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ON L^2-PROJECTIONS ON A SPACE OF STOCHASTIC INTEGRALS

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Let X be an $\mathbb{R}^d$-valued continuous semimartingale, T a fixed time horizon and Θ the space of all $\mathbb{R}^d$-valued predictable X-integrable processes such that the stochastic integral $G(\vartheta) = \int \vartheta\, dX$ is a square-integrable semimartingale. A recent paper gives necessary and sufficient conditions on X for $G_T(\Theta)$ to be closed in $L^2(P)$. In this paper, we describe the structure of the L^2-projection mapping an $\mathscr{F}_T$-measurable random variable $H \in L^2(P)$ on $G_T(\Theta)$ and provide the resulting integrand $\vartheta^H \in \Theta$ in feedback form. This is related to variance-optimal hedging strategies in financial mathematics and generalizes previous results imposing very restrictive assumptions on X. Our proofs use the variance-optimal martingale measure $\widetilde{P}$ for X and weighted norm inequalities relating $\widetilde{P}$ to the original measure P.

0. Introduction. Let X be an $\mathbb{R}^d$-valued semimartingale and Θ the space of all $\mathbb{R}^d$-valued predictable X-integrable processes such that the stochastic integral $G(\vartheta) = \int \vartheta\, dX$ is a square-integrable semimartingale. For a fixed time horizon T, $G_T(\Theta)$ is then a linear subspace of $L^2(P)$, and so one can ask if there is an L^2-projection on $G_T(\Theta)$, that is, if $G_T(\Theta)$ is closed in $L^2(P)$. If X is a local martingale, the answer is of course positive since the stochastic integral is then an isometry. For a continuous semimartingale X, necessary and sufficient conditions for the closedness of $G_T(\Theta)$ in $L^2(P)$ have recently been established by Delbaen, Monat, Schachermayer, Schweizer and Stricker (1996), subsequently abbreviated as DMSSS; see also Grandits and Krawczyk (1996) for a generalization to the case of $L^p(P)$ with $p > 1$.

In this paper, we describe the structure of the L^2-projection mapping an $\mathscr{F}_T$-measurable random variable $H \in L^2(P)$ on $G_T(\Theta)$ and show how to obtain the integrand $\vartheta^H \in \Theta$ appearing in this projection. If X is a local martingale, this is a classical question whose answer is given by the well-known Galtchouk–Kunita–Watanabe projection theorem. The more general semimartingale case comes up naturally in hedging problems from financial mathematics, and some partial results have been obtained by Duffie and Richardson (1991), Hipp (1993, 1996), Schweizer (1994), Wiese (1995) and Pham, Rheinländer and Schweizer (1996), among others. But all these papers imposed unnatural and very restrictive conditions on X which do not hold in typical financial models; this is discussed in more detail in Pham, Rheinländer and Schweizer (1996). Moreover, no paper so far gives a solution for $H \in L^2(P)$; at least

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$H \in L^{2+\varepsilon}(P)$ is always assumed. The present paper gives the solution in the general continuous L^2-case.

What do we mean by "general continuous L^2-case"? First of all, we assume that X is a *continuous semimartingale*; any extensions to a discontinuous process are for the moment postponed to future research. Moreover, we only suppose that $H \in L^2(P)$. The basic idea for attacking the problem is to connect the semimartingale to the martingale case in some way, and this is achieved by assuming that there exists an equivalent local martingale measure (ELMM, for short) for X, that is, a probability measure Q equivalent to P such that X is a local Q-martingale. This is a well-known condition in financial mathematics, which states that X should not allow arbitrage opportunities. By Girsanov's theorem, the existence of an ELMM implies that the canonical decomposition of X must have the form

$$X = X_0 + M + \int d\langle M\rangle\,\lambda$$

for some predictable process λ. Again by Girsanov's theorem, a natural candidate for an ELMM is then given by the so-called *minimal martingale measure* $\widehat{P}$ with density

$$\frac{d\widehat{P}}{dP} = \mathscr{E}\Big(-\int \lambda\, dM\Big)_T.$$

The main results in the existing literature show that the integrand ϑ^H of X in the projection of H on $G_T(\Theta)$ can be written in feedback form as

$$\vartheta^H = \widehat{\xi}^H - \frac{\widehat{\zeta}}{\widehat{Z}}\Big(\widehat{V}^H_- - \int \vartheta^H\, dX\Big), \tag{0.1}$$

where $\widehat{V}^H$ is the $\widehat{P}$-martingale

$$\widehat{V}^H_t = \widehat{E}[H|\mathscr{F}_t], \qquad 0 \le t \le T \tag{0.2}$$

and $\widehat{\xi}^H$ is the integrand of X in the Galtchouk–Kunita–Watanabe decomposition of H under $\widehat{P}$. The *crucial assumption* for this to be true is that the density of $\widehat{P}$ can be written as a constant plus a stochastic integral of X,

$$\frac{d\widehat{P}}{dP} = \widehat{E}\Big[\frac{d\widehat{P}}{dP}\Big] + \int_0^T \widehat{\zeta}_s\, dX_s \tag{0.3}$$

for some $\widehat{\zeta} \in \Theta$, and the process $\widehat{Z}$ in (0.1) is then

$$\widehat{Z}_t = \widehat{E}\Big[\frac{d\widehat{P}}{dP}\Big|\mathscr{F}_t\Big] = \widehat{E}\Big[\frac{d\widehat{P}}{dP}\Big] + \int_0^t \widehat{\zeta}_s\, dX_s, \qquad 0 \le t \le T. \tag{0.4}$$

In addition, one has to impose moment conditions on H and $d\widehat{P}/dP$ since (0.1) is proved by switching from P to $\widehat{P}$ and back, and one needs square integrability under $\widehat{P}$ for this method to work.

As pointed out in Pham, Rheinländer and Schweizer (1996), the minimal martingale measure $\widehat{P}$ will typically not satisfy (0.3) so that the preceding result has a very limited scope. But there is another ELMM whose density almost by definition does satisfy the requirement (0.3). This is the *variance-optimal martingale measure* $\widetilde{P}$ defined by the property that its density with respect to P has minimal $L^2(P)$-norm among all ELMMs for X. Due to a result of Delbaen and Schachermayer (1996), $\widetilde{P}$ always exists if X is continuous and if there is at least one ELMM for X with density in $L^2(P)$. In this paper, we show that these two conditions plus closedness of $G_T(\Theta)$ in $L^2(P)$ are already sufficient to obtain ϑ^H in feedback form. More precisely, we show that under these assumptions, (0.1)–(0.4) always hold if we replace the minimal martingale measure throughout by the variance-optimal martingale measure and every hat $\widehat{}$ by a tilde $\widetilde{}$. Moreover, no assumption on H is needed except, of course, $H \in L^2(P)$.

The main tools to obtain these results are *weighted norm inequalities* which allow us to obtain estimates in $L^2(P)$ for processes which are local martingales under $\widetilde{P}$. This is possible thanks to the main result of DMSSS, which characterizes the closedness of $G_T(\Theta)$ by the validity of such inequalities. Section 1 contains a precise formulation of the basic problem and a brief survey of those results of DMSSS that we use in this paper. In Section 2, we study the properties of the Galtchouk–Kunita–Watanabe decomposition of H under an ELMM Q, and we show that the terms in this decomposition have good properties in $L^2(P)$ if one has weighted norm inequalities linking P and Q. Any such Q then leads to a decomposition of H into a constant, an integral in $G_T(\Theta)$ and a certain orthogonal term, and it remains to project constants and those orthogonal terms on $G_T(\Theta)$. By the definition of $\widetilde{P}$, the density $d\widetilde{P}/dP$ is a multiple of the projection of the constant 1 on the orthogonal complement of $G_T(\Theta)$ in $L^2(P)$, and this suggests working with $Q = \widetilde{P}$ to effect the decomposition of H. In Section 3, we show that this does indeed give the solution and leads to the representation of ϑ^H as in (0.1). An alternative approach to determine the integrand ϑ^H has recently been proposed by Gouriéroux, Laurent and Pham (1996). We briefly discuss their main result in Section 4, and, because this is not clear from their formulation, we prove that they do indeed solve the same problem as in our paper.

