\nabla\hat{W}^\varepsilon|^2 + \hat{l}^\varepsilon = \lambda^\varepsilon,$$

the required estimate for $|\nabla\hat{W}^\varepsilon(\cdot)|$ follows from a uniform upper bound of λ^ε and $-\Delta\hat{W}^\varepsilon$. Here we note that

$$|\hat{g}^\varepsilon(x)|^2 + |\hat{l}^\varepsilon(x)| \le c|x|^2$$

for some $c > 0$. Estimation (4.25) gives a uniform bound for λ^ε. On the other hand, we can derive a uniform upper bound for $-\Delta\hat{W}^\varepsilon$ by using the argument in Section 2. This completes the proof. □

PROOF OF LEMMA 4.6. Let

$$\hat{I}^\varepsilon(x) = \frac{1}{\varepsilon} I(\sqrt{\varepsilon}x).$$

Again, in the following we will omit ε whenever it is convenient and does not cause confusion. We fix a positive δ such that $I(\cdot)$ is smooth in the region $\{x; |x| \le \delta\}$.

We consider the function $\hat{W} - \hat{I}$ in the set $\{x; |x| \le \delta/\sqrt{\varepsilon}\}$. By some calculations,

$$\begin{aligned} &\tfrac{1}{2}\Delta(\hat{W} - \hat{I}) + (\hat{g} + \nabla\hat{I})(\nabla(\hat{W} - \hat{I})) + \tfrac{1}{2}|\nabla(\hat{W} - \hat{I})|^2 \\ &\qquad = \tfrac{1}{2}|\nabla\hat{I}|^2 - \hat{l} + \lambda^\varepsilon - \tfrac{1}{2}\Delta\hat{I} \\ &\qquad = \hat{V} + \lambda^\varepsilon - \tfrac{1}{2}\Delta\hat{I}. \end{aligned} \tag{4.27}$$

Let z_t be the diffusion governed by the stochastic differential equation

$$dz_t = (\hat{g} + \nabla\hat{I})(z_t)\,dt + db_t$$

before the exit time

$$\tau = \inf\left\{t > 0; |z_t| = \frac{\delta}{\sqrt{\varepsilon}}\right\}$$

if $z_0 = x$ with $|x| \le \delta/\sqrt{\varepsilon}$. Applying Itô's differential rule,

$$\exp\left\{(\hat{W} - \hat{I})(z_{t\wedge\tau}) - \int_0^{t\wedge\tau} (\hat{V} + \lambda^\varepsilon - \tfrac{1}{2}\Delta\hat{I})(z_s)\,ds\right\}$$

is a martingale. Therefore, for $|x| \le \delta/\sqrt{\varepsilon}$,

$$\begin{aligned} &E_x\left[\exp\left\{(\hat{W} - \hat{I})(z_{t\wedge\tau}) - \int_0^{t\wedge\tau} (\hat{V} + \lambda^\varepsilon - \tfrac{1}{2}\Delta\hat{I})(z_s)\,ds\right\}\right] \\ &\qquad = \exp\{(\hat{W} - \hat{I})(x)\}. \end{aligned} \tag{4.28}$$

By (4.25), $|\lambda^\varepsilon|$ is bounded. Note that this part of proof of Theorem 4.3 did not depend on the lemmas which follow it. Since $\hat{V}+\lambda^\varepsilon$ is bounded from below and $\Delta\hat{I}$ is bounded for $|x|\le\delta/\sqrt{\varepsilon}$, we have

$$\begin{aligned}\exp\{(\hat{W}-\hat{I})(x)\} &\le E_x[\exp\{(\hat{W}-\hat{I})(z_{t\wedge\tau})\}]e^{ct}\\ &= E_x[\exp\{(\hat{W}-\hat{I})(z_t)\}; t\le\tau]e^{ct}\\ &\quad + E_x[\exp\{(\hat{W}-\hat{I})(z_\tau)\}; t>\tau]e^{ct}.\end{aligned} \tag{4.29}$$

The second term on the right has an upper bound $P_x[t>\tau]e^{ct}e^{c\delta/\varepsilon}$. Here we use (4.9). To get an upper bound for $P_x[t>\tau]$, we apply Itô's differential rule to $\hat{I}(z_t)$ and use (4.3),

$$d\hat{I}(z_t)=\tfrac{1}{2}\Delta\hat{I}(z_t)\,dt+\nabla\hat{I}(z_t)\,db_t. \tag{4.30}$$

Using

$$\hat{I}(z_\tau)-\hat{I}(x)=\int_0^\tau \tfrac{1}{2}\Delta\hat{I}(z_t)\,dt+\int_0^\tau\nabla\hat{I}(z_t)\,db_t,$$

as well as the fact that

$$E\biggl[\exp\biggl(\int_0^{\tau\wedge t}\alpha\nabla\hat{I}(z_s)\,db_s-\tfrac{1}{2}\alpha^2|\nabla\hat{I}(z_s)|^2ds\biggr)\biggr]=1,$$

then by choosing $\alpha=c1/t$ for some suitable c and using

$$E\biggl[\exp\biggl(\int_0^{\tau\wedge t}\alpha\nabla\hat{I}(z_s)\,db_s-\tfrac{1}{2}\alpha^2|\nabla\hat{I}(z_s)|^2\,ds\biggr),\tau\le t\biggr]\le 1,$$

we can prove

$$P_x\{\tau<t\}\le\exp\biggl(-c\frac{\delta^2}{t\varepsilon}\biggr)\quad\text{if } |x|\le\frac{\delta_1}{\varepsilon}, \tag{4.31}$$

where $\delta_1<\delta$ is sufficiently small. We conclude that the second term on the right of (4.29) is bounded by $\frac{1}{2}\exp(\hat{W}-\hat{I})(x)$ if $|x|\le\delta_1/\varepsilon$ and t is small. Therefore, for such x, t,

$$\begin{aligned}\exp\{(\hat{W}-\hat{I})(x)\} &\le 2E_x[\exp\{(\hat{W}-\hat{I})(z_t)\}; t<\tau]e^{ct}\\ &\le c\biggl(\frac{1}{\sqrt{t}}\biggr)^n\int_{|y|\le\delta/\sqrt{\varepsilon}}\exp\{(\hat{W}-\hat{I})(y)\}\,dy\\ &= c\int_{|y|\le\delta/\sqrt{\varepsilon}}\exp\{(\hat{W}-\hat{I})(y)\}\,dy\end{aligned} \tag{4.32}$$

if we fix t. Here we use the fact that the diffusion z_t stopped before τ has transition density bounded by $c(1/\sqrt{t})^n$. See [31], Corollary 3.9. This is little different from the estimate cited above where the constant is found to depend on x. The dependence of c on x comes because the diffusion coefficient may grow linearly at infinity. A careful reexamining of the proof, in particular Lemma 3.2 and Theorem 3.5 (refer to [31]), assures us that in our case we may choose c which is independent of x. We note that the constant depends on the derivatives of the drift up to the second order. See also a related result in [40], Corollary 3.37.

