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OPTIMAL STOPPING OF THE MAXIMUM PROCESS: THE MAXIMALITY PRINCIPLE

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The solution is found to the optimal stopping problem with payoff

$$\sup_\tau E\left(S_\tau - \int_0^\tau c(X_t)\,dt\right),$$

where $S = (S_t)_{t\ge 0}$ is the maximum process associated with the one-dimensional time-homogeneous diffusion $X = (X_t)_{t\ge 0}$, the function $x \mapsto c(x)$ is positive and continuous, and the supremum is taken over all stopping times τ of X for which the integral has finite expectation. It is proved, under no extra conditions, that this problem has a solution; that is, the payoff is finite and there is an optimal stopping time, if and only if the following *maximality principle* holds: the first-order nonlinear differential equation

$$g'(s) = \frac{\sigma^2(g(s))L'(g(s))}{2c(g(s))(L(s)-L(g(s)))}$$

admits a maximal solution $s \mapsto g_*(s)$ which stays strictly below the diagonal in $\mathbb{R}^2$. [In this equation $x \mapsto \sigma(x)$ is the diffusion coefficient and $x \mapsto L(x)$ the scale function of X.] In this case the stopping time

$$\tau_* = \inf\{t > 0 \mid X_t \le g_*(S_t)\}$$

is proved optimal, and explicit formulas for the payoff are given. The result has a large number of applications and may be viewed as the cornerstone in a general treatment of the maximum process.

1. Introduction. Our main aim in this paper is to present the solution to a problem of optimal stopping for the maximum process associated with a one-dimensional time-homogeneous diffusion. The solution found has a large number of applications and may be viewed as the cornerstone in a general treatment of the maximum process.

In the setting of (2.1)–(2.3) we consider the optimal stopping problem (2.4), where the supremum is taken over all stopping times τ satisfying (2.5), and the cost function c is positive and continuous. The main result of the paper is presented in Theorem 3.1, where it is proved that this problem has a solution (the payoff is finite and there is an optimal stopping strategy) if and only if *the maximality principle* holds; that is, the first-order nonlinear differential equation (3.21) admits a maximal solution which stays strictly below the diagonal

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in $\mathbb{R}^2$ (see Figure 1). The maximal solution is proved to be an optimal stopping boundary; that is, the stopping time (3.31) is optimal, and the payoff is given explicitly by (3.30). Moreover, this stopping time is shown to be pointwise the smallest possible optimal stopping time. If there is no such maximal solution of (3.21), the payoff is proved to be infinite and there is no optimal stopping time. The paper finishes with four examples in Section 4 which are aimed to illustrate some applications of the result proved.

The optimal stopping problem (2.4) has been considered in some special cases earlier. Jacka [16] treats the case of reflected Brownian motion, while Dubins, Shepp and Shiryaev [6] treat the case of Bessel processes. In these papers the problem was solved very effectively by guessing the nature of the optimal stopping boundary and making use of the principle of smooth fit. The same is true for the "discounted" problem (3.60) with $c \equiv 0$ in the case of geometric Brownian motion which, in the framework of option pricing theory (Russian option), was solved by Shepp and Shiryaev in [26] (see also [27] and [10]). For the first time, a strong need for additional arguments was felt in [11], where the problem (2.4) for geometric Brownian motion was considered with the cost function $c(x) \equiv c > 0$. There, by use of Picard's method of successive approximations, it was proved that the maximal solution of (3.21) is an optimal stopping boundary, and since this solution could not be expressed in closed form, it really showed the full power of the method. Such nontrivial solutions were also obtained in [6] by a method which relies on estimates of the payoff obtained a priori. Motivated by similar ideas, sufficient conditions for the maximality principle to hold for general diffusions are given in [12]. The method of proof used there relies on a transfinite induction argument. In order to solve the problem in general, the fundamental question was how to relate the maximality principle to the superharmonic characterization of the payoff, which is the key result in the general theory. This fact has been indicated by Shiryaev.

The most interesting point in our solution of the optimal stopping problem (2.4) relies on the fact that we have now described this connection and actually proved that the maximality principle is equivalent to the superharmonic characterization of the payoff (for a three-dimensional process). The crucial observations in this direction are (3.28) and (3.29), which show that the only possible optimal stopping boundary is the maximal solution [see (3.38) in the proof of Theorem 3.1]. In the next step of proving that the maximal solution is indeed an optimal stopping boundary, it was crucial to make use of so-called "bad–good" solutions of (3.21), "bad" in the sense that they hit the diagonal in $\mathbb{R}^2$ and "good" in the sense that they are not too large (see Figure 1). These "bad–good" solutions are used to approximate the maximal solution in a desired manner [see the proof of Theorem 3.1 starting from (3.40) onwards], and this turns out to be the key argument in completing the proof.

Our methodology adopts and extends earlier results of Dubins, Shepp and Shiryaev [6], and is, in fact, quite standard in the business of solving particular optimal stopping problems: (i) One tries to guess the nature of the optimal stopping boundary as a member of a "reasonable" family; (ii) computes the expected reward; (iii) maximizes this over the family; (iv) and then tries to argue that the resulting stopping time is optimal in general. This process is often facilitated by ad hoc principles, such as the famous "principle of smooth fit." This procedure is used very effectively in this paper, too, as opposed to results from the general theory of optimal stopping, and, as suggested by the referee, we should like to stress this fact. We would also like to point out, however, that the maximality principle of the present paper should rather be seen as a convenient reformulation of the basic principle on a superharmonic characterization from the general theory than a new principle on its own. Shiryaev has also noticed a similar maximality property of his solution a long while ago (see [28], Figure 3, page 85), and similar tricks were used by other people too; see also [20] for a related result.

2. Formulation of the problem. Let $X=(X_t)_{t\geq 0}$ be a one-dimensional time-homogeneous *diffusion* process associated with the infinitesimal generator

$$\mathbb{L}_X = \mu(x)\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2}{\partial x^2}, \tag{2.1}$$

where the *drift* coefficient $x \mapsto \mu(x)$ and the *diffusion* coefficient $x \mapsto \sigma(x) > 0$ are continuous. Assume, moreover, that there exists a standard *Brownian motion* $B=(B_t)_{t\geq 0}$ defined on $(\Omega, \mathscr{F}, P)$ such that X solves the stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \tag{2.2}$$

with $X_0 = x$ under $P_x := P$ for $x \in \mathbb{R}$. The state space of X is assumed to be $\mathbb{R}$.

With X we associate the maximum process

$$S_t = \Big(\max_{0\leq r\leq t} X_r\Big) \vee s \tag{2.3}$$

started at $s \geq x$ under $P_{x,s} := P$. The main objective of this paper is to present the solution to the optimal stopping problem with payoff

$$V_*(x,s) = \sup_{\tau} E_{x,s}\Big(S_\tau - \int_0^\tau c(X_t)\,dt\Big), \tag{2.4}$$

where the supremum is taken over stopping times τ of X satisfying

$$E_{x,s}\Big(\int_0^\tau c(X_t)\,dt\Big) < \infty, \tag{2.5}$$

and the *cost* function $x \mapsto c(x) > 0$ is continuous.

2.1. To state and prove the initial observation about (2.4) and for further reference, we need to recall a few general facts about one-dimensional diffusions (see [25], pages 270–303).

The *scale function* of X is given by

$$L(x) = \int^x \exp\left(-\int^y \frac{2\mu(z)}{\sigma^2(z)}\,dz\right)dy \tag{2.6}$$

for $x \in \mathbb{R}$. Throughout we denote

$$\tau_x = \inf\{t > 0 \mid X_t = x\} \tag{2.7}$$

and set $\tau_{x,y} = \tau_x \wedge \tau_y$. Then we have

$$P_x(X_{\tau_{a,b}} = a) = \frac{L(b) - L(x)}{L(b) - L(a)}, \tag{2.8}$$

$$P_x(X_{\tau_{a,b}} = b) = \frac{L(x) - L(a)}{L(b) - L(a)}, \tag{2.9}$$

whenever $a \le x \le b$.

The *speed measure* of X is given by

$$m(dx) = \frac{2\,dx}{L'(x)\sigma^2(x)}. \tag{2.10}$$

The *Green function* of X on $[a, b]$ is defined by

$$G_{a,b}(x, y) = \begin{cases} \dfrac{\big(L(b) - L(x)\big)\big(L(y) - L(a)\big)}{\big(L(b) - L(a)\big)}, & \text{if } a \le y \le x, \\[2ex] \dfrac{\big(L(b) - L(y)\big)\big(L(x) - L(a)\big)}{\big(L(b) - L(a)\big)}, & \text{if } x \le y \le b. \end{cases} \tag{2.11}$$

If $f: \mathbb{R} \mapsto \mathbb{R}$ is a measurable function, then

$$E_x\left(\int_0^{\tau_{a,b}} f(X_t)\,dt\right) = \int_a^b f(y) G_{a,b}(x, y) m(dy). \tag{2.12}$$

2.2. Due to the specific form of the optimal stopping problem (2.4), the following observation is nearly evident (see [6], pages 237 and 238).

PROPOSITION 2.1. *The process $\bar{X}_t = (X_t, S_t)$ cannot be optimally stopped on the diagonal of $\mathbb{R}^2$.*

PROOF. Fix $x \in \mathbb{R}$ and set $l_n = x - 1/n$ and $r_n = x + 1/n$. Denoting $\tau_n = \tau_{l_n, r_n}$ it will be enough to show that

$$E_{x,x}\left(S_{\tau_n} - \int_0^{\tau_n} c(X_t)\,dt\right) > x \tag{2.13}$$

for $n \ge 1$ large enough.