1. Preliminaries.

Let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a filtered probability space with a filtration $\mathbb{F} = (\mathscr{F}_t)_{0 \le t \le T}$ satisfying the usual conditions, where $T \in (0, \infty]$ is a fixed time horizon. For simplicity, we assume that $\mathscr{F}_0$ is trivial and $\mathscr{F} = \mathscr{F}_T$. All stochastic processes will be indexed by $t \in [0, T]$. Let X be a continuous $\mathbb{R}^d$-valued semimartingale with canonical decomposition $X = X_0 + M + A$. For any $\mathbb{R}^d$-valued predictable X-integrable process ϑ, we denote by $G(\vartheta)$ the (real-valued) stochastic integral process $G(\vartheta) := \int \vartheta \, dX$. Unexplained terminology and notation from martingale theory can be found in Dellacherie and Meyer (1982). Throughout the paper, C denotes a generic constant in $(0, \infty)$ which may vary from line to line.

DEFINITION. For any RCLL process Y, we denote by $Y_t^* := \sup_{0\le s\le t}|Y_s|$ the supremum process of Y. The space $\mathscr{R}^2(P)$ consists of all adapted RCLL processes Y such that

$$\|Y\|_{\mathscr{R}^2(P)} := \|Y_T^*\|_{L^2(P)} < \infty.$$

DEFINITION. Let $L^2(M)$ be the space of all $\mathbb{R}^d$-valued predictable processes ϑ such that

$$\|\vartheta\|_{L^2(M)}^2 := E\biggl[\int_0^T \vartheta_t^{\mathrm{tr}}\, d\langle M\rangle_t\, \vartheta_t\biggr] < \infty.$$

Let $L^2(A)$ be the space of all $\mathbb{R}^d$-valued predictable processes ϑ such that

$$\|\vartheta\|_{L^2(A)}^2 := E\biggl[\biggl(\int_0^T \bigl|\vartheta_t^{\mathrm{tr}}\, dA_t\bigr|\biggr)^2\biggr] < \infty.$$

Finally, we set $\Theta := L^2(M) \cap L^2(A)$.

If ϑ is in Θ, the continuous semimartingale $G(\vartheta)$ is in $\mathscr{R}^2(P)$ so that in particular its terminal value $G_T(\vartheta)$ is in $L^2(P)$. For any given $H \in L^2(P)$, we may thus consider the optimization problem

$$\text{Minimize } \|H - G_T(\vartheta)\|_{L^2(P)} \text{ over all } \vartheta \in \Theta. \tag{1.1}$$

To ensure that (1.1) has a solution for every $H \in L^2(P)$, we impose throughout this paper the *standing assumption*

$$G_T(\Theta) \text{ is closed in } L^2(P). \tag{1.2}$$

Necessary and sufficient conditions on X to guarantee (1.2) were established in DMSSS, and we briefly summarize here those results we shall use in the present paper.

DEFINITION. Let Z be a uniformly integrable martingale with $Z_0 = 1$ and $Z_T > 0$. We say that Z satisfies the *reverse Hölder inequality* with exponent $p \in (1, \infty)$ under P, denoted by $R_p(P)$, if there is a constant C such that for every stopping time $S \le T$, we have

$$E\biggl[\biggl(\frac{Z_T}{Z_S}\biggr)^p \bigg| \mathscr{F}_S\biggr] \le C.$$

DEFINITION. Let Z be an adapted RCLL process. We say that Z satisfies *condition* (J) if there is a constant C such that

$$\frac{1}{C} Z_- \le Z \le C Z_-.$$

DEFINITION. If Q is a probability measure equivalent to P, we denote by Z^Q an RCLL version of the strictly positive P-martingale

$$Z_t^Q := E_P\biggl[\frac{dQ}{dP}\bigg|\mathscr{F}_t\biggr], \qquad 0 \le t \le T.$$

With these definitions in place, we can now recall two fundamental *weighted norm inequalities*. The first one is a consequence of Propositions 4 and 5 and the Corollary on page 318 of Doléans-Dade and Meyer (1979); the second one follows by a localization argument from Theorem 2 of Bonami and Lépingle (1979), combined with Proposition 5 of Doléans-Dade and Meyer (1979).

PROPOSITION 1. *Let Q be a probability measure equivalent to P and assume that Z^Q satisfies $R_2(P)$ and (J). Then we have the following.*

(i) *There exists a constant C such that*

$$E\big[(N_S^*)^2\big] \le CE\big[N_S^2\big]$$

for all uniformly integrable Q-martingales N and all stopping times $S \le T$.

(ii) *There exist two constants c and C in $(0, \infty)$ such that*

$$cE\big[(N_S^*)^2\big] \le E\big[[N]_S\big] \le CE\big[(N_S^*)^2\big]$$

for all local Q-martingales N and all stopping times $S \le T$.

Note that (i) and (ii) are generalizations of the Doob and Burkholder–Davis–Gundy inequalities, respectively, since we have estimates in the L^2-norm under P for processes which are local martingales under Q.

To relate Proposition 1 to the closedness of $G_T(\Theta)$ in $L^2(P)$, we recall the concept of the variance-optimal martingale measure which was studied in Delbaen and Schachermayer (1996) and Schweizer (1996). Let $\mathscr{V}$ denote the linear subspace of $L^\infty(\Omega, \mathscr{F}, P)$ spanned by the simple stochastic integrals of the form $Y = h^{\mathrm{tr}}(X_{T_2} - X_{T_1})$, where $T_1 \le T_2 \le T$ are stopping times such that the stopped process X^{T_2} is bounded and h is a bounded $\mathbb{R}^d$-valued $\mathscr{F}_{T_1}$-measurable random variable.

DEFINITION. Let $\mathscr{M}^s(P)$ be the space of all signed measures $Q \ll P$ with $Q[\Omega] = 1$ and

$$E\left[\frac{dQ}{dP}\, Y\right] = 0 \quad \text{for all } Y \in \mathscr{V}.$$

Let $\mathscr{M}^e(P)$ denote the subset of all probability measures $Q \in \mathscr{M}^s(P)$ such that Q is equivalent to P. Finally, we define two sets of densities by

$$\mathscr{D}^x := \left\{ \frac{dQ}{dP} \,\middle|\, Q \in \mathscr{M}^x(P) \right\} \quad \text{for } x \in \{e, s\}.$$

It is clear that X is a local Q-martingale for any $Q \in \mathscr{M}^e(P)$ and that $\mathscr{D}^s \cap L^2(P)$ is a closed convex set.

DEFINITION. The *variance-optimal martingale measure* $\widetilde{P}$ is the unique element of $\mathscr{M}^s(P)$ such that $\widetilde{D} = d\widetilde{P}/dP$ is in $L^2(P)$ and minimizes $\|D\|_{L^2(P)}$ over all $D \in \mathscr{D}^s \cap L^2(P)$.

Note that $\widetilde{P}$ exists if and only if $\mathscr{D}^s \cap L^2(P)$ is nonempty. In that case, we define Z and $\widetilde{Z}$ as RCLL versions of

$$Z_t := E\left[\frac{d\widetilde{P}}{dP}\middle|\mathscr{F}_t\right] = Z_t^{\widetilde{P}}, \qquad 0 \le t \le T$$

and

$$\widetilde{Z}_t := \widetilde{E}\left[\frac{d\widetilde{P}}{dP}\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T,$$

where $\widetilde{E}$ denotes expectation with respect to $\widetilde{P}$. Since X is continuous, Theorem 1.3 of Delbaen and Schachermayer (1996) implies that $\widetilde{P}$ is actually in $\mathscr{M}^e(P)$ as soon as it exists. In particular, $\mathscr{D}^e \cap L^2(P)$ is nonempty as soon as $\mathscr{D}^s \cap L^2(P)$ is. The following result is then a partial statement of Theorem 4.1 of DMSSS, combined with their Lemma 2.17, Theorem 3.7, Lemma 3.2, Theorem 2.22 and Theorem 1.3 of Delbaen and Schachermayer (1996); $L^\infty_+(P)$ denotes the space of all nonnegative bounded random variables.

THEOREM 2. *For a continuous semimartingale X, the following conditions are equivalent.*

(i) *$G_T(\Theta)$ is closed in $L^2(P)$, and $G_T(\Theta) \cap L^\infty_+(P) = \{0\}$.*
(ii) *$G_T(\Theta)$ is closed in $L^2(P)$, and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$.*
(iii) *The variance–optimal martingale measure $\widetilde{P}$ exists and is in $\mathscr{M}^e(P)$, and $Z = Z^{\widetilde{P}}$ satisfies the reverse Hölder inequality $R_2(P)$.*

Moreover, each of these conditions implies that Z satisfies condition (J) and that $\Theta = L^2(M)$.