Let

$$\tau_1 = \inf\left\{t > 0; |z_t| = \frac{\delta_1}{\sqrt{\varepsilon}}\right\}.$$

Another positive number is $\delta_0 < \delta_1$ which is sufficiently small. For $|x| \le \delta_0/\sqrt{\varepsilon}$, repeat the above argument, and note that we now have (4.28), (4.32),

$$\begin{aligned}\exp\{(\hat W - \hat I)(x)\} &\le c\int_{|y|\le\delta/\sqrt{\varepsilon}} \exp\{(\hat W - \hat I)(y)\}\,dy \\ &\quad\times E_x\left[\exp\left\{-\int_0^t (\hat V + \lambda^\varepsilon - \tfrac{1}{2}\Delta\hat I)(z_s)\,ds\right\}; t < \tau_1\right].\end{aligned} \tag{4.33}$$

For a fixed t, the expectation in (4.33) is bounded by $c_1\exp(-c_2|x|^2)$ which can be proved by estimating the expectation in the sets $\{\alpha|x| \le |z_s| \le \alpha^{-1}|x|$ for all $s \in [0, t]\}$ and its complement. Here α is a very small positive number. The expectation in the first set has an upper bound $\exp(-c|x|^2)$ if we use the property $\hat V(x) \ge c|x|^2$. Using a similar argument for (4.31), the expectation in the complement also has the same upper bound. We thus conclude:

$$\begin{aligned}&\exp\{(\hat W - \hat I)(x)\} \\ &\quad\le c\exp(-c|x|^2)\cdot\int_{|y|\le\delta/\sqrt{\varepsilon}} \exp\{(\hat W - \hat I)(y)\}\,dy, \qquad |x| \le \frac{\delta_0}{\sqrt{\varepsilon}}.\end{aligned} \tag{4.34}$$

Let β be a small positive number. Theorem 4.2 implies, for any $M > 0$, $\rho > 0$,

$$\hat W(x) - (1-\beta)\hat I(x) \le -c|x|^2 + \frac{\rho}{\varepsilon}, \qquad |x| \le \frac{M}{\sqrt{\varepsilon}},$$

if ε is small. Therefore, the maximum of the function $\hat W(\cdot) - (1-\beta)\hat I(\cdot)$ in $\{|x| \le \delta/\sqrt{\varepsilon}\}$ is attained in $\{|x| \le \delta_0/\sqrt{\varepsilon}\}$ if β is small enough. We fix one such maximal point x^*. Similarly to (4.27),

$$\begin{aligned}&\tfrac{1}{2}\Delta(\hat W - (1-\beta)\hat I) + (\hat g + (1-\beta)\nabla\hat I)(\nabla\hat W - (1-\beta)\nabla\hat I) \\ &\quad + \tfrac{1}{2}|\nabla\hat W - (1-\beta)\nabla\hat I|^2 \\ &\qquad = \tfrac{1}{2}(1-\beta^2)|\nabla\hat I|^2 - \hat l + \lambda^\varepsilon - \tfrac{1}{2}(1-\beta)\Delta\hat I.\end{aligned}$$

Then

$$(\tfrac{1}{2}(1-\beta^2)|\nabla\hat I|^2 - \hat l + \lambda^\varepsilon - \tfrac{1}{2}(1-\beta)\Delta\hat I)(x^*) \le 0,$$

which implies $|x^*| \le r$ for some positive r independent of ε. Here we use

$$\tfrac{1}{2}(1-\beta^2)|\nabla\hat I|^2 - \hat l \ge c|x|^2$$

for some $c > 0$ if β is small.

From

$$\hat{W}(x) - \hat{I}(x) = \hat{W}(x) - (1-\beta)\hat{I}(x) - \beta\hat{I}(x)$$
$$\leq \hat{W}(x^*) - (1-\beta)\hat{I}(x^*) - \beta\hat{I}(x)$$

for x in $\{|x| \leq \delta/\sqrt{\varepsilon}\}$ and (4.32), we obtain

$$c_1 \exp(\hat{W}(x^*)) \leq \int_{|y|\leq\delta/\sqrt{\varepsilon}} \exp((\hat{W} - \hat{I})(y))\,dy \leq c_2 \exp(\hat{W}(x^*)).$$

Then by (4.34) we have

$$\exp((\hat{W} - \hat{I})(x)) \leq c_1 \exp(\hat{W}(x^*)) \exp(-c|x|^2), \qquad |x| \leq \frac{\delta_0}{\sqrt{\varepsilon}},$$

which implies the result.

PROOF OF LEMMA 4.7. Recall that $x_t^{\varepsilon *}$ is the process satisfying

$$dx_t^{\varepsilon *} = (g + \nabla W^{\varepsilon})(x_t^{\varepsilon *})\,dt + \varepsilon^{1/2}\,db_t.$$

Applying Itô's differential rule to $\exp\{c|x_t^{\varepsilon *}|^2/\varepsilon\}$,

$$d\exp\left\{\frac{c}{\varepsilon}|x_t^{\varepsilon *}|^2\right\} = \frac{c}{\varepsilon}[2c|x_t^{\varepsilon *}|^2 + 2x_t^{\varepsilon *}g(x_t^{\varepsilon *}) + 2x_t^{\varepsilon *}\cdot\nabla W^{\varepsilon}(x_t^{\varepsilon *}) + n\varepsilon]$$
$$\times \exp\left\{\frac{c}{\varepsilon}|x_t^{\varepsilon *}|^2\right\} dt + dM_1(t).$$

Then

$$d\exp\left\{\frac{c}{\varepsilon}|x_t^{\varepsilon *}|^2\right\}\exp\{ct\} = \frac{c}{\varepsilon}[2c|x_t^{\varepsilon *}|^2 + 2x_t^{\varepsilon *}g(x_t^{\varepsilon *}) + 2x_t^{\varepsilon *}\cdot\nabla W^{\varepsilon}(x_t^{\varepsilon *}) + n\varepsilon]$$
$$\times \exp\left\{\frac{c}{\varepsilon}|x_t^{\varepsilon *}|^2 + ct\right\} dt + dM_2(t).$$

In the above, $M_1(t)$, $M_2(t)$ are local martingales. By using (4.9) and condition (4.1), the coefficient of the dt term is bounded by $\exp\{ct + c_1/\varepsilon\}$ for some c_1 if c is sufficiently small, say, we may take $c = \frac{1}{4}c_0$ with c_0 in (4.1). Therefore

$$E_x\left[\exp\left\{\frac{c}{\varepsilon}|x_t^*|^2\right\}\right] \leq \exp\left\{\frac{c}{\varepsilon}|x|^2 - ct\right\} + c_1\exp\left\{\frac{c_1}{\varepsilon}\right\}.$$

Letting $t \to \infty$, we have

$$\int p^{\varepsilon *}(x)\exp\left\{\frac{c}{\varepsilon}|x|^2\right\} dx \leq e^{c_1/\varepsilon},$$

which implies the result.