For this, note first by the strong Markov property and (2.8), (2.9) that

$$
\begin{aligned}
E_{x,x}(S_{\tau_n}) &\geq x P_x(X_{\tau_n} = l_n) + r_n P_x(X_{\tau_n} = r_n) \\
&= x \frac{L(r_n) - L(x)}{L(r_n) - L(l_n)} + r_n \frac{L(x) - L(l_n)}{L(r_n) - L(l_n)} \\
&= x + (r_n - x) \frac{L(x) - L(l_n)}{L(r_n) - L(l_n)} \\
&= x + (r_n - x) \frac{L'(\xi_n)(x - l_n)}{L'(\eta_n)(r_n - l_n)} \geq x + K/n
\end{aligned}
\tag{2.14}
$$

since $L \in C^1$. On the other hand, $K_1 := \sup_{l_n \leq z \leq r_n} c(z) < \infty$. Thus by (2.10)–(2.12) we get

$$
\begin{aligned}
E_{x,x}\bigg(\int_0^{\tau_n} c(X_t)\,dt\bigg) &\leq K_1 E_x(\tau_n) = 2K_1 \int_{l_n}^{r_n} G_{a,b}(x,y) \frac{dy}{\sigma^2(y) L'(y)} \\
&\leq K_2 \bigg(\int_{l_n}^{x} \big(L(y) - L(l_n)\big)\,dy + \int_x^{r_n} \big(L(r_n) - L(y)\big)\,dy\bigg) \\
&\leq K_3\big((x - l_n)^2 + (r_n - x)^2\big) = 2K_3/n^2,
\end{aligned}
\tag{2.15}
$$

since σ is continuous and $L \in C^1$. Combining (2.14) and (2.15), we clearly obtain (2.13) for $n \geq 1$ large enough. The proof is complete. □

For a survey and the definitive results of Engelbert and Schmidt on existence, uniqueness, and various other aspects of solutions of one-dimensional stochastic differential equations, we refer to Karatzas and Shreve [17], Chapter 5.

3. Optimal stopping of the maximum process.

In the setting of (2.1)–(2.3), consider the optimal stopping problem (2.4) where the supremum is taken over all stopping times τ of X satisfying (2.5). Our main aim in this section is to present the solution to this problem (Theorem 3.1). We begin our exposition with a few observations on the underlying structure of (2.4) with a view to the Markovian theory of optimal stopping.

3.1. Note that $\bar{X}_t = (X_t, S_t)$ is a two-dimensional Markov process with the state space $D = \{(x,s) \in \mathbb{R}^2 \,|\, x \leq s\}$, which can change (increase) in the second coordinate only after hitting the diagonal $x = s$ in $\mathbb{R}^2$. Off the diagonal, the process $\bar{X} = (\bar{X}_t)_{t \geq 0}$ changes only in the first coordinate and may be identified with X. Due to its form and behavior at the diagonal, we claim that the infinitesimal generator of $\bar{X}$ may thus be formally described as follows:

$$
\begin{aligned}
&\mathbb{L}_{\bar{X}} = \mathbb{L}_X \quad \text{in } x < s, \\
&\frac{\partial}{\partial s} = 0 \quad \text{at } x = s,
\end{aligned}
\tag{3.1}
$$

with $\mathbb{L}_X$ as in (2.1). This means that the infinitesimal generator of $\bar{X}$ is acting on a space of C^2-functions f on D satisfying $(\partial f/\partial s)(s,s) = 0$. Observe that we do not tend to specify the domain of $\mathbb{L}_{\bar{X}}$ precisely, but will only verify that if $f\colon D \to \mathbb{R}$ is a C^2-function which belongs to the domain, then $(\partial f/\partial s)(s,s)$ must be zero.

To see this, we shall apply Itô's formula to the process $f(X_t, S_t)$ and take the expectation under $P_{s,s}$. By applying the optional sampling theorem to the continuous local martingale which appears in this process (localized if needed), we obtain

$$\begin{aligned}\frac{E_{s,s}\big(f(X_t,S_t)\big)-f(s,s)}{t} &= E_{s,s}\bigg(\frac{1}{t}\int_0^t (\mathbb{L}_X f)(X_r,S_r)\,dr\bigg)\\ &\quad + E_{s,s}\bigg(\frac{1}{t}\int_0^t \frac{\partial f}{\partial s}(X_r,S_r)\,dS_r\bigg) \to \mathbb{L}_X f(s,s)\\ &\quad + \frac{\partial f}{\partial s}(s,s)\bigg(\lim_{t\downarrow 0}\frac{E_{s,s}(S_t-s)}{t}\bigg)\end{aligned} \tag{3.2}$$

as $t \downarrow 0$. Due to $\sigma > 0$, we have $t^{-1}E_{s,s}(S_t - s) \to \infty$ as $t \downarrow 0$, and therefore the limit above is infinite, unless $(\partial f/\partial s)(s,s) = 0$. This completes the claim (see also [6], pages 238 and 239).

3.2. Problem (2.4) can be related to the Markovian theory of optimal stopping by introducing the functional

$$A_t = a + \int_0^t c(X_r)\,dr, \tag{3.3}$$

with $a \geq 0$ given and fixed, and noting that $Z_t = (A_t, X_t, S_t)$ is a Markov process which starts at (a,x,s) under P. Its infinitesimal generator is obtained by adding $c(x)(\partial/\partial a)$ to the infinitesimal generator of $\bar{X}$, which combined with (3.1) leads to the formal description

$$\begin{aligned}&\mathbb{L}_Z = c(x)(\partial/\partial a) + \mathbb{L}_X \quad \text{in } x < s,\\ &\frac{\partial}{\partial s} = 0 \quad \text{at } x = s,\end{aligned} \tag{3.4}$$

with $\mathbb{L}_X$ as in (2.1). Given $Z = (Z_t)_{t\geq 0}$, introduce the *gain function* $\Gamma(a,x,s) = s - a$, note that the payoff (2.4) viewed in terms of the general theory ought to be defined as

$$\widetilde{V}_*(a,x,s) = \sup_{\tau} E\big(\Gamma(Z_\tau)\big), \tag{3.5}$$

where the supremum is taken over all stopping times τ of Z satisfying $E(A_\tau) < \infty$ and observe that

$$\widetilde{V}_*(a,x,s) = V_*(x,s) - a. \tag{3.6}$$

This identity is the main reason that we abandon the general formulation (3.5) and simplify it to the form (2.4) and that we speak of optimal stopping for the process $\bar{X}_t = (X_t, S_t)$ rather than the process $Z_t = (A_t, X_t, S_t)$.

Let us point out that the contents of this subsection are used in the sequel merely to clarify the result and method in terms of the general theory.

3.3. From now on our main aim will be to show that (2.4) reduces to the problem of solving a first-order nonlinear differential equation (for the optimal stopping boundary). To derive this equation we shall first try to get a feeling for the points in the state space $\{(x, s) \in \mathbb{R}^2 \mid x \le s\}$ at which the process $\bar{X}_t = (X_t, S_t)$ can be optimally stopped.

When on the vertical level s, the process $\bar{X}_t = (X_t, S_t)$ stays at the same level until it hits the diagonal $x = s$ in $\mathbb{R}^2$. During that time $\bar{X}$ does not change (increase) in the second coordinate. Due to the strictly positive cost in (2.4), it is clear that we should not let the process $\bar{X}$ run too much to the left, since it could be "too expensive" to get back to the diagonal in order to offset the "cost" spent to travel all that way. More specifically, given s, there should exist a point $g_*(s) \le s$ such that if the process (X, S) reaches the point $(g_*(s), s)$ we should stop it instantly. In other words, the stopping time

$$\tau_* = \inf\{t > 0 \mid X_t \le g_*(S_t)\} \tag{3.7}$$

should be optimal for problem (2.4). For this reason we call $s \mapsto g_*(s)$ an *optimal stopping boundary*, and our aim will be to prove its existence and to characterize it. Observe by Proposition 2.1 that we must have $g_*(s) < s$ for all s, and that $V_*(x, s) = s$ for all $x \le g_*(s)$.

3.4. To compute the payoff $V_*(x, s)$ for $g_*(s) < x \le s$, and to find the optimal stopping boundary $s \mapsto g_*(s)$, we are led to formulate the following system:

$$(\mathbb{L}_X V)(x, s) = c(x) \quad \text{for } g(s) < x < s \text{ with } s \text{ fixed}, \tag{3.8}$$

$$\frac{\partial V}{\partial s}(x, s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}, \tag{3.9}$$

$$V(x, s)\big|_{x=g(s)+} = s \quad \text{(instantaneous stopping)}, \tag{3.10}$$

$$\frac{\partial V}{\partial x}(x, s)\Big|_{x=g(s)+} = 0 \quad \text{(smooth fit)} \tag{3.11}$$

with $\mathbb{L}_X$ as in (2.1). Note that (3.8) + (3.9) are in accordance with the general theory upon using (3.4) and (3.6) above: the infinitesimal generator of the process being applied to the payoff must be zero in the continuation region. The condition (3.10) is evident. The condition (3.11) is not part of the general theory; it is imposed, since we believe that in the "smooth" setting of problem

(2.4) the principle of smooth fit should hold. This belief will be vindicated after the fact, when we show in Theorem 3.1 that the solution of the system (3.8)–(3.11) leads to the payoff of (2.4). The system (3.8)–(3.11) constitutes a *Stephan problem with moving* (*free*) *boundary* (see [29], pages 157–162). It was derived for the first time by Dubins, Shepp and Shiryaev [6] in the case of Bessel processes.