We conclude this section with a simple observation from DMSSS, which turns out to be extremely useful in the sequel. If $\widetilde{P}$ exists, the Bayes rule yields

$$\widetilde{Z}_t = \widetilde{E}\big[\widetilde{Z}_T\big|\mathscr{F}_t\big] = \frac{1}{Z_t}E\big[\widetilde{Z}_T^2\big|\mathscr{F}_t\big] = \frac{1}{Z_t}E\big[Z_T^2\big|\mathscr{F}_t\big].$$

If Z satisfies $R_2(P)$, we have from Jensen's inequality

$$1 \le \frac{1}{Z_t^2}E\big[Z_T^2\big|\mathscr{F}_t\big] \le C,$$

and therefore

$$Z_t \le \widetilde{Z}_t \le CZ_t. \tag{1.3}$$

The importance of this comparison lies in the fact that it will allow us to switch freely between Z and $\widetilde{Z}$ for the purposes of estimation.

2. Kunita–Watanabe decompositions under a change of measure. Let Q be an equivalent local martingale measure for X, that is, a probability measure equivalent to P such that X is a local Q-martingale. Since X is continuous, every local Q-martingale admits a Galtchouk–Kunita–Watanabe decomposition with respect to X under Q into a stochastic integral of X and a local Q-martingale strongly Q-orthogonal to X; see Ansel and Stricker (1993). Our main result in this section shows that a control on the density process Z^Q allows us to obtain good integrability properties under the original measure P for this decomposition.

THEOREM 3. *Assume (1.2) as well as $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. Let $dQ/dP \in \mathscr{D}^e \cap L^2(P)$ be such that the associated density process Z^Q satisfies $R_2(P)$ and (J). For any $H \in L^2(P)$, define the Q-martingale $V^{H,Q}$ as an RCLL-version of $V_t^{H,Q} := E_Q[H|\mathscr{F}_t]$. Then there exist a process $\xi^{H,Q} \in \Theta$ and a Q-martingale $L^{H,Q}$ null at 0 with $L^{H,Q} \in \mathscr{R}^2(P)$ and*

$$[L^{H,Q}, X^i] = 0 \quad \text{for } i = 1, \dots, d \tag{2.1}$$

such that $V^{H,Q}$ can be uniquely written as

$$V_t^{H,Q} = E_Q[H] + \int_0^t \xi_s^{H,Q}\, dX_s + L_t^{H,Q}, \qquad 0 \le t \le T.$$

PROOF. Since X is a continuous local Q-martingale, we know from Ansel and Stricker (1993) that $V^{H,Q}$ has a unique Galtchouk–Kunita–Watanabe decomposition with respect to X under Q. More precisely, there exist an $\mathbb{R}^d$-valued predictable X-integrable process $\xi^{H,Q}$ and a local Q-martingale $L^{H,Q}$ null at 0 with

$$V^{H,Q} = E_Q[H] + \int \xi^{H,Q}\, dX + L^{H,Q}$$

and such that $[L^{H,Q}, X^i]$ is a local Q-martingale for $i = 1, \dots, d$. Since X is continuous, we have

$$[L^{H,Q}, X^i] = \langle L^{H,Q}, X^i \rangle = 0 \quad \text{for } i = 1, \dots, d$$

and therefore (2.1). By definition, $V^{H,Q}$ is a uniformly integrable Q-martingale. Because Z^Q satisfies $R_2(P)$ and (J), Proposition 1 implies that

$$E\big[[V^{H,Q}]_T\big] \le CE\Big[\sup_{0\le t\le T} |V_t^{H,Q}|^2\Big] \le CE\big[(V_T^{H,Q})^2\big] = CE[H^2] < \infty.$$

By (2.1) and the continuity of X,

$$[V^{H,Q}] = \int (\xi^{H,Q})^{\mathrm{tr}}\, d\langle M\rangle\, \xi^{H,Q} + [L^{H,Q}],$$

and so we conclude that $\xi^{H,Q}$ is in $L^2(M)$, hence in Θ by Theorem 2. Moreover, $L^{H,Q}$ is a local Q-martingale with $[L^{H,Q}]_T \le [V^{H,Q}]_T \in L^1(P)$, and so $L^{H,Q}$ is in $\mathscr{R}^2(P)$ by part (ii) of Proposition 1. Since dQ/dP is in $L^2(P)$, this finally implies that the local Q-martingale $L^{H,Q}$ is in fact a true Q-martingale. □

REMARK. If the minimal martingale measure $\widehat{P}$ happens to satisfy the assumptions of Theorem 3, the above decomposition for $Q = \widehat{P}$ will coincide with the Föllmer–Schweizer decomposition of H; see Schweizer (1995a). For $Q \neq \widehat{P}$, we obtain in general a different decomposition. Moreover, it may happen that $G_T(\Theta)$ is closed and that the variance-optimal martingale measure $\widetilde{P}$ satisfies $R_2(P)$, while $\widehat{P}$ fails to satisfy $R_2(P)$; see Example 3.12 of DMSSS. Together with the development in the next section, this shows that the Föllmer–Schweizer decomposition is in general not the appropriate tool to solve the optimization problem (1.1).

3. The integrand in the L^2-projection on $G_T(\Theta)$.

Consider now a fixed random variable $H \in L^2(P)$. Thanks to the standing assumption (1.2), we can project H in $L^2(P)$ on $G_T(\Theta)$ so that (1.1) has a solution which we denote by $\vartheta^H \in \Theta$. Although the random variable $G_T(\vartheta^H)$ is uniquely determined, ϑ^H itself need not be unique, but it will be as soon as the mapping $\vartheta \mapsto G_T(\vartheta)$ is injective. According to Lemma 3.5 of DMSSS, this is the case if $\mathscr{D}^e \cap L^2(P)$ is nonempty, and so we shall adopt this assumption in addition to (1.2).

In order to determine ϑ^H, we can use Theorem 3 to decompose H into three terms and to project these on $G_T(\Theta)$ separately. The middle term is already in $G_T(\Theta)$ for any suitable choice of Q in Theorem 3. The first term is a constant, and so its projection will be directly related to the density of the variance-optimal martingale measure $\widetilde{P}$. This suggests working with $Q = \widetilde{P}$ in Theorem 3, an intuition supported by the results obtained in Pham, Rheinländer and Schweizer (1996), and we shall see that $Q = \widetilde{P}$ is indeed the right choice.

According to the projection theorem, a process $\vartheta^H \in \Theta$ solves (1.1) if and only if

$$E\big[\big(H - G_T(\vartheta^H)\big)G_T(\vartheta)\big] = 0 \quad \text{for all } \vartheta \in \Theta. \tag{3.1}$$

By Theorem 2, the density process $Z = Z^{\widetilde{P}}$ of $\widetilde{P}$ satisfies $R_2(P)$ and condition (J), and so Theorem 3 allows us to write H as

$$H = \widetilde{E}[H] + \int_0^T \widetilde{\xi}_s^H \, dX_s + \widetilde{L}_T^H \tag{3.2}$$

for a process $\widetilde{\xi}^H \in \Theta$ and a $\widetilde{P}$-martingale $\widetilde{L}^H$ null at 0 with $\widetilde{L}^H \in \mathscr{R}^2(P)$ and

$$\big[\widetilde{L}^H, X^i\big] = 0 \quad \text{for } i = 1, \dots, d. \tag{3.3}$$

By Lemma 1 of Schweizer (1996), the density of $\widetilde{P}$ with respect to P can be written as

$$\frac{d\widetilde{P}}{dP} = \widetilde{E}\bigg[\frac{d\widetilde{P}}{dP}\bigg] + \int_0^T \widetilde{\zeta}_s \, dX_s \quad \text{for some } \widetilde{\zeta} \in \Theta,$$

and so we have

$$\widetilde{Z}_t = \widetilde{E}\left[\frac{d\widetilde{P}}{dP}\middle|\mathscr{F}_t\right] = \widetilde{E}[\widetilde{Z}_T] + \int_0^t \widetilde{\zeta}_s\, dX_s, \qquad 0 \le t \le T. \tag{3.4}$$

This shows in particular that the $\widetilde{P}$-martingale $\widetilde{Z}$ is continuous and strongly $\widetilde{P}$-orthogonal to a local $\widetilde{P}$-martingale $\widetilde{L}$ if and only if $\widetilde{L}$ is strongly $\widetilde{P}$-orthogonal to X.

LEMMA 4. *Assume (1.2) as well as $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. Then we have the following.*

(i) *For $H \equiv 1$, the solution ϑ^H of (1.1) is given by*

$$\vartheta^H = -\widetilde{Z}_0^{-1}\widetilde{\zeta}.$$

(ii) *For $H = \int_0^T \widetilde{\xi}_s^H\, dX_s$ with $\widetilde{\xi}^H \in \Theta$, the solution ϑ^H of (1.1) is given by*

$$\vartheta^H = \widetilde{\xi}^H.$$

PROOF. Since (ii) is obvious, we only have to prove (i). Property (3.4) of the variance-optimal martingale measure implies that

$$H = 1 = \widetilde{Z}_0^{-1}\widetilde{Z}_T - \int_0^T \widetilde{Z}_0^{-1}\widetilde{\zeta}_s\, dX_s,$$

and by the definition of $\widetilde{P}$, $\widetilde{Z}_T$ is in the orthogonal complement of $G_T(\Theta)$ in $L^2(P)$. Since $\widetilde{Z}_0^{-1}\widetilde{\zeta}$ is in Θ, the assertion follows from (3.1). □

In view of the preceding discussion, it now remains to consider the case where $H = \widetilde{L}_T^H$. This is actually the hardest case, and the next theorem can in a sense be viewed as the main result of this paper.

THEOREM 5. *Assume (1.2) and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. Let $H \in L^2(P)$ be such that the $\widetilde{P}$-martingale $\widetilde{L}$ defined by $\widetilde{L}_t := \widetilde{E}[H|\mathscr{F}_t]$ is null at 0 and satisfies $[\widetilde{L}, X^i] = 0$ for $i = 1, \dots, d$. Then the solution ϑ^H of (1.1) is given by*

$$\vartheta_t^H = -\widetilde{\zeta}_t \int_0^{t-} \widetilde{Z}_s^{-1}\, d\widetilde{L}_s.$$

PROOF. Since $[\widetilde{L}, X^i] = 0$ for $i = 1, \dots, d$, (3.4) implies that $[\widetilde{L}, \widetilde{Z}] = 0$. If we define the $\mathbb{R}^d$-valued predictable X-integrable process $\bar{\vartheta}$ by

$$\bar{\vartheta}_t := -\widetilde{\zeta}_t \int_0^{t-} \widetilde{Z}_s^{-1}\, d\widetilde{L}_s,$$

the product rule and (3.4) therefore imply that

$$\int \bar{\vartheta}\, dX = \widetilde{L} - \widetilde{Z} \int \widetilde{Z}^{-1}\, d\widetilde{L}. \tag{3.5}$$

The subsequent Lemma 7 will show that

$$\widetilde{Z}\int \widetilde{Z}^{-1}\,d\widetilde{L} \in \mathscr{R}^2(P). \tag{3.6}$$

By Theorem 3, $\widetilde{L} \in \mathscr{R}^2(P)$, and so (3.5) and (3.6) show that the local $\widetilde{P}$-martingale $\int \bar{\vartheta}\,dX$ is also in $\mathscr{R}^2(P)$. Proposition 1 and the continuity of X thus imply that

$$\int_0^T \bar{\vartheta}_t^{\mathrm{tr}}\,d\langle M\rangle_t\,\bar{\vartheta}_t = \int_0^T \bar{\vartheta}_t^{\mathrm{tr}}\,d[X]_t\,\bar{\vartheta}_t = \biggl[\int \bar{\vartheta}\,dX\biggr]_T \in L^1(P),$$

and so $\bar{\vartheta}$ is in $L^2(M) = \Theta$ by Theorem 2. To complete the proof, it thus remains to show that $\bar{\vartheta}$ satisfies (3.1). Now the product rule and $[\widetilde{L}, X^i] = 0$ for $i = 1, \dots, d$ imply that for $\vartheta \in \Theta$, the process $G(\vartheta)\int \widetilde{Z}^{-1}\,d\widetilde{L}$ is a local $\widetilde{P}$-martingale, and so $ZG(\vartheta)\int \widetilde{Z}^{-1}\,d\widetilde{L}$ is a local P-martingale. We now use (1.3) to replace Z by $\widetilde{Z}$, then (3.6), the fact that $G(\vartheta) \in \mathscr{R}^2(P)$ and the Cauchy–Schwarz inequality to finally obtain

$$\sup_{0\le t\le T}\biggl|Z_t G_t(\vartheta)\int_0^t \widetilde{Z}_s^{-1}\,d\widetilde{L}_s\biggr| \in L^1(P),$$

and so $ZG(\vartheta)\int \widetilde{Z}^{-1}\,d\widetilde{L}$ is even a true P-martingale for every $\vartheta \in \Theta$. Since $\widetilde{Z}_T = Z_T$, (3.5) and $\widetilde{L}_T = H$ imply that

$$E\bigl[(H - G_T(\bar{\vartheta}))G_T(\vartheta)\bigr] = E\biggl[\widetilde{Z}_T G_T(\vartheta)\int_0^T \widetilde{Z}_s^{-1}\,d\widetilde{L}_s\biggr] = 0 \quad \text{for all } \vartheta \in \Theta,$$

which proves that $\bar{\vartheta}$ solves the optimization problem (1.1). □

Now define the process $\widetilde{V}^H$ by setting

$$\widetilde{V}_t^H := \widetilde{E}[H] + \int_0^t \widetilde{\xi}_s^H\,dX_s + \widetilde{L}_t^H = \widetilde{E}[H|\mathscr{F}_t], \qquad 0 \le t \le T. \tag{3.7}$$

Putting everything together, we then obtain the following result.

THEOREM 6. *Assume (1.2) and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. For any $H \in L^2(P)$, the solution of (1.1) takes the form*

$$\vartheta_t^H = \widetilde{\xi}_t^H - \widetilde{\zeta}_t\biggl(\widetilde{E}[H]\widetilde{Z}_0^{-1} + \int_0^{t-} \widetilde{Z}_s^{-1}\,d\widetilde{L}_s^H\biggr) = \widetilde{\xi}_t^H - \frac{\widetilde{\zeta}_t}{\widetilde{Z}_t}\biggl(\widetilde{V}_{t-}^H - \int_0^t \vartheta_s^H\,dX_s\biggr). \tag{3.8}$$

PROOF. Due to the linearity of $H \mapsto \vartheta^H$, the first equality is immediate from Lemma 4 and Theorem 5. Since $[\widetilde{L}^H, \widetilde{Z}] = 0$ by (3.4) and (3.3), we can use the product rule, (3.4) and the first equality in (3.8) to obtain

$$\begin{aligned}\widetilde{Z}\left(\widetilde{E}[H]\widetilde{Z}_0^{-1}+\int\widetilde{Z}^{-1}\,d\widetilde{L}^H\right)&=\widetilde{E}[H]+\int\widetilde{E}[H]\widetilde{Z}_0^{-1}\widetilde{\zeta}\,dX\\&\quad+\int\left(\int\widetilde{Z}^{-1}\,d\widetilde{L}^H\right)_-\widetilde{\zeta}\,dX+\widetilde{L}^H\\&=\widetilde{E}[H]+\widetilde{L}^H+\int(\widetilde{\xi}^H-\vartheta^H)\,dX\\&=\widetilde{V}^H-\int\vartheta^H\,dX,\end{aligned}$$

and this yields the second equality in (3.8). □

REMARK. The second expression for ϑ^H in (3.8) gives us the optimal integrand in feedback form, with a correction term which is proportional to the amount by which the cumulative gains from trade $\int\vartheta^H\,dX$ deviate from the current intrinsic $\widetilde{P}$-value $\widetilde{V}^H$ of H in (3.7). This generalizes results of various authors where this representation was only obtained under very restrictive additional conditions. Duffie and Richardson (1991) and Schweizer (1994) worked with a "deterministic mean-variance tradeoff," while Hipp (1993, 1996), Wiese (1995) and Pham, Rheinländer and Schweizer (1996) assumed somewhat more generally that the minimal martingale measure $\widehat{P}$ coincides with the variance-optimal martingale measure $\widetilde{P}$. But all these assumptions are quite unnatural and will fail in most typical situations; see Pham, Rheinländer and Schweizer (1996) for an amplification of this point.

It now remains to prove the crucial estimate (3.6), and this is indeed where the main work has to be done. The key observation in the following proof is that the stochastic integral $\int\widetilde{Z}^{-1}\,d\widetilde{L}$ can equivalently be written as a *backward integral*, which is possible thanks to the orthogonality of $\widetilde{L}$ and X and the property (3.4) of the variance-optimal martingale measure. This alternative representation allows us in turn to apply the reverse Hölder inequality $R_2(P)$ backward in time to obtain the desired estimate by an approximation procedure. The original motivation for looking at the problem in this way comes from Schweizer (1995b) where a backward induction argument is used to solve the optimization problem (1.1) in finite discrete time. By using a suitable change of measure, we are able to give an alternative shorter proof in Section 4.2. On the other hand, the subsequent argument has the advantage that all computations and estimates are made under the original measure P, and this appears more promising in view of possible generalizations to a discontinuous process X.

LEMMA 7. *With the assumptions and notations of Theorem 5, we have*

$$\sup_{0\le t\le T}\left|\widetilde{Z}_t\int_0^t\widetilde{Z}_s^{-1}d\widetilde{L}_s\right|\in L^2(P). \tag{3.6}$$

PROOF. For brevity, let us write $N := \widetilde{Z}\int \widetilde{Z}^{-1} d\widetilde{L}$. By (3.5), N is a local $\widetilde{P}$-martingale so that we can choose an increasing sequence (T_n) of stopping times such that N^{T_n} is a uniformly integrable $\widetilde{P}$-martingale. From part (i) of Proposition 1, we get

$$E\Big[\sup_{0\le t\le T}\big|N_t^{T_n}\big|^2\Big] \le CE\big[N_{T_n}^2\big]$$

for a constant C which does not depend on n, and so it is enough to show that

$$\sup_S E\big[N_S^2\big] < \infty, \tag{3.9}$$

where the supremum runs over all stopping times $S \le T$. The assertion then follows by letting n tend to infinity and applying the monotone convergence theorem.

To prove (3.9), we shall first show that

$$E\big[N_S^2\big] \le CE[[\widetilde{L}]_S] \tag{3.10}$$

for any stopping time $S \le T$, with a constant C which does not depend on S. Theorem 2 and Proposition 1 then imply that

$$\sup_S E\big[N_S^2\big] \le CE[[\widetilde{L}]_T] \le CE\Big[\sup_{0\le t\le T}\widetilde{L}_t^2\Big] \le CE\big[\widetilde{L}_T^2\big] = CE[H^2] < \infty,$$

which gives (3.9).

As a preparation for the proof of (3.10), we now fix a stopping time $S \le T$ and approximate the stochastic integral $\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u$ appearing in N_S. A random partition of $[[0,S]]$ is a finite family σ of stopping times T_i such that $0 = T_0 \le T_1 \le \cdots \le T_k = S$ P-a.s.; its (random) grid size is $|\sigma| := \max_{i=1,\dots,k}|T_i - T_{i-1}|$. According to Theorems II.21 and II.23 of Protter (1990), there exists a sequence $(\sigma_m)_{m\in\mathbb{N}}$ of random partitions of $[[0,S]]$ with $\lim_{m\to\infty}|\sigma_m| = 0$ P-a.s. such that

$$\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u = \lim_{m\to\infty}\sum_{T_i\in\sigma_m}\widetilde{Z}_{T_i}^{-1}\big(\widetilde{L}_{T_{i+1}} - \widetilde{L}_{T_i}\big) \quad \text{in probability}$$

as well as

$$\big[\widetilde{Z}^{-1},\widetilde{L}\big]_S = \lim_{m\to\infty}\sum_{T_i\in\sigma_m}\big(\widetilde{Z}_{T_{i+1}}^{-1} - \widetilde{Z}_{T_i}^{-1}\big)\big(\widetilde{L}_{T_{i+1}} - \widetilde{L}_{T_i}\big) \quad \text{in probability.}$$

But $[\widetilde{Z}^{-1},\widetilde{L}] = 0$ by Itô's formula since $\widetilde{Z}$ is continuous and $[\widetilde{Z},\widetilde{L}] = 0$ as in the proof of Theorem 5. Hence we get by addition

$$\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u = \lim_{m\to\infty}\sum_{T_i\in\sigma_m}\widetilde{Z}_{T_{i+1}}^{-1}\big(\widetilde{L}_{T_{i+1}} - \widetilde{L}_{T_i}\big) \quad \text{in probability,} \tag{3.11}$$

and this shows that the forward integral $\int \widetilde{Z}^{-1}\,d\widetilde{L}$ can also be written as a backward integral $\int \widetilde{Z}^{-1}\,d^*\widetilde{L}$.

According to (1.3) and the definition of N, proving (3.10) is equivalent to showing that

$$E\left[Z_S\widetilde{Z}_S\left(\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u\right)^2\right] \le CE[[\widetilde{L}]_S]. \tag{3.12}$$

If $T_i, T_{i+1}, T_j, T_{j+1}$ are stopping times with $0 \le T_i \le T_{i+1} \le T_j \le T_{j+1} \le S$, we have

$$\begin{aligned}
&E\left[Z_S\widetilde{Z}_S\frac{(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})}{\widetilde{Z}_{T_{i+1}}}\frac{(\widetilde{L}_{T_{j+1}}-\widetilde{L}_{T_j})}{\widetilde{Z}_{T_{j+1}}}\right]\\
&\qquad = E\left[\frac{(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})}{\widetilde{Z}_{T_{i+1}}}E\left[Z_S\widetilde{Z}_S\frac{(\widetilde{L}_{T_{j+1}}-\widetilde{L}_{T_j})}{\widetilde{Z}_{T_{j+1}}}\middle|\mathscr{F}_{T_j}\right]\right]=0.
\end{aligned}$$

In fact, $Z\widetilde{Z}$ is a P-martingale because $\widetilde{Z}$ is a $\widetilde{P}$-martingale; thus we obtain

$$E\left[Z_S\widetilde{Z}_S\frac{(\widetilde{L}_{T_{j+1}}-\widetilde{L}_{T_j})}{\widetilde{Z}_{T_{j+1}}}\middle|\mathscr{F}_{T_j}\right]=E\big[Z_{T_{j+1}}(\widetilde{L}_{T_{j+1}}-\widetilde{L}_{T_j})\big|\mathscr{F}_{T_j}\big]=0$$

by first conditioning on $\mathscr{F}_{T_{j+1}}$ and then using the fact that $Z\widetilde{L}$ is a P-martingale because $\widetilde{L}$ is a $\widetilde{P}$-martingale. If we approximate $\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u$ as in (3.11), the mixed terms appearing in the corresponding approximation of (3.12) thus have expectation 0, and so we obtain

$$\begin{aligned}
&\sup_m E\left[Z_S\widetilde{Z}_S\left(\sum_{T_i\in\sigma_m(S)}\widetilde{Z}_{T_{i+1}}^{-1}(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})\right)^2\right]\\
&\qquad = \sup_m E\left[\sum_{T_i\in\sigma_m(S)}\frac{Z_S\widetilde{Z}_S}{\widetilde{Z}_{T_{i+1}}^2}(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})^2\right]\\
&\qquad \le C\sup_m E\left[\sum_{T_i\in\sigma_m(S)}\frac{Z_S^2}{Z_{T_{i+1}}^2}(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})^2\right]\\
&\qquad = C\sup_m E\left[\sum_{T_i\in\sigma_m(S)}(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})^2E\left[\frac{Z_S^2}{Z_{T_{i+1}}^2}\middle|\mathscr{F}_{T_{i+1}}\right]\right]\\
&\qquad \le C\sup_m E\left[\sum_{T_i\in\sigma_m(S)}(\widetilde{L}_{T_{i+1}}-\widetilde{L}_{T_i})^2\right]\\
&\qquad \le C\sup_m E\left[\sum_{T_i\in\sigma_m(S)}\big([\widetilde{L}]_{T_{i+1}}-[\widetilde{L}]_{T_i}\big)\right]\\
&\qquad \le CE[[\widetilde{L}]_S],
\end{aligned}$$

where we have used (1.3), the reverse Hölder inequality $R_2(P)$ and Proposition 1. In particular, the third inequality is obtained by applying part (ii) of

Proposition 1 to the finitely many $\widetilde{P}$-martingales $N^i := \widetilde{L}^{T_{i+1}} - \widetilde{L}^{T_i}$. Note also that none of the appearing constants depends on m or on the stopping time S. By (3.11),

$$\lim_{m\to\infty} Z_S \widetilde{Z}_S \bigg(\sum_{T_i \in \sigma_m(S)} \widetilde{Z}^{-1}_{T_{i+1}} \big(\widetilde{L}_{T_{i+1}} - \widetilde{L}_{T_i}\big)\bigg)^2$$
$$= Z_S \widetilde{Z}_S \bigg(\int_0^S \widetilde{Z}_s^{-1}\, d\widetilde{L}_s \bigg)^2 \quad \text{in probability,}$$

and so Fatou's lemma yields (3.12). This completes the proof. □

4. A second solution.

A very elegant different method of attacking the basic problem (1.1) has recently been proposed by Gouriéroux, Laurent and Pham (1996), subsequently abbreviated as GLP. Their idea is to combine a change of measure with a change of coordinates to transform the problem in such a way that it can be solved directly by means of the Galtchouk–Kunita–Watanabe projection theorem. But a priori, GLP are only able to solve a weaker problem by their approach, and one contribution of the present paper is to prove that they actually obtain the solution to the same question that we consider here.

4.1. *The alternative approach.*

This subsection briefly explains the results of GLP. Their basic model is a multidimensional diffusion model with a Brownian filtration. The $\mathbb{R}^{d+1}$-valued process S is given by

$$\frac{dS_t^0}{S_t^0} = r_t\, dt, \qquad S_0^0 = 1$$

and

$$\frac{dS_t^i}{S_t^i} = \mu_t^i\, dt + \sum_{j=1}^n \sigma_t^{ij}\, dW_t^j, \qquad S_0^i > 0$$

for $i = 1, \dots, d \le n$, with predictable processes r, μ, σ satisfying appropriate integrability conditions. The process X is then the $\mathbb{R}^d$-valued process with components $X^i := S^i/S^0$ for $i = 1, \dots, d$. To facilitate comparisons and to avoid some technical problems, we consider in the sequel the discounted case where $r \equiv 0$ so that $S^0 \equiv 1$. Our subsequent arguments do not need the diffusion structure, but only the continuity of X.

Denote as above by $\widetilde{P}$ the variance-optimal martingale measure for X so that X is a continuous local $\widetilde{P}$-martingale. GLP then consider the optimization problem

$$\text{(4.1)} \qquad \text{Minimize } \|H - G_T(\vartheta)\|_{L^2(P)} \text{ over all } \vartheta \in \widetilde{\Theta},$$

where the space $\widetilde{\Theta}$ consists of all $\mathbb{R}^d$-valued predictable X-integrable processes ϑ such that the stochastic integral $G(\vartheta)$ is a $\widetilde{P}$-martingale satisfying $G_T(\vartheta) \in$

$L^2(P)$. It is easy to check (and will be proved in Lemma 9) that Θ is then contained in $\widetilde{\Theta}$ so that (4.1) is more likely to have a solution than (1.1).

Now consider the strictly positive $\widetilde{P}$-martingale $\widetilde{Z}$ given by (3.4) and define a new probability measure $\widetilde{R} \approx P$ by setting

$$\frac{d\widetilde{R}}{d\widetilde{P}} := \frac{\widetilde{Z}_T}{\widetilde{Z}_0} = 1 + \int_0^T \widetilde{Z}_0^{-1} \widetilde{\zeta}_s \, dX_s.$$

Since X is a continuous local $\widetilde{P}$-martingale, the $\mathbb{R}^{d+1}$-valued process Y with $Y^0 := \widetilde{Z}^{-1}$ and $Y^i := X^i \widetilde{Z}^{-1}$ for $i = 1, \dots, d$ is a continuous local $\widetilde{R}$-martingale. Moreover,

$$\frac{d\widetilde{R}}{dP} = \frac{d\widetilde{R}}{d\widetilde{P}} \frac{d\widetilde{P}}{dP} = \frac{\widetilde{Z}_T^2}{\widetilde{Z}_0}, \tag{4.2}$$

and so we obtain

$$\|H - G_T(\vartheta)\|_{L^2(P)} = \sqrt{\widetilde{Z}_0} \left\| \frac{H}{\widetilde{Z}_T} - \frac{G_T(\vartheta)}{\widetilde{Z}_T} \right\|_{L^2(\widetilde{R})}. \tag{4.3}$$

A generalized version of the crucial result of GLP is then

PROPOSITION 8. *Assume that X is a continuous semimartingale which satisfies (1.2) and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. Then*

$$\frac{1}{\widetilde{Z}_T} G_T(\widetilde{\Theta}) = \left\{ \int_0^T \psi_u \, dY_u \,\middle|\, \psi \in L^2(Y, \widetilde{R}) \right\}, \tag{4.4}$$

where $L^2(Y, \widetilde{R})$ is the space of all $\mathbb{R}^{d+1}$-valued predictable Y-integrable processes ψ such that $\int \psi \, dY$ is in the space $\mathscr{M}^2(\widetilde{R})$ of martingales. Moreover, the relation between $\vartheta \in \widetilde{\Theta}$ and $\psi \in L^2(Y, \widetilde{R})$ is given by

$$\begin{aligned} \psi^i &:= \vartheta^i \quad \text{for } i = 1, \dots, d, \\ \psi^0 &:= G(\vartheta) - \vartheta^{\mathrm{tr}} X \end{aligned} \tag{4.5}$$

and

$$\vartheta^i := \psi^i + \widetilde{\zeta}^i \left(\int \psi \, dY - \psi^{\mathrm{tr}} Y \right) \quad \text{for } i = 1, \dots, d. \tag{4.6}$$

PROOF. The crucial step of the argument is to show that

$$\begin{aligned} &\{ G(\vartheta) \,|\, \vartheta \text{ is } \mathbb{R}^d\text{-valued, predictable and } X\text{-integrable} \} \\ &\quad = \left\{ \widetilde{Z} \int \psi \, dY \,\middle|\, \psi \text{ is } \mathbb{R}^{d+1}\text{-valued, predictable and } Y\text{-integrable} \right\} \end{aligned} \tag{4.7}$$

with the relation between ϑ and ψ given by (4.5) and (4.6). As a preparation for this, note first that the product rule yields

$$d(X\widetilde{Z}^{-1}) = \widetilde{Z}^{-1} \, dX + X \, d\widetilde{Z}^{-1} + d[X, \widetilde{Z}^{-1}], \tag{4.8}$$

$$d\bigl(G(\vartheta)\widetilde{Z}^{-1}\bigr) = \widetilde{Z}^{-1}\,dG(\vartheta) + G(\vartheta)\,d\widetilde{Z}^{-1} + d\bigl[G(\vartheta), \widetilde{Z}^{-1}\bigr] \\ = \widetilde{Z}^{-1}\vartheta\,dX + G(\vartheta)\,d\widetilde{Z}^{-1} + \vartheta^{\mathrm{tr}}\,d\bigl[X, \widetilde{Z}^{-1}\bigr] \tag{4.9}$$

and

$$d(\widetilde{Z}Y) = Y\,d\widetilde{Z} + \widetilde{Z}\,dY + d[\widetilde{Z}, Y]. \tag{4.10}$$

Suppose first that ϑ is X-integrable and define $\vartheta^n := \vartheta I_{\{|\vartheta|\le n\}}$. Then (4.9), (4.8) and the definition of Y imply that

$$\begin{aligned} d\bigl(G(\vartheta^n)\widetilde{Z}^{-1}\bigr) &= \widetilde{Z}^{-1}\vartheta^n\,dX + G(\vartheta^n)\,d\widetilde{Z}^{-1} + (\vartheta^n)^{\mathrm{tr}} d\bigl[X, \widetilde{Z}^{-1}\bigr] \\ &= \vartheta^n\,d\bigl(X\widetilde{Z}^{-1}\bigr) + \bigl(G(\vartheta^n) - (\vartheta^n)^{\mathrm{tr}}X\bigr)\,d\widetilde{Z}^{-1} \\ &= \bigl(\psi^{(n)}\bigr)\,dY, \end{aligned}$$

where the Y-integrable process $\psi^{(n)}$ is given by

$$\begin{aligned} (\psi^{(n)})^0 &:= G(\vartheta^n) - (\vartheta^n)^{\mathrm{tr}}X, \\ (\psi^{(n)})^i &:= (\vartheta^n)^i \quad \text{for } i = 1, \dots, d. \end{aligned}$$

As n tends to infinity, $G(\vartheta^n)$ converges to $G(\vartheta)$ in the semimartingale topology because ϑ is X-integrable. This implies that $\int \psi^{(n)}\,dY = \widetilde{Z}^{-1}G(\vartheta^n)$ also converges in the semimartingale topology since multiplication with a fixed semimartingale is a continuous operation; see Proposition 4 of Emery (1979). By Theorem V.4 of Mémin (1980), the subspace $\{\int \psi\,dY | \psi$ is Y-integrable$\}$ is closed in the semimartingale topology, and so we conclude that

$$\widetilde{Z}^{-1}G(\vartheta) = \int \bar{\psi}\,dY \quad \text{for some } Y\text{-integrable process } \bar{\psi}.$$

But since $\psi^{(n)}$ converges for $n \to \infty$ (P-a.s. uniformly in t, at least along a subsequence) to ψ given by (4.5), we deduce from Theorem V.4 of Mémin (1980) that $\bar{\psi} = \psi$, and this establishes the inclusion "$\subseteq$" in (4.7).

The proof of the converse is very similar. If ψ is Y-integrable, we define $\psi^n := \psi I_{\{|\psi|\le n\}}$ and use the product rule, (3.4), (4.10) and the definition of Y to obtain

$$\begin{aligned} d\biggl(\widetilde{Z}\int \psi^n\,dY\biggr) &= \biggl(\int \psi^n\,dY\biggr)\,d\widetilde{Z} + \widetilde{Z}\psi^n\,dY + (\psi^n)^{\mathrm{tr}}\,d[\widetilde{Z}, Y] \\ &= \biggl(\int \psi^n\,dY\biggr)\widetilde{\zeta}\,dX + \psi^n\,d(\widetilde{Z}Y) - \bigl((\psi^n)^{\mathrm{tr}}Y\bigr)\,d\widetilde{Z} \\ &= \vartheta^{(n)}\,dX \end{aligned}$$

with the X-integrable process

$$(\vartheta^{(n)})^i := (\psi^n)^i + \widetilde{\zeta}^i\biggl(\int \psi^n\,dY - (\psi^n)^{\mathrm{tr}}Y\biggr) \quad \text{for } i = 1, \dots, d.$$

An analogous argument as above then yields for $n \to \infty$ that

$$\widetilde{Z} \int \psi \, dY = G(\vartheta)$$

with ϑ given by (4.6), and this establishes the inclusion "$\supseteq$" in (4.7).

The proof of (4.4) is now easy. For $\psi \in L^2(Y, \widetilde{R})$, the stochastic integral $\int \psi \, dY$ is an $\widetilde{R}$-martingale so that the product $\widetilde{Z} \int \psi \, dY = G(\vartheta)$ is a $\widetilde{P}$-martingale. Moreover, (4.2) and (4.7) yield

$$E[(G_T(\vartheta))^2] = \widetilde{Z}_0 E_{\widetilde{R}}\left[\left(\int_0^T \psi_u \, dY_u\right)^2\right] < \infty$$

since $\int \psi \, dY \in \mathscr{M}^2(\widetilde{R})$, and so $G_T(\vartheta)$ is in $L^2(P)$. Conversely, let $G(\vartheta)$ be a $\widetilde{P}$-martingale with terminal value $G_T(\vartheta) \in L^2(P)$. Then (4.7) shows that $\int \psi \, dY$ is an $\widetilde{R}$-martingale whose terminal value $G_T(\vartheta)/\widetilde{Z}_T$ is in $L^2(\widetilde{R})$ due to (4.2). Hence ψ must be in $L^2(Y, \widetilde{R})$, and this completes the proof. □

In view of Proposition 8 and (4.3), (4.1) is equivalent to the optimization problem

(4.11) $$\text{Minimize } \left\| \frac{H}{\widetilde{Z}_T} - \int_0^T \psi_u \, dY_u \right\|_{L^2(\widetilde{R})} \text{ over all } \psi \in L^2(Y, \widetilde{R}).$$

But this is a much easier problem. In fact, since Y is an $\widetilde{R}$-martingale, the solution ψ^* of (4.11) is simply given by the integrand of Y in the Galtchouk–Kunita–Watanabe decomposition under $\widetilde{R}$ of the random variable $H/\widetilde{Z}_T \in L^2(\widetilde{R})$. The solution ϑ^* of (4.1) is then obtained via (4.6). The transformation from X to Y and back is the change of coordinates alluded to above.

REMARKS. (i) A result similar to Proposition 8 is given in GLP for the multidimensional diffusion case under the assumptions that $\sigma\sigma^{\mathrm{tr}}$ is invertible and that $\sigma^{\mathrm{tr}}(\sigma\sigma^{\mathrm{tr}})^{-1}(\mu - r\underline{1})$ is bounded. This amounts to saying that X has a bounded mean-variance tradeoff which is a well-known convenient condition. It is sufficient (but not necessary) to ensure that $G_T(\Theta)$ is closed in $L^2(P)$ and that $\mathscr{D}^s \cap L^2(P)$ contains the density of the minimal martingale measure $\widehat{P}$ and is therefore nonempty; see DMSSS for more details.

(ii) A closer look at the proof of Proposition 8 shows that we do not really need the assumption (1.2) that $G_T(\Theta)$ is closed in $L^2(P)$. All we require is the existence of the variance-optimal martingale measure $\widetilde{P}$ and the representation

$$\widetilde{Z}_t = \widetilde{E}[\widetilde{Z}_T] + \int_0^t \widetilde{\zeta}_s \, dX_s, \qquad 0 \le t \le T$$

for some X-integrable process $\widetilde{\zeta}$ (which need not even be in Θ). By Lemma 2.2 of Delbaen and Schachermayer (1996), this is satisfied as soon as $\mathscr{D}^e \cap L^2(P)$ is nonempty. In particular, Proposition 8 then implies that $G_T(\widetilde{\Theta})$ is closed in

$L^2(P)$ so that (4.1) is indeed easier to solve than (1.1). (We are grateful to L. Krawczyk for this remark.)

4.2. *The relation to our results.* Let us now compare our results to those of GLP. As pointed out in GLP, the solution ϑ^* of (4.1) is only in the space $\widetilde{\Theta}$ which is a priori bigger than Θ. The first result in this subsection shows that under our assumptions, the two spaces actually coincide so that the GLP solution is also a solution to (1.1). Although the proof below is very short, it is worth pointing out that it relies crucially on the weighted norm inequalities used in the present paper.

LEMMA 9. *Assume (1.2) and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. Then $\widetilde{\Theta} = \Theta$.*

PROOF. The inclusion "$\supseteq$" is easy and already pointed out in GLP. In fact, if ϑ is in Θ, then $G(\vartheta)$ is in $\mathscr{R}^2(P)$ as well as a local $\widetilde{P}$-martingale so that $Z^{\widetilde{P}} G(\vartheta)$ is a local P-martingale. Since $d\widetilde{P}/dP \in L^2(P)$, the density process $Z^{\widetilde{P}}$ is also in $\mathscr{R}^2(P)$ by Doob's inequality so that $Z^{\widetilde{P}} G(\vartheta)$ is actually a true P-martingale; hence $G(\vartheta)$ is a $\widetilde{P}$-martingale. Note that this argument uses no further properties of $\widetilde{P}$ except that $d\widetilde{P}/dP \in L^2(P)$.

Conversely, suppose now that ϑ is in $\widetilde{\Theta}$ so that $G(\vartheta)$ is a $\widetilde{P}$-martingale with terminal value $G_T(\vartheta) \in L^2(P)$. Since $Z^{\widetilde{P}}$ satisfies the reverse Hölder inequality $R_2(P)$ and condition (J) by Theorem 2, we conclude from Proposition 1 and the continuity of X that

$$\int_0^T \vartheta_s^{\text{tr}} \, d\langle M\rangle_s \, \vartheta_s = [G(\vartheta)]_T \in L^1(P)$$

so that ϑ is in $L^2(M)$, hence in Θ by Theorem 2. This shows the inclusion "$\subseteq$" and thus completes the proof. □

Of course, Lemma 9 implies that the solution ϑ^* of (4.1) and the solution ϑ^H of (1.1) in Theorem 6 must actually coincide. One can ask if this could not be seen directly by comparing the two decompositions of H used for obtaining the two solutions. Recall that the decomposition used for Theorem 6 is

$$H = \widetilde{E}[H] + \int_0^T \widetilde{\xi}_s^H \, dX_s + \widetilde{L}_T^H \tag{3.2}$$

from Theorem 3, where $\widetilde{\xi}^H \in \Theta$ and $\widetilde{L}^H$ is a $\widetilde{P}$-martingale null at 0 with $\widetilde{L}^H \in \mathscr{R}^2(P)$ and $[\widetilde{L}^H, X^i] = 0$ for $i = 1, \dots, d$. On the other hand, the GLP solution uses the Galtchouk–Kunita–Watanabe decomposition

$$\frac{H}{\widetilde{Z}_T} = E_{\widetilde{R}}\left[\frac{H}{\widetilde{Z}_T}\right] + \int_0^T \psi_u \, dY_u + L_T \tag{4.12}$$

under $\widetilde{R}$, where $\psi \in L^2(Y, \widetilde{R})$ and L is in $\mathscr{M}_0^2(\widetilde{R})$ and strongly $\widetilde{R}$-orthogonal to Y. The next result explicitly describes the connection between (3.2) and (4.12).