PROOF OF LEMMA 4.8. First we observe, with x_t defined in (1.1),

$$\begin{aligned}\int p_t^*(x, y) f(y)\, dy &= E_x[f(x_t^*)]\\ &= E_x\biggl[f(x_t)\exp\biggl\{\frac{1}{\sqrt{\varepsilon}}\int_0^t \nabla W^\varepsilon(x_s)\, db_s - \frac{1}{2\varepsilon}\int_0^t |\nabla W^\varepsilon(x_s)|^2\, ds\biggr\}\biggr]\\ &= E_x\biggl[f(x_t)\exp\biggl\{\frac{1}{\varepsilon}(W^\varepsilon(x_t) - W^\varepsilon(x)) + \frac{1}{\varepsilon}\int_0^t l(x_t)\, dt - \lambda^\varepsilon t\biggr\}\biggr].\end{aligned}$$

Note that here the drift of the process x_t does not depend on ε and W^ε converges uniformly on compact sets. This implies the large deviation properties for the process $x^{\varepsilon *}$. The result, (4.13), follows from this also by using the argument in [36] (see also [29, 43]).

Equation (4.15) can be proved by using the argument in [20]. See Chapter 4, Theorem 4.3 therein in particular. Here we only add a few words for the convenience of the reader. The estimate (4.20) in [20], page 131, follows from the large deviation properties and still holds in our case. The estimate (4.19), in [20], page 131, can also be proved since it only uses the following: large deviation properties of the process, the estimate (4.20) there and condition A in [20], page 128, which follows from condition (4.1), and Theorem 4.2 in our case.

Finally, (4.14) can be proved by adopting an argument of Day [3], page 133 and using properties (4.13) and (4.15). We only mention that estimate (5.2) in [3] has to be verified in our case. Instead, we will give an upper bound for the integral

$$\int_{|z|>M} p^{\varepsilon *}(z) p_T^{\varepsilon *}(z, x)\, dz.$$

This is enough for our purpose. For this, we use the relation

$$\begin{aligned}p_t^{\varepsilon *}(x, y) = q_t^\varepsilon(x, y)&\exp\biggl(-\lambda^\varepsilon t + \frac{W^\varepsilon(y) - W^\varepsilon(x)}{\varepsilon}\biggr)\\ &\times E_x\biggl[\exp\biggl(\frac{1}{\varepsilon}\int_0^t l(x_s)\, ds\biggr)\bigg| x_t = y\biggr]\end{aligned}$$

and the boundness of l and λ^ε to obtain

$$p_1^{\varepsilon *}(x, y) \le c\biggl(\frac{1}{\sqrt{\varepsilon}}\biggr)^n \exp\biggl(\frac{c}{\varepsilon}\biggr)\exp\biggl(\frac{W^\varepsilon(y) - W^\varepsilon(x)}{\varepsilon}\biggr)$$

for some $c > 0$. Here we use the property

$$q_1^\varepsilon(x, y) \le c\biggl(\frac{1}{\sqrt{\varepsilon}}\biggr)^n.$$

For the process x_t, $q_t^\varepsilon(x, y)$ is the transition density. See [31], Corollary 3.9 and some explanation after (4.32). By the semigroup property

$$p_{t+1}^{\varepsilon *}(x, y) = \int p_t^{\varepsilon *}(x, z) p_1^{\varepsilon *}(z, y)\, dz$$

and the boundness of $|\nabla W^\varepsilon|$, we have

$$p_{t+1}^{\varepsilon*}(x,y) \le c\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \exp\left(\frac{c}{\varepsilon}\right)\exp\left(c\frac{|y|}{\varepsilon}\right)\int p_t^{\varepsilon*}(x,z)\exp\left(c\frac{|z|}{\varepsilon}\right)dz.$$

To estimate the last integral, we use

$$dx_t^{\varepsilon*} = (g+\nabla W^\varepsilon)(x_t^{\varepsilon*})\,dt + \varepsilon^{1/2}\,db_t$$

and apply Itô's differential rule to $\exp(c(1+|x_t^{\varepsilon*}|^2)^{1/2}/\varepsilon)$. After some simple calculation, using condition (4.1) and the boundedness of ∇W^ε, we have

$$\int p_t^{\varepsilon*}(x,z)\exp\left(c\frac{(1+|z|^2)^{1/2}}{\varepsilon}\right)dz \le \exp(-c_1 t)\exp\left(c\frac{(1+|x|^2)^{1/2}}{\varepsilon}\right) + \exp\left(\frac{c_2}{\varepsilon}\right)$$

for some positive constants c_1, c_2. Therefore, we obtain, for some $c>0$,

$$p_{t+1}^{\varepsilon*}(x,y) \le c\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \exp\left(\frac{c}{\varepsilon}\right)\exp\left(c\frac{|x|+|y|}{\varepsilon}\right).$$

By the above argument we can also get the following estimate. For any $c_3>0$, there is $c_4>0$ such that

$$\int p^{\varepsilon*}(z)\exp\left(c_3\frac{(1+|z|^2)^{1/2}}{\varepsilon}\right)dz \le \exp\left(\frac{c_4}{\varepsilon}\right).$$

We take $c_3 = 2c$. Then

$$\begin{aligned}
&\int_{|z|>M} p^{\varepsilon*}(z)p_{t+1}^{\varepsilon*}(z,x)\,dz\\
&\quad\le c\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \exp\left(\frac{c}{\varepsilon}\right)\int_{|z|>M} p^{\varepsilon*}(z)\exp\left(c\frac{|z|+|x|}{\varepsilon}\right)dz\\
&\quad\le c\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \exp\left(\frac{c}{\varepsilon}\right)\exp\left(-c\frac{M}{\varepsilon}\right)\exp\left(c\frac{|x|}{\varepsilon}\right)\int_{|z|>M} p^{\varepsilon*}(z)\exp\left(2c\frac{|z|}{\varepsilon}\right)dz\\
&\quad\le c\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \exp\left(\frac{c+c_4}{\varepsilon}\right)\exp\left(-\frac{M}{\varepsilon}\right)\exp\left(c\frac{|x|}{\varepsilon}\right)
\end{aligned}$$

for all t. This is the estimate we need in order to apply the argument in [3]. This completes the sketch of the proof of the lemma.

PROOF OF LEMMA 4.9. Let

$$q(x) = p^{\varepsilon*}(x)\exp\left\{-\frac{W^\varepsilon(x)}{\varepsilon}\right\}.$$

Then

$$\frac{\varepsilon}{2}\Delta q - \mathrm{div}(gq) + \left(\frac{l}{\varepsilon} - \lambda^\varepsilon\right)q = 0.$$

By a simple calculation using integration by parts,

$$
\begin{aligned}
0 &= \int \varpi \exp\left(\frac{I}{\varepsilon}\right)\left(\frac{\varepsilon}{2}\Delta q - \operatorname{div}(gq) + \left(\frac{l}{\varepsilon} - \lambda^{\varepsilon}\right) q\right) dx \\
&= \int q\, \varpi \exp\left(\frac{I}{\varepsilon}\right)\left(\frac{1}{2}\Delta I - \frac{1}{2\varepsilon}|\nabla I|^2 + \frac{l}{\varepsilon} - \lambda^{\varepsilon}\right) dx \\
&\quad + \int q \exp\left(\frac{I}{\varepsilon}\right)\left(\frac{\varepsilon}{2}\Delta \varpi + g\nabla\varpi + \frac{1}{2}\nabla I \nabla \varpi\right) dx.
\end{aligned} \tag{4.35}
$$

Equation (4.14) implies that the second term is bounded above by a constant. In fact, by an argument in Appendix 3, we can show

$$
I^*(x) - (I(x) - W^0(x)) \geq c|x|^2
$$

for some $c > 0$. Using the property of ϖ, we can easily prove the boundedness of the second term in (4.35).

From (4.35),

$$
\begin{aligned}
\int q\, \varpi \exp\left(\frac{I}{\varepsilon}\right)\frac{V}{\varepsilon}\, dx &\leq c + \int q \varpi \exp\left(\frac{I}{\varepsilon}\right)\left(\frac{1}{2}\Delta I - \lambda^{\varepsilon}\right) dx \\
&\leq c + c\int q \varpi \exp\left(\frac{I}{\varepsilon}\right) dx.
\end{aligned} \tag{4.36}
$$

To estimate the last term, let M be a large number and be fixed. The integral is evaluated separately in two regions: $\{|x| \leq M\sqrt{\varepsilon}\}$ and the complement. Here we choose a large M. The integral in the first region can be bounded by a constant for the following reason: $(I(x))/\varepsilon$ is bounded above by a constant in this region. Also by Lemma 4.5, $|\nabla W^{\varepsilon}(x)| \leq c\sqrt{\varepsilon}$ holds in this region. By the mean value theorem, we have $|W^{\varepsilon}(x)| \leq c\varepsilon$ in this region. These imply the result.

For the integral in the region $\{|x| \geq M\sqrt{\varepsilon}\}$, we remark that

$$
\frac{V(x)}{\varepsilon} \geq cM^2
$$

holds in this region. Therefore, this integral can be dominated by the half of the integral on the left-hand side of (4.36) if M is large enough. Then the left-hand side of (4.36) is bounded by a constant. This completes the proof of the lemma. □

5. Robust and risk sensitive control.

We recall the definition of the H_∞ norm of a nonlinear system. As in Section 2, consider ξ_t^0 satisfying

$$
\dot{\xi}_t^0 = g(\xi_t^0) + v_t, \qquad t \geq 0 \tag{5.1}
$$

with initial state $\xi_0^0 = x$. The function $v_\cdot$ is interpreted as a disturbance entering the dynamical system (5.1). Let z_t denote a state dependent output,

$z_t = h(\xi_t^0)$, with values in R^d for some d. System (5.1) has H_∞-norm $\le \gamma$ if and only if there exists $W(x)$ with $W(0) = 0$ such that

$$\int_0^T |z_t|^2\,dt \le \gamma^2 \int_0^T |v_t|^2\,dt + W(x) \tag{5.2}$$

for every $T > 0$ and $v_\cdot \in L_2([0, T]; R^n)$; W is called a storage function [45].

Let

$$l = \gamma^{-2} l_1, \qquad l_1 = \tfrac{1}{2}|h|^2. \tag{5.3}$$

Assume that g satisfies (2.1)(a) and (4.1) and that l_1 satisfies (2.1)(c) with $l_1(x) \ge 0$, $l_1(0) = 0$. This guarantees the existence of $\hat{\lambda}^0 \ge 0$ and Lipschitz W^0 as in Theorem 2.3. The lower bound on $|\nabla I|$ in (4.5) implies that there exists γ_1 such that $V_\gamma = \frac{1}{2}|\nabla I|^2 - \gamma^{-2} l_1$ satisfies (4.6) for all $\gamma > \gamma_1$. If $\gamma > \gamma_1$, then $\hat{\lambda}^0 = 0$. Moreover, W^0 is a storage function. This follows from (2.10) and the fact that $W^0(\xi_T^0) \ge 0$. The nonnegativity of W^0 follows from an easy argument using (4.1). See the proof of [12], Corollary 3.3. Thus γ_1 is an upper bound for the H_∞ norm. In the gradient case $g = -\nabla U$ with U satisfying (3.3)(a), (b), the H_∞-norm $\le \gamma$ if and only if $V_\gamma \ge 0$, that is,

$$\gamma|\nabla U(x)| \ge |h(x)| \qquad \text{for all } x.$$

The exponential in (1.4) can be interpreted as a risk-sensitive cost criterion and λ^ε represents a long-term growth rate of expected exponential cost. Theorems 4.3 and 4.4 are more precise approximations for λ^ε, $W^\varepsilon(x)$ than those in [14]. Theorem 4.4 states that

$$W^\varepsilon(x) = W^0(x) + \varepsilon Z(x) + o(\varepsilon), \tag{5.4}$$

where $Z(x)$ is the integral in this theorem. However (5.4) holds only in the region where W^0 is smooth.

In the case of controlled dynamical systems, instead of (1.1) let the state dynamics be governed by

$$dx_t = f(x_t, u_t)\,dt + \varepsilon^{1/2}\,db_t, \tag{5.5}$$

where u_t is the control applied at time t, with $u_t \in \mathscr{U}$ (the control space). We consider state feedback (complete state information). For a more precise formulation, see [17]. Let

$$\mu = \frac{1}{\gamma^2}.$$

The risk-sensitive control problem considered in [14] is to find a stationary feedback control policy $u_t = \underline{u}(x_t)$ for which a criterion of the form

$$\frac{\varepsilon}{\mu} \limsup_{T\to\infty} \frac{1}{T} \log E_x \exp\left[\frac{\mu}{\varepsilon}\int_0^T L(x_t, u_t)\,dt\right] \tag{5.6}$$

is minimized. The counterpart of the linear eigenvalue problem for $\lambda^\varepsilon, \psi^\varepsilon$ in Section 1 is a nonlinear eigenvalue problem considered in [14] by stochastic control methods. The solution depends on finding $\hat\lambda^{\varepsilon,\mu}$, $W^{\varepsilon,\mu}$ satisfying

$$(5.7^\varepsilon) \qquad \hat\lambda^{\varepsilon,\mu} = \frac{\varepsilon}{2}\Delta W^{\varepsilon,\mu} + \min_{u\in\mathscr{U}}[f(x,u)\nabla W^{\varepsilon,\mu} + L(x,u)] + \frac{\mu}{2}|\nabla W^{\varepsilon,\mu}|^2.$$

In the cases without control, $\hat\lambda^{\varepsilon,\mu} = (\varepsilon/\mu)\lambda^{\varepsilon,\mu}$, $W^{\varepsilon,\mu} = (\varepsilon/\mu)\log\psi^{\varepsilon,\mu}$ where

$$\lambda^{\varepsilon,\mu}\psi^{\varepsilon,\mu} = \frac{\varepsilon}{2}\Delta\psi^{\varepsilon,\mu} + g\nabla\psi^{\varepsilon,\mu} + \frac{\mu}{\varepsilon}l\psi^{\varepsilon,\mu}.$$

Under suitable assumptions on f, L and compact $\mathscr{U}$, $\hat\lambda^{\varepsilon,\mu}, W^{\varepsilon,\mu}$ exist, with $\hat\lambda^{\varepsilon,\mu}, \nabla W^{\varepsilon,\mu}$ uniformly bounded. As $\varepsilon \to 0$ and μ fixed, they tend to $\hat\lambda^0$, W^0 satisfying (5.7^0) in the viscosity sense. See [14], Sections 7, 8. Moreover, $\hat\lambda^0$ is the value of a differential game with average cost per unit time payoff and with (5.7^0) as the Isaacs PDE. In this game, the minimizing control corresponds to u_t and the disturbance v_t has the role of a maximizing control.

It is of interest to seek more precise asymptotic results like Theorems 4.3 and 4.4 for controlled dynamical systems. This can be done in the following special case. Let

$$f(x,u) = g(x) + bu, \qquad b \neq 1,\ b > 0,$$
$$L(x,u) = l_1(x) + \tfrac{1}{2}|u|^2$$

and $\mathscr{U} = R^n$. Assume that g, l_1 are as above. If we impose the artificial bound $|u| \le M$, then (5.7^ε) has a solution $\hat\lambda^{\varepsilon,\mu}$, $W^{\varepsilon,\mu}$, with

$$0 \le \hat\lambda^{\varepsilon,\mu} \le ||l_1||, \qquad ||\nabla W^{\varepsilon,\mu}|| \le c_0^{-1}||\nabla l_1||$$

with c_0 as in (4.1). See [14], Theorem 7.1. Let $M \ge c_0^{-1}||\nabla l_1||$. Then the minimum in ($5.7^\varepsilon$) is the same as the minimum for $u \in R^n$. Thus in this special case,

$$(5.8^\varepsilon) \qquad \hat\lambda^{\varepsilon,\mu} = \frac{\varepsilon}{2}\Delta W^{\varepsilon,\mu} + g\nabla W^{\varepsilon,\mu} + l + \frac{\mu - b^2}{2}|\nabla W^{\varepsilon,\mu}|^2.$$

Assume $\mu = 1$. If we let

$$\tilde\lambda^\varepsilon = (1-b^2)\hat\lambda^{\varepsilon,1}, \qquad \tilde W^\varepsilon = (1-b^2)W^{\varepsilon,1}, \qquad \tilde l = (1-b^2)l,$$

then this reduces to (4.8) and Theorems 4.3 and 4.4 apply. Note that in Assumption (4.6) we have not required $l \ge 0$, so that both $b < 1$ and $b > 1$ are allowed.

In [12] the asymptotic behavior of $\lambda^{\varepsilon,\mu}$ and $W^{\varepsilon,\mu}$ was considered as $\varepsilon \to 0, \gamma \to \infty$ so that $\gamma^2\varepsilon$ is constant. The asymptotic series obtained involve both H_2 and H_∞ formulations of the disturbance attenuation problem. The results of [12] are for the uncontrolled system with dynamics (1.1). It would be interesting to find corresponding results for controlled dynamical systems, governed by (5.5). We consider this special case with

$$\mu = \beta\varepsilon,$$

where β is a constant. Let

$$\tilde{\lambda}^{\varepsilon} = (\beta\varepsilon - b^2)\hat{\lambda}^{\varepsilon,\,\beta\varepsilon}, \qquad \tilde{W}^{\varepsilon} = (\beta\varepsilon - b^2)W^{\varepsilon,\,\beta\varepsilon}, \qquad \tilde{l}^{\varepsilon} = (\beta\varepsilon - b^2)l.$$

Then

$$\tilde{\lambda}^{\varepsilon} = \frac{\varepsilon}{2}\Delta\tilde{W}^{\varepsilon} + g\nabla\tilde{W}^{\varepsilon} + \tfrac{1}{2}|\nabla\tilde{W}^{\varepsilon}|^2 + \tilde{l}^{\varepsilon}.$$

The same argument can be applied and we have the following asymptotic result.

THEOREM 5.1. *Assume condition (4.1) and let l satisfy (4.6)*(a) *and $l \geq 0$. Then $W^{\varepsilon,\,\beta\varepsilon}$ converges to W^0 given by*

$$W^0(x) = \frac{1}{b^2}\inf_{\substack{\phi_0 = x,\\ \phi_\infty = 0}} \int_0^\infty \left(\frac{1}{2}|\phi_t - g(\phi_t)|^2 + b^2 l(\phi_t)\right) dt.$$

Moreover, we have

$$\lim_{\varepsilon\to 0}\frac{\hat{\lambda}^{\varepsilon,\,\beta\varepsilon}}{\varepsilon} = \kappa = \frac{1}{2}\Delta W^0(0).$$

If W^0 is smooth at x, then

$$\lim_{\varepsilon\to 0}\frac{W^{\varepsilon,\,\beta\varepsilon}(x) - W^0(x)}{\varepsilon} = \frac{\beta}{b^2}W^0(x) + \int_0^\infty \left(\frac{1}{2}\Delta W^0(\phi_t^*) - \kappa\right) dt,$$

where ϕ_t^* is the unique minimizing curve of the above calculus variations problem, which defines W^0. The limit exists uniformly for x in compact subsets of the smooth region of W^0.

APPENDIX 1

The purpose of this appendix is to sketch a proof of (3.8) following the argument in [4]. The interested reader should consult [4] for the details. Recall the conditions in (3.3).

The calculus of variations problem

$$C(x) = \inf_{\varphi_0 = x}\int_0^\infty \left(\tfrac{1}{2}|\dot{\varphi}_t|^2 + V(\varphi_t)\right) dt \tag{A1.1}$$

has Hamiltonian

$$H(x, p) = \tfrac{1}{2}|p|^2 - V(x).$$

An optimal trajectory of (A1.1) satisfies the equation

$$\begin{aligned} \dot{\varphi} &= H_p(\varphi, p),\\ \dot{p} &= -H_x(\varphi, p). \end{aligned} \tag{A1.2}$$

That is,

$$\begin{aligned} \dot{\varphi} &= p,\\ \dot{p} &= \nabla V(\varphi). \end{aligned}$$

As in [4], they also have the property

$$\varphi_t,\, p_t \to 0 \quad \text{as } t \to \infty.$$

For discussing the property of the system (A1.2) near the origin, we linearize the system (A1.2) at $(0, 0)$. We have the matrix

$$\Lambda = \begin{bmatrix} 0, & I \\ D, & 0 \end{bmatrix}.$$

It is not difficult to see that eigenvalues of Λ are given by $\pm\sqrt{\mu}$, where μ is an eigenvalue of D. The eigenvectors of Λ corresponding to $-\sqrt{\mu}$ and $\sqrt{\mu}$ are given by

$$\begin{bmatrix} -\dfrac{1}{\sqrt{D}}v \\ v \end{bmatrix}, \qquad \begin{bmatrix} \dfrac{1}{\sqrt{D}}v \\ v \end{bmatrix},$$

with v being an eigenvector of D corresponding to μ. Therefore, the stable manifold of (A1.2) near (0,0) is given by

$$p = \Phi(x), \qquad \Phi_x(0) = -\sqrt{D}.$$

Moreover,

$$\Phi(x) = -\nabla C(x)$$

for x near 0. Since $W^0 = U - C$, the above relation implies (3.8) after an easy computation.

APPENDIX 2

In this appendix we will prove (3.15).

By (3.13) and Itô's differential rule, we have

$$\text{(A2.1)} \qquad f^\varepsilon(x) = E_x\left[f^\varepsilon(b_1)\exp\left\{\left(-\int_0^1 (\hat{V}^\varepsilon(b_s) + \lambda^\varepsilon)\, ds\right)\right\}\right],$$

where b_t is a Brownian motion. It is easy to see that $-\lambda^\varepsilon$ has a uniform upper bound by (3.10). Also, $-\hat{V}^\varepsilon \le (\varepsilon/2)\Delta U$, which is bounded above. Therefore, f^ε is bounded by

$$c\int\left(\frac{1}{\sqrt{\pi}}\right)^n \exp\left(-\frac{|x-y|^2}{2}\right) f^\varepsilon(x)\, dx$$

which in turn is bounded by a constant, since $\int (f^\varepsilon)^2\, dx = 1$.

Using (A2.1) again and Condition (3.3)(d),

$$f^\varepsilon(x) \le cE_x\left[\exp\left(-c\int_0^1 |b_s|^2\, ds\right)\right].$$

Note the following identity,

$$E_x\left[\exp\left(-c\int_0^1 |b_s|^2\,ds\right)\right] = \exp(-\alpha n)u(x)E_x\left[\frac{1}{u(y_1)}\right], \tag{A2.2}$$

with

$$u(x) = \exp(-\alpha|x|^2), \qquad \alpha = \sqrt{\frac{c}{2}},$$

where y_t is the Ornstein–Uhlenbeck process satisfying

$$dy_t = -2\alpha y_t\,dt + db_t$$

with initial state $y_0 = x$. This is an easy consequence of the Girsanov theorem. Finally, by some computation, the expectation in (A2.2) is bounded by a constant. Thus, (3.15) is proved.

APPENDIX 3

In this appendix, we prove (4.5), (3.6) and Theorem 4.2.

PROOF OF (4.5). Let $\phi_0 = x$, $\phi_\infty = 0$, $u_t = \dot\phi_t + g(\phi_t)$. Then

$$\begin{aligned}\frac{d}{dt}|\phi|^2 &= -2\phi g(\phi) + 2\phi u\\ &\geq c|\phi|^2 + 2\phi u\\ &= c\left|\phi + \frac{1}{c}u\right|^2 - \frac{1}{c}|u|^2.\end{aligned}$$

Here $c = 2c_0$ with c_0 in (4.1). It is easy to deduce $I(x) \geq c_1|x|^2$ from this.

On the other hand, using ϕ_t as a test function, where

$$\dot\phi = g(\phi), \qquad \phi_0 = x,$$

then, for some constant c,

$$\begin{aligned}\tfrac{1}{2}\int_0^\infty |\dot\phi + g(\phi)|^2\,dt &= 2\int_0^\infty |g(\phi)|^2\,dt\\ &\leq c\int_0^\infty |\phi|^2\,dt\\ &\leq c|x|^2\end{aligned}$$

which implies $I(x) \leq c_2|x|^2$ for some constant c_2. Here we use

$$|\phi_t| \leq |\phi_0|\exp(-c_0 t)$$

by an easy argument.

Let I be differentiable at x. We first prove that

$$|\nabla I(x)| \geq c|x|$$

holds for some positive constant c which is independent of x. We know from [4] that there is a unique minimizing curve of (4.4) which satisfies

$$\dot{\phi} = -g(\phi) + p,$$
$$\dot{p} = g_x(\phi)p,$$

with

$$\phi_0 = x, \qquad p_0 = -2\nabla I(x).$$

By (4.1),

$$|p_t| \le \exp(-c_0 t)|p_0|.$$

Then

$$\begin{aligned} 2I(x) &= \frac{1}{2}\int_0^\infty |\dot{\phi} + g(\phi)|^2\,dt \\ &= \frac{1}{2}\int_0^\infty |p_t|^2\,dt \\ &\le \frac{1}{4c_0}|p_0|^2 \\ &= \frac{1}{c_0}|\nabla I(x)|^2. \end{aligned}$$

This implies the result with $c = \sqrt{2c_1 c_0}$.

On the other hand, we have $|\nabla I(x)| \le \bar{c}|x|$ for some constant $\bar{c}$ which follows easily from

$$|\nabla I(x)|^2 + g(x)\nabla I(x) = 0.$$

PROOF OF (3.6) AND THEOREM 4.2. Since (3.6) is a special case of (4.11), we only prove Theorem 4.2.

We first prove that $\hat{\lambda}^0 = 0$. Following the same calculation for (A3.2) below, we have

$$\int_0^T (l(\phi_t) - \tfrac{1}{2}|v_t|^2)\,dt \le I(x)$$

if $\phi_0 = x$, $\dot{\phi}_t = g(\phi_t) + v_t$. In particular,

$$\int_0^T (l(\phi_t) - \tfrac{1}{2}|v_t|^2)\,dt \le 0$$

if $\phi_0 = 0$, $\dot{\phi}_t = g(\phi_t) + v_t$.

From relation (2.10) with $x = 0$, $v = 0$, we have $\hat{\lambda}^0 T \ge 0$ for all T. Therefore,

$$\hat{\lambda}^0 \ge 0.$$

On the other hand, there is an optimal trajectory of (2.10) with $\phi_0 = 0$, satisfying

$$\dot{\phi}_t = g(\phi_t) + v_t, \qquad \phi_0 = 0. \tag{A3.1}$$

such that $|v_t| \le M$, $0 \le t \le T$, for some M which is independent of T. See [14], Section 5. Then we can use this property to deduce that $|\phi_t| \le r$, $0 \le t \le T$, for some $r > 0$ independent of T. From

$$\begin{aligned} 0 = W^0(0) &= W^0(\phi_T) + \int_0^T (l(\phi_t) - \tfrac{1}{2}|v_t|^2)\,dt - \hat{\lambda}^0 T \\ &\le W^0(\phi_T) - \hat{\lambda}^0 T, \end{aligned}$$

we have

$$\hat{\lambda}^0 T \le \sup_{|x|\le r} W^0(x)$$

for any $T > 0$. Then $\hat{\lambda}^0 \le 0$, that is, $\hat{\lambda}^0 = 0$.

We now prove (4.11). First, we remark that there is $\mu < 1$, μ near 1, such that

$$\int_0^T (l(\phi_t) - \tfrac{1}{2}\mu|v_t|^2)\,dt \le \mu I(x)$$

if $\phi_0 = x$, $\dot{\phi}_t = g(\phi_t) + v_t$. This follows by noting that (4.6) still holds for $\mu^{-1} l$ and using an inequality above with l, I replaced by $\mu^{-1}l, \mu^{-1}I$.

Fix $T > 0$. Let ϕ be an optimal trajectory of (2.10). Let $\dot{\phi}_t = g(\phi_t) + v_t$. Then, as above, there are M and r independent of T such that

$$|v_t| \le M, \qquad |\phi_t| \le r,$$

for all $0 \le t \le T$. By assumption,

$$\begin{aligned} W^0(x) &= W^0(\phi_T) + \int_0^T (l(\phi_t) - \tfrac{1}{2}|v_t|^2)\,dt \\ &= W^0(\phi_T) + \int_0^T (l(\phi_t) - \tfrac{1}{2}\mu|v_t|^2)\,dt - \tfrac{1}{2}(1-\mu)\int_0^T |v_t|^2\,dt \\ &\le W^0(\phi_T) + \mu I(x) - \tfrac{1}{2}(1-\mu)\int_0^T |v_t|^2\,dt. \end{aligned}$$

From this, there is c independent of T such that

$$\int_0^T |v_t|^2\,dt \le c.$$

From

$$\dot{\phi}_t = g(\phi_t) + v_t$$

and (4.1),

$$\begin{aligned} \frac{d}{dt}|\phi_t|^2 &= 2\langle \phi_t, g(\phi_t)\rangle + 2\langle \phi_t, v_t\rangle \\ &\le -2c_0|\phi_t|^2 + 2|\phi_t||v_t| \\ &\le -c_0|\phi_t|^2 + \frac{1}{c_0}|v_t|^2, \end{aligned}$$

we have

$$\int_0^T |\phi_t|^2\,dt \le \frac{1}{c_0}\left(|\phi_0|^2 + \frac{c}{c_0}\right).$$

This implies, for any $\delta > 0$, if T is large enough there is $S < T$ such that $|\phi_S| < \delta$ for ϕ, an optimal trajectory of (2.10). Remark that we have

$$W^0(x) = W^0(\phi_S) + \int_0^S (l(\phi_t) - \tfrac{1}{2}|v_t|^2)\,dt.$$

Denote $\tilde{v}$ by

$$\begin{aligned}\tilde{v}_t &= v_t, \qquad t \le S,\\ &= 0, \qquad t > S,\end{aligned}$$

and $\tilde{\phi}$ the solution of

$$\dot{\tilde{\phi}}_t = g(\tilde{\phi}_t) + \tilde{v}_t.$$

Using (4.1), we can prove that there is c independent of δ such that

$$\int_S^\infty l(\tilde{\phi}_t)\,dt \le c\delta.$$

Therefore,

$$\begin{aligned}W^0(x) &\le \int_0^\infty (l(\tilde{\phi}_t) - \tfrac{1}{2}|\tilde{v}_t|^2)\,dt + c\delta\\ &\le \sup_{\substack{\phi_0 = x,\\ \phi_\infty = 0}} \int_0^\infty (l(\phi_t) - \tfrac{1}{2}|\dot{\phi}_t - g(\phi_t)|^2)\,dt + c\delta.\end{aligned}$$

Since δ is arbitrary, we have

$$W^0(x) \le \sup_{\substack{\phi_0 = x,\\ \phi_\infty = 0}} \int_0^\infty (l(\phi_t) - \tfrac{1}{2}|\dot{\phi}_t - g(\phi_t)|^2)\,dt.$$

The reverse inequality is obvious by using (2.10) and the properties that $\hat{\lambda}^0 = 0$ and $W^0(0) = 0$. Thus we have the equality which is (4.11) as claimed. This completes the proof. □

PROOF OF (4.12). Given ϕ with $\phi_0 = x$, $\phi_\infty = 0$. We may assume that ϕ_t is in the smooth region of I for almost all t. In general, we can approximate a given curve by those satisfying such property for the following reason. Let A be the complement of the smooth region of I. We know A has Lebesgue measure zero. We take smooth α: $[0,\infty) \to [0,1]$ such that $\alpha(t) \to 0$ as $t \to \infty$, $\alpha(0) = 0$ and $\alpha(t) \ne 0$ for $t \ne 0$. Since $\int \chi_A(\phi_t + \alpha(t)y)\,dy = 0$ holds for all $t \ne 0$, by Fubini's theorem, we have $\int_0^\infty \chi_A(\phi_t + \alpha(t)y)\,dt = 0$ for almost all y. We can take $\phi_t + \alpha(t)y$ as an approximation for ϕ_t by choosing y with $|y|$ small and satisfying $\int_0^\infty \chi_A(\phi_t + \alpha(t)y)\,dt = 0$.

Now observing the relation

$$l(\phi) - \tfrac{1}{2}|\dot{\phi} - g(\phi)|^2 = -V(\phi) - \tfrac{1}{2}|\dot{\phi} - \theta(\phi)|^2 - \phi\nabla I(\phi),$$

where $\theta(x) = g(x) + \nabla I(x)$ and $V = \frac{1}{2}|\nabla I|^2 - l$, we have

$$\text{(A3.2)} \quad \int_0^\infty (l(\phi) - \tfrac{1}{2}|\dot{\phi} - g(\phi)|^2)\,dt = I(x) - \int_0^\infty (V(\phi) + \tfrac{1}{2}|\dot{\phi} - \theta(\phi)|^2)\,dt.$$

Let

$$u_t = \dot{\phi}_t - \theta_t.$$

Then

$$\frac{dI(\phi_t)}{dt} = u_t \nabla I(\phi_t).$$

Here we use $\theta\nabla I = 0$.

$$\begin{aligned} I(x) &= \int_0^\infty -u_t \nabla I(\phi_t)\,dt \\ &\le \int_0^\infty |u_t|^2\,dt + \int_0^\infty |\nabla I(\phi_t)|^2\,dt. \end{aligned}$$

Since $|\nabla I(\phi_t)| \le c_2|\phi_t|$,

$$c_1|x|^2 \le I(x) \le \int_0^\infty \left[|u_t|^2 + \frac{c_2}{c}V(\phi_t)\right]dt$$

with c as in (4.6)(b) and hence

$$c|x|^2 \le \int_0^\infty [|u_t|^2 + V(\phi_t)]\,dt$$

with another c. This gives the second inequality in (4.12). Note that $||\nabla W|| \le c_0^{-1}||\nabla l||$ was proved in [14]. See also Section 2.

APPENDIX 4

In this appendix, we shall derive some estimates for the process defined by (1.1) under condition (2.1), from which we can deduce that the semigroup generated by G^ε is bounded in B_c for each c and the eigenfunction ψ^ε is of polynomial growth, therefore is in B_1. The uniqueness of ψ^ε can be proved using an argument in [12], Theorem 3.1. In the following, we shall assume $\varepsilon = 1$.

Let $x(t)$ be the process defined by (1.1).

LEMMA A4.1. *Assume* (2.1). *Let* c_0 *be the constant in* (2.1). *Then for any* $c_1 > 0$ *there is* c_2 *such that*

$$E_x[\exp(c_1|x(t)|)] \le \exp(c_1|x|\exp(-c_0 t) + c_2).$$

PROOF. Let

$$y(t) = x(t)\exp(c_0 t).$$

Then

$$dy(t) = (c_0 y(t) + \exp(c_0 t) g(\exp(-c_0 t) y(t)))\,dt + \exp(c_0 t)\,db_t.$$

Therefore, by applying Itô's formula to the function

$$f(y) = \exp((1+|y|^2)^{1/2}),$$

we have

$$\begin{aligned} df(y(t)) = \bigg(& c_0 |y(t)|^2 + \exp(c_0 t) y g(\exp(-c_0 t) y(t)) \\ & + \tfrac{1}{2}\exp(2c_0 t) - \tfrac{1}{2}\exp(2c_0 t)\frac{|y(t)|^2}{1+|y(t)|^2}\bigg)\frac{1}{(1+|y(t)|^2)^{1/2}} f(y(t))\,dt \\ & + \tfrac{1}{2}\exp(2c_0 t)\frac{|y(t)|^2}{1+|y(t)|^2} f(y(t))\,dt + dM(t). \end{aligned}$$

Here $M(t)$ is a martingale. Since

$$|\exp(-c_0 t) y(t)| \geq R_0$$

implies

$$\exp(c_0 t) y(t) g(\exp(-c_0 t) y(t)) \leq -c_0 |y(t)^2$$

by (2.1), we have

$$df(y(t)) \leq c\exp(2c_0 t + R_0 \exp(c_0 t))\,dt + dM(t).$$

for some constant c. This implies

$$\begin{aligned} E_x[f(y(t))] &\leq f(y(0)) + \int_0^t c\exp(2c_0 s + R_0\exp(c_0 s))\,ds \\ &\leq f(y(0)) + \frac{c}{R_0}\exp(c_0 t + R_0 \exp(c_0 t)) \\ &\leq \exp(|x| + c_0 t + R_0 \exp(c_0 t) + c). \end{aligned}$$

Here c can be a different constant. Thus,

$$E_x[\exp(\exp(c_0 t)|x(t)|)] \leq \exp(|x| + c_0 t + R_0 \exp(c_0 t) + c).$$

By Hölder's inequality,

$$\begin{aligned} E_x[\exp(c_1|x|)] &\leq (E_x[\exp(\exp(c_0 t)|x(t)|)])^{c_1 \exp(-c_0 t)} \\ &\leq \exp(c_1 |x| \exp(-c_0 t) + c_0 c_1 \exp(-c_0 t) t + R_0 + c\exp(-c_0 t)) \\ &\leq \exp(c_1|x|\exp(-c_0 t) + c_2) \end{aligned}$$

for some constant c_2. This completes the proof. □

Let G be the operator in (1.2) and with $\varepsilon = 1$. The space B_c is defined by

$$B_c = \{f \in C(R^n); f(x)\exp(-c|x|)\text{is bounded}\}$$

COROLLARY A4.2. *The operator G generates a semigroup T_t on B_c.*

PROOF. T_t is given by

$$T_t f(x) = E_x\left[f(x(t))\exp\left(\int_0^t l(x(s))\,ds\right)\right]$$

by the Feynman–Kac formula. See [28]. Therefore,

$$|T_t f(x)| \le \exp(\|l\|t)E_x[|f(x(t))|].$$

The result follows from Lemma A4.1 immediately.

COROLLARY A4.3 *Let ψ be the principal eigenfunction for G with $\psi(0) = 1$. Then there are m, c_1, c_2 such that*

$$\psi(x) \le c_1|x|^m + c_2.$$

PROOF. By Theorem 2.2, there are c, r such that

$$\psi(x) \le \exp(c|x| + r).$$

Since ψ satisfies

$$G\psi(x) = \lambda_0\psi,$$

we have

$$\psi(x) = E_x\left[\psi(x(t))\exp\left(\int_0^t (\lambda_0 + l(x(s)))\,ds\right)\right].$$

Therefore, with $\hat{c} = \|l\| + |\lambda_0|$,

$$\begin{aligned}\psi(x) &\le \exp(\hat{c}t + r)E_x[\exp(c|x(t)|)]\\ &\le \exp(\hat{c}t + c|x|\exp(-c_0 t) + \hat{r}).\end{aligned}$$

Minimizing this w.r.t. t, we get the result with

$$m = \frac{\hat{c}}{c_0}, \qquad c_1 = \left(\frac{c_0 c}{\hat{c}}\right)^{\hat{c}/c_0}\exp\left(\frac{\hat{c}}{c_0} + \hat{r}\right)$$

for $|x| > \hat{c}/c_0 c$. Since ψ is continuous, we have

$$\psi(x) \le c_2 \quad \text{if } |x| \le \frac{\hat{c}}{c_0 c}.$$

The lemma is proved with this choice of m, c_1, c_2.

LEMMA A4.4. *Let $\lambda > \lambda_0$; λ_0 is the principal eigenvalue for G. Then the inverse of $\lambda - G$ exits on B_c and is given by*

$$(\lambda - G)^{-1} f(x) = \int_0^\infty e^{-\lambda t} T_t f(x)\, dt.$$

PROOF. We only need to prove that the integral defined by the right-hand side of this equality converges.

Let $W = \log \psi$. Then,

$$\tfrac{1}{2}\Delta W + g \cdot \nabla W + \tfrac{1}{2}|\nabla W|^2 + l = \lambda_0.$$

Now,

$$\begin{aligned} T_t f(x) &= E_x\left[f(x(t)) \exp\left(\int_0^t l(x(s))\, ds \right)\right] \\ &= E_x\left[f(x(t)) \exp\left(\int_0^t -(\tfrac{1}{2}\Delta W + g \cdot \nabla W + \tfrac{1}{2}|\nabla W|^2)(x(s))\, ds \right)\right] \exp(\lambda_0 t), \end{aligned}$$

by Itô's formula and we have

$$\begin{aligned} T_t f(x) = \exp(\lambda_0 t) E_x\Bigg[& f(x(t)) \exp(-(W(x(t)) - W(x))) \\ & \times \exp\left(\int_0^t \nabla W(x(s))\, db_s - \tfrac{1}{2}\int_0^t |\nabla W(x(s))|^2\, ds \right)\Bigg] \\ = \exp(\lambda_0 t) & E_x[f(x^*(t)) \exp(-(W(x^*(t)) - W(x)))], \end{aligned}$$

where $x^*(t)$ is the process defined by

$$dx^*(t) = (g + \nabla \log \psi)(x^*(t))\, dt + db_t.$$

Here we apply the Girsanov theorem for changing the measure. We remark that Lemma A4.1 also applies to $x^*(t)$. The assertion follows from these easily.

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