3.5. To solve the system (3.8)–(3.11) we shall consider a stopping time of the form

$$\tau_g = \inf\{t > 0 \mid X_t \le g(S_t)\} \tag{3.12}$$

and the map

$$V_g(x, s) = E_{x,s}\left(S_{\tau_g} - \int_0^{\tau_g} c(X_t)\,dt\right) \tag{3.13}$$

associated with it, where $s \mapsto g(s)$ is a given function such that both $E_{x,s}(S_{\tau_g})$ and $E_{x,s}(\int_0^{\tau_g} c(X_t)\,dt)$ are finite. Set $V_g(s) := V_g(s, s)$ for all s. Considering $\tau_{g(s),s} = \inf\{t > 0 \mid X_t \notin]g(s), s[\}$ and using the strong Markov property of X at $\tau_{g(s),s}$, by (2.8)–(2.12) we find

$$\begin{aligned} V_g(x, s) &= s\frac{L(s) - L(x)}{L(s) - L(g(s))} + V_g(s)\frac{L(x) - L(g(s))}{L(s) - L(g(s))} \\ &\quad - \int_{g(s)}^{s} G_{g(s),s}(x, y)c(y)m(dy) \end{aligned} \tag{3.14}$$

for all $g(s) < x < s$.

In order to determine $V_g(s)$, we shall rewrite (3.14) as follows:

$$\begin{aligned} V_g(s) - s &= \frac{L(s) - L(g(s))}{L(x) - L(g(s))} \\ &\quad \times \left(\big(V_g(x, s) - s\big) + \int_{g(s)}^{s} G_{g(s),s}(x, y)c(y)m(dy)\right) \end{aligned} \tag{3.15}$$

and then divide and multiply through by $x - g(s)$ to obtain

$$\lim_{x \downarrow g(s)} \frac{V_g(x, s) - s}{L(x) - L(g(s))} = \frac{1}{L'(g(s))}\frac{\partial V_g}{\partial x}(x, s)\bigg|_{x=g(s)+}. \tag{3.16}$$

It is easily seen by (2.11) that

$$\begin{aligned} &\lim_{x \downarrow g(s)} \frac{L(s) - L(g(s))}{L(x) - L(g(s))} \int_{g(s)}^{s} G_{g(s),s}(x, y)c(y)m(dy) \\ &\quad = \int_{g(s)}^{s} \big(L(s) - L(y)\big)c(y)m(dy). \end{aligned} \tag{3.17}$$

Thus, if *the condition of smooth fit*,

$$\frac{\partial V_g}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \tag{3.18}$$

is satisfied, we see from (3.15)–(3.17) that the following identity holds:

$$V_g(s) = s + \int_{g(s)}^{s} \big(L(s) - L(y)\big)c(y)m(dy). \tag{3.19}$$

Inserting this into (3.14), and using (2.11) and (2.12), we get

$$V_g(x,s) = s + \int_{g(s)}^{x} \big(L(x) - L(y)\big)c(y)m(dy) \tag{3.20}$$

for all $g(s) \le x \le s$.

If we now forget the origin of $V_g(x,s)$ in (3.13), and consider it purely as defined by (3.20), then it is straightforward to verify that $(x,s) \mapsto V_g(x,s)$ solves the system (3.8)–(3.11) in the region $g(s) < x < s$ if and only if the C^1-function $s \mapsto g(s)$ solves the following first-order nonlinear differential equation:

$$g'(s) = \frac{\sigma^2\big(g(s)\big)L'\big(g(s)\big)}{2c\big(g(s)\big)\big(L(s) - L\big(g(s)\big)\big)}. \tag{3.21}$$

Thus, to each solution $s \mapsto g(s)$ of (3.21) corresponds a function $(x,s) \mapsto V_g(x,s)$ defined by (3.20), which solves the system (3.8)–(3.11) in the region $g(s) < x < s$ and coincides with the expectation in (3.13) whenever $E_{x,s}(S_{\tau_g})$ and $E_{x,s}(\int_0^{\tau_g} c(X_t)\,dt)$ are finite (the latter is easily verified by Itô's formula). We shall use this fact in the proof of Theorem 3.1 below upon approximating the selected solution of (3.21) by solutions which hit the diagonal in $\mathbb{R}^2$.

3.6. Observe that among all possible functions $s \mapsto g(s)$, only those which satisfy (3.21) have the smooth-fit property (3.18) for $V_g(x,s)$ of (3.13), and vice versa. Thus *the differential equation* (3.21) *is obtained by the principle of smooth fit* in the problem (2.4). The fundamental question to be answered is how to chose the optimal stopping boundary $s \mapsto g_*(s)$ among all admissible candidates which solve (3.21).

Before passing to answer this question let us also observe from (3.20) that

$$\frac{\partial V_g}{\partial x}(x,s) = L'(x)\int_{g(s)}^{x} c(y)m(dy), \tag{3.22}$$

$$V_g'(s) = L'(s)\int_{g(s)}^{s} c(y)m(dy). \tag{3.23}$$

These equations show that, in addition to the continuity of the derivative of $V_g(x,s)$ along the vertical line across $g(s)$ in (3.18), we have obtained the continuity of $V_g(x,s)$ along the vertical line and the diagonal in $\mathbb{R}^2$ across the point where they meet. In fact, we see that the latter condition is equivalent to the former and thus may be used as an alternative way of looking at the principle of smooth fit in this problem.

3.7. In view of the analysis about (3.7), we assign a constant value to $V_g(x,s)$ at all $x < g(s)$. The following properties of the solution $V_g(x,s)$ obtained are then straightforward:

(3.24) $$V_g(x,s) = s \quad \text{for } x \le g(s);$$

(3.25) $$x \mapsto V_g(x,s) \text{ is (strictly) increasing on } [g(s), s];$$

(3.26) $$(x,s) \mapsto V_g(x,s) \text{ is } C^2 \text{ outside } \{(g(s),s) \mid s \in \mathbb{R}\};$$

(3.27) $$x \mapsto V_g(x,s) \text{ is } C^1 \text{ at } g(s).$$

Let us also make the following observations:

(3.28) $g \mapsto V_g(x,s)$ is (strictly) *decreasing*

(3.29) The function $(a,x,s) \mapsto V_g(x,s) - a$ is *superharmonic* for the Markov process $Z_t = (A_t, X_t, S_t)$ [with respect to stopping times τ satisfying (2.5)].

Property (3.28) is evident from (3.20), whereas (3.29) is derived in the proof of Theorem 3.1 [see (3.37) below].

3.8. Combining (3.6)+(3.28)+(3.29) with the superharmonic characterization of the payoff from the Markovian theory (see [29], page 124) and recalling the result of Proposition 2.1, we are led to the following Markovian principle for determining the optimal stopping boundary. We say that $s \mapsto g_*(s)$ is *an optimal stopping boundary* for the problem (2.4), if the stopping time τ_* defined in (3.7) is optimal for this problem.

The Maximality principle: The optimal stopping boundary $s \mapsto g_*(s)$ for the problem (2.4) *is the maximal solution of the differential equation (3.21) which stays strictly below the diagonal in $\mathbb{R}^2$ (see Figure 1).*

This principle is equivalent to the superharmonic characterization of the payoff [for the process $Z_t = (A_t, X_t, S_t)$], and may be viewed as its alternative (analytic) description. The proof of its validity is given in the next theorem, the main result of the paper.

THEOREM 3.1 (Optimal stopping of the maximum process). *In the setting of (2.1)–(2.3) consider the optimal stopping problem (2.4) where the supremum is taken over all stopping times τ of X satisfying (2.5).*

(I) *Let $s \mapsto g_*(s)$ denote the maximal solution of (3.21) which stays strictly below the diagonal in $\mathbb{R}^2$ (whenever such a solution exists; see Figure 1). Then we have the following.*

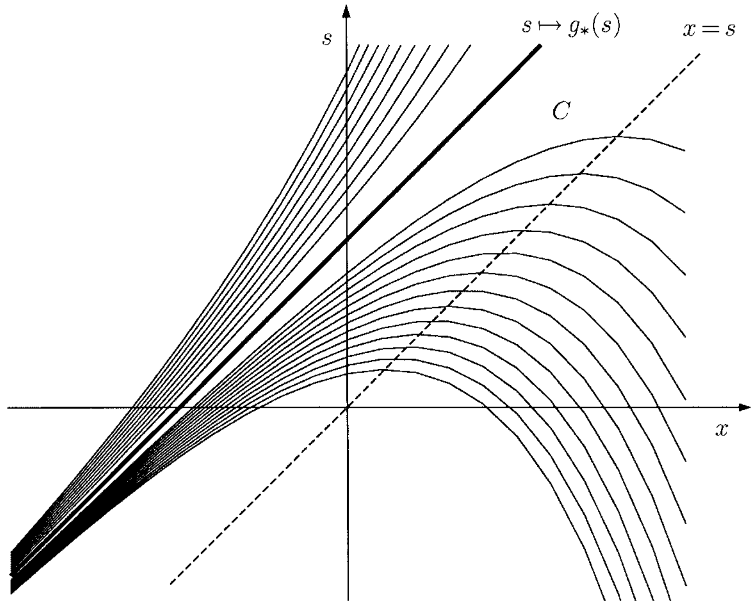

FIG. 1. *A computer drawing of solutions of the differential equation* (3.21) *in the case when* $\mu \equiv 0$, $\sigma \equiv 1$ [*thus* $L(x) = x$] *and* $c \equiv 1/2$. *The bold line* $s \mapsto g_*(s)$ *is the* maximal *solution which stays strictly below the diagonal in* $\mathbb{R}^2$. [*In this particular case* $s \mapsto g_*(s)$ *is a linear function.*] *By the maximality principle proved below*, *this solution is the optimal stopping boundary* [*the stopping time* τ_* *from* (3.7) *is optimal for the problem* (2.4)].