PROPOSITION 10. *Assume (1.2) and $\mathscr{D}^s \cap L^2(P) \neq \varnothing$. For every $H \in L^2(P)$, the decompositions (3.2) and (4.12) are then related by*

$$\widetilde{L}^H = \int \widetilde{Z}\, dL \tag{4.13}$$

and

$$\widetilde{\xi}^H = \widetilde{E}[H]\widetilde{Z}_0^{-1}\widetilde{\zeta} + \vartheta + L_-\widetilde{\zeta}, \tag{4.14}$$

where ϑ is given from ψ via (4.6).

PROOF. Since the density of $\widetilde{R}$ with respect to $\widetilde{P}$ is $\widetilde{Z}_T/\widetilde{Z}_0$, the representation (3.4) of $\widetilde{Z}_T$ implies that

$$\widetilde{Z}_T E_{\widetilde{R}}\biggl[\frac{H}{\widetilde{Z}_T}\biggr] = \biggl(\widetilde{Z}_0 + \int_0^T \widetilde{\zeta}_s\, dX_s\biggr)\widetilde{E}\biggl[\frac{H}{\widetilde{Z}_0}\biggr] = \widetilde{E}[H]\biggl(1 + \int_0^T \widetilde{Z}_0^{-1}\widetilde{\zeta}_s\, dX_s\biggr).$$

By (4.4),

$$\widetilde{Z}_T \int_0^T \psi_u\, dY_u = G_T(\vartheta) = \int_0^T \vartheta_u\, dX_u$$

for some $\vartheta \in \Theta$ given from ψ via (4.6), and so it only remains to study the product $\widetilde{Z}_T L_T$. Since L is strongly $\widetilde{R}$-orthogonal to the continuous process Y, we have $[L, Y^i] = 0$ for $i = 0, 1, \dots, d$. For $i = 0$, this yields $[L, \widetilde{Z}^{-1}] = 0$ and therefore by Itô's formula and the continuity of $\widetilde{Z}$ that

$$[L, \widetilde{Z}] = 0. \tag{4.15}$$

For $i = 1, \dots, d$, we have $X^i = \widetilde{Z}Y^i$, and so (4.10) and the preceding arguments imply that

$$[L, X^i] = 0 \quad \text{for } i = 1, \dots, d, \tag{4.16}$$

because $[\widetilde{Z}, Y^i]$ is continuous and of finite variation. Thanks to (4.15) and (3.4), the product rule now gives

$$\widetilde{Z}_T L_T = \int_0^T L_{s-}\, d\widetilde{Z}_s + \int_0^T \widetilde{Z}_s\, dL_s = \int_0^T L_{s-}\widetilde{\zeta}_s\, dX_s + \int_0^T \widetilde{Z}_s\, dL_s$$

and so we conclude from (4.12) that H can be decomposed as

$$H = \widetilde{E}[H] + \int_0^T (\widetilde{E}[H]\widetilde{Z}_0^{-1}\widetilde{\zeta}_s + \vartheta_s + L_{s-}\widetilde{\zeta}_s)\, dX_s + \int_0^T \widetilde{Z}_s\, dL_s.$$

But we already know that $\widetilde{\zeta}$ and ϑ are in Θ; thanks to the uniqueness in Theorem 3, (4.13) and (4.14) will thus follow once we show that $L_-\widetilde{\zeta}$ is in Θ and that the process $N := \int \widetilde{Z}\, dL$ is a $\widetilde{P}$-martingale null at 0 with $N \in \mathscr{R}^2(P)$ and $[N, X^i] = 0$ for $i = 1, \dots, d$.

The last assertion is immediate from (4.16) and the definition of N. Since L is strongly $\widetilde{R}$-orthogonal to $Y^0 = \widetilde{Z}^{-1}$, the product $L\widetilde{Z}^{-1}$ is a local $\widetilde{R}$-martingale. Thus L is a local $\widetilde{P}$-martingale, and so is N since $\widetilde{Z}$ is continuous, hence locally bounded. Because $Z^{\widetilde{P}}$ satisfies $R_2(P)$ and (J) by Theorem 2, part (ii) of Proposition 1 implies that N will be in $\mathscr{R}^2(P)$ if we can show that $[N]_T \in L^1(P)$. To prove that this is true, we use successively (1.3), Theorem VI.57 of Dellacherie and Meyer (1982), the definition of $\widetilde{P}$, again Theorem VI.57 of Dellacherie and Meyer (1982) and the definition of $\widetilde{R}$ to obtain

$$\begin{aligned} E[[N]_T] &= E\left[\int_0^T \widetilde{Z}_s^2\, d[L]_s\right] \\ &\le CE\left[\int_0^T Z_s \widetilde{Z}_s\, d[L]_s\right] \\ &= CE\left[Z_T \int_0^T \widetilde{Z}_s\, d[L]_s\right] \\ &= \widetilde{E}\left[\int_0^T \widetilde{Z}_s\, d[L]_s\right] \\ &= C\widetilde{E}[\widetilde{Z}_T[L]_T] \\ &= C\widetilde{Z}_0 E_{\widetilde{R}}[[L]_T] < \infty, \end{aligned}$$

because L is in $\mathscr{M}^2(\widetilde{R})$. Now N is a local $\widetilde{P}$-martingale with $N \in \mathscr{R}^2(P)$ and the density process Z of $\widetilde{P}$ with respect to P is in $\mathscr{R}^2(P)$; hence we conclude that ZN is a true P-martingale so that N is a true $\widetilde{P}$-martingale. This shows that N has all the properties claimed above and implies that N satisfies the assumptions of Theorem 5. Therefore

$$L_-\widetilde{\zeta} = \widetilde{\zeta}\left(\int \widetilde{Z}^{-1}\, dN\right)_-$$

is in Θ by Theorem 5, and this completes the proof. □

Proposition 10 allows us to see quite easily that the solutions ϑ^* of (4.1) and ϑ^H of (1.1) coincide. In fact, (4.14) and (4.13) imply that

$$\vartheta^* = \widetilde{\xi}^H - \widetilde{E}[H]\widetilde{Z}_0^{-1}\widetilde{\zeta} - L_-\widetilde{\zeta} = \widetilde{\xi}^H - \widetilde{\zeta}\left(\widetilde{E}[H]\widetilde{Z}_0^{-1} + \left(\int \widetilde{Z}^{-1}\, d\widetilde{L}^H\right)_-\right)$$

which equals ϑ^H according to Theorem 6.

Interestingly, the probability measure $\widetilde{R}$ introduced by GLP also allows us to give a shorter proof of the crucial Lemma 7. As in the first proof, it is enough to show that

$$E[N_S^2] \le CE[[\widetilde{L}]_S] \tag{3.10}$$

for any stopping time $S \le T$, with a constant C which does not depend on S. We first observe that $\widetilde{Z}\widetilde{L}$ is a local $\widetilde{P}$-martingale since both $\widetilde{Z}$ and $\widetilde{L}$ are, and since $[\widetilde{Z}, \widetilde{L}] = 0$. Thus $\widetilde{L}$ is a local $\widetilde{R}$-martingale, and so is $\int \widetilde{Z}^{-1}\, d\widetilde{L}$ because $\widetilde{Z}^{-1}$ is continuous, hence locally bounded. Using successively the definition of

N, (1.3), the definitions of Z and $\widetilde{R}$, the Burkholder–Davis–Gundy inequality under $\widetilde{R}$, the definitions of $\widetilde{R}$ and Z and (1.3) yields

$$\begin{aligned}
E[N_S^2] &= E\left[\widetilde{Z}_S^2\left(\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u\right)^2\right] \\
&\le CE\left[Z_S\widetilde{Z}_S\left(\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u\right)^2\right] \\
&= CE_{\widetilde{R}}\left[\left(\int_0^S \widetilde{Z}_u^{-1}\,d\widetilde{L}_u\right)^2\right] \\
&\le CE_{\widetilde{R}}\left[\int_0^S \widetilde{Z}_u^{-2}\,d[\widetilde{L}]_u\right] \\
&= CE\left[Z_S\widetilde{Z}_S\int_0^S \widetilde{Z}_u^{-2}\,d[\widetilde{L}]_u\right] \\
&\le CE\left[Z_S\widetilde{Z}_S\int_0^S Z_u^{-1}\widetilde{Z}_u^{-1}\,d[\widetilde{L}]_u\right].
\end{aligned}$$

But $\widetilde{Z}$ is a $\widetilde{P}$-martingale, hence $Z\widetilde{Z}$ is a P-martingale, and so the last term equals $E[[\widetilde{L}]_S]$ by Theorem VI.57 of Dellacherie and Meyer (1982). Since none of the appearing constants depends on S, this again proves (3.10) and therefore Lemma 7. □

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