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THE LIMITS OF SINAI'S SIMPLE RANDOM WALK IN RANDOM ENVIRONMENT

BY YUEYUN HU AND ZHAN SHI

Université Paris VI

We study the sample path asymptotics of a class of recurrent diffusion processes with random potentials, including examples of Sinai's simple random walk in random environment and Brox's diffusion process with Brownian potential. The main results consist of several integral criteria which completely characterize all the possible Lévy classes, therefore providing a very precise image of the almost sure asymptotic behaviors of these processes.

1. Introduction.

1.1. *Simple random walk in random environment.* Problems related to random environments arise naturally in several branches of physics (cf. [2] for an overview) and receive much attention from both mathematicians and physicists. The most elementary model is Sinai's simple random walk in random environment, which can be described as follows: let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of random variables taking values in $(0,1)$. Define a random walk $\{S_n\}_{n\geq 0}$ by $S_0 = 0$ and

$$\mathbb{P}\big(S_{n+1} = j \mid S_n = i;\, \Xi\big) = \begin{cases} \xi_i, & \text{if } j = i+1,\\ 1-\xi_i, & \text{if } j = i-1,\\ 0, & \text{otherwise}\end{cases}$$

for any $n \geq 1$ and $i \in \mathbb{N}$, where $\Xi = \{\xi_j\}_{j\in\mathbb{N}}$ as defined is the so-called random environment. Observe that both the environment and the walk are random under $\mathbb{P}$. In physics, it is often the case that little is known about the realization of random environment. It is therefore convenient to formulate results under the absolute probability $\mathbb{P}$ (the so-called annealed setting). Throughout the paper, it is under probability $\mathbb{P}$ we shall be working, and anything like "with probability 1" or "almost surely" is to be understood with respect to $\mathbb{P}$. (In the literature, there is also much interest in the "quenched" setting, i.e., the random walk under the conditional probability $\mathbb{P}(\cdot|\Xi)$; cf. for example [15] and [13]).

A remarkable result in the study of random walk in random environment (RWRE) is Sinai's theorem [30], which establishes convergence in distribu-

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tion, denoted by "$\xrightarrow{\text{law}}$" in the sequel, of $S_n/(\log n)^2$ to some nondegenerate variable. The limit law was later determined in [20] and [14] independently.

THEOREM A (Sinai [30], Kesten [20], Golosov [14]). *Assuming that*

$$\{\xi_i\}_{i\in\mathbb{Z}} \text{ are independent and identically distributed,} \tag{1.1}$$

$$\mathbb{P}(\nu < \xi_0 < 1-\nu) = 1, \quad \text{for some } \nu > 0, \tag{1.2}$$

$$\mathbb{E}\log\frac{\xi_0}{1-\xi_0} = 0, \tag{1.3}$$

$$0 < \sigma^2 = \mathbb{E}\left(\log\frac{\xi_0}{1-\xi_0}\right)^2 < \infty \quad \text{as defined,} \tag{1.4}$$

we have

$$\frac{\sigma^2}{(\log n)^2} S_n \xrightarrow{\text{law}} b_\infty, \tag{1.5}$$

$$\frac{\sigma^2}{(\log n)^2} \max_{0\le k\le n} S_n \xrightarrow{\text{law}} \overline{b}_\infty, \tag{1.6}$$

where b_∞ is a symmetric variable, and $\overline{b}_\infty$ is positive. Moreover, their respective laws are characterized via Laplace transforms

$$\mathbb{E}\exp(-\lambda|b_\infty|) = \frac{\cosh(\sqrt{2\lambda}) - 1}{\lambda\cosh\sqrt{2\lambda}},$$

$$\mathbb{E}\exp(-\lambda\overline{b}_\infty) = \frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}}, \qquad \lambda > 0.$$

REMARK 1.1. Loosely speaking, Theorem A tells us that, for large n, a "typical" value of S_n or $\max_{0\le k\le n} S_k$ is of order $(\log n)^2$, which is far smaller than $n^{1/2}$, the magnitude order of a usual simple symmetric random walk [in a nonrandom environment, i.e., $\sigma = 0$ in (1.4)]. An explanation for this is that it takes a long time for S_n to go through the deep "valleys" of the random environment Ξ. Simple heuristic arguments for getting the $(\log n)^2$ rate can be found in [30] or [26], page 276. We also mention that conditions bearing different natures in (1.1)–(1.4) are also adopted in the literature; compare, for example, [21] and [9] in the discrete-time setting and [18] and [4] in the continuous-time setting.

In contrast to the huge number of results concerning random environments, relatively little is known about the *almost sure* asymptotic behavior of Sinai's RWRE S_n. To the best of our knowledge, this problem was first attacked in [8].

THEOREM B (Deheuvels and Révész [8]). *Assuming* (1.1)–(1.4), *for any* $\varepsilon > 0$ *and* $p \geq 3$, *the following inequalities hold almost surely for all but finitely many* n:

$$\frac{(\log n)^2}{(\log\log n)^{2+\varepsilon}} \leq \max_{0\leq k\leq n} |S_k| \tag{1.7}$$

$$\leq (\log n)^2 (\log_2 n)^2 \cdots (\log_{p-1} n)^2 (\log_p n)^{2+\varepsilon}, \tag{1.8}$$

where $\log_j n$ *denotes the jth iterative logarithmic function.*

REMARK 1.2. In the "reflecting" setting (i.e., a positive random walk with a reflecting barrier at 0), Theorem B was recently recovered in [6], using Lyapunov functions.

Natural questions arise here: what is the exact amount of almost sure asymptotic behavior of S_n? Are (1.7) and/or (1.8) sharp? More generally, we suggest studying the following problems.

1. How big can S_n (resp. $\max_{0\leq k\leq n} S_n$, $\max_{0\leq k\leq n}|S_k|$) be?
2. How small can $\max_{0\leq k\leq n}|S_k|$ be?
3. How small can $\max_{0\leq k\leq n} S_k$ be?

Observe that by symmetry, the corresponding "how small" problem for S_n is equivalent to (1).

We provide complete solutions to problems (1)–(3), via three integral tests which characterize all the possible Lévy classes of $\{S_n\}_{n\geq 0}$. It is noted that $\max_{0\leq k\leq n}|S_k|$ and $\max_{0\leq k\leq n} S_k$ have very different lower functions. This can be understood as follows: when $\max_{0\leq k\leq n} S_k$ is small, the RWRE makes some extraordinarily large negative excursions.

Before stating our main results, we give some precision about conditions of regularity. Though the "classical" conditions (1.1)–(1.4) are adopted by many mathematicians and physicists in the study of Sinai's recurrent RWRE, we assume in this paper a somewhat weaker condition (possibly in an enlarged probability space): there exists a coupling for Ξ and standard "two-sided" Brownian motion $\{W(y);\ y \in \mathbb{R}\}$, such that, for all $n \geq 1$,

$$\mathbb{P}\left[\sup_{1\leq |m|\leq n}\left|\sum_{j=1}^{m}\log\left(\frac{1-\xi_j}{\xi_j}\right) - \sigma W(m)\right| \geq C_1 \log n\right] \leq \frac{C_2}{n^{C_3}}, \tag{1.9}$$

where $C_i > 0$ $(1 \leq i \leq 3)$ and $\sigma > 0$ are finite constants (for negative m's, $\sum_{j=1}^{m} x_j = x_{-1} + \cdots + x_m$) as defined. By the well-known Komlós–Major–Tusnády [24] strong approximation theorem, (1.1)–(1.4) together imply (1.9). Recall that according to [31], Theorem 1.7(iii), (1.9) also ensures the recurrence of S_n.

THEOREM 1.3. *Let* $\{a_n\}_{n\ge 1}$ *be a sequence of positive nondecreasing numbers. We have, under* (1.9),

$$\mathbb{P}\Big[S_n > (\log n)^2 a_n \ i.o.\Big] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \sum_{n\ge 2} \frac{a_n}{n \log n} \exp\left(-\frac{\pi^2\sigma^2}{8} a_n\right) \begin{cases} < \infty \\ = \infty, \end{cases} \tag{1.10}$$

where we adopt the usual symbol "i.o." denoting "infinitely often" as the relevant variable tends to infinity. In particular,

$$\limsup_{n\to\infty} \frac{S_n}{(\log n)^2 \log\log\log n} = \frac{8}{\pi^2\sigma^2} \quad a.s. \tag{1.11}$$

We can replace S_n *by either* $\max_{0\le k\le n}|S_k|$ *or* $\max_{0\le k\le n} S_k$ *in* (1.10) *and* (1.11).

THEOREM 1.4. *Let* $\{a_n\}_{n\ge 1}$ *be a sequence of positive nondecreasing numbers. Assuming* (1.9),

$$\mathbb{P}\left[\max_{0\le k\le n}|S_k| \le \frac{(\log n)^2}{a_n} i.o.\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \sum_{n\ge 2} \frac{\sqrt{a}_n}{n \log n} \exp\left(-\frac{a_n}{\sigma^2}\right) \begin{cases} < \infty \\ = \infty. \end{cases}$$

As a consequence, we obtain the following Chung-type iterated logarithm law:

$$\liminf_{n\to\infty} \frac{\log\log\log n}{(\log n)^2} \max_{0\le k\le n}|S_k| = \frac{1}{\sigma^2} \quad a.s.$$

THEOREM 1.5. *If* $\{a_n\}_{n\ge 1}$ *is positive nondecreasing, and if* (1.9) *holds,*

$$\mathbb{P}\left[\max_{0\le k\le n} S_k \le \frac{(\log n)^2}{a_n} i.o.\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \sum_{n} \frac{1}{n\sqrt{a_n} \log n} \begin{cases} < \infty \\ = \infty. \end{cases}$$

Therefore, with probability 1,

$$\liminf_{n\to\infty} \frac{(\log\log n)^a}{(\log n)^2} \max_{0\le k\le n} S_k = \begin{cases} 0, & \text{if } a \le 2, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorems 1.3–1.5 are proved in Section 10.

1.2. *Diffusion with Brownian potential.* The continuous-time analogue of Sinai's RWRE is Brox's diffusion process $\{X(t);\ t\ge 0\}$, formally defined by

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}W'(X(t))\,dt, \\ X(0) = 0, \end{cases} \tag{1.12}$$

where $\{\beta(t);\ t\ge 0\}$ and $\{W(x);\ x\in\mathbb{R}\}$ are independent one-dimensional Brownian motions (W being "two-sided") with $\beta(0) = W(0) = 0$. Strictly speaking, instead of writing the formal derivative of W in (1.12), we should consider X as a diffusion process with generator

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

The analogue of the weak convergence (1.5) for X is established in Brox [3]. Again, it results from the intuitive idea that X spends a long time in the "valleys" of W. The study of X usually relies on rigorous treatment of the valleys, which, originally due to Sinai and Brox, is now much developed for a large class of processes. See, for example, [32] and [25] together with their references. In Section 8, we solve the problem of determining the almost sure asymptotics of X. The answer is a continuous-time analogue of that to Sinai's RWRE.

THEOREM 1.6. *For any nondecreasing function $f > 0$,*

$$\mathbb{P}\Big[X(t) > (\log t)^2 f(t)\ i.o.\Big] = \begin{cases}0\\1\end{cases} \Leftrightarrow \int^{\infty}\frac{f(t)}{t\log t}\exp\left(-\frac{\pi^2}{8}f(t)\right)dt\begin{cases}<\infty\\=\infty.\end{cases}$$

In particular,

$$\limsup_{t\to\infty}\frac{X(t)}{(\log t)^2\log\log\log t} = \limsup_{t\to\infty}\frac{\sup_{0\le s\le t}|X(s)|}{(\log t)^2\log\log\log t} = \frac{8}{\pi^2}\quad a.s.$$

THEOREM 1.7. *For any nondecreasing function $f > 0$,*

$$\mathbb{P}\left[\sup_{0\le s\le t}|X(s)| \le \frac{(\log t)^2}{f(t)}\ i.o.\right] = \begin{cases}0\\1\end{cases} \Leftrightarrow \int^{\infty}\frac{\sqrt{f(t)}}{t\log t}\exp(-f(t))\,dt\begin{cases}<\infty\\=\infty.\end{cases}$$

In particular,

$$\liminf_{t\to\infty}\frac{\log\log\log t}{(\log t)^2}\sup_{0\le s\le t}|X(s)| = 1\quad a.s.$$

THEOREM 1.8. *For any nondecreasing function $f > 0$,*

$$\mathbb{P}\left[\sup_{0\le s\le t}X(s) \le \frac{(\log t)^2}{f(t)}\ i.o.\right] = \begin{cases}0\\1\end{cases} \Leftrightarrow \int^{\infty}\frac{dt}{t\sqrt{f(t)}\,\log t}\begin{cases}<\infty\\=\infty.\end{cases}$$

The rest of the paper is organized as follows. In Section 2, a distributional result concerning one-dimensional Brownian motion is obtained, which may be of independent interest and which ultimately plays an important role, in Sections 5–8, in the proofs of our main theorems as well as in the forthcoming key estimates and technical lemmas. In Section 3, we present some key estimates (Propositions 3.1–3.3) for tail probabilities in the general setting of a diffusion process with random potential. Two lemmas are stated in Section 4, and they are proved in Sections 6 and 7, respectively. Section 5 is devoted to the proof of the key estimates. Based on the key estimates, Theorems

1.6–1.8 are proved in Section 8, where we shall actually prove a general result implying Theorems 1.6–1.8 as special cases. In Section 9, a Skorokhod-type embedding is presented, which relates Sinai's RWRE to a Brox-type diffusion process with random potential. We prove Theorems 1.3–1.5 in Section 10. Finally, to illustrate how our approach allows dealing with other aspects of Sinai's RWRE, we study weak convergence in Section 11. In particular, the limit law of $\max_{0\le k\le n} S_k$ stated in Theorem A is recovered.

2. One-dimensional Brownian motion. In the sequel, for any stochastic process ξ and $u \in \mathbb{R}$, we write indifferently $\xi(u)$ or ξ_u, and define,

$$\bar{\xi}(u) = \sup_{s\in[0,u]} \xi(s) \quad \text{as defined,} \tag{2.1}$$

$$\underline{\xi}(y) = \inf_{s\in[0,u]} \xi(s) \quad \text{as defined,} \tag{2.2}$$

$$\xi^*(u) = \sup_{s\in[0,u]} |\xi(s)| \quad \text{as defined,} \tag{2.3}$$

where, by abuse of notation, $[0, u]$ means $[u, 0]$ for negative u's.

Let $\{W(t);\ t \ge 0\}$ be one-dimensional Brownian motion, starting from 0. Define

$$\begin{aligned} W^\#(t) &= \sup_{0\le r\le s\le t} (W(s) - W(r)) \\ &= \sup_{0\le s\le t} \big(W(s) - \underline{W}(s)\big) \quad \text{as defined,} \end{aligned} \tag{2.4}$$

for $t \ge 0$. Here is the main result of the section.