(a) *The payoff is finite and is given by*

$$V_*(x, s) = s + \int_{g_*(s)}^{x} \big(L(x) - L(y)\big)c(y)m(dy) \tag{3.30}$$

for $g_*(s) \le x \le s$, *and* $V_*(x, s) = s$ *for* $x \le g_*(s)$.

(b) *The stopping time*

$$\tau_* = \inf\big\{t > 0 \,|\, X_t \le g_*(S_t)\big\} \tag{3.31}$$

is optimal for the problem (2.4) whenever it satisfies (2.5); otherwise it is "approximately" optimal in the sense described in the proof below.

(c) *If there exists an optimal stopping time* σ *in (2.4) satisfying (2.5), then* $P_{x,s}(\tau_* \le \sigma) = 1$ *for all* (x, s), *and* τ_* *is an optimal stopping time for (2.4) as well.*

(II) *If there is no (maximal) solution of (3.21) which stays strictly below the diagonal in* $\mathbb{R}^2$, *then* $V_*(x, s) = +\infty$ *for all* (x, s), *and there is no optimal stopping time.*

PROOF. (I) Let $s \mapsto g(s)$ be any solution of (3.21) satisfying $g(s) < s$ for all s. Then, as indicated above, the function $V_g(x, s)$ defined by (3.20) solves the system (3.8)–(3.11) in the region $g(s) < x < s$. Due to (3.26) and (3.27), Itô's formula can be applied to the process $V_g(X_t, S_t)$, and in this way by (2.1), (2.2) we get

$$
\begin{aligned}
V_g(X_t,S_t) &= V_g(x,s) + \int_0^t \frac{\partial V_g}{\partial x}(X_r,S_r)\,dX_r \\
&\quad + \int_0^t \frac{\partial V_g}{\partial s}(X_r,S_r)\,dS_r + \frac{1}{2}\int_0^t \frac{\partial^2 V_g}{\partial x^2}(X_r,S_r)\,d\langle X,X\rangle_r \\
&= V_g(x,s) + \int_0^t \sigma(X_r)\frac{\partial V_g}{\partial x}(X_r,S_r)\,dB_r + \int_0^t (\mathbb{L}_X V_g)(X_r,S_r)\,dr,
\end{aligned}
\tag{3.32}
$$

where the integral with respect to dS_r is zero, since the increment ΔS_r outside the diagonal in $\mathbb{R}^2$ equals zero, while at the diagonal we have (3.9).

The process $M = (M_t)_{t\ge 0}$, defined by

$$M_t = \int_0^t \sigma(X_r)\frac{\partial V_g}{\partial x}(X_r,S_r)\,dB_r \tag{3.33}$$

is a continuous local martingale. Introducing the increasing process

$$P_t = \int_0^t c(X_r) 1_{(X_r \le g(S_r))}\,dr \tag{3.34}$$

and using the fact that the set of all t for which X_t is either $g(S_t)$ or S_t is of Lebesgue measure zero, the identity (3.32) can be rewritten as

$$V_g(X_t,S_t) - \int_0^t c(X_r)\,dr = V_g(x,s) + M_t - P_t \tag{3.35}$$

by means of (3.8) with (3.24). From this representation we see that the process $V_g(X_t,S_t) - \int_0^t c(X_r)\,dr$ is a local supermartingale.

Let τ be any stopping time of X satisfying (2.5). Choose a localization sequence $(\sigma_n)_{n\ge 1}$ of bounded stopping times for M. By means of (3.24) and (3.25) we see that $V_g(x,s) \ge s$ for all (x,s), so from (3.35) it follows that

$$
\begin{aligned}
&E_{x,s}\left(S_{\tau\wedge\sigma_n} - \int_0^{\tau\wedge\sigma_n} c(X_t)\,dt\right) \\
&\quad \le E_{x,s}\left(V_g\big(X_{\tau\wedge\sigma_n}, S_{\tau\wedge\sigma_n}\big) - \int_0^{\tau\wedge\sigma_n} c(X_t)\,dt\right) \\
&\quad \le V_g(x,s) + E_{x,s}(M_{\tau\wedge\sigma_n}) = V_g(x,s).
\end{aligned}
\tag{3.36}
$$

Letting $n \to \infty$ and using Fatou's lemma with (2.5), we get

$$E_{x,s}\left(S_\tau - \int_0^\tau c(X_t)\,dt\right) \le V_g(x,s). \tag{3.37}$$

This proves (3.29). Taking the supremum over all such τ and then the infimum over all such g, by means of (3.28) we may conclude that

$$V_*(x,s) \le \inf_g V_g(x,s) = V_{g_*}(x,s) \tag{3.38}$$

for all (x, s). From these considerations it clearly follows that the only possible candidate for the optimal stopping boundary is the maximal solution $s \mapsto g_*(s)$ of (3.21).

To prove that we have the equality in (3.38) and that the payoff $V_*(x, s)$ is given by (3.30), assume first that the stopping time τ_*, defined by (3.31), satisfies (2.5). Then, as pointed out when deriving (3.20), we have

$$V_{g_*}(x, s) = E_{x,s}\left(S_{\tau_{g_*}} - \int_0^{\tau_{g_*}} c(X_t)\,dt\right) \tag{3.39}$$

so that $V_{g_*}(x, s) = V_*(x, s)$ in (3.38) and τ_* is an optimal stopping time. The explicit expression given in (3.30) is obtained by (3.20).

Assume now that τ_* fails to satisfy (2.5). Let $(g_n)_{n\geq 1}$ be a decreasing sequence of solutions of (3.21) satisfying $g_n(s) \downarrow g_*(s)$ as $n \to \infty$ for all s. Note that each such solution must hit the diagonal in $\mathbb{R}^2$, so the stopping times τ_{g_n} defined as in (3.12) must satisfy (2.5). Moreover, since $S_{\tau_{g_n}}$ is bounded by a constant, we see that $V_{g_n}(x, s)$ defined as in (3.13) is given by (3.20) with $g = g_n$ for $n \geq 1$. By letting $n \to \infty$, we get

$$V_{g_*}(x, s) = \lim_{n\to\infty} V_{g_n}(x, s) = \lim_{n\to\infty} E_{x,s}\left(S_{\tau_{g_n}} - \int_0^{\tau_{g_n}} c(X_t)\,dt\right). \tag{3.40}$$

This shows that the equality in (3.38) is attained through the sequence of stopping times $(\tau_{g_n})_{n\geq 1}$, and the explicit expression in (3.30) is easily obtained as already indicated above.

To prove the final (uniqueness) statement, assume that σ is an optimal stopping time in (2.4) satisfying (2.5). Suppose that $P_{x,s}(\sigma < \tau_*) > 0$. Note that τ_* can be written in the form

$$\tau_* = \inf\{t > 0 \mid V_*(X_t, S_t) = S_t\} \tag{3.41}$$

so that $S_\sigma < V_*(X_\sigma, S_\sigma)$ on $\{\sigma < \tau_*\}$, and thus

$$\begin{aligned} E_{x,s}\left(S_\sigma - \int_0^\sigma c(X_t)\,dt\right) &< E_{x,s}\left(V_*(X_\sigma, S_\sigma) - \int_0^\sigma c(X_t)\,dt\right) \\ &\leq V_*(x, s), \end{aligned} \tag{3.42}$$

where the latter inequality is derived as in (3.37), since the process $V_*(X_t, S_t) - \int_0^t c(X_r)\,dr$ is a local supermartingale. The strict inequality in (3.42) shows that $P_{x,s}(\sigma < \tau_*) > 0$ fails, so we must have $P_{x,s}(\tau_* \leq \sigma) = 1$ for all (x, s).

To prove the optimality of τ_* in such a case, it is enough to note that if σ satisfies (2.5), then τ_* must satisfy it as well. Therefore (3.39) is satisfied, and thus τ_* is optimal. A straightforward argument can also be given by using the local supermartingale property of the process $V_*(X_t, S_t) - \int_0^t c(X_r)\,dr$; since

$P_{x,s}(\tau_* \le \sigma) = 1$, we get

$$\begin{aligned} V_*(x,s) &= E_{x,s}\left(S_\sigma - \int_0^\sigma c(X_t)\,dt\right) \le E_{x,s}\left(V_*(X_\sigma, S_\sigma) - \int_0^\sigma c(X_t)\,dt\right) \\ &\le E_{x,s}\left(V_*(X_{\tau_*}, S_{\tau_*}) - \int_0^{\tau_*} c(X_t)\,dt\right) \\ &= E_{x,s}\left(S_{\tau_*} - \int_0^{\tau_*} c(X_t)\,dt\right), \end{aligned} \tag{3.43}$$

so τ_* is optimal for (2.4). The proof of the first part of the theorem is complete.

(II) Let $(g_n)_{n\ge1}$ be a decreasing sequence of solutions of (3.21) which satisfy $g_n(0) = -n$ for $n \ge 1$. Then each g_n must hit the diagonal in $\mathbb{R}^2$ at some $s_n > 0$ for which we have $s_n \uparrow \infty$ when $n \to \infty$. Since there is no solution of (3.21) which stays below the diagonal, we must have $g_n(s) \downarrow -\infty$ as $n \to \infty$ for all s. Let τ_{g_n} denote the stopping time defined by (3.12) with $g = g_n$. Then τ_{g_n} satisfies (2.5), and since $S_{\tau_{g_n}} \le s \vee s_n$, we see that $V_{g_n}(x,s)$ defined by (3.13) with $g = g_n$ is given as in (3.20),

$$V_{g_n}(x,s) = s + \int_{g_n(s)}^{x} \big(L(x) - L(y)\big)c(y)m(dy) \tag{3.44}$$

for all $g_n(s) \le x \le s$. Letting $n \to \infty$ in (3.44), we see that the following integral,

$$I := \int_{-\infty}^{x} \big(L(x) - L(y)\big)c(y)m(dy), \tag{3.45}$$

plays a crucial role in the proof (independently of the given x and s).

Assume first that $I = +\infty$ [this is the case whenever $c(y) \ge \varepsilon > 0$ for all y, and $-\infty$ is a natural boundary point for X]. Then from (3.44) we clearly get

$$V_*(x,s) \ge \lim_{n\to\infty} V_{g_n}(x,s) = +\infty, \tag{3.46}$$

so the payoff must be infinite.

On the other hand, if $I < \infty$, then $(2.11) + (2.12)$ imply

$$E_{x,s}\left(\int_0^{\tau_{\hat{s}}} c(X_t)\,dt\right) \le \int_{-\infty}^{\hat{s}} \big(L(\hat{s}) - L(y)\big)c(y)m(dy) < \infty, \tag{3.47}$$

where $\tau_{\hat{s}} = \inf\{t > 0 \mid X_t = \hat{s}\}$ for $\hat{s} \ge s$. Thus, if we let the process (X_t, S_t) first hit $(\hat{s}, \hat{s})$ and then the boundary $\{(g_n(s), s) \mid s \in \mathbb{R}\}$ with $n \to \infty$, then by (3.44) (with $x = s = \hat{s}$) we see that the payoff equals at least $\hat{s}$. More precisely, if the process (X_t, S_t) starts at (x,s), consider the stopping times $\tau_n = \tau_{\hat{s}} + \tau_{g_n} \circ \theta_{\tau_{\hat{s}}}$ for $n \ge 1$. Then by (3.47) we see that each τ_n satisfies (2.5), and by the strong Markov property of X we easily get

$$V_*(x,s) \ge \limsup_{n\to\infty} E_{x,s}\left(S_{\tau_n} - \int_0^{\tau_n} c(X_t)\,dt\right) \ge \hat{s}. \tag{3.48}$$

By letting $\hat{s} \uparrow \infty$, we again find $V_*(x,s) = +\infty$. The proof of the theorem is complete. □

3.9. *On equation* (3.21). Theorem 3.1 shows that the optimal stopping problem (2.4) reduces to the problem of solving the first-order nonlinear differential equation (3.21). If this equation admits a maximal solution which stays strictly below the diagonal in $\mathbb{R}^2$, then this solution is an optimal stopping boundary. We may note that this equation is of the following *normal* form:

$$y' = \frac{F(y)}{G(x) - G(y)} \tag{3.49}$$

for $x > y$, where $y \mapsto F(y)$ is strictly positive, and $x \mapsto G(x)$ is strictly increasing. To the best of our knowledge, (3.49) has not been studied before, and in view of the result proved above we want to point out the need for its investigation. It turns out that its treatment depends heavily on the behavior of the map G.

(i) If the process X is in natural scale, that is $L(x) = x$ for all x, we can completely characterize and describe the maximal solution of (3.21). This can be done in terms of (3.49) with $G(x) = x$ and $F(y) = \sigma^2(y)/2c(y)$ as follows. Note that by passing to the inverse $z \mapsto y^{-1}(z)$, the equation (3.49) in this case can be rewritten as

$$(y^{-1})'(z) - \frac{1}{F(z)} y^{-1}(z) = -\frac{z}{F(z)}. \tag{3.50}$$

This is a first-order linear equation and its general solution is given by

$$y_\alpha^{-1}(z) = \exp\biggl(\int_0^z \frac{dy}{F(y)}\biggr)\biggl(\alpha - \int_0^z \frac{y}{F(y)} \exp\biggl(-\int_0^y \frac{du}{F(u)}\biggr)\,dy\biggr), \tag{3.51}$$

where α is a constant. Hence we see that, with $G(x) = x$, *the necessary and sufficient condition* for (3.49) to admit a maximal solution which stays strictly below the diagonal in $\mathbb{R}^2$ is that

$$\begin{aligned} \alpha_* := \sup_{z\in\mathbb{R}} \biggl(& z \exp\biggl(-\int_0^z \frac{dy}{F(y)}\biggr) \\ & + \int_0^z \frac{y}{F(y)} \exp\biggl(-\int_0^y \frac{du}{F(u)}\biggr)\,dy\biggr) < \infty, \end{aligned} \tag{3.52}$$

and that this supremum is not attained at any $z \in \mathbb{R}$. In this case the maximal solution $x \mapsto y_*(x)$ of (3.49) can be expressed explicitly through its inverse $z \mapsto y_{\alpha_*}^{-1}(z)$ given by (3.51).

Note also when $L(x) = G(x) = x^2 \operatorname{sign}(x)$ that the same argument transforms (3.49) into a *Riccati equation*, which then can be further transformed into a linear homogeneous equation of second order by means of standard techniques. The trick of passing to the inverse in (3.21) is further used in [24] where a natural connection between the result of the present paper and the Azéma–Yor solution of the Skorokhod-embedding problem [1] is described.

(ii) If the process X is not in natural scale, then the treatment of (3.49) is much harder, due to the lack of closed-form solutions. In such cases it is possible to prove (or disprove) the existence of the maximal solution by using Picard's method of successive approximations. The idea is to use Picard's theorem locally, step-by-step, and in this way show the existence of some global solution which stays strictly below the diagonal. Then, by passing to the equivalent integral equation and using a monotone convergence theorem, one can argue that this implies the existence of the maximal solution. This technique is described in detail in Section 3 of [11] in the case of $G(x) = x^p$ and $F(y) = y^{p+1}$ when $p > 1$. It is also seen there that during the construction one obtains tight bounds on the maximal solution, which makes it possible to compute it numerically as accurately as desired (see [11] for details). In this process it is desirable to have a local existence and uniqueness of the solution, and these are provided by the following general facts.

From the general theory (Picard's method) we know that if the direction field $(x, y) \mapsto f(x, y) := F(y)/(G(x) - G(y))$ is (locally) continuous and (locally) Lipschitz in the second variable, then (3.49) admits (locally) a unique solution. For instance, this will be so if along a (local) continuity of $(x, y) \mapsto f(x, y)$, we have a (local) continuity of $(x, y) \mapsto (\partial f/\partial y)(x, y)$. In particular, upon differentiating over y in $f(x, y)$, we see that (3.21) admits (locally) a unique solution whenever the map $y \mapsto \sigma^2(y)L'(y)/c(y)$ is (locally) C^1. It is also possible to prove that (3.49) admits (locally) a solution, if only the (local) continuity of the direction field $(x, y) \mapsto F(y)/(G(x) - G(y))$ is verified. However, such a solution may fail to be (locally) unique.

Instead of entering further into such abstract considerations here, we shall rather confine ourselves to some concrete examples with applications in the next section.

3.10. We have proved in Theorem 3.1 that τ_* is optimal for (2.4) whenever it satisfies (2.5). In Example 4.1 below we will exhibit a stopping time τ_* which fails to satisfy (2.5), but nevertheless its payoff is given by (3.30) as proved above. In this case τ_* is "approximately" optimal in the sense that (3.40) holds with $\tau_{g_n} \uparrow \tau_*$ as $n \to \infty$.

3.11. *Other state spaces.* The result of Theorem 3.1 extends to diffusions with other state spaces in $\mathbb{R}$. In view of many applications, we will indicate such an extension for nonnegative diffusions.

In the setting of (2.1)–(2.3), assume that the diffusion X is nonnegative, consider the optimal stopping problem (2.4) where the supremum is taken over all stopping times τ of X satisfying (2.5) and note that the result of Proposition 2.1 extends to this case provided that the diagonal is taken in $]0, \infty[^2$. In this context it is natural to assume that $\sigma(x) > 0$ for $x > 0$, and $\sigma(0)$ may be equal 0. Similarly, we shall see that the case of strictly positive cost function c differs from the case when c is strictly positive only on $]0, \infty[$. In any case, both $x \mapsto \sigma(x)$ and $x \mapsto c(x)$ are assumed continuous on $[0, \infty[$.

In addition to the infinitesimal characteristics from (2.1) which govern X in $]0, \infty[$, we must specify the boundary behavior of X at 0. For this we shall consider the cases when 0 is a natural, exit, regular (instantaneously reflecting), and entrance boundary point (see [18], pages 226–250).

The relevant fact in the case when 0 is either a *natural* or *exit* boundary point is that

$$\int_0^s \big(L(s) - L(y)\big)c(y)m(dy) = +\infty \tag{3.53}$$

for all $s > 0$ whenever $c(0) > 0$. In view of (3.30) this shows that for the maximal solution of (3.21) we must have $0 < g_*(s) < s$ for all $s > 0$ unless $V_*(s, s) = +\infty$. If $c(0) = 0$, then the integral in (3.53) can be finite, and we cannot state a similar claim, but from our method used below it will be clear how to handle such a case, too, and therefore the details in this direction will be omitted for simplicity.

The relevant fact in the case when 0 is either a *regular* (*instantaneously reflecting*) or *entrance* boundary point is that

$$E_{0,s}\bigg(\int_0^{\tau_{s_*}} c(X_t)\,dt\bigg) = \int_0^{s_*} \big(L(s_*) - L(y)\big)c(y)m(dy) \tag{3.54}$$

for all $s_* \geq s > 0$ where $\tau_{s_*} = \inf\{t > 0 \mid X_t = s_*\}$. In view of (3.30) this shows that it is never optimal to stop at $(0, s)$. Therefore, if the maximal solution of (3.21) satisfies $g_*(s_*) = 0$ for some $s_* > 0$ with $g_*(s) > 0$ for all $s > s_*$, then $\tau_* = \inf\{t > 0 \mid X_t \leq g_*(S_t)\}$ is to be the optimal stopping time, since X does not take negative values. If, moreover, $c(0) = 0$, then the value of $m(\{0\})$ does not play any role, and all regular behavior [from absorption $m(\{0\}) = +\infty$, over sticky barrier phenomenon $0 < m(\{0\}) < +\infty$, to instantaneous reflection $m(\{0\}) = 0$] can be treated in the same way.

For simplicity in the next result we will assume that $c(0) > 0$ if 0 is either a natural (attracting or unattainable) or an exit boundary point and will only consider the instantaneously reflecting regular case. The remaining cases can be treated similarly.

COROLLARY 3.2 (Optimal stopping for nonnegative diffusions). *In the setting of (2.1)–(2.3), assume that the diffusion X is nonnegative, and that* 0 *is a natural, exit, instantaneously reflecting regular or entrance boundary point. Consider the optimal stopping problem (2.4) where the supremum is taken over all stopping times τ of X satisfying (2.5).*

(I) *Let $s \mapsto g_*(s)$ denote the maximal solution of (3.21) in the following sense (whenever such a solution exists; see Figure 2): There exists a point $s_* \geq 0$ (with $s_* = 0$ if 0 is either a natural or an exit boundary point) such that $g_*(s_*) = 0$ and $g_*(s) > 0$ for all $s > s_*$; the map $s \mapsto g_*(s)$ solves (3.21) for $s > s_*$ and stays strictly below the diagonal in $]0, \infty[^2$; the map $s \mapsto g_*(s)$ is the maximal solution satisfying these two properties (the comparison of two maps is taken pointwise wherever they are both strictly positive). Then we have the following.*

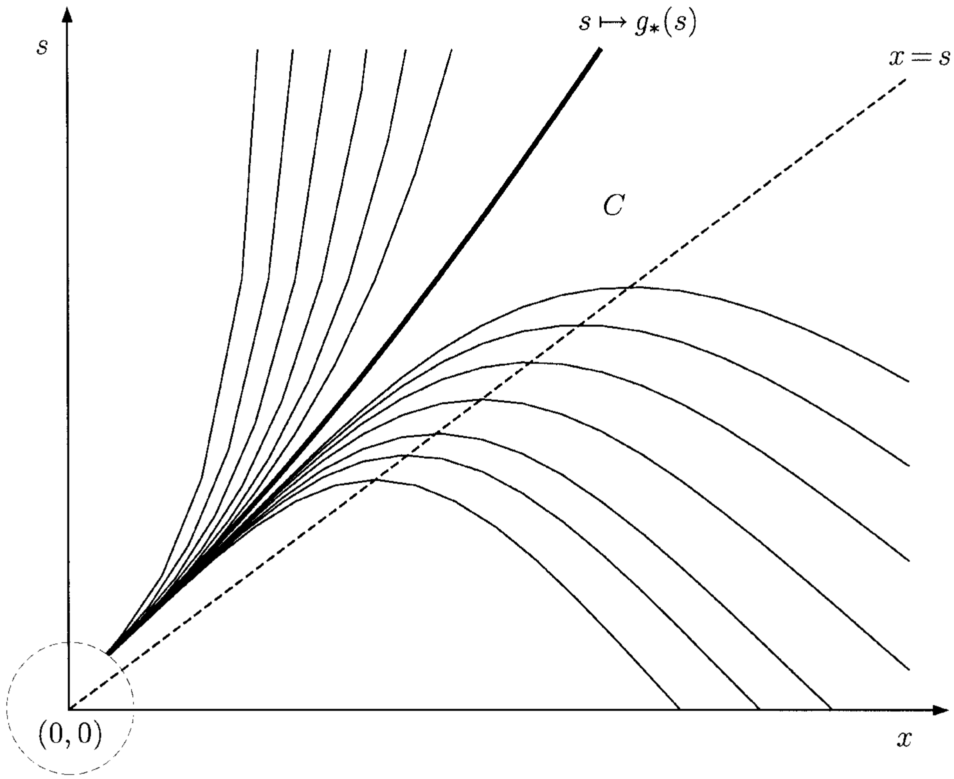

FIG. 2. *A computer drawing of solutions of the differential equation* (3.21) *in the case when* X *is a geometric Brownian motion from Example* 4.4 *with* $\mu = -1$, $\sigma^2 = 2$ (*thus* $\Delta = 2$) *and* $c = 50$. *The bold line* $s \mapsto g_*(s)$ *is the* maximal *solution, which stays strictly below the diagonal in* $\mathbb{R}^2_+$. [*In this particular case there is no closed formula for* $s \mapsto g_*(s)$. *By the maximality principle proved, this solution is the optimal stopping boundary* [*see also* (4.30)].

(i) *The payoff is finite and for* $s \geq s_*$ *is given by*

$$V_*(x,s) = s + \int_{g_*(s)}^{x} \big(L(x) - L(y)\big) c(y) m(dy) \tag{3.55}$$

for $g_*(s) \leq x \leq s$ *with* $V_*(x,s) = s$ *for* $0 \leq x \leq g_*(s)$, *and for* $s \leq s_*$ (*when* 0 *is either an instantaneously reflecting regular or an entrance boundary point) is given by*

$$V_*(x,s) = s_* + \int_{0}^{x} \big(L(x) - L(y)\big) c(y) m(dy) \tag{3.56}$$

for $0 \leq x \leq s$.

(ii) *The stopping time*

$$\tau_* = \inf\big\{t > 0 \,|\, S_t \geq s_*, \, X_t \leq g_*(S_t)\big\} \tag{3.57}$$

is optimal for the problem (2.4) whenever it satisfies (2.5); otherwise, it is "approximately" optimal.

(iii) *If there exists an optimal stopping time σ in (2.4) satisfying (2.5), then $P_{x,s}(\tau_* \le \sigma) = 1$ for all (x, s), and τ_* is an optimal stopping time for (2.4) as well.*

(II) *If there is no (maximal) solution of (3.21) in the sense of* (I) *above, then $V_*(x, s) = +\infty$ for all (x, s), and there is no optimal stopping time.*

PROOF. With only minor changes, the proof can be carried out in exactly the same way as the proof of Theorem 3.1 upon using the additional facts about (3.53) and (3.54) stated above, and the details will be omitted; note, however, that in the case when 0 is either an instantaneously reflecting regular or an entrance boundary point, the strong Markov property of X at $\tau_{s_*} = \inf\{t > 0 \mid X_t = s_*\}$ gives

$$V_*(x, s) = s_* + \int_0^{s_*} \big(L(s_*) - L(y)\big)c(y)m(dy) - E_{x,s}\bigg(\int_0^{\tau_{s_*}} c(X_t)\,dt\bigg) \tag{3.58}$$

for all $0 \le x \le s \le s_*$. Hence (3.56) follows by applying (2.11) and (2.12) to the last term in (3.58). [In the instantaneous reflecting case one can make use of τ_{s_*,s_*} after extending L to $\mathbb{R}_-$ by setting $L(x) := -L(-x)$ for $x < 0$.] The proof is complete. □

3.12. *The "discounted" problem.* One is often more interested in the discounted version of the optimal stopping problem (2.4). Such a problem can be reduced to the initial problem (2.4) by changing the underlying diffusion process.

Given a continuous function $x \mapsto \lambda(x) \ge 0$ called the *discounting rate*, in the setting of (2.1)–(2.3) introduce the functional

$$\Lambda(t) = \int_0^t \lambda(X_r)\,dr, \tag{3.59}$$

and consider the optimal stopping problem with payoff

$$V_*(x, s) = \sup_\tau E_{x,s}\bigg(\exp(-\Lambda(\tau))S_\tau - \int_0^\tau \exp(-\Lambda(t))c(X_t)\,dt\bigg), \tag{3.60}$$

where the supremum is taken over all stopping times τ of X for which the integral has finite expectation, and the *cost* function $x \mapsto c(x) > 0$ is continuous.

The standard argument shows that the problem (3.60) is equivalent to the problem

$$V_*(x, s) = \sup_\tau E_{x,s}\bigg(\widetilde{S}_\tau - \int_0^\tau c(\widetilde{X}_t)\,dt\bigg), \tag{3.61}$$

where $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ is a diffusion process which corresponds to the "killing" of the sample paths of X at the "rate" $\lambda(X)$. The infinitesimal generator of $\widetilde{X}$ is given by

$$\mathbb{L}_{\widetilde{X}} = -\lambda(x) + \mu(x)\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2}{\partial x^2}. \tag{3.62}$$

We conjecture that the maximality principle proved above also holds for this problem (see [26]). The main technical difficulty in a general treatment of this problem is the fact that the infinitesimal generator $\mathbb{L}_{\widetilde{X}}$ has the constant term $-\lambda(x)$, so that $\mathbb{L}_{\widetilde{X}} = 0$ may have no simple solution. Nonetheless, it is clear that the corresponding system (3.8)–(3.11) must be valid, and this system defines the (maximal) boundary $s \mapsto g_*(s)$ implicitly.

3.13. *The "Markovian" cost problem*. Yet another class of optimal stopping problems reduces to problem (2.4). Suppose that in the setting of (2.1)–(2.3) we are given a smooth function $x \mapsto D(x)$ and consider the optimal stopping problem with payoff

$$V_*(x, s) = \sup_\tau E_{x,s}\big(S_\tau - D(X_\tau)\big), \tag{3.63}$$

where the supremum is taken over a class of stopping times τ of X. Then a variant of Itô's formula applied to $D(X_t)$, the optional sampling theorem applied to the continuous local martingale $M_t = \int_0^t D'(X_s)\,\sigma(X_s)\,dB_s$ localized if necessary, and uniform integrability conditions enable one to conclude that

$$E_{x,s}\big(D(X_\tau)\big) = D(x) + E_{x,s}\bigg(\int_0^\tau (\mathbb{L}_X D)(X_s)\,ds\bigg). \tag{3.64}$$

Hence we see that (3.63) reduces to (2.4) with $x \mapsto c(x)$ replaced by $x \mapsto (\mathbb{L}_X D)(x)$ whenever nonnegative. The conditions assumed above to make such a transfer possible are not restrictive in general (see Example 4.2).

4. Examples and applications.

There is a large number of applications of the optimal stopping results (Theorem 3.1 and Corollary 3.2) from the previous section. In this section we present some of them (see also [24]). Our main aim is to derive sharp versions of some known classical inequalities, as well as to deduce some new closely related sharp inequalities. It should be noted that the method applies to all diffusions. Throughout, $B = (B_t)_{t\geq 0}$ denotes the standard Brownian motion started at zero.

EXAMPLE 4.1 (The Doob inequality). Consider the optimal stopping problem (2.4) with $X_t = |B_t + x|^p$ and $c(x) = cx^{(p-2)/p}$ for $p > 1$. Then X is a nonnegative diffusion having 0 as an instantaneously reflecting regular boundary point, and the infinitesimal generator of X in $]0, \infty[$ is given by the expression

$$\mathbb{L}_X = \frac{p(p-1)}{2}x^{1-2/p}\frac{\partial}{\partial x} + \frac{p^2}{2}x^{2-2/p}\frac{\partial^2}{\partial x^2}. \tag{4.1}$$

Equation (3.21) takes the form

(4.2) $$g'(s) = \frac{pg^{1/p}(s)}{2c(s^{1/p} - g^{1/p}(s))},$$

and its maximal solution of (4.2) is given by

(4.3) $$g_*(s) = \alpha s,$$

where $0 < \alpha < 1$ is the maximal root (out of two possible ones) of

(4.4) $$\alpha - \alpha^{1-1/p} + p/2c = 0.$$

It is easily verified that (4.4) admits such a root if and only if $c \geq p^{p+1}/2(p-1)^{(p-1)}$. Then by the result of Corollary 3.2, upon using (3.64) and letting $c \downarrow p^{p+1}/2(p-1)^{(p-1)}$, we get

(4.5) $$E\Bigl(\max_{0\le t\le \tau} |B_t + x|^p\Bigr) \le \left(\frac{p}{p-1}\right)^p E|B_\tau + x|^p - \left(\frac{p}{p-1}\right)x^p$$

for all stopping times τ of B such that $E(\tau^{p/2}) < \infty$. The constants $(p/(p-1))^p$ and $p/(p-1)$ are the best possible, and the equality in (4.5) is attained in the limit through the stopping times $\tau_* = \inf\{t > 0 \,|\, X_t \le \alpha S_t\}$ when $c \downarrow p^{p+1}/2(p-1)^{(p-1)}$. These stopping times are pointwise the smallest possible with this property, and they satisfy $E(\tau_*^{p/2}) < \infty$ if and only if $c > p^{p+1}/2(p-1)^{(p-1)}$. For more information and remaining details we refer to [13].

The inequality (4.5) can be further extended (for simplicity we state this extension only for $x = 0$) by using the result of Corollary 3.2 as follows:

(4.6) $$E\Bigl(\max_{0\le t\le \tau} |B_t|^p\Bigr) \le \gamma^*_{p,q}\left(E\int_0^\tau |B_t|^{q-1}\,dt\right)^{p/(q+1)}$$

for all stopping times τ of B, all $0 < p < 1 + q$ and all $q > 0$, with the best possible value for the constant $\gamma^*_{p,q}$ being equal to

(4.7) $$\gamma^*_{p,q} = (1+\kappa)\left(\frac{s_*}{\kappa^\kappa}\right)^{1/(1+\kappa)},$$

where $\kappa = p/(q - p + 1)$, and s_* is the zero point of the maximal solution $s \mapsto g_*(s)$ of

(4.8) $$g'(s) = \frac{pg^{(1-q/p)}(s)}{2\bigl(s^{1/p} - g^{1/p}(s)\bigr)},$$

satisfying $0 < g_*(s) < s$ for all $s > s_*$. [This solution is also characterized by $g_*(s)/s \to 1$ for $s \to \infty$.] The equality in (4.6) is attained at the stopping time $\tau_* = \inf\{t > 0 \,|\, X_t = g_*(S_t)\}$, which is pointwise the smallest possible with this property. In the case $p = 1$ the closed form for $s \mapsto g_*(s)$ is found as

(4.9) $$\begin{aligned} &s\exp\left(-\frac{2}{pq}g_*^q(s)\right) + \frac{2}{p}\int_0^{g_*(s)} t^q \exp\left(-\frac{2}{pq}t^q\right)dt \\ &\quad = \left(\frac{pq}{2}\right)^{1/q}\Gamma\left(\frac{q+1}{q}\right) \end{aligned}$$

for $s \geq s_*$. This, in particular, yields

$$\gamma^*_{1,q} = \left(\frac{q(1+q)}{2}\right)^{1/(1+q)} \left(\Gamma\left(2+\frac{1}{q}\right)\right)^{q/(1+q)} \tag{4.10}$$

for all $q > 0$. In the case $p \neq 1$, no closed form for $s \mapsto g_*(s)$ seems to exist. For more information and remaining details in this direction, as well as for the extension of inequality (4.6) to $x \neq 0$, we refer to [23] (see also [21]). To give a more familiar form to (4.6), note by Itô's formula and the optional sampling theorem that

$$E\left(\int_0^\tau |B_t|^{q-1}\, dt\right) = \frac{2}{q(q+1)} E|B_\tau|^{q+1}, \tag{4.11}$$

whenever τ is a stopping time of B satisfying $E(\tau^{(q+1)/2}) < \infty$ for $q > 0$. Hence we see that the right-hand side in (4.6) is the well-known Doob's bound. The advantage of formulation (4.6) lies in its validity for all stopping times.

While (4.6) (with some constant $\gamma_{p,q} > 0$) can be derived quite easily, the question of its sharpness has gained interest. The case $p = 1$ was treated independently by Jacka [16] (probabilistic methods) and Gilat [8] (analytic methods), who both found the best possible value $\gamma^*_{1,q}$ for $q > 0$. This in particular yields $\gamma^*_{1,1} = \sqrt{2}$, which was independently obtained by Dubins and Schwarz [5] and later again by Dubins, Shepp and Shiryaev [6], who studied a more general case of Bessel processes. (A simple probabilistic proof for $\gamma^*_{1,1} = \sqrt{2}$ is given in [9]). The Bessel processes results are further extended in [19]. In the case $p = 1 + q$ with $q > 0$, (4.6) reduces to the Doob's maximal inequality (4.5). I learned from Burkholder that this inequality can be obtained as a by-product from his new proof of Doob's inequality for discrete nonnegative submartingales (see [2], page 14). The proof given there in essence relies on a submartingale property, while the proof presented above in essence relies on a strong Markov property. Cox [3] also derived the analogue of this inequality for discrete martingales by a method which is based on results from the theory of moments. That the equality in Doob's maximal inequality (4.5) cannot be attained by a nonzero (sub)martingale was observed by Cox [3]. It should be noted that this fact also follows from the method and results above [the equality in (4.5) is attained only in the limit]. The best values $\gamma^*_{p,q}$ in (4.6) and the corresponding optimal stopping times τ^* for all $0 < p \leq 1 + q$ and all $q > 0$ are given in [23]. The main novelty about (4.5) and (4.6), which is realized here, is that the optimal τ_* from (3.57) is pointwise the smallest possible stopping time at which the equalities in (4.5) (in the limit) and in (4.6) can be attained. The results about (4.5) and (4.6) extend to all nonnegative submartingales. This can be obtained by using the maximal embedding result of Jacka [15] (for details see [13] and [23]).

EXAMPLE 4.2 (The Hardy–Littlewood inequality). Consider the "Markovian" cost problem (3.63) with $X_t = |B_t + x|$ and $D(x) = x \log x$ for $x \geq 0$. Then X is a nonnegative diffusion having 0 as an instantaneously reflecting regular boundary point, and the infinitesimal generator of X in $]0, \infty[$ is given by

(4.1) with $p = 1$. The main difficulty in this problem is that we cannot apply Itô's formula directly to $D(X_t)$ as suggested in Subsection 13 of Section 3. Thus we truncate $D(x)$ by setting $\widetilde{D}(x) = D(x)$ for $x \geq 1/e$ and $\widetilde{D}(x) = -1/e$ for $0 \leq x \leq 1/e$. Then $\widetilde{D} \in C^1$ and $\widetilde{D}''$ exists and is continuous everywhere but at $1/e$. Thus the Itô–Tanaka formula can be applied and, since the time spent by X at $1/e$ is of Lebesgue measure zero, this formula reduces to Itô's formula. In this way problem (3.63) reduces to problem (2.4). Equation (3.21) takes the form

$$g'(s) = \frac{g(s)}{2c\big(s - g(s)\big)}, \tag{4.12}$$

and its maximal solution of (4.12) is given by

$$g_*(s) = \alpha s, \tag{4.13}$$

where $\alpha = (c-1)/c$. By applying the result of Corollary 2.2, we get

$$E\Big(\max_{0\leq t\leq\tau} |B_t + x|\Big) \leq V(x;c) + cE\big(|B_\tau + x| \log |B_\tau + x|\big) \tag{4.14}$$

for all $c > 1$ and all stopping times τ of B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$, where

$$V(x;c) = \begin{cases} \dfrac{c^2}{e(c-1)}, & \text{if } 0 \leq x \leq u_*, \\ cx \log\Big(\dfrac{c}{x(c-1)}\Big), & \text{if } x \geq u_*, \end{cases} \tag{4.15}$$

with $u_* = c/e(c-1)$. This inequality is sharp and, for each $c > 1$ and $x \geq 0$ given and fixed, the equality in (4.14) is attained at the stopping time

$$\tau_* = \inf\big\{t > 0 \mid S_t \geq u_*, X_t = \alpha S_t\big\}, \tag{4.16}$$

which is pointwise the smallest possible with this property.

The same problem with more familiar $D(x) = x \log^+ x$ brings the local time of X at 1 into consideration (see [14] for details), and the analogue of (4.14) may be stated as follows:

$$E\Big(\max_{0\leq t\leq\tau} |B_t + x|\Big) \leq V_+(x;c) + cE\big(|B_\tau + x| \log^+ |B_\tau + x|\big) \tag{4.17}$$

for all $c > 1$ and all stopping times τ of B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$, where

$$V_+(x;c) = \begin{cases} 1 + \dfrac{1}{e^c(c-1)}, & \text{if } 0 \leq x \leq v_*, \\ x + (1-x)\log(x-1) & \\ \qquad -(c + \log(c-1))(x-1), & \text{if } v_* \leq x \leq z_*, \\ cx \log\Big(\dfrac{c}{x(c-1)}\Big), & \text{if } x \geq z_*, \end{cases} \tag{4.18}$$

with $v_* = 1 + 1/e^c(c-1)$ and $z_* = c/(c-1)$. This inequality is sharp, and for each $c > 1$ and $x \geq 0$ given and fixed, the equality in (4.17) is attained at the stopping time

$$\sigma_* = \inf\{t > 0 \mid S_t \geq v_*, X_t = 1 \vee \alpha S_t\}, \tag{4.19}$$

which is pointwise the smallest possible with this property. For remaining details and more information on (4.14) and (4.17) we refer to [14]. Note that (4.14)–(4.19) contain and refine the results of Gilat [7], which settle a question raised by Dubins and Gilat [4], and later again by Cox [3] and which are obtained by analytic methods.

EXAMPLE 4.3 (A sharp integral inequality of the $L \log L$-type). Consider the optimal stopping problem (2.4) with $X_t = |B_t + x|$ and $c(x) = 1/(1+x)$ for $x \geq 0$. Then X is a nonnegative diffusion having 0 as an instantaneously reflecting regular boundary point, and the infinitesimal generator of X in $]0, \infty[$ is given by (4.1) with $p = 1$. Equation (3.21) takes the form

$$g'(s) = \frac{1 + g(s)}{2c\big(s - g(s)\big)}, \tag{4.20}$$

and its maximal solution of (4.20) is given by

$$g_*(s) = \alpha s - \beta, \tag{4.21}$$

where $\alpha = (2c-1)/2c$ and $\beta = 1/2c$. By applying the result of Corollary 2.2 we get

$$E\Big(\max_{0\leq t\leq\tau} |B_t + x|\Big) \leq W(x;c) + cE\left(\int_0^\tau \frac{dt}{1 + |B_t + x|}\right) \tag{4.22}$$

for all stopping times τ of B, all $c > 1/2$ and all $x \geq 0$, where

$$W(x;c) = \begin{cases} \dfrac{1}{2c-1} + 2c\big((1+x)\log(1+x) - x\big), & \text{if } x \leq 1/(2c-1), \\ 2c(1+x)\log\left(1 + \dfrac{1}{2c-1}\right) - 1, & \text{if } x > 1/(2c-1). \end{cases} \tag{4.23}$$

This inequality is sharp, and for each $c > 1/2$ and $x \geq 0$ given and fixed, the equality in (4.23) is attained at the stopping time

$$\tau_* = \inf\{t > 0 \,|\, S_t - \alpha X_t \geq \beta\}, \tag{4.24}$$

which is pointwise the smallest possible with this property. By minimizing over all $c > 1/2$ on the right-hand side in (4.22), we get a sharp inequality [the equality is attained at each stopping time τ_* from (4.24) whenever $c > 1/2$ and $x \geq 0$]. In particular, this for $x = 0$ yields

$$E\Big(\max_{0\leq t\leq\tau} |B_t|\Big) \leq \frac{1}{2}E\left(\int_0^\tau \frac{dt}{1+|B_t|}\right) + \sqrt{2}\left(E\int_0^\tau \frac{dt}{1+|B_t|}\right)^{1/2} \tag{4.25}$$

for all stopping times τ of B. This inequality is sharp, and the equality in (4.25) is attained at each stopping time τ_* from (4.24). Note that by Itô's formula and the optional sampling theorem,

$$E\left(\int_0^\tau \frac{dt}{1+|B_t|}\right) = 2E\big((1+|B_\tau|)\log(1+|B_\tau|) - |B_\tau|\big) \tag{4.26}$$

for all stopping times τ of B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. This shows that the inequality (4.25) in essence is of the $L \log L$-type. The advantage of (4.25) over the classical Hardy–Littlewood $L \log L$-inequality is its sharpness for small stopping times as well [note that the equality in (4.25) is attained for $\tau \equiv 0$]. For more information on this inequality and remaining details we refer to [22].

EXAMPLE 4.4. (A sharp maximal inequality for geometric Brownian motion). Consider the optimal stopping problem (2.4) where X is geometric Brownian motion and $c(x) \equiv c$. Recall that X is a nonnegative diffusion having 0 as an entrance boundary point, and the infinitesimal generator of X in $]0,\infty[$ is given by the expression

$$\mathbb{L}_X = \mu x \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2}, \tag{4.27}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The process X may be realized as

$$X_t = x \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right) \tag{4.28}$$

with $x \geq 0$. Equation (3.21) takes the form

$$g'(s) = \frac{\Delta \sigma^2 g^{\Delta+1}(s)}{2c\big(s^\Delta - g^\Delta(s)\big)}, \tag{4.29}$$

where $\Delta = 1 - 2\mu/\sigma^2$. By using Picard's method of successive approximations, it is possible to prove that for $\Delta > 1$ the equation (4.29) admits the maximal solution $s \mapsto g_*(s)$ satisfying

$$g_*(s) \sim s^{1-1/\Delta} \quad \text{(see Figure 2)} \tag{4.30}$$

for $s \to \infty$ (see [11] for details). There seems to be no closed form for this solution. In the case $\Delta = 1$, it is possible to find the general solution of (4.29) in a closed form, and this shows that the only nonnegative solution is zero-function (see [11]). By the result of Corollary 3.2 we may conclude that the payoff (2.4) is finite if and only if $\Delta > 1$ (note that another argument was used in [11] to obtain this equivalence), and in this case it is given by

$$V_*(x,s) = \begin{cases} \dfrac{2c}{\Delta^2\sigma^2}\left(\left(\dfrac{x}{g_*(s)}\right)^\Delta - \log\left(\dfrac{x}{g_*(s)}\right)^\Delta - 1\right) + s, & \text{if } g_*(s) < x \leq s, \\ s, & \text{if } 0 < x \leq g_*(s). \end{cases} \tag{4.31}$$

The optimal stopping time is given by (3.57) with $s_* = 0$. By using explicit estimates from (4.30) on $s \to g_*(s)$ in (4.31), and then minimizing over all $c > 0$, we obtain

$$E\left(\max_{0\le t\le \tau} \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right) \le 1 - \frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu}\exp\left(-\frac{(\sigma^2-2\mu)^2}{2\sigma^2}E(\tau) - 1\right) \tag{4.32}$$

for all stopping times τ of B. This inequality extends the well-known estimates of Doob in a sharp manner from deterministic times to stopping times. For more information and remaining details we refer to [11]. Observe that the cost function $c(x) = cx$ in the optimal stopping problem (2.4) would imply that the maximal solution of (3.21) is linear. This shows that such a cost function suits better the maximum process and therefore is more natural. Explicit formulas for the payoff and the maximal inequality obtained by minimizing over $c > 0$ are also obtained easily in this case from the result of Corollary 3.2.

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