THEOREM 2.1. *Let $\overline{W}$ and $W^\#$ be as in* (2.1) *and* (2.4). *For* $0 < a \le b$ *and* $t > 0$,

$$\begin{aligned} &\mathbb{P}\big(\overline{W}(t) < a;\ W^\#(t) < b\big) \\ &\quad = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2 t}{8b^2}\right)\sin\left(\left(k+\frac{1}{2}\right)\frac{\pi a}{b}\right). \end{aligned} \tag{2.5}$$

PROOF. By scaling, we only have to treat the case $t = 1$. For each fixed $x \in \mathbb{R}$, let

$$H_x = \inf\{t > 0 : W(t) = x\} \quad \text{as defined.}$$

Consider the measurable events

$$\begin{aligned} E_1 &= \big\{\overline{W}(1) < a\big\} \quad \text{(as defined)} \\ &= \{H_a > 1\}, \\ E_2 &= \big\{W^\#(1) < b\big\} \quad \text{(as defined)} \\ &= \Big\{\sup_{0\le s\le 1} (W(s) - \underline{W}(s)) < b\Big\}. \end{aligned}$$

Recall that $a \le b$. When $\omega \in E_1 \cap E_2$, there are two possible situations, namely:

(i) either $a - b < W(s) < a$ for $0 \le s \le 1$;

(ii) or W hits $a - b$ before hitting a and before time 1, and for $\{\alpha(s) = W(s + H_{a-b}) + (b - a);\ s \ge 0\}$, the process $s \mapsto \alpha(s) - \inf_{0 \le r \le s} \alpha(r)$ stays below b until $1 - H_{a-b}$.

Hence

$$\begin{aligned}\mathbb{P}(E_1 \cap E_2) &= \mathbb{P}(H_{a-b} > 1;\ H_a > 1)\\ &\quad + \mathbb{P}\Bigl(H_{a-b} < \min(1, H_a);\\ &\qquad\qquad \sup_{0 \le s \le 1 - H_{a-b}} \Bigl(\alpha(s) - \inf_{0 \le r \le s} \alpha(r)\Bigr) < b\Bigr)\\ &= \mathbb{P}\Bigl(\sup_{0 \le u \le 1} \gamma(u) < b\Bigr),\end{aligned}$$

where the process $\{\gamma(u);\ u \ge 0\}$ is defined by

$$\gamma(u) = \begin{cases} W(u) + (b - a), & \text{if } 0 \le u \le H_{a-b},\\ \alpha(u - H_{a-b}) - \inf_{0 \le r \le u - H_{a-b}} \alpha(r), & \text{if } u \ge H_{a-b}.\end{cases}$$

In words, the process γ begins life as the shifted Brownian motion $W + (b - a)$, and when hitting 0 for the first time, it follows the path of $s \mapsto \alpha(s) - \inf_{0 \le r \le s} \alpha(r)$. Observe that α is Brownian motion starting from 0, independent of $F_{H_{a-b}}$ (F denoting the natural filtration of W). On the other hand, it follows from Lévy's identity that $\{\alpha(s) - \inf_{0 \le r \le s} \alpha(r);\ s \ge 0\}$ is reflecting Brownian motion (i.e., behaving like $|W|$). Therefore, by the strong Markov property,

$$\{\gamma(u);\ u \ge 0\} \overset{\text{law}}{=} \{|W(u) + (b - a)|;\ u \ge 0\} \quad \text{by law},$$

where "$\overset{\text{law}}{=}$" stands for identity in distribution. Consequently,

$$\begin{aligned}\mathbb{P}(E_1 \cap E_2) &= \mathbb{P}\Bigl(\sup_{0 \le u \le 1} |W(u) + (b - a)| < b\Bigr)\\ &= \mathbb{P}\bigl(\overline{W}(1) < a;\ \underline{W}(1) > -(2b - a)\bigr).\end{aligned}$$

Theorem 2.1 now follows from the well-known joint distribution of $\overline{W}(1)$ and $\underline{W}(1)$ (cf. [12], page 342). □

COROLLARY 2.2. *The joint distribution-density function of* $(\overline{W}, W^{\#})$ *is given by*

$$\frac{1}{db}\mathbb{P}\bigl(\overline{W}(1) < a;\ W^{\#}(1) \in db\bigr) = \left(\frac{8}{\pi}\right)^{1/2} \sum_{k=-\infty}^{\infty} (-1)^k k \exp\left(-\frac{(a + 2kb)^2}{2}\right),$$

for $0 < a < b$. *Consequently, there exist universal constants* $C_4 > 0$ *and* $C_5 > 0$ *such that*

$$
\begin{aligned}
\frac{C_4}{b}\exp\left(-\frac{(2b-a)^2}{2}\right) &\le \mathbb{P}\big(\overline{W}(1) < a;\, W^{\#}(1) > b\big) \\
&\le \frac{C_5}{b}\exp\left(-\frac{(2b-a)^2}{2}\right), \qquad 1 \le a < b.
\end{aligned}
\tag{2.6}
$$

For the proof, take $t = 1$ in (2.5), differentiate on both sides with respect to b and use the Poisson summation formula.

3. Key estimates. Our main concern is Sinai's RWRE *and* Brox's diffusion process. Therefore, we develop a method which can be applied to *both* processes.

Consider a Brox-type diffusion process $\{\mathbb{X}(t);\ t \ge 0\}$ formally defined by

$$
\begin{cases} d\mathbb{X}(t) = d\beta(t) - \frac{1}{2}\mathbb{W}'(\mathbb{X}(t))\, dt, \\ \mathbb{X}(0) = 0, \end{cases}
$$

where $\{\beta(t);\ t \ge 0\}$ is real-valued Brownian motion, independent of the random potential $\{W(x);\ x \in \mathbb{R}\}$ and $\beta(0) = \mathbb{W}(0) = 0$. We call $\mathbb{X}$ "diffusion with potential $\mathbb{W}$."

We shall assume $\mathbb{W}$ to be a cadlag process (i.e., right continuous with limits on the left), satisfying the following condition of regularity: there exists a coupling for $\mathbb{W}$ and the standard two-sided Brownian motion W such that for all $t \ge 1$,

$$
\mathbb{P}\left[\sup_{|s|\le t}\left|\mathbb{W}(s) - \sigma W(s)\right| \ge C_1 \log t\right] \le \frac{C_2}{t^{C_3}}, \tag{3.1}
$$

where $C_i > 0$ $(1 \le i \le 3)$ and $\sigma > 0$ are finite constants. As is pointed out in [3], it is easily seen, using diffusion theory, that $\mathbb{X}$ can be represented as

$$
\mathbb{X}(t) = \mathbb{A}^{-1}\big(B(\mathbb{T}^{-1}(t))\big), \qquad t \ge 0, \tag{3.2}
$$

where $\{B(t);\ t \ge 0\}$ is Brownian motion starting from 0, independent of $(\mathbb{W}, W)$, and

$$
\mathbb{A}(x) = \int_0^x e^{\mathbb{W}(y)}\, dy, \qquad x \in \mathbb{R}, \text{ as defined,} \tag{3.3}
$$

$$
\mathbb{T}(r) = \int_0^r \exp\big[-2\mathbb{W}\big(\mathbb{A}^{-1}(B(s))\big)\big]\, ds, \qquad r \ge 0, \text{ as defined} \tag{3.4}
$$

($\mathbb{A}^{-1}$ and $\mathbb{T}^{-1}$ denote the respective inverse functions of $\mathbb{A}$ and $\mathbb{T}$). Actually, $\mathbb{A}$ is the scale function of $\mathbb{X}$, and is for this reason denoted by S in [3]. Observe that under (3.1), the random potential $\mathbb{W}$ is bounded by a finite random constant in each compact interval, which ensures that $\mathbb{A}$ is well defined, with $\mathbb{A}(\infty) = \infty$ and $\mathbb{A}(-\infty) = -\infty$ almost surely.

The Brownian motion W in (3.1) being two-sided, we write

$$H(x) = \inf\{t > 0: W(t) > x\} \quad (x \geq 0, \text{ as defined}) \\ = \inf\{t > 0: W(t) < x\} \quad (x < 0, \text{ as defined}) \tag{3.5}$$

and

$$H_-(x) = \inf\{t > 0: W(-t) > x\} \quad (x \geq 0, \text{ as defined}) \\ = \inf\{t > 0: W(-t) < x\} \quad (x < 0, \text{ as defined}). \tag{3.6}$$

The subscript "$-$" is to insist that (3.6) represents the first hitting time processes for W indexed by $\mathbb{R}_-$. Here are the key estimates of the paper.

PROPOSITION 3.1. *Assume* (3.1). *There exist sufficiently large constants* C_6 *and* t_0, *whose values depend on* (σ, C_1, C_2, C_3), *such that for*

$$t > t_0 \quad \text{and} \quad 4 \leq \lambda \leq (\log\log t)^{1/2}, \tag{3.7}$$

we have

$$\mathbb{P}\big(\overline{\mathbb{X}}(t) > \lambda \log^2 t\big) \leq C_6 \exp\left(-\frac{\pi^2\sigma^2\lambda}{8}\right). \tag{3.8}$$

Moreover, under (3.7),

$$\big\{\overline{\mathbb{X}}(t) > \lambda \log^2 t\big\} \supseteq E_3 \cap E_4, \tag{3.9}$$

where

$$E_3 = E_3(\lambda, t) = \left\{H_-\left(-\frac{\log t}{4\sigma}\right) > H_-\left(\frac{\log t}{4\sigma}\right)\right\} \\ \cap \left\{\overline{W}(\lambda \log^2 t) < \frac{\log t}{5\sigma}\right\} \\ \cap \left\{W^{\#}(\lambda \log^2 t) < \left(1 - \frac{3}{\lambda}\right)\frac{\log t}{\sigma}\right\} \quad \text{as defined} \tag{3.10}$$

and $E_4 = E_4(\lambda, t)$ *is a measurable event such that*

$$\mathbb{P}(E_4^c) \leq C_6 \exp(-\lambda^2), \tag{3.11}$$

E_4^c *denoting the complement of* E_4.

PROPOSITION 3.2. *Under* (3.1), *there exist large* C_7 *and* t_0, *depending on* (σ, C_1, C_2, C_3), *such that for all* (λ, t) *satisfying* (3.7),

$$\mathbb{P}\left(\mathbb{X}^*(t) \leq \frac{\log^2 t}{\lambda}\right) \leq \frac{C_7}{\sqrt{\lambda}} \exp\left(-\frac{\lambda}{\sigma^2}\right). \tag{3.12}$$

Furthermore, assuming (3.7), *there exists* $E_5 = E_5(\lambda, t)$ *satisfying*

$$\left\{\mathbb{X}^*(t) \leq \frac{\log^2 t}{\lambda}\right\} \supseteq E_5 \cap \big\{\overline{W}(-v) > \overline{W}(u) + \log^4 v\big\} \\ \cap \big\{\sigma W^{\#}(v) > \log t + \log^4 v\big\}, \tag{3.13}$$

$$\mathbb{P}(E_5^c) \leq C_7 \exp(-\lambda^2), \tag{3.14}$$

with $v = (\log t)^2/\lambda$ *as defined.*

PROPOSITION 3.3. *Assume* (3.1). *There exist large* C_8 *and* t_0, *depending on* (σ, C_1, C_2, C_3), *such that for*

$$t > t_0 \quad \text{and} \quad 1 \le \lambda \le (\log\log t)^3, \tag{3.15}$$

we have

$$\mathbb{P}\left(\overline{\mathbb{X}}(t) \le \frac{\log^2 t}{\lambda}\right) \le \frac{C_8}{\sqrt{\lambda}}. \tag{3.16}$$

Furthermore, under (3.15),

$$\begin{aligned}\left\{\overline{\mathbb{X}}(t) \le \frac{\log^2 t}{\lambda}\right\} &\supseteq E_6 \cap \left\{H_-\left(\frac{\log t}{\sqrt{\lambda}}\right) > H_-\left(-\frac{\log t}{\sigma}\right)\right\} \\ &\quad \cap \left\{\overline{W}\left(\frac{\log^2 t}{\lambda}\right) \ge \frac{2\log t}{\sqrt{\lambda}}\right\},\end{aligned} \tag{3.17}$$

with $E_6 = E_6(\lambda, t)$ *satisfying*

$$\mathbb{P}(E_6^c) \le C_8 \exp(-\sqrt{\lambda}). \tag{3.18}$$

Although it may not be obvious from their statements, inequalities (3.8), (3.12) and (3.16) are all two-sided, namely, for (λ, t) satisfying (3.7) [or (3.15) for Proposition 3.3], we have (with possibly enlarged values of the constants),

$$\frac{1}{C_6}\exp\left(-\frac{\pi^2\sigma^2\lambda}{8}\right) \le \mathbb{P}\left(\overline{\mathbb{X}}(t) > \lambda\log^2 t\right) \le C_6\exp\left(-\frac{\pi^2\sigma^2\lambda}{8}\right),$$

$$\frac{1}{C_7\sqrt{\lambda}}\exp\left(-\frac{\lambda}{\sigma^2}\right) \le \mathbb{P}\left(\mathbb{X}^*(t) \le \frac{\log^2 t}{\lambda}\right) \le \frac{C_7}{\sqrt{\lambda}}\exp\left(-\frac{\lambda}{\sigma^2}\right),$$

$$\frac{1}{C_8\sqrt{\lambda}} \le \mathbb{P}\left(\overline{\mathbb{X}}(t) \le \frac{\log^2 t}{\lambda}\right) \le \frac{C_8}{\sqrt{\lambda}}.$$

The proofs of Propositions 3.1–3.3 are postponed until Section 5. They are based on the technical lemmas stated in the next section.

4. Two lemmas.

Throughout the paper, unless stated otherwise, $C_j > 0$ $(9 \le j \le 70)$ denote (finite) constants, depending on (σ, C_1, C_2, C_3). We continue using (2.1)–(2.4). Recall the notation for the Brox-type diffusion process $\mathbb{X}$ from (3.2)–(3.4). We assume (3.1). Define

$$\begin{aligned}\varrho_x &= \inf\{t > 0 \colon B(t) > x\}, \qquad (x \ge 0, \text{ as defined}) \\ &= \inf\{t > 0 \colon B(t) < x\}, \qquad (x < 0, \text{ as defined})\end{aligned} \tag{4.1}$$

the processes of first hitting times for B. Let $\{L(t, x);\ t \in \mathbb{R}_+,\ x \in \mathbb{R}\}$ be the local time processes of B. By the occupation time formula, for $v > 0$ and $t > 0$,

$$
\begin{aligned}
\{\overline{\mathbb{X}}(t) > v\} &= \left\{\int_0^{\varrho_{\mathbb{A}(v)}} \exp\left[-2\mathbb{W}\left(\mathbb{A}^{-1}(B(s))\right)\right] ds < t\right\} \\
&= \left\{\int_{-\infty}^{\mathbb{A}(v)} \exp\left(-2\mathbb{W}\left(\mathbb{A}^{-1}(y)\right)\right) L(\varrho_{\mathbb{A}(v)}, y)\, dy < t\right\} \\
&= \left\{\int_{-\infty}^{v} \exp(-\mathbb{W}(z))\, L\left(\varrho_{\mathbb{A}(v)}, \mathbb{A}(z)\right) dz < t\right\},
\end{aligned}
\tag{4.2}
$$

using a change of variable $y = \mathbb{A}(z)$. Observe that (4.2) is an ω-by-ω identity.

Accordingly, by writing

$$
I_1(v) = \int_0^{v} e^{-\mathbb{W}(s)} L\left(\varrho_{\mathbb{A}(v)}, \mathbb{A}(s)\right) ds \quad \text{as defined,} \tag{4.3}
$$

$$
I_2(v) = \int_0^{\infty} e^{-\mathbb{W}(-s)} L\left(\varrho_{\mathbb{A}(v)}, \mathbb{A}(-s)\right) ds \quad \text{as defined} \tag{4.4}
$$

we have

$$
\{\overline{\mathbb{X}}(t) > v\} = \{I_1(v) + I_2(v) < t\}, \qquad v > 0, t > 0. \tag{4.5}
$$

For brevity, we shall write, throughout the paper,

$$
\begin{aligned}
U_-(x) &= -\underline{W}(-H_-(x)) + x \qquad \text{as defined} \\
&= \sup_{0 \le s \le \inf\{u > 0:\, W(-u) > x\}} (-W(-s)) + x, \qquad x > 0,
\end{aligned}
\tag{4.6}
$$

where W is the Brownian motion in (3.1). We present some technical bounds for $I_1(v)$ and $I_2(v)$. They will play important roles in the proofs of Propositions 3.1–3.3 in Section 5.

LEMMA 4.1. *Assume* (3.1). *There exists $C_9 > 0$ such that for all sufficiently large v, we can find measurable events $E_7 = E_7(v)$ and $E_8 = E_8(v)$, satisfying*

$$\log I_1(v) \le \sigma W^{\#}(v) + \log^4 v \quad \text{on } E_7, \tag{4.7}$$

$$\log I_1(v) \ge \sigma W^{\#}(v) - \log^4 v \quad \text{on } E_8, \tag{4.8}$$

$$\mathbb{P}(E_7^c) \le C_9 \exp\left(-\log^2 v\right), \tag{4.9}$$

$$\mathbb{P}(E_8^c) \le C_9 \exp\left(-\log^2 v\right). \tag{4.10}$$

LEMMA 4.2. *Assume* (3.1). *For a large constant $C_{10} > 0$ and all $v > v_0 = v_0(C_{10})$, there exists a measurable event $E_9 = E_9(v)$ such that,*

$$\log I_2(v) \le \sigma U_-\left(\overline{W}(v) + \log^4 v\right) \quad \text{on } E_9, \tag{4.11}$$

$$\log I_2(v) \ge \sigma U_-\left(\overline{W}(v) - \log^4 v\right) \quad \text{on } E_9 \cap \left\{\overline{W}(v) \ge 2\log^4 v\right\}, \tag{4.12}$$

$$\mathbb{P}(E_9^c) \le C_{10} \exp\left(-\log^2 v\right). \tag{4.13}$$

The proofs of Lemmas 4.1 and 4.2 are presented in Sections 6 and 7, respectively.

5. Proofs of Propositions 3.1–3.3. By admitting Lemmas 4.1 and 4.2 for the moment, we prove Propositions 3.1−3.3 in this section.

Some preliminaries first. For $t > 0$, define

$$\eta_W(t) = \inf\{0 \le s \le t: W(s) = \overline{W}(t)\}, \tag{5.1}$$

the (first) location of the maximum of W over $[0, t]$. Write, for $0 \le x \le t$,

$$\omega_W(x, t) = \sup_{0 \le r \le s \le t;\ s-r<x} |W(s) - W(r)| \quad \text{as defined,} \tag{5.2}$$

the oscillation modulus of W over $[0, t]$. Here is a collection of results we shall make use of

LEMMA 5.1. *For $t > 0$, $x > 0$, $0 < a < t$ and $0 < b < \sqrt{t}$,*

$$\mathbb{P}(\omega_W(a, t) > b) \le C_{11} \frac{t}{a} \exp\left(-\frac{b^2}{3a}\right), \tag{5.3}$$

$$\mathbb{P}(W^{\#}(t) < x) \le 2 \exp\left(-\frac{\pi^2 t}{8x^2}\right), \tag{5.4}$$

$$\mathbb{P}(W^{\#}(t) > x) \le \frac{2\sqrt{t}}{x} \exp\left(-\frac{x^2}{2t}\right), \tag{5.5}$$

$$\mathbb{P}(\eta_W(t) > t - a) \le \sqrt{\frac{a}{t}}\,. \tag{5.6}$$

PROOF. The first inequality is borrowed from a well-known estimate of the Brownian oscillation, and actually more is true (cf. [7], page 24). To see why (5.4) holds, it suffices to observe that according to Lévy's identity, $W - \underline{W}$ is reflecting Brownian motion. Thus the processes $W^{\#}$ and W^* have the same distribution. Now (5.4) is a straightforward consequence of the exact distribution of $W^*(1)$ evaluated by Chung ([5], page 221), and (5.5) follows from the usual estimate of Gaussian tails. Finally, by Lévy's classical arcsine law,

$$\mathbb{P}(\eta_W(1) > 1 - a) = \frac{2}{\pi} \arcsin\sqrt{a}\,,$$

which implies (5.6), using the relation that $\arcsin(y) \le \pi y/2$ for $0 \le y \le 1$. $\square$

Define

$$C_{12} = \frac{3}{C_3} + 4 \quad \text{as defined,}$$

where C_3 is the constant in (3.1). For large s, consider the event

$$\Omega(s) = \left\{ \sup_{|r| \le \exp(C_{12} \log^2 s)} |\mathbb{W}(r) - \sigma W(r)| \le \log^3 s \right\} \quad \text{as defined.} \tag{5.7}$$

By (3.1),

$$\mathbb{P}(\Omega^c(s)) \le C_2 \exp(-3\log^2 s) \le \exp(-2\log^2 s), \tag{5.8}$$

for all sufficiently large s.

PROOF OF PROPOSITION 3.1. Recalling (4.3)–(4.5), for all $t > 0$ and large $v > 0$,

$$\begin{aligned}\mathbb{P}(\overline{\mathbb{X}}(t) > v) &\le \mathbb{P}(I_1(v) < t)\\ &\le C_9 \exp(-\log^2 v) + \mathbb{P}(\sigma W^{\#}(v) < \log t + \log^4 v),\end{aligned}$$

by means of (4.8) and (4.10). Taking $v = \lambda(\log t)^2$ as defined and using (5.4), this yields the upper bound (3.8) in Proposition 3.1.

To prove the lower bound, define $E_4 = E_7 \cap E_9$ as defined, where E_7 and E_9 are as in Lemmas 4.1 and 4.2, with $v = \lambda(\log t)^2$ as defined. It is easily seen from (4.9) and (4.13) that

$$\mathbb{P}(E_4^c) \le \mathbb{P}(E_7^c) + \mathbb{P}(E_9^c) \le C_{13} \exp(-\log^2 v) \le C_{14} \exp(-\lambda^2),$$

the last inequality following from (3.7). This yields (3.11). It remains to verify (3.9), with E_3 defined in (3.10). By (4.7) and (4.11), on E_4, $I_1(v) + I_2(v)$ is smaller than

$$\exp[\sigma W^{\#}(v) + \log^4 v] + \exp[\sigma U_-(\overline{W}(v) + \log^4 v)],$$

where U_- is defined via (4.6).

On E_3, we have $\overline{W}(v) + \log^4 v \le (\log t)/4\sigma$ and $H_-((\log t)/4\sigma) < H_-(-(\log t)/4\sigma)$, hence $U_-(\overline{W}(v) + \log^4 v) \le U_-((\log t)/4\sigma) \le (\log t)/2\sigma$, whereas $\sigma W^{\#}(v) \le (1 - 3/\lambda)\log t$. Accordingly, on $E_3 \cap E_4$,

$$\begin{aligned}I_1(v) + I_2(v) &\le \exp\left[\left(1 - \frac{3}{\lambda}\right)\log t + \log^4 v\right] + \exp\left[\frac{\log t}{2}\right]\\ &\le \exp\left[\left(1 - \frac{2}{\lambda}\right)\log t\right] + \sqrt{t}\\ &< t,\end{aligned}$$

which, in view of (4.5), yields (3.9). □

PROOF OF PROPOSITION 3.3. Take $v = (\log t)^2/\lambda$ as defined this time and assume (3.15). Use (4.5) and Lemma 4.2 to see that

$$\begin{aligned}\{\overline{X}(t) \le v\} &\supseteq \{\log I_2(v) \ge \log t\}\\ &\supseteq \{\sigma U_-(\overline{W}(v) - \log^4 v) \ge \log t; \overline{W}(v) \ge 2\sqrt{v}\} \cap E_9\\ &\supseteq \left\{H_-\left(\frac{\log t}{\sqrt{\lambda}}\right) > H_-\left(-\frac{\log t}{\sigma}\right); \overline{W}\left(\frac{\log^2 t}{\lambda}\right) \ge \frac{2\log t}{\sqrt{\lambda}}\right\} \cap E_9,\end{aligned}$$

which implies (3.17) with E_9 in place of E_6.

For the upper bound, use once more Lemmas 4.1 and 4.2 and (4.5) to arrive at the following estimate:

$$\begin{aligned}
\mathbb{P}\big(\overline{\mathbb{X}}(t) \le v\big) &\le \mathbb{P}\left(I_1(v) \ge \frac{t}{2}\right) + \mathbb{P}\left(I_2(v) \ge \frac{t}{2}\right) \\
&\le \mathbb{P}(E_7^c \cup E_9^c) + \mathbb{P}\left(\sigma W^{\#}(v) + \log^4 v \ge \log\frac{t}{2}\right) \\
&\quad + \mathbb{P}\left(\sigma U_-\big(\overline{W}(v) + \log^4 v\big) \ge \log\frac{t}{2}\right) \\
&\le C_{15}\exp(-\log^2 v) + \mathbb{P}\left(W^{\#}(1) \ge \frac{\sqrt{\lambda}}{\sigma} - \frac{\log 2 + \log^4 v}{\sigma\sqrt{v}}\right) \\
&\quad + \mathbb{P}\left[U_-\left(\overline{W}(1) + \frac{\log^4 v}{\sqrt{v}}\right) \ge \frac{\sqrt{\lambda}}{\sigma} - \frac{\log 2}{\sigma\sqrt{v}}\right],
\end{aligned}$$

by the scaling property. Since $\mathbb{P}(U_-(r) \ge a) = r/a$ for all $a \ge r > 0$, the above is, by (5.5) and (3.15), less than or equal to

$$C_{15}\exp(-\log^2 v) + C_{16}\exp\left(-\frac{\lambda}{2\sigma^2}\right) + \mathbb{E}\frac{\overline{W}(1) + (\log^4 v)/\sqrt{v}}{\sqrt{\lambda}/\sigma - (\log 2)/\sigma\sqrt{v}} \le \frac{C_{17}}{\sqrt{\lambda}},$$

yielding (3.16). □

The proof of Proposition 3.2 is slightly more technical and needs some preliminaries.

LEMMA 5.2. *Let ϱ be as in* (4.1). *For $a > 0$, $b > 0$ and $t > 0$,*

$$\begin{aligned}
&\frac{1}{dt}\mathbb{P}(\varrho_{-b} < \varrho_a; \varrho_a \in dt) \\
&\qquad = \sum_{k=-\infty,\, k\neq 0}^{\infty} \frac{2k(a+b) - a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2k(a+b)-a)^2}{2t}\right).
\end{aligned}$$

Consequently, for all $a > 0$ and $b > 0$ such that $a + b \ge 1$,

$$\begin{aligned}
\frac{C_{18}}{a+2b}\exp\left(-\frac{(a+2b)^2}{2}\right) &\le \mathbb{P}(\varrho_a < 1; \varrho_{-b} < \varrho_a) \\
&\le \frac{C_{19}}{a+2b}\exp\left(-\frac{(a+2b)^2}{2}\right),
\end{aligned} \tag{5.9}$$

where $C_{18} > 0$ and $C_{19} > 0$ are absolute constants.

The proof follows from formula 2.1.4(1) in [1], together with the symmetry and density function of ϱ_a.

LEMMA 5.3. *Let U_- be the process defined in* (4.6). *For all* $0 < r < s < y$ *and* $y \geq 1$,

$$\mathbb{P}\big(U_-(s) > y; \overline{W}(-1) > r\big) \leq \frac{C_{19}}{y} \exp\left(-\frac{(2y-s)^2}{2}\right) + (s-r). \tag{5.10}$$

PROOF. The probability term on the left-hand side of (5.10) is less than or equal to

$$\begin{aligned}
&\mathbb{P}\big(U_-(s) > y; \overline{W}(-1) > s\big) + \mathbb{P}\big(r < \overline{W}(-1) \leq s\big) \\
&\quad = \mathbb{P}\big(H_-(s) < 1; H_-(-(y-s)) < H_-(s)\big) + \mathbb{P}\big(r < \overline{W}(-1) \leq s\big),
\end{aligned}$$

which yields Lemma 5.3 upon using (5.9) and the fact that the density of $\overline{W}(-1)$ is uniformly bounded by 1. □

LEMMA 5.4. *Let $\mathbb{A}$ be as in* (3.3). *For sufficiently large* v,

$$\begin{aligned}
\mathbb{P}\big(E(v); \varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)}\big) &\leq \mathbb{P}\big(E(v); \overline{W}(-v) > \overline{W}(v) - \log^4 v\big) \\
&\quad + C_{20} \exp\big(-\log^2 v\big),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\mathbb{P}\big(E(v); \varrho_{\mathbb{A}(-v)} < \varrho_{\mathbb{A}(v)}\big) &\leq \mathbb{P}\big(E(v); \overline{W}(v) > \overline{W}(-v) - \log^4 v\big) \\
&\quad + C_{21} \exp\big(-\log^2 v\big),
\end{aligned} \tag{5.12}$$

for any event $E(v)$ (depending on v) which is measurable with respect to $\{W(x), \mathbb{W}(x);\ x \in \mathbb{R}\}$.

PROOF. Since two inequalities bear the same nature, we only prove (5.11). Write I_3 for the probability term on the left-hand side of (5.11). Since ϱ is the process of first hitting times for B [which is independent of $(\mathbb{W}, W)$], by conditioning on $(\mathbb{W}, W)$,

$$I_3 = \mathbb{E}\left[\frac{|\mathbb{A}(-v)|}{\mathbb{A}(v) + |\mathbb{A}(-v)|} \mathbb{1}_{E(v)}\right].$$

Let $\omega_W(\cdot,\cdot)$ and $\eta_W(\cdot)$ be, respectively, as in (5.2) and (5.1). Let $\delta = \exp(-\log^3 v)$ as defined. Define [recalling (5.7)],

$$\begin{aligned}
E_{10} &= \big\{\omega_W(\delta v, v) < \log^3 v; \eta_W(v) < (1-\delta)v\big\} \cap \Omega(v) \quad \text{as defined}, \\
F(v) &= \left\{\overline{W}(-v) \leq \overline{W}(v) - \frac{\sigma+4}{\sigma} \log^3 v\right\} \quad \text{as defined}.
\end{aligned}$$

By (5.3), (5.6) and (5.8), for large v, $\mathbb{P}(E_{10}^c) \le C_{22} \exp(-\log^2 v)$. On the other hand, on E_{10},

$$\begin{aligned}\mathbb{A}(v) &= \int_0^v e^{\mathbb{W}(s)}\,ds \ge \int_0^v \exp\bigl(\sigma W(s) - \log^3 v\bigr)\,ds\\ &\ge \int_{\eta_W(v)}^{\eta_W(v)+\delta v} \exp\bigl(\sigma W(s) - \log^3 v\bigr)\,ds\\ &\ge \delta v \exp\bigl(\sigma \overline{W}(v) - \sigma\omega_W(\delta v, v) - \log^3 v\bigr)\\ &\ge v \exp\bigl(\sigma \overline{W}(v) - (\sigma+2)\log^3 v\bigr).\end{aligned}$$

Therefore,

$$\begin{aligned}I_3 \le{}& \mathbb{P}(E(v) \cap F^c(v)) + C_{22}\exp(-\log^2 v)\\ &+ \mathbb{E}\left[\frac{|\mathbb{A}(-v)|}{v\exp\bigl(\sigma\overline{W}(v) - (\sigma+2)\log^3 v\bigr)} \mathbb{1}_{E_{10}\cap F(v)}\right].\end{aligned}$$

Since $|\mathbb{A}(-v)| \le v\exp(\sigma\overline{W}(-v) + \log^3 v)$ on $\Omega(v)$, this yields

$$\begin{aligned}I_3 &\le \mathbb{P}(E(v) \cap F^c(v)) + C_{22}\exp(-\log^2 v) + \exp(-\log^3 v)\\ &\le \mathbb{P}\bigl(E(v); \overline{W}(-v) > \overline{W}(v) - \log^4 v\bigr) + C_{23}\exp(-\log^2 v),\end{aligned}$$

as desired. □

LEMMA 5.5. *There exist universal constants $C_{24} > 0$ and $C_{25} > 0$ such that for all $x \ge 25$, $|a| \le 1/x$ and $|b| \le 1/x$,*

$$\frac{C_{24}}{x}\exp(-x^2) \le \mathbb{P}\bigl(W^{\#}(1) > x - a;\ x > \overline{W}(-1) > \overline{W}(1) - b\bigr) \tag{5.13}$$

$$\le \frac{C_{25}}{x}\exp(-x^2). \tag{5.14}$$

PROOF. Let I_4 denote the probability term in question. We begin with the proof of (5.14). Clearly, we can assume $a \ge 0$ and $b \ge 0$ without loss of generality. In this case,

$$\begin{aligned}I_4 \le{}& \mathbb{P}\left(W^{\#}(1) > x - a;\ x > \overline{W}(-1) > \overline{W}(1) - b;\ \overline{W}(1) \ge \frac{x}{2}\right)\\ &+ \mathbb{P}\left(W^{\#}(1) > x - a;\ \overline{W}(1) < \frac{x}{2}\right).\end{aligned}$$

The second probability term on the right-hand side is, by (2.6), less than or equal to $(C_{26}/x)\exp(-x^2)$ (we have used the fact that $3x/2 \ge 2a + \sqrt{2x}$), whereas the first term is, by independence, (2.6) and (5.5), less than or equal to

$$\begin{aligned}
&\int_{x/2-b}^{x} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{u^2}{2}\right) \mathbb{P}\big(W^{\#}(1) > x-a;\, \overline{W}(1) < u+b\big)\,du \\
&\quad \le \int_{x-a-b}^{x} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{u^2}{2}\right) \mathbb{P}\big(W^{\#}(1) > x-a\big)\,du \\
&\qquad + \int_{x/2-b}^{x-a-b} \frac{C_{27}}{x} \exp\left(-\frac{u^2}{2} - \frac{(2(x-a)-(u+b))^2}{2}\right) du \\
&\quad \le \frac{C_{28}}{x} \exp\left(-\frac{(x-a-b)^2}{2} - \frac{(x-a)^2}{2}\right) \\
&\qquad + \frac{C_{27}}{x} \exp\left(-\left(x-a-\frac{b}{2}\right)^2\right) \int_{-(x-2a+b)/2}^{-b/2} \exp(-z^2)\,dz \\
&\quad \le \frac{C_{29}}{x} \exp(-x^2),
\end{aligned}$$

which yields (5.14).

To prove (5.13), we assume without loss of generality that $a \le 0$ and $b \le 0$. Now by (2.6),

$$\begin{aligned}
I_4 &\ge \int_{x/2-b}^{x} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{u^2}{2}\right) \mathbb{P}\big(W^{\#}(1) > x-a;\, \overline{W}(1) < u+b\big)\,du \\
&\ge \frac{C_{30}}{x} \exp(-x^2).
\end{aligned}$$

Lemma 5.5 is proved. □

We now prove Proposition 3.2 in two steps, showing first the upper bound in (3.12), then the lower bound in (3.13).

PROOF OF PROPOSITION 3.2 [Upper bound (3.12)]. As in (4.2) for the one-sided case, for all $t > 0$ and $v > 0$,

$$\{\mathbb{X}^*(t) \le v\} = \left\{ \int_{-v}^{v} e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(v)} \wedge \varrho_{\mathbb{A}(-v)}, \mathbb{A}(z)\big)\,dz \ge t \right\}. \tag{5.15}$$

Let $v = (\log t)^2/\lambda$ as defined, with (λ, t) satisfying (3.7). Consider

$$\begin{aligned}
&\mathbb{P}\left(\int_0^v e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(v)} \wedge \varrho_{\mathbb{A}(-v)}, \mathbb{A}(z)\big)\,dz \ge \frac{t}{2} \right) \\
&\qquad = \mathbb{P}\left(\int_0^v e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(v)}, \mathbb{A}(z)\big)\,dz \ge \frac{t}{2};\, \varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)} \right) \\
&\qquad\quad + \mathbb{P}\left(\int_0^v e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(-v)}, \mathbb{A}(z)\big)\,dz \ge \frac{t}{2};\, \varrho_{\mathbb{A}(v)} > \varrho_{\mathbb{A}(-v)} \right) \\
&\qquad = I_5 + I_6 \quad \text{as defined.}
\end{aligned} \tag{5.16}$$

with obvious notation. Let us bound above I_5 and I_6. In what follows, it is to be understood that we work on sufficiently large t, though this is not always indicated.

Recall the notation for $I_1(v)$ from (4.3). By (4.7) and (4.9),

$$I_5 = \mathbb{P}\left(I_1(v) \geq \frac{t}{2}; \varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)} \right) \\ \leq \mathbb{P}\left(\sigma W^{\#}(v) \geq \log\frac{t}{2} - \log^4 v; \varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)} \right) + C_9 \exp\bigl(-\log^2 v\bigr).$$

Applying Lemma 5.4 to $E(v) = \{W^{\#}(v) > \log(\frac{t}{2}) - \log^4 v\}$ as defined yields

$$I_5 \leq \mathbb{P}\left(\sigma W^{\#}(v) \geq \log\frac{t}{2} - \log^4 v; \overline{W}(-v) > \overline{W}(v) - \log^4 v \right) \\ + C_{31} \exp\bigl(-\log^2 v\bigr) \\ = \mathbb{P}\left(W^{\#}(1) \geq \frac{\sqrt{\lambda}}{\sigma} - \frac{\log^4 v + \log 2}{\sigma\sqrt{v}}; \overline{W}(-1) > \overline{W}(1) - \frac{\log^4 v}{\sqrt{v}} \right) \\ + C_{31} \exp\bigl(-\log^2 v\bigr).$$

Since $4 \leq \lambda \leq (\log\log t)^{1/2}$, $v \geq (\log t)^2/(\log\log t)^{1/2}$ and $\log 2 < \log^4 v$,

$$I_5 \leq C_{31} \exp\bigl(-\log^2 v\bigr) + \mathbb{P}\left(W^{\#}(1) > \frac{\sqrt{\lambda}}{\sigma} - \frac{2\log^4 v}{\sigma\sqrt{v}}; \overline{W}(-1) \geq \frac{\sqrt{\lambda}}{\sigma} \right) \\ + \mathbb{P}\left(W^{\#}(1) > \frac{\sqrt{\lambda}}{\sigma} - \frac{2\log^4 v}{\sigma\sqrt{v}}; \frac{\sqrt{\lambda}}{\sigma} > \overline{W}(-1) > \overline{W}(1) - \frac{\log^4 v}{\sqrt{v}} \right).$$

On the right-hand side, the first probability term equals, by independence, $\mathbb{P}(W^{\#}(1) > \sqrt{\lambda}/\sigma - 2(\log^4 v)/\sigma\sqrt{v})\mathbb{P}(\overline{W}(-1) \geq \sqrt{\lambda}/\sigma)$, which, according to (5.5) and the Gaussian tail estimate, is smaller than $(C_{32}/\sqrt{\lambda})\exp(-\lambda/\sigma^2)$, whereas the second probability term is bounded above by $(C_{33}/\sqrt{\lambda})\exp(-\lambda/\sigma^2)$ by applying Lemma 5.5 to $x = \sqrt{\lambda}/\sigma$, $a = 2(\log^4 v)/\sigma\sqrt{v}$ and $b = (\log^4 v)/\sqrt{v}$. Consequently, for all (t, λ) satisfying (3.7) and $v = (\log t)^2/\lambda$,

$$I_5 \leq \frac{C_{34}}{\sqrt{\lambda}} \exp\left(-\frac{\lambda}{\sigma^2}\right). \tag{5.17}$$

We now estimate the term I_6 in (5.16), for sufficiently large t. Let $\tilde{\mathbb{W}}(x) = \mathbb{W}(-x)$ as defined, $\tilde{W}(x) = W(-x)$ as defined, for $x \in \mathbb{R}$, and $\tilde{B}(t) = -B(t)$ as defined, for $t \geq 0$. Define $(\tilde{L}, \tilde{\varrho})$, $\tilde{\mathbb{A}}$ and $\tilde{U}_-$, which relate to $\tilde{B}$, $\tilde{\mathbb{W}}$ and $\tilde{W}$ exactly in the same ways (L, ϱ), $\mathbb{A}$ and U_- do to B, $\mathbb{W}$ and W. Then

$$I_6 \leq \mathbb{P}\left(\int_0^\infty e^{-\mathbb{W}(z)} L\bigl(\varrho_{\mathbb{A}(-v)}, \mathbb{A}(z)\bigr)\, dz \geq \frac{t}{2}; \varrho_{\mathbb{A}(v)} > \varrho_{\mathbb{A}(-v)} \right) \\ = \mathbb{P}\left(\int_0^\infty e^{-\tilde{\mathbb{W}}(-s)} \tilde{L}\bigl(\tilde{\varrho}_{\tilde{\mathbb{A}}(v)}, \tilde{\mathbb{A}}(-s)\bigr)\, ds \geq \frac{t}{2}; \tilde{\varrho}_{\tilde{\mathbb{A}}(v)} < \tilde{\varrho}_{\tilde{\mathbb{A}}(-v)} \right).$$

Since the coupling (3.1) holds for $(\tilde{\mathbb{W}}, \tilde{W})$ in place of $(\mathbb{W}, W)$, we can apply Lemma 4.2 to $(\tilde{\mathbb{W}}, \tilde{W})$ to arrive at

$$I_6 \le \mathbb{P}\Big(\sigma \tilde{U}_-\Big(\overline{\tilde{W}}(v) + \log^4 v\Big) \ge \log\frac{t}{2};\, \tilde{\varrho}_{\tilde{\mathbb{A}}(v)} < \tilde{\varrho}_{\tilde{\mathbb{A}}(-v)}\Big) + C_{10}\exp\big(-\log^2 v\big).$$

Applying Lemma 5.4 to $(\tilde{\mathbb{W}}, \tilde{W}, \tilde{\varrho})$ and $E(v) = \{\sigma \tilde{U}_-(\overline{\tilde{W}}(v) + \log^4 v) \ge \log(t/2)\}$ gives

$$\begin{aligned} I_6 &\le \mathbb{P}\Big(\sigma \tilde{U}_-\Big(\overline{\tilde{W}}(v) + \log^4 v\Big) \ge \log\frac{t}{2};\, \overline{\tilde{W}}(-v) > \overline{\tilde{W}}(v) - \log^4 v\Big) \\ &\qquad + C_{35}\exp\big(-\log^2 v\big) \\ &= \mathbb{P}\Big(\sigma U_-\big(\overline{W}(v) + \log^4 v\big) \ge \log\frac{t}{2};\, \overline{W}(-v) > \overline{W}(v) - \log^4 v\Big) \\ &\qquad + C_{35}\exp\big(-\log^2 v\big) \\ &\le \mathbb{P}\bigg(U_-\bigg(\overline{W}(1) + \frac{\log^4 v}{\sqrt{v}}\bigg) > \frac{\sqrt{\lambda}}{\sigma} - \frac{\log^4 v}{\sigma\sqrt{v}};\, \overline{W}(-1) > \overline{W}(1) - \frac{\log^4 v}{\sqrt{v}}\bigg) \\ &\qquad + C_{35}\exp\big(-\log^2 v\big) \end{aligned}$$

[recalling $\lambda = (\log t)^2/v$ as defined]. Write $\tau = \overline{W}(1) + (\log^4 v)/\sqrt{v}$ as defined and $E_{11} = \{2(\log^4 v)/\sqrt{v} < \tau < \sqrt{\lambda}/\sigma - (\log^4 v)/\sigma\sqrt{v}\}$ as defined for brevity. We have

$$\begin{aligned} I_6 &\le \mathbb{P}\bigg(U_-(\tau) > \frac{\sqrt{\lambda}}{\sigma} - \frac{\log^4 v}{\sigma\sqrt{v}};\, \overline{W}(-1) > \tau - \frac{2\log^4 v}{\sqrt{v}};\, E_{11}\bigg) \\ &\quad + \mathbb{P}\bigg(\overline{W}(1) \ge \frac{\sqrt{\lambda}}{\sigma} - \frac{(\sigma+1)\log^4 v}{\sigma\sqrt{v}}; \\ &\qquad \overline{W}(-1) > \frac{\sqrt{\lambda}}{\sigma} - \frac{(2\sigma+1)\log^4 v}{\sigma\sqrt{v}}\bigg) \\ &\quad + \mathbb{P}\bigg(\overline{W}(1) \le \frac{\log^4 v}{\sqrt{v}}\bigg) + C_{35}\exp\big(-\log^2 v\big). \end{aligned} \tag{5.18}$$

Consider the first probability term on the right-hand side, denoted by I_7, say. Since $\{U_-(x);\ x \in \mathbb{R}\}$ and $\overline{W}(-1)$ are independent of τ, by conditioning on τ, applying Lemma 5.3 to $s = \tau$, $y = \sqrt{\lambda}/\sigma - (\log^4 v)/\sigma\sqrt{v}$ and $r = \tau - 2(\log^4 v)/\sqrt{v}$, recalling that $4 \le \lambda \le (\log\log t)^{1/2}$ and $v \ge (\log t)^2/\log\log t)^{1/2}$,

$$\begin{aligned} I_7 &\le \frac{C_{36}}{\sqrt{\lambda}}\mathbb{E}\bigg[\exp\bigg(-\frac{(2\sqrt{\lambda} - \sigma\tau)^2}{2\sigma^2}\bigg)\mathbb{1}_{E_{11}}\bigg] + \frac{2\log^4 v}{\sqrt{v}} \\ &\le \frac{C_{37}}{\sqrt{\lambda}}\mathbb{E}\exp\bigg(-\frac{(2\sqrt{\lambda} - \sigma\overline{W}(1))^2}{2\sigma^2}\bigg) + \frac{1}{\sqrt{\lambda}}\exp\bigg(-\frac{\lambda}{\sigma^2}\bigg). \end{aligned}$$

Since $\overline{W}(1)$ is distributed as the modulus of a Gaussian $N(0,1)$ variable, it is easily checked that $I_7 \le (C_{38}/\sqrt{\lambda})\exp(-\lambda/\sigma^2)$. Going back to (5.18), we have bounded above the first probability term on the right-hand side. The third probability term presents no problem, since the density function of $\overline{W}(1)$ is smaller than 1. For the second, let us note that by the independence of $\overline{W}(1)$ and $\overline{W}(-1)$, and a well-known Gaussian tail property, it is again smaller than $(C_{39}/\lambda)\exp(-\lambda/\sigma^2)$. Summarizing the situation, we have proved that, for $v=(\log t)^2/\lambda$,

$$I_6 \le \frac{C_{40}}{\sqrt{\lambda}}\exp\left(-\frac{\lambda}{\sigma^2}\right). \tag{5.19}$$

Combining (5.16), (5.17) and (5.19) yields

$$\mathbb{P}\left(\int_0^v e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(v)}\wedge\varrho_{\mathbb{A}(-v)},\mathbb{A}(z)\big)\,dz \ge \frac{t}{2}\right) \le \frac{C_{41}}{\sqrt{\lambda}}\exp\left(-\frac{\lambda}{\sigma^2}\right)$$

[with $v=(\log t)^2/\lambda$, of course]. Similarly, we have

$$\mathbb{P}\left(\int_{-v}^0 e^{-\mathbb{W}(z)} L\big(\varrho_{\mathbb{A}(v)}\wedge\varrho_{\mathbb{A}(-v)},\mathbb{A}(z)\big)\,dz \ge \frac{t}{2}\right) \le \frac{C_{42}}{\sqrt{\lambda}}\exp\left(-\frac{\lambda}{\sigma^2}\right).$$

In view of (5.15), we have proved the desired conclusion in (3.12). □

PROOF OF PROPOSITION 3.2 [Lower bound (3.13)]. Pick large numbers $t>0$ and $v>0$. By (5.15) and the definition of $I_1(v)$,

$$\{\mathbb{X}^*(t)\le v\} \supseteq \left\{\varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)};\ I_1(v)\ge t\right\}.$$

Using Lemma 4.1 gives

$$\{\mathbb{X}^*(t)\le v\} \supseteq \left\{\varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)};\ \sigma W^{\#}(v) > \log t + \log^4 v\right\} \cap E_8,$$

with $\mathbb{P}(E_8^c)\le C_9\exp(-\log^2 v)$. Define

$$E_{12} = \{\varrho_{\mathbb{A}(v)} > \varrho_{\mathbb{A}(-v)}\} \cap \left\{\overline{W}(-v) > \overline{W}(v) + \log^4 v\right\} \quad \text{as defined.}$$

Applying (5.12) to $E(v)=\{\overline{W}(-v)>\overline{W}(v)+\log^4 v\}$ yields $\mathbb{P}(E_{12})\le C_{21}\exp(-\log^2 v)$. Let $E_5=E_8\cap E_{12}^c$ as defined. Obviously, $\mathbb{P}(E_5^c)\le C_{43}\exp(-\log^2 v)$. Moreover,

$$\begin{aligned}\{\mathbb{X}^*(t)\le v\} &\supseteq \{\varrho_{\mathbb{A}(v)} < \varrho_{\mathbb{A}(-v)}\} \cap \left\{\sigma W^{\#}(v) > \log t + \log^4 v\right\} \cap E_8 \\ &\qquad \cap \left\{\overline{W}(-v) > \overline{W}(v) + \log^4 v\right\} \\ &= \left\{\overline{W}(-v) > \overline{W}(v) + \log^4 v\right\} \\ &\qquad \cap \left\{\sigma W^{\#}(v) > \log t + \log^4 v\right\} \cap E_5.\end{aligned}$$

This proves (3.13) by taking $v=(\log t)^2/\lambda$ as defined. □

6. Proof of Lemma 4.1. The proof of Lemma 4.1 is based on the following elementary estimates concerning two-dimensional Bessel processes. Recall that a Bessel process of dimension 2 can be realized as the Euclidean modulus of Brownian motion in dimension 2.

LEMMA 6.1. *Let* $\{R(t);\ t \geq 0\}$ *be a Bessel process of dimension* 2, *starting from* 0. *For all* $0 < a < b$ *and* $x > 0$,

$$\mathbb{P}\Big(\inf_{a \leq t \leq b} R(t) \leq x\sqrt{b}\Big) \leq 2x + 2\exp\left(-\frac{x^2}{2(1 - a/b)}\right), \tag{6.1}$$

$$\mathbb{P}\left(\sup_{0 < t \leq 1} \frac{R^*(t)}{\sqrt{t \log(8/t)}} > x\right) \leq C_{44} \exp\left(-\frac{x^2}{2}\right), \tag{6.2}$$

where R^* *is as in* (2.3).

PROOF. To check (6.1), let us note that R is stochastically greater than the linear reflecting Brownian motion $|B|$. By scaling, the probability term on the left-hand side of (6.1) is less than or equal to

$$\begin{aligned}
&\mathbb{P}\Big(\inf_{a \leq t \leq b} |B(t)| \leq x\sqrt{b}\Big)\\
&\quad = \mathbb{P}\Big(\inf_{a/b \leq r \leq 1} |B(r)| \leq x\Big)\\
&\quad \leq P(|B(1)| \leq 2x) + \mathbb{P}\Big(\sup_{a/b \leq r \leq 1} |B(r) - B(1)| \geq x\Big)\\
&\quad = \mathbb{P}(|B(1)| \leq 2x) + \mathbb{P}\Big(\sup_{0 \leq t \leq 1 - a/b} |B(t)| \geq x\Big),
\end{aligned}$$

where in the last equality, we have used time reversal for Brownian motion. Now (6.1) follows from the form of Gaussian densities and Mill's ratio for Gaussian tails.

To verify (6.2), recall the well-known estimate

$$\mathbb{P}(R^*(t) > y) \leq C_{45} \exp(-y^2/2t)$$

(for all positive y and t), where C_{45} is an absolute constant. Hence, the expression on the left-hand side of (6.2) is less than or equal to

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{1/(n+1) \leq t \leq 1/n} \frac{R^*(t)}{\sqrt{t \log(8/t)}} > x\right)\\
&\qquad \leq \sum_{n=1}^{\infty} \mathbb{P}\left(R^*\left(\frac{1}{n}\right) > \frac{x\sqrt{\log(8n)}}{\sqrt{n+1}}\right)\\
&\qquad \leq C_{45} \sum_{n=1}^{\infty} \exp\left(-\frac{nx^2 \log(8n)}{2(n+1)}\right)\\
&\qquad \leq C_{45} \sum_{n=1}^{\infty} \exp\left(-\frac{x^2}{2} - \frac{x^2 \log n}{4}\right),
\end{aligned}$$

which implies (6.2) in case $x \geq 3$. But for $0 < x < 3$, (6.2) holds trivially, possibly with an enlarged value of C_{44}. □

The proof of Lemma 4.1 is split into two parts. We first prove the upper estimate in (4.7) with E_7 defined in (6.4) below and then the lower bound in (4.8) with an appropriate E_8.

PROOF OF LEMMA 4.1 [upper bound (4.7)]. Fix a large number v. Write as before $L(\cdot,\cdot)$ for the local time of B. According to the classical Ray–Knight theorem (cf. [28], Theorem VI.52.1), for any given $a>0$, $\{a^{-1}L(\varrho(a), a-ta); 0\le t\le 1\}$ is a squared Bessel process of dimension 2, starting from 0. By scaling and the independence of $(\mathbb{W}, W)$ and B, letting

$$R^2(t) = \frac{L\big(\varrho_{\mathbb{A}(v)}, \mathbb{A}(v) - t\mathbb{A}(v)\big)}{\mathbb{A}(v)}, \qquad 0\le t\le 1, \text{ as defined,} \tag{6.3}$$

$\{R(t); 0\le t\le 1\}$ is also a two-dimensional Bessel process with $R(0)=0$ (clearly R is to be taken as the positive square root of R^2), and is, moreover, independent of $(\mathbb{W}, W)$ [we shall need the independence later in the proof of (4.8)]. For brevity, write

$$l = \log v \quad \text{as defined,}$$

("l" for logarithm). Recall W^* and $\Omega(v)$ from (2.3) and (5.7), respectively, and define

$$\begin{aligned} E_7 = {} & \left\{ \sup_{0<t\le 1} \frac{R(t)}{\sqrt{t\log(8/t)}} \le v \right\} \\ & \cap \big\{W^*(v) \le \exp(l^2)\big\} \cap \Omega(v) \quad \text{as defined.} \end{aligned} \tag{6.4}$$

By means of (6.2), Mill's ratio for Gaussian tails and (5.8),

$$\mathbb{P}(E_7^c) \le C_{44}\exp\left(-\frac{v^2}{2}\right) + 4\exp\left(-\frac{\exp(2l^2)}{2v}\right) + \exp(-2l^2) \le \exp(-l^2).$$

This shows E_7 satisfies (4.9).

Now recall $I_1(v)$ from (4.3). On E_7,

$$\begin{aligned} I_1(v) &= \int_0^v \exp(-\mathbb{W}(s))\mathbb{A}(v)\,R^2\left(\frac{\mathbb{A}(v)-\mathbb{A}(s)}{\mathbb{A}(v)}\right) ds \\ &\le \int_0^v \exp(-\sigma W(s) + l^3)v^2(\mathbb{A}(v)-\mathbb{A}(s))\log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v)-\mathbb{A}(s)}\right) ds. \end{aligned}$$

Since on E_7, $\exp(-\sigma W(s))(\mathbb{A}(v)-\mathbb{A}(s)) \le \int_s^v \exp(\sigma W(y) - \sigma W(s) + l^3)\,dy \le v\exp(\sigma W^{\#}(v) + l^3)$ for all $0\le s\le v$, this yields that, on E_7,

$$I_1(v) \le v^3\exp(\sigma W^{\#}(v) + 2l^3)\int_0^v \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v)-\mathbb{A}(s)}\right) ds.$$

On E_7, for $0 \le s \le v$, $\mathbb{A}(v) - \mathbb{A}(s) \ge (v-s)\exp(\sigma \underline{W}(v) - l^3) \ge (v-s)\exp(-\sigma W^*(v) - l^3)$, whereas $\mathbb{A}(v) \le v\exp(\sigma W^*(v) + l^3)$. Hence, on E_7,

$$I_1(v) \le v^3 \exp\big(\sigma W^{\#}(v) + 2l^3\big) \int_0^v \log\left(\frac{8v\exp\big(2\sigma W^*(v) + 2l^3\big)}{v-s}\right) ds$$
$$= v^4 \exp\big(\sigma W^{\#}(v) + 2l^3\big)\big(1 + 2\sigma W^*(v) + 2l^3 + \log 8\big).$$

By definition, on E_7, $W^*(v) \le \exp(l^2)$, which yields $I_1(v) \le \exp(\sigma W^{\#}(v) + l^4)$. □

PROOF OF LEMMA 4.1 [Lower bound (4.8)]. Fix a large v, and write again $l = \log v$ as defined as before. Write $\delta = \exp(-l^2)$ as defined for brevity. From the definition of $W^{\#}$ [cf. (2.4)], for each $t > 0$, there exist a couple of random times $(\theta_W^{(-)}(t), \theta_W^{(+)}(t))$ such that $0 \le \theta_W^{(-)}(t) < \theta_W^{(+)}(t) \le t$ and that

$$W^{\#}(t) = W\big(\theta_W^{(+)}(t)\big) - W\big(\theta_W^{(-)}(t)\big).$$

Recall ω_W from (5.2). Let R be as in (6.3). Define

$$E_{13} = \big\{\omega_W(\delta v, v) \le l^3\big\} \cap \big\{W^{\#}(v) \ge l^4\big\} \quad \text{as defined,}$$
$$E_{14} = \left\{ \inf_{\theta_W^{(-)}(v) \le s \le \theta_W^{(-)}(v) + \delta v} R\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) \ge \delta \sqrt{\frac{\mathbb{A}(v) - \mathbb{A}\big(\theta_W^{(-)}(v)\big)}{\mathbb{A}(v)}} \right\} \quad \text{as defined,}$$
$$E_8 = E_{13} \cap E_{14} \cap \Omega(v) \quad \text{as defined.}$$

Note that, on E_{13},

$$v \ge \theta_W^{(+)}(v) \ge \theta_W^{(-)}(v) + \delta v, \tag{6.5}$$

which confirms $\theta_W^{(-)}(v) \le (1-\delta)v$. Hence $E_{13} \cap E_{14}$ is well defined. According to (5.3) and (5.4),

$$\mathbb{P}(E_{13}^c) \le \frac{C_{11}}{\delta}\exp\left(-\frac{l^6}{3\delta v}\right) + 2\exp\left(-\frac{\pi^2 v}{8 l^8}\right) \le \exp(-l^2). \tag{6.6}$$

On the other hand, since R is independent of $(\mathbb{W}, W)$, by conditioning on $(\mathbb{W}, W)$ and applying (6.1) to $a = (\mathbb{A}(v) - \mathbb{A}(\theta_W^{(-)}(v) + \delta v))/\mathbb{A}(v)$, $b = (\mathbb{A}(v) - \mathbb{A}(\theta_W^{(-)}(v)))/\mathbb{A}(v)$ and $x = \delta$,

$$\mathbb{P}\big(E_{14}^c \cap E_{13} \cap \Omega(v)\big) \le 2\delta + 2\mathbb{E}\left[\exp\left(-\frac{\delta^2}{2} I_8(v)\right) \mathbb{1}_{E_{13} \cap \Omega(v)}\right],$$

where

$$I_8(v) = \frac{\mathbb{A}(v) - \mathbb{A}\big(\theta_W^{(-)}(v)\big)}{\mathbb{A}\big(\theta_W^{(-)}(v) + \delta v\big) - \mathbb{A}\big(\theta_W^{(-)}(v)\big)} \quad \text{as defined.}$$

By (6.5), on $E_{13} \cap \Omega(v)$,

$$\begin{aligned} \mathbb{A}(v) - \mathbb{A}\big(\theta_W^{(-)}(v)\big) &= \int_{\theta_W^{(-)}(v)}^{v} e^{\mathbb{W}(s)}\, ds \\ &\geq \int_{\theta_W^{(+)}(v)-\delta v}^{\theta_W^{(+)}(v)} \exp\big(\sigma W(s) - l^3\big)\, ds \\ &\geq \delta v \exp\Big(\sigma\Big(W\big(\theta_W^{(+)}(v)\big) - \omega_W(\delta v, v)\Big) - l^3\Big), \end{aligned} \tag{6.7}$$

whereas

$$\begin{aligned} &\mathbb{A}\big(\theta_W^{(-)}(v) + \delta v\big) - \mathbb{A}\big(\theta_W^{(-)}(v)\big) \\ &\quad = \int_{\theta_W^{(-)}(v)}^{\theta_W^{(-)}(v)+\delta v} \exp(\mathbb{W}(s))\, ds \\ &\quad \leq \delta v \exp\Big(\sigma\Big(W\big(\theta_W^{(-)}(v)\big) + \omega_W(\delta v, v)\Big) + l^3\Big). \end{aligned}$$

Hence, on $E_{13} \cap \Omega(v)$, $I_8(v) \geq \exp(\sigma W^{\#}(v) - 2\sigma\omega_W(\delta v, v) - 2l^3) \geq \exp(l^3)$, which yields

$$\mathbb{P}\big(E_{14}^c \cap E_{13} \cap \Omega(v)\big) \leq C_{46} \exp(-l^2). \tag{6.8}$$

Since by (5.8), $\mathbb{P}(\Omega^c(v)) \leq \exp(-2l^2)$, combining (6.6) with (6.8) gives $\mathbb{P}(E_8^c) \leq C_{47} \exp(-l^2)$. This ensures that the event E_8 satisfies condition (4.10).

To verify (4.8), observe that on $E_{13} \cap \Omega(v)$,

$$\begin{aligned} I_1(v) &= \int_0^v \exp(-\mathbb{W}(s))\mathbb{A}(v) R^2\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds \\ &\geq \int_{\theta_W^{(-)}(v)}^{\theta_W^{(-)}(v)+\delta v} \exp\big(-\sigma W(s) - l^3\big)\mathbb{A}(v) R^2\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds \\ &\geq \delta v \exp\Big(-\sigma W\big(\theta_W^{(-)}(v)\big) - \sigma\omega_W(\delta v, v) - l^3\Big)\mathbb{A}(v) \\ &\qquad \times \inf_{\theta_W^{(-)}(v) \leq s \leq \theta_W^{(-)}(v)+\delta v} R^2\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right). \end{aligned}$$

Hence, with the aid of (6.7), on $E_8 = E_{13} \cap E_{14} \cap \Omega(v)$,

$$\begin{aligned} I_1(v) &\geq \delta^3 v \exp\Big(-\sigma W\big(\theta_W^{(-)}(v)\big) - \sigma\omega_W(\delta v, v) - l^3\Big)\Big(\mathbb{A}(v) - \mathbb{A}\big(\theta_W^{(-)}(v)\big)\Big) \\ &\geq \delta^4 v^2 \exp\big(\sigma W^{\#}(v) - 2\sigma\omega_W(\delta v, v) - 2l^3\big) \\ &\geq \exp\big(\sigma W^{\#}(v) - l^4\big), \end{aligned}$$

as desired. □

REMARK 6.2. Our argument also yields the following estimate: for any $0 < \varepsilon < 1$, there exists a constant $\tilde{C}_9$ (possibly depending on ε) satisfying that for all large v, we can find a measurable event $\tilde{E}_7 = \tilde{E}_7(v)$ with

$\mathbb{P}(\tilde{E}_7^c) \le \tilde{C}_9 \exp(-\log^2 v)$ such that on $\tilde{E}_7$,

$$\left|\log \sup_{0\le s\le(1-\varepsilon)v} e^{-\mathbb{W}(s)} L\big(\varrho_{\mathbb{A}(v)}, \mathbb{A}(s)\big) - \sigma \sup_{0\le s\le(1-\varepsilon)v,\, s\le t\le v} \big(W(t) - W(s)\big)\right|$$
$$\le \log^4 v, \left|\log \sup_{s\in\mathbb{Z}_+} e^{-\mathbb{W}(s)} L\big(\varrho_{\mathbb{A}(v)}, \mathbb{A}(s)\big) - \sigma W^{\#}(v)\right| \le \log^4 v.$$

7. Proof of Lemma 4.2. Throughout the section, v is a very large number. Let $L(\cdot,\cdot)$ be as before the local time of B, and define the process

$$Z(t) = \frac{L\big(\varrho_{\mathbb{A}(v)}, -t\mathbb{A}(v)\big)}{\mathbb{A}(v)}, \qquad t \ge 0 \text{ as defined},$$

which, by Brownian scaling, is independent of $(\mathbb{W}, W)$, and is distributed as $\{L(\varrho_1, -t);\ t \ge 0\}$. According to the Ray–Knight theorem (cf. [28], Theorem VI.52.1), Z is a squared Bessel process of dimension 0, such that $Z(0)$ has an exponential distribution of mean 2. In particular, 0 is an absorbing state for Z. Let

$$\zeta = \inf\{t > 0: Z(t) = 0\} \quad \text{as defined} \tag{7.1}$$

be the absorption time. The next is a collection of elementary facts about Z.

LEMMA 7.1. *For all positive t and x,*

$$\mathbb{P}(\zeta > t) = \frac{1}{1+t}, \tag{7.2}$$

$$\mathbb{P}\Big(\inf_{0\le s\le t} Z(s) < x\Big) \le x + \frac{8t}{x^2}, \tag{7.3}$$

$$\mathbb{P}\Big(\sup_{s\ge 0} Z(s) > x\Big) \le \frac{4}{x}. \tag{7.4}$$

PROOF. Since Z is distributed as $s \mapsto L(\varrho_1, -s)$,

$$\mathbb{P}(\zeta > t) = \mathbb{P}(\varrho_{-t} < \varrho_1) = \frac{1}{1+t},$$

which yields (7.2). The variable $Z(0)$ having an exponential law with mean 2, its density function is bounded by $1/2$. Applying Doob's inequality to the continuous martingale $\{Z(t) - Z(0);\ t \ge 0\}$ gives

$$\mathbb{P}\Big(\sup_{0\le s\le t} |Z(s) - Z(0)| > x\Big) \le \frac{1}{x^2}\mathbb{E}\big(Z(t) - Z(0)\big)^2.$$

To compute the expectation term on the right-hand side, note that according again to the Ray–Knight theorem (cf. [28], Theorem VI.52.1), $Z(\cdot) - Z(0)$ is a martingale whose associated increasing process equals $t \mapsto 4\int_0^t Z(s)\,ds$. Hence

$$\mathbb{E}(Z(t) - Z(0))^2 = 4\int_0^t \mathbb{E}(Z(s))\,ds = 4t\mathbb{E}Z(0) = 8t.$$

This shows $\mathbb{P}(\sup_{0\le s\le t}|Z(s) - Z(0)| > x) \le 8t/x^2$. As a consequence,

$$\mathbb{P}\Big(\inf_{0\le s\le t} Z(s) < x\Big) \le \mathbb{P}(Z(0) < 2x) + \mathbb{P}\Big(\sup_{0\le s\le t}|Z(s) - Z(0)| > x\Big)$$
$$\le x + \frac{8t}{x^2},$$

proving (7.3). To check (7.4), recall (cf. [33]) that for a nonnegative continuous local martingale M tending to 0, the conditional law of $\sup_{s\ge 0} M(s)$ given $M(0)$ is the same as $M(0)/V$, where V denotes a uniform $(0,1)$ variable independent of M. Accordingly,

$$\mathbb{P}\Big(\sup_{s\ge 0} Z(s) > x\Big) = \mathbb{P}(Z(0) \ge x) + \mathbb{E}\bigg(\frac{Z(0)}{x}\mathbb{1}_{\{Z(0)<x\}}\bigg) \le \exp\Big(-\frac{x}{2}\Big) + \frac{2}{x},$$

which implies (7.4). □

PROOF OF LEMMA 4.2. Let v be sufficiently large. Write as before $l = \log v$ as defined. Let $W_-(t) = W(-t)$ as defined and $\mathbb{W}_-(t) = \mathbb{W}(-t)$ as defined for $t \ge 0$. Hence $\{W_-(t);\ t \ge 0\}$ is a Brownian motion independent of $\{W(t);\ t\ge 0\}$ and $\{B(t);\ t \ge 0\}$. Let

$$\mathbb{A}_-(t) = \int_0^t \exp(\mathbb{W}_-(s))\,ds, \qquad t \ge 0 \text{ as defined},$$

which is to $\mathbb{W}_-$ as $\mathbb{A}$ to $\mathbb{W}$. Define

$$\zeta(v) = \inf\bigg\{s > 0: Z\bigg(\frac{\mathbb{A}_-(s)}{\mathbb{A}(v)}\bigg) = 0\bigg\} \quad \text{as defined.} \tag{7.5}$$

Let $I_2(v)$ be as in (4.4). By the definitions of Z and $\mathbb{A}_-$, it can be rewritten in a more convenient form:

$$I_2(v) = \mathbb{A}(v)\int_0^\infty \exp(-\mathbb{W}_-(s))Z\bigg(\frac{\mathbb{A}_-(s)}{\mathbb{A}(v)}\bigg)\,ds$$

$$= \mathbb{A}(v)\int_0^{\zeta(v)} \exp(-\mathbb{W}_-(s))Z\bigg(\frac{\mathbb{A}_-(s)}{\mathbb{A}(v)}\bigg)\,ds \tag{7.6}$$

$$\le v\zeta(v)\exp\Big(\overline{\mathbb{W}}(v) + \sup_{0\le s\le \zeta(v)}(-\mathbb{W}_-(s))\Big)\sup_{u\ge 0} Z(u), \tag{7.7}$$

where in the last inequality, we have used the trivial estimate $\mathbb{A}(v) \le v\exp(\overline{\mathbb{W}}(v))$. Consider the events [recalling H_- from (3.6), which denotes the process of first hitting times for W_-]

$$E_{15} = \Big\{\sup_{u\ge 0} Z(u) \le \exp\big(\sigma l^4/2\big)\Big\} \quad \text{as defined},$$

$$E_{16} = \Big\{\zeta(v) \le H_-\big(\overline{W}(v) + l^4\big) \le \exp(4l^2)\Big\} \quad \text{as defined}.$$

On $E_{16} \cap \Omega(v)$,

$$\begin{aligned}\overline{\mathbb{W}}(v) + \sup_{0\le s\le \zeta(v)} \big(-\mathbb{W}_-(s)\big) &\le \sigma\overline{W}(v) + \sigma \sup_{0\le s\le H_-(\overline{W}(v)+l^4)} \big(-W_-(s)\big) + 2l^3\\ &= \sigma U_-\big(\overline{W}(v) + l^4\big) - \sigma l^4 + 2l^3,\end{aligned}$$

where U_- is as in (4.6). By (7.7), on $E_{15} \cap E_{16} \cap \Omega(v)$,

$$\begin{aligned}I_2(v) &\le v\exp(4l^2)\exp\big[\sigma U_-\big(\overline{W}(v) + l^4\big) - \sigma l^4 + 2l^3\big]\exp\left(\frac{\sigma l^4}{2}\right)\\ &\le \exp\big[\sigma U_-\big(\overline{W}(v) + l^4\big)\big].\end{aligned}$$

This proves the upper bound (4.11) with $E_9 = E_{15} \cap E_{16} \cap E_{17} \cap \Omega(v)$ as defined, where E_{17} can be an arbitrary event, which is to be chosen ultimately.

To show the lower bound (4.12), write $\delta = \exp(-5l^2)$ as defined for brevity, and define

$$E_{18} = \left\{\omega_W(\delta v, v) \le \frac{l^4}{5};\ \eta_W(v) \le (1-\delta)v\right\} \quad \text{as defined},$$

where $\omega_W(\cdot,\cdot)$ and $\eta_W(\cdot)$ are as in (5.2) and (5.1), respectively. By (5.3) and (5.6),

$$\mathbb{P}(E_{18}^c) \le C_{48}\exp(-l^2), \tag{7.8}$$

which has been observed in the proof of Lemma 5.4 in Section 5. On $E_{18} \cap \Omega(v)$,

$$\begin{aligned}\mathbb{A}(v) = \int_0^v \exp(\mathbb{W}(s))\,ds &\ge \int_{\eta_W(v)}^{\eta_W(v)+\delta v} \exp\big(\sigma W(s) - l^3\big)\,ds\\ &\ge \delta v\exp\big(\sigma\overline{W}(v) - \sigma\omega_W(\delta v, v) - l^3\big) \qquad (7.9)\\ &\ge \exp\left(\sigma\overline{W}(v) - \frac{\sigma l^4}{4}\right).\end{aligned}$$

Write $\Theta(v) = H_-((\overline{W}(v) - l^4)^+)$ as defined for brevity (x^+ standing for the positive part of x), and introduce, for $t > x > 0$,

$$\eta_-(t) = \inf\Big\{0 \le s \le t; -W_-(s) = \sup_{0 \le r \le t}(-W_-(r))\Big\} \quad \text{as defined},$$
$$\omega_-(x, t) = \sup_{0 \le r \le s \le t;\ s-r<x} |W_-(s) - W_-(r)| \quad \text{as defined},$$

which relate to $-W_-$ in the same way as η_W and ω_W do to W. Consider the events

$$E_{19} = \{\zeta(v) \ge \Theta(v)\} \quad \text{as defined},$$
$$E_{20} = \left\{H_-(\overline{W}(v)) \le \exp\left(\frac{\sigma l^4}{5}\right)\right\} \quad \text{as defined},$$
$$E_{21} = \left\{\inf_{0 \le r \le \exp(-\sigma l^4/2)} Z(r) \ge \exp\left(-\frac{\sigma l^4}{6}\right)\right\} \quad \text{as defined},$$
$$E_{22} = \{\eta_-(H_-(l^4)) > \delta v\} \quad \text{as defined},$$
$$E_{23} = \left\{\omega_-\left(\delta v, \exp\left(\frac{\sigma l^4}{5}\right)\right) \le \frac{l^4}{4}\right\} \quad \text{as defined}.$$

Pick a (random) $s \in [0, \Theta(v)]$. On $E_{19} \cap E_{20} \cap \{\overline{W}(v) > l^4\} \cap E_{16} \cap \Omega(v)$,

$$\begin{aligned}
\mathbb{A}_-(s) &\le \int_0^{H_-((\overline{W}(v)-l^4)^+)} \exp(\sigma W_-(z) + l^3)\, dz \\
&= \int_0^{H_-(\overline{W}(v)-l^4)} \exp(\sigma W_-(z) + l^3)\, dz \\
&\le H_-(\overline{W}(v)) \exp(\sigma(\overline{W}(v) - l^4) + l^3) \\
&\le \exp\left(\sigma \overline{W}(v) - \frac{4\sigma l^4}{5} + l^3\right) \\
&\le \exp\left(\sigma \overline{W}(v) - \frac{3\sigma l^4}{4}\right).
\end{aligned}$$

By (7.9), on $E_{18} \cap E_{19} \cap E_{20} \cap \{\overline{W}(v) > l^4\} \cap E_{16} \cap \Omega(v)$, for $s \in [0, \Theta(v)]$,

$$\mathbb{A}(v) \ge \exp\left(\sigma \overline{W}(v) - \frac{\sigma l^4}{4}\right),$$
$$\frac{\mathbb{A}_-(s)}{\mathbb{A}(v)} \le \exp\left(-\frac{\sigma l^4}{2}\right).$$

Using the representation (7.6), on $E_{18} \cap E_{19} \cap E_{20} \cap E_{21} \cap \{\overline{W}(v) > \lambda^4\} \cap E_{16} \cap \Omega(v)$,

$$
\begin{aligned}
I_2(v) &\geq \mathbb{A}(v) \int_0^{\Theta(v)} \exp\bigl(-\sigma W_-(s) - l^3\bigr) Z\left(\frac{\mathbb{A}_-(s)}{\mathbb{A}(v)}\right) ds \\
&\geq \exp\left(\sigma \overline{W}(v) - \frac{\sigma l^4}{4}\right) \int_0^{\Theta(v)} \exp\bigl(-\sigma W_-(s) - l^3\bigr)\, ds \\
&\qquad \times \inf_{0 \leq r \leq \exp(-\sigma l^4/2)} Z(r) \\
&\geq \exp\left(\sigma \overline{W}(v) - \frac{5\sigma l^4}{12} - l^3\right) \int_0^{\Theta(v)} \exp\bigl(-\sigma W_-(s)\bigr)\, ds.
\end{aligned} \tag{7.10}
$$

On $E_{22} \cap \{\overline{W}(v) \geq 2l^4\}$, we have $\eta_-(\Theta(v)) \geq \eta_-(H_-(l^4)) > \delta v$, whereas on E_{20}, $\Theta(v) \leq H_-(\overline{W}(v)) \leq \exp(\sigma l^4/5)$. Hence, on $E_{20} \cap E_{22} \cap E_{23} \cap \{\overline{W}(v) \geq 2l^4\}$,

$$
\begin{aligned}
&\int_0^{\Theta(v)} \exp\bigl(-\sigma W_-(s)\bigr)\, ds \\
&\quad \geq \int_{\eta_-(\Theta(v)) - \delta v}^{\eta_-(\Theta(v))} \exp\bigl(-\sigma W_-(s)\bigr)\, ds \\
&\quad \geq \delta v \exp\left[\sigma \sup_{0 \leq s \leq H_-(\overline{W}(v) - l^4)} \bigl(-W_-(s)\bigr) \right. \\
&\qquad\qquad \left. - \sigma \omega_-\left(\delta v, \exp\left(\frac{\sigma l^4}{5}\right)\right)\right] \\
&\quad \geq \delta v \exp\left[\sigma U_-\bigl(\overline{W}(v) - l^4\bigr) - \sigma\bigl(\overline{W}(v) - l^4\bigr) - \frac{\sigma l^4}{4}\right] \\
&\quad \geq \exp\left[\sigma U_-\bigl(\overline{W}(v) - l^4\bigr) - \sigma \overline{W}(v) + \frac{\sigma l^4}{2}\right].
\end{aligned} \tag{7.11}
$$

Now choose $E_{17} = \bigcap_{j=18}^{23} E_j$ as defined and let $E_9 = E_{15} \cap E_{16} E_{17} \cap \Omega(v)$ as defined. According to (7.10) and (7.11),

$$
\log I_2(v) \geq \sigma U_-\bigl(\overline{W}(v) - l^4\bigr) \quad \text{on } E_9 \cap \bigl\{\overline{W}(v) \geq 2l^4\bigr\},
$$

proving the desired lower bound in (4.12).

It remains only to check (4.13), that is $\mathbb{P}(E_9^c) \leq C_{10} \exp(-l^2)$. To this end, observe that, by Lemma 7.1,

$$
\mathbb{P}(E_{15}^c) \leq 4 \exp\bigl(\sigma l^4/2\bigr) \leq \exp(-l^2), \tag{7.12}
$$

$$
\mathbb{P}(E_{21}^c) \leq \exp\bigl(-\sigma l^4/6\bigr) + 8 \exp\bigl(-\sigma l^4/6\bigr) \leq \exp(-l^2), \tag{7.13}
$$

whereas by applying (5.3) to $a = \delta v = v\exp(-5l^2)$, $t = \exp(\sigma l^4/5)$ and $b = l^4/4$,

$$\mathbb{P}(E_{23}^c) \le \frac{C_{11}}{v}\exp\left(\frac{\sigma l^4}{5} + 5l^2 - \frac{l^8}{48\delta v}\right) \le C_{49}\exp(-l^2). \tag{7.14}$$

Since $H_-(\overline{W}(v)) \overset{\text{law}}{=} vC^2$, where C stands for a standard Cauchy variable,

$$\mathbb{P}(E_{20}^c) \le \pi^{-1}v^{1/2}\exp\bigl(-\sigma l^4/10\bigr) \le \exp(-l^2). \tag{7.15}$$

To estimate $\mathbb{P}(E_{22}^c)$, recall that for each fixed $t > 0$, by Paul Lévy's classical arcsine law, $\eta_-(t)$ has the density distribution $\mathbb{P}(\eta_-(t) \in dx)/dx = (1/\pi\sqrt{x(t-x)})\mathbb{1}_{\{0<x<t\}}$. Hence

$$\begin{aligned}\mathbb{P}(E_{22}^c) &\le \mathbb{P}\bigl(H_-(l^4) < \sqrt{\delta}v\bigr) + \mathbb{P}\bigl(\eta_-(\sqrt{\delta}v) \le \delta v\bigr)\\ &\le 2\exp\left(-\frac{l^8}{2\sqrt{\delta}v}\right) + \delta^{1/4}\\ &\le \exp(-l^2).\end{aligned} \tag{7.16}$$

Recall from (5.8) that $\mathbb{P}(\Omega^c(v)) \le \exp(-2l^2)$. In view of (7.8) and (7.12)–(7.16), the proof of (4.13) is reduced to showing the following estimates:

$$\mathbb{P}(E_{16}^c \cap \Omega(v)) \le C_{50}\exp(-l^2), \tag{7.17}$$

$$\mathbb{P}(E_{19}^c \cap \Omega(v)) \le C_{51}\exp(-l^2). \tag{7.18}$$

Let us check (7.17) first. We have

$$\begin{aligned}\mathbb{P}(E_{16}^c \cap \Omega(v)) &\le \mathbb{P}\bigl(\zeta(v) > H_-\bigl(\overline{W}(v) + l^4\bigr);\\ &\qquad\qquad \Omega(v);\ H_-\bigl(\overline{W}(v) + l^4\bigr) \le \exp(4l^2)\bigr)\\ &\qquad + \mathbb{P}\bigl(H_-\bigl(\overline{W}(v) + l^4\bigr) > \exp(4l^2)\bigr)\\ &= I_9 + I_{10} \quad \text{as defined,}\end{aligned}$$

with obvious notation. By definition,

$$\begin{aligned}I_{10} &= \mathbb{P}\bigl(\overline{W}(v) + l^4 > \overline{W}(-\exp(4l^2))\bigr)\\ &\le \mathbb{P}\bigl(\overline{W}(v) > \tfrac{1}{2}\overline{W}(-\exp(4l^2))\bigr) + \mathbb{P}\bigl(\overline{W}(-\exp(4l^2)) < 2l^4\bigr)\\ &= \mathbb{P}\left(\frac{\overline{W}(v)}{\overline{W}(-v)} > \frac{\exp(2l^2)}{2\sqrt{v}}\right) + \mathbb{P}\bigl(\overline{W}(-\exp(4l^2)) < 2l^4\bigr),\end{aligned}$$

using the scaling property. Since $\overline{W}(v)/\overline{W}(-v)$ is distributed as the modulus of a standard Cauchy variable, this, jointly considered with the form of Gaussian densities, yields

$$I_{10} \le \frac{2\sqrt{v}}{\pi\exp(2l^2)} + \frac{2l^4}{\exp(2l^2)} \le \exp(-l^2). \tag{7.19}$$

To estimate I_9, consider the events

$$E_{24} = \left\{ H_-\big(\overline{W}(v) + l^4\big) - H_-\left(\overline{W}(v) + \frac{l^4}{2}\right) \geq \delta v \right\} \quad \text{as defined},$$

$$E_{25} = \left\{ \sup_{H_-(\Delta_W(v)) \leq s \leq H_-(\Delta_W(v)) + \delta v} \big| W_-(s) - \Delta_W(v) \big| < \frac{l^4}{3} \right\} \quad \text{as defined},$$

with $\Delta_W(v) = \overline{W}(v) + l^4/2$ as defined. By the independence of H_- and $\overline{W}(v)$, and the strong Markov property,

$$\begin{aligned} \mathbb{P}(E_{24}^c) &= \mathbb{P}\left(H_-\left(\frac{l^4}{2}\right) < \delta v \right) = \mathbb{P}\left(\overline{W}(-\delta v) > \frac{l^4}{2} \right) \\ &\leq 2 \exp\left(-\frac{l^8}{8 \delta v} \right) \\ &\leq \exp(-l^2), \end{aligned}$$

and

$$\mathbb{P}(E_{25}^c) = \mathbb{P}\left(\sup_{0 \leq t \leq \delta v} |W_-(t)| \geq \frac{l^4}{3} \right) \leq 4 \exp\left(-\frac{l^8}{18 \delta v} \right) \leq \exp(-l^2).$$

Consequently,

$$\begin{aligned} I_9 &\leq 2 \exp(-l^2) \\ &\quad + \mathbb{P}\big(\zeta(v) > H_-\big(\overline{W}(v) + l^4\big); E_{24}; E_{25}; \Omega(v); \\ &\qquad\qquad H_-\big(\overline{W}(v) + l^4\big) \leq \exp(4 l^2) \big). \end{aligned} \tag{7.20}$$

By the definitions of ζ and $\zeta(v)$ [cf. (7.1) and (7.5), respectively], for any $a \geq 0$,

$$\{\zeta(v) > a\} = \{\mathbb{A}_-(a) < \zeta \mathbb{A}(v)\}. \tag{7.21}$$

Of course, on $\Omega(v)$, $\mathbb{A}(v) \leq v \exp(\sigma \overline{W}(v) + l^3)$. On $E_{24} \cap E_{25} \cap \Omega(v) \cap \{H_-(\overline{W}(v) + l^4) \leq \exp(4 l^2)\}$,

$$\begin{aligned} \mathbb{A}_-\big(H_-\big(\overline{W}(v) + l^4\big)\big) &\geq \int_{H_-(\overline{W}(v) + l^4/2)}^{H_-(\overline{W}(v) + l^4/2) + \delta v} \exp\big(\sigma W_-(s) - l^3\big)\, ds \\ &\geq \delta v \exp\left(\sigma \left(\overline{W}(v) + \frac{l^4}{2} \right) - \frac{\sigma l^4}{3} - l^3 \right) \\ &\geq v \exp\left(\sigma \overline{W}(v) + \frac{\sigma l^4}{7} \right). \end{aligned}$$

Going back to (7.20), and then using (7.2),

$$\begin{aligned} I_9 &\le 2\exp(-l^2) + \mathbb{P}\big(\zeta > \exp(\sigma l^4/7 - l^3)\big) \\ &\le 2\exp(-l^2) + \exp(-\sigma l^4/7 + l^3) \\ &\le 3\exp(-l^2). \end{aligned}$$

Since $\mathbb{P}(E_{16}^c \cap \Omega(v)) \le I_9 + I_{10}$, this estimate, jointly considered with (7.19), yields (7.17).

To verify (7.18), note from (7.21) that for $b \ge 0$, on $\Omega(v) \cap \{H_-(b) \le \exp(4l^2)\}$,

$$\begin{aligned} \{\zeta(v) \le H_-(b)\} &= \{\mathbb{A}_-(H_-(b)) \ge \zeta\mathbb{A}(v)\} \\ &\subseteq \{\exp(\sigma b + l^3) H_-(b) \ge \zeta\mathbb{A}(v)\}. \end{aligned}$$

Accordingly,

$$\begin{aligned} \mathbb{P}(E_{19}^c \cap \Omega(v)) &= \mathbb{P}\big(\zeta(v) < H_-(\overline{W}(v) - l^4); \overline{W}(v) \ge l^4; \Omega(v)\big) \\ &\le \mathbb{P}\big(H_-(\overline{W}(v)) > \exp(4l^2)\big) \\ &\quad + \mathbb{P}\big(\exp(\sigma\overline{W}(v) - \sigma l^4 + l^3) H_-(\overline{W}(v)) \ge \zeta\mathbb{A}(v); \\ &\qquad \Omega(v); H_-(\overline{W}(v)) \le \exp(4l^2)\big), \end{aligned}$$

which, by virtue of (7.8) and (7.9), is less than or equal to

$$\begin{aligned} &\mathbb{P}\big(H_-(\overline{W}(v)) > \exp(4l^2)\big) + C_{48}\exp(-l^2) \\ &\quad + \mathbb{P}\big(\exp(-\sigma l^4/2) H_-(\overline{W}(v)) \ge \zeta; E_{18}; \Omega(v); H_-(\overline{W}(v)) \le \exp(4l^2)\big) \\ &\qquad \le \mathbb{P}\big(H_-(\overline{W}(v)) > \exp(4l^2)\big) + C_{48}\exp(-l^2) + \mathbb{P}\left(\zeta \le \exp\left(-\frac{\sigma l^4}{2} + 4l^2\right)\right). \end{aligned}$$

Since $H_-(\overline{W}(v)) \overset{\text{law}}{=} vC^2$ (C denoting a standard Cauchy variable), the above estimate, jointly considered with (7.2), yields

$$\begin{aligned} \mathbb{P}(E_{19}^c \cap \Omega(v)) &\le C_{48}\exp(-l^2) + 2\pi^{-1}v^{1/2}\exp(-2l^2) + \exp\left(-\frac{\sigma l^4}{2} + 4l^2\right) \\ &\le C_{52}\exp(-l^2). \end{aligned}$$

This implies the desired estimate in (7.18), hence Lemma 4.2. □

8. Lévy classes for Brox-type diffusions. Theorems 1.6–1.8 are particular cases of the following general result. Its proof is based on the key estimates in Section 3.

THEOREM 8.1. *Let* $\{\mathbb{W}(x);\ x \in \mathbb{R}\}$ *be a cadlag process satisfying* (3.1), *and* $\{\mathbb{X}(t);\ t \ge 0\}$ *a diffusion process with potential* $\mathbb{W}$ [*cf.* (3.2)]. *Let*

$$J_1(f) = \int^{\infty} \frac{f(t)}{t \log t} \exp\left(-\frac{\pi^2\sigma^2}{8} f(t)\right) dt \quad \textit{as defined},$$

$$J_2(f) = \int^{\infty} \frac{\sqrt{f(t)}}{t \log t} \exp\left(-\frac{f(t)}{\sigma^2}\right) dt \quad \textit{as defined},$$

$$J_3(f) = \int^{\infty} \frac{dt}{t\sqrt{f(t)}\ \log t} \quad \textit{as defined}.$$

For any nondecreasing function $f > 0$,

$$\mathbb{P}\Big[\mathbb{X}(t) > (\log t)^2 f(t)\ i.o.\Big] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \quad J_1(f)\begin{cases} < \infty \\ = \infty, \end{cases} \tag{8.1}$$

$$\mathbb{P}\left[\sup_{0 \le s \le t} |\mathbb{X}(s)| \le \frac{(\log t)^2}{f(t)} i.o.\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \quad J_2(f)\begin{cases} < \infty \\ = \infty, \end{cases} \tag{8.2}$$

$$\mathbb{P}\left[\sup_{0 \le s \le t} \mathbb{X}(s) \le \frac{(\log t)^2}{f(t)} i.o.\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \quad J_3(f)\begin{cases} < \infty \\ = \infty. \end{cases} \tag{8.3}$$

In particular, with probability 1,

$$\limsup_{t \to \infty} \frac{\mathbb{X}(t)}{(\log t)^2 \log\log\log t} = \frac{8}{\pi^2\sigma^2};$$

$$\liminf_{t \to \infty} \frac{\log\log\log t}{(\log t)^2} \sup_{0 \le s \le t} |\mathbb{X}(s)| = \frac{1}{\sigma^2};$$

$$\limsup_{t \to \infty} \frac{(\log\log t)^a}{(\log t)^2} \sup_{0 \le s \le t} \mathbb{X}(s) = \begin{cases} 0, & \textit{if } a \le 2, \\ \infty, & \textit{otherwise}. \end{cases}$$

PROOF OF (8.1). It is known that $\mathbb{X}$, $\overline{\mathbb{X}}$ and $\mathbb{X}^*$ have the same upper functions. This can be shown using an ω-by-ω argument; compare [7], page 28, or [22]. We only need to prove (8.1) for $\overline{\mathbb{X}}$.

As is well known for such results, we only have to limit ourselves to the study of the "critical case,"

$$\frac{4}{\pi^2\sigma^2} \log\log\log t \le f(t) \le \frac{16}{\pi^2\sigma^2} \log\log\log t, \tag{8.4}$$

for sufficiently large t. For a rigorous justification, compare the pioneer paper of Erdös [11].

In view of (3.8), the convergence part of (8.1) is straightforward. Indeed, pick an arbitrary nondecreasing function $f > 0$ satisfying $J_1(f) < \infty$. In this case, $f(t)$ clearly tends to infinity. Define the sequence $(t_i)_{i \geq 0}$ by recurrence: choose a large t_0 and let $\log t_{i+1} = (1 + 1/f(t_i))\log t_i$ as defined for $i \geq 0$. It is easily seen that t_i increases to infinity and that $\sum_i \exp(-\pi^2\sigma^2 f(t_i)/8) < \infty$. By Proposition 3.1,

$$\mathbb{P}\Big(\overline{\mathbb{X}}(t_{i+1}) > (\log t_i)^2 f(t_i)\Big) \leq C_6 \exp\left(-\frac{\pi^2\sigma^2(\log t_i)^2 f(t_i)}{8(\log t_{i+1})^2}\right)$$

$$\leq C_{53} \exp\left(-\frac{\pi^2\sigma^2}{8} f(t_i)\right),$$

which is summable for i. By the Borel–Cantelli lemma, almost surely for all sufficiently large i, $\overline{\mathbb{X}}(t_{i+1}) \leq (\log t_i)^2 f(t_i)$. Let $t \in [t_i, t_{i+1}]$, then

$$\overline{\mathbb{X}}(t) \leq \overline{\mathbb{X}}(t_{i+1}) \leq (\log t_i)^2 f(t_i) \leq (\log t)^2 f(t),$$

proving the convergence part in (8.1).

It remains to verify the divergence part. Let $f > 0$ be nondecreasing such that $J_1(f) = \infty$. Fix a large i_0 and let $t_i = \exp(\exp(i/\log i))$ for $i \geq i_0$. Elementary computations show that

$$\sum_i \exp\left(-\frac{\pi^2\sigma^2}{8} f(t_i)\right) = \infty, \tag{8.5}$$

$$\sum_i \exp\big(-f^2(t_i)\big) < \infty. \tag{8.6}$$

Recall the hitting time process H_- from (3.6). Write $v_i = (\log t_i)^2 f(t_i)$ as defined for $i \geq i_0$. Applying Proposition 3.1 to $\lambda = f(t_i)$ and $t = t_i$, there exists a sequence of events $(G_i)_{i \geq i_0}$, with $\mathbb{P}(G_i^c) \leq C_6 \exp(-f^2(t_i))$, such that

$$\Big\{\overline{\mathbb{X}}(t_i) > (\log t_i)^2 f(t_i)\Big\} \supseteq F_i \cap G_i, \qquad i \geq i_0, \tag{8.7}$$

where

$$F_i = \left\{H_-\left(-\frac{\log t_i}{4\sigma}\right) > H_-\left(\frac{\log t_i}{4\sigma}\right);\ \overline{W}(v_i) < \frac{\log t_i}{5\sigma};\right.$$

$$\left. W^{\#}(v_i) < \left(1 - \frac{3}{f(t_i)}\right)\frac{\log t_i}{\sigma}\right\} \quad \text{as defined.}$$

Since by (8.6), $\sum_i \mathbb{P}(G_i^c) < \infty$, we have $\mathbb{P}(G_i;\ \text{eventually}) = 1$ according to the Borel–Cantelli lemma. In view of (8.7), it remains only to prove that

$$\mathbb{P}(F_i\ i.o.) = 1. \tag{8.8}$$

To this end, noting that by the independence of $H_-(\cdot)$ (which depends only on $\{W(t);\ t\in\mathbb{R}_-\}$ and $(\overline{W}, W^{\#})$ (which depends on $\{W(t);\ t\in\mathbb{R}_+\}$), we have

$$\mathbb{P}(F_i) = \frac{1}{2}\mathbb{P}\left[\overline{W}(v_i) < \frac{\log t_i}{5\sigma};\ W^{\#}(v_i) < \left(1-\frac{3}{f(t_i)}\right)\frac{\log t_i}{\sigma}\right]$$
$$\geq C_{54}\exp\left(-\frac{\pi^2\sigma^2 f(t_i)}{8(1-3/f(t_i))^2}\right),$$

by virtue of (2.5). Hence,

$$\mathbb{P}(F_i) \geq C_{55}\exp\left(-\frac{\pi^2\sigma^2}{8}f(t_i)\right),\qquad i\geq i_0, \tag{8.9}$$

which by (8.5) implies $\sum_i \mathbb{P}(F_i)=\infty$. To apply the Borel–Cantelli lemma, we have to estimate the second moment. Pick $i_0 \leq i < j$. Let $\hat{W}(t) = W(t+v_i) - W(v_i)$, and $\hat{W}^{\#}(t) = \sup_{0\leq u\leq s\leq t}(\hat{W}(s)-\hat{W}(u))$. By the independence of Brownian increments,

$$\mathbb{P}(F_i\cap F_j) \leq \mathbb{P}\left[F_i;\ \hat{W}^{\#}(v_j - v_i) \leq \left(1-\frac{3}{f(t_j)}\right)\frac{\log t_j}{\sigma}\right]$$
$$\leq 2\mathbb{P}(F_i)\exp\left[-\frac{\pi^2\sigma^2(v_j-v_i)}{8(\log t_j)^2(1-3/f(t_j))^2}\right],$$

using (5.4). From here, several lines of elementary calculation using (8.4) and (8.9) show that (cf. [11] for details)

$$\mathbb{P}(F_i\cap F_j) \leq \begin{cases} C_{56}\mathbb{P}(F_i)\mathbb{P}(F_j), & \text{if } j-i\geq(\log i)^2,\\ C_{57}\mathbb{P}(F_i)\, j^{-C_{58}}, & \text{if } \log i < j-i < (\log i)^2,\\ C_{59}\mathbb{P}(F_i)\exp(-C_{60}(j-i)), & \text{if } 2\leq j-i\leq \log i,\end{cases}$$

which implies

$$\liminf_{n\to\infty} \sum_{i=i_0}^{n}\sum_{j=i_0}^{n}\mathbb{P}(F_i\cap F_j)\Big/\left(\sum_{i=i_0}^{n}\mathbb{P}(F_i)\right)^2 \leq C_{61}.$$

Using the Borel–Cantelli lemma of [23], $\mathbb{P}(F_i \text{ i.o.}) \geq 1/C_{61} > 0$.

We now apply a 0–1 argument. For any integer $n\geq 0$, let ${}^nW$ and ${}^nW_-$ be the increment processes, of W and W_-, respectively, on the time interval $[n, n+1]$:

$${}^nW = \{W(n+t) - W(n);\ 0\leq t\leq 1\}\quad \text{as defined},$$
$${}^nW_- = \{W_-(n+t) - W_-(n);\ 0\leq t\leq 1\}\quad \text{as defined},$$

Observe that the random variables (with values in a space of paths) $({}^nW, {}^nW_-)_{n\geq 0}$ are iid. Let Σ be a finite permutation on $\mathbb{N}$, that is, for some $N > 0$, $\Sigma(n) = n$ whenever $n \geq N$. Let ${}^{\Sigma}W$ be the Brownian motion obtained by the permutation of the increments of W; that is, the increment of ${}^{\Sigma}W$ on the time interval $[n, n+1]$ is ${}^{\Sigma(n)}W$. Define similarly ${}^{\Sigma}W_-$ by the permutation of the increments of W_-. According to our construction, $W(t) = {}^{\Sigma}W(t)$ and $W_-(t) = {}^{\Sigma}W_-(t)$ for all $t \geq N+1$. Clearly, the event $\{F_i \text{ i.o.}\}$ remains unchanged if we replace W and W_- by ${}^{\Sigma}W$ and ${}^{\Sigma}W_-$, respectively. As a consequence, we can apply the Hewitt–Saveage 0–1 law, to see that $\{F_i \text{ i.o.}\}$ is a trivial event. We have proved (8.8). □

PROOF OF (8.2). The convergence part is a direct consequence of Proposition 3.2, requiring only standard techniques. For more details, compare the proof of (8.1).

To prove the divergence part, let $f > 0$ be a nondecreasing function such that $J_2(f) = \infty$. Without loss of generality, we assume

$$\frac{\sigma^2}{2}\log\log\log t \leq f(t) \leq 2\sigma^2 \log\log\log t, \tag{8.10}$$

for sufficiently large t. Fix a large initial value of i_0 and define $t_i = \exp(\exp(\theta i/\log i))$ as defined for $i \geq i_0$, with $\theta > 0$ such that $2C_{25}\exp(-\theta/3) \leq C_{24}$, where C_{24} and C_{25} are the constants in (5.13) and (5.14). Elementary computations using (8.10) give

$$\sum_i f(t_i)^{-1/2}\exp\left(-\frac{f(t_i)}{\sigma^2}\right) = \infty, \tag{8.11}$$

$$\sum_i \exp\bigl(-f^2(t_i)\bigr) < \infty. \tag{8.12}$$

Write $v_i = (\log t_i)^2/f(t_i)$ as defined. Applying Proposition 3.2 to $t = t_i$ and $\lambda = f(t_i)$,

$$\bigl\{\mathbb{X}^*(t_i) \leq v_i\bigr\} \supseteq F_i \cap G_i, \qquad i \geq i_0, \tag{8.13}$$

where

$$F_i = \left\{\frac{\log t_i}{\sigma} > \overline{W}(-v_i) > \overline{W}(v_i) + \log^4 v_i;\ W(v_i) > -\frac{2\log t_i}{\sigma};\right.$$
$$\left.\frac{\log t_{i+1}}{\sigma} \geq W^{\#}(v_i) > \frac{\log t_i}{\sigma} + \frac{\log^4 v_i}{\sigma}\right\} \quad \text{as defined,}$$

and $\mathbb{P}(G_i^c) \leq C_7\exp(-f^2(t_i))$. Using (8.12), $\sum_i \mathbb{P}(G_i^c) < \infty$, which according to the Borel–Cantelli lemma implies $\mathbb{P}(G_i;\text{ eventually}) = 1$. In view of (8.13), we only have to show that

$$\mathbb{P}(F_i \text{ i.o.}) = 1. \tag{8.14}$$

Note that $\log t_{i+1}/\log t_i - 1 \sim \theta/\log i$ and that $\sigma^2(\log i)/3 \le f(t_i) \le 3\sigma^2 \log i$ for $i \ge i_0$. Applying scaling and Lemma 5.5, it is readily seen that

$$\begin{aligned}
\mathbb{P}(F_i) \ge \mathbb{P}\bigg(&\frac{\log t_i}{\sigma} > \overline{W}(-v_i) > \overline{W}(v_i) + \log^4 v_i;\\
&\qquad W^{\#}(v_i) > \frac{\log t_i + \log^4 v_i}{\sigma}\bigg)\\
&- \mathbb{P}\bigg(\frac{\log t_{i+1}}{\sigma} > \overline{W}(-v_i) > \overline{W}(v_i) + \log^4 v_i;\\
&\qquad W^{\#}(v_i) > \frac{\log t_{i+1}}{\sigma}\bigg)\\
&- \mathbb{P}\bigg(W(v_i) \le -\frac{2\log t_i}{\sigma}\bigg)\\
\ge{}& \frac{C_{24}\sigma}{\sqrt{f(t_i)}} \exp\bigg(-\frac{f(t_i)}{\sigma^2}\bigg) - \frac{C_{25}\sigma}{\sqrt{f(t_i)}}\\
&\times \exp\bigg(-\frac{f(t_i)}{\sigma^2} - \theta/3\bigg) - 2\exp\bigg(-\frac{2f(t_i)}{\sigma^2}\bigg)\\
\ge{}& \frac{C_{62}}{\sqrt{f(t_i)}} \exp\bigg(-\frac{f(t_i)}{\sigma^2}\bigg),
\end{aligned} \tag{8.15}$$

by our choice of θ. To apply the Borel–Cantelli lemma, let $\hat{W}(s) = W(s + v_i) - W(v_i)$ as defined and $\hat{W}(-s) = W(-s - v_i) - W(-v_i)$ as defined for $s \ge 0$. Let $i_0 \le i \le j - 2$. On $F_i \cap F_j$,

$$\begin{aligned}
\hat{\mathbb{W}}^{\#}(v_j - v_i) &= \sup_{v_i \le s \le r \le v_j} (W(r) - W(s))\\
&\ge W^{\#}(v_j) - W^{\#}(v_i)\\
&\ge \frac{\log t_j - \log t_{i+1}}{\sigma},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\hat{W}}(v_j - v_i) &= \sup_{v_i \le r \le v_j} W(r) - W(v_i)\\
&\le \overline{W}(v_j) + \frac{2\log t_i}{\sigma}\\
&\le \overline{W}(-v_j) + \frac{2\log t_i}{\sigma}\\
&\le \overline{\hat{W}}\big(-(v_j - v_i)\big) + \overline{W}(-v_i) + \frac{2\log t_i}{\sigma}\\
&\le \overline{\hat{W}}\big(-(v_j - v_i)\big) + \frac{3\log t_i}{\sigma}.
\end{aligned}$$

By the independence of Brownian increments,

$$\mathbb{P}(F_i \cap F_j) \le \mathbb{P}\bigg(F_i;\ \hat{W}^{\#}(v_j - v_i) \ge \frac{\log t_j - \log t_{i+1}}{\sigma};\ \overline{\hat{W}}(-(v_j - v_i)) \ge \overline{\hat{W}}(v_j - v_i) - \frac{3\log t_i}{\sigma}\bigg) = \mathbb{P}(F_i)\, I_{11}, \tag{8.16}$$

where

$$\begin{aligned} I_{11} &= \mathbb{P}\bigg(W^{\#}(v_j - v_i) \ge \frac{\log t_j - \log t_{i+1}}{\sigma};\\ &\qquad \overline{W}(-(v_j - v_i)) \ge \overline{W}(v_j - v_i) - \frac{3\log t_i}{\sigma}\bigg) \quad \text{(as defined)}\\ &= \mathbb{P}\bigg(W^{\#}(1) \ge \frac{\log t_j - \log t_{i+1}}{\sigma\sqrt{v_j - v_i}};\ \overline{W}(-1) \ge \overline{W}(1) - \frac{3\log t_i}{\sigma\sqrt{v_j - v_i}}\bigg). \end{aligned}$$

From here, the proof becomes routine. Indeed, for $j - i \ge (\log i)^2$,

$$\begin{aligned} I_{11} &\le \mathbb{P}\bigg(W^{\#}(1) \ge \frac{\log t_j}{\sigma\sqrt{v_j - v_i}} - \frac{\log t_{i+1}}{\sigma\sqrt{v_j - v_i}};\\ &\qquad \frac{\log t_j}{\sigma\sqrt{v_j - v_i}} \ge \overline{W}(-1) \ge \overline{W}(1) - \frac{3\log t_i}{\sigma\sqrt{v_j - v_i}}\bigg)\\ &\quad + \mathbb{P}\bigg(\overline{W}(-1) > \frac{\log t_j}{\sigma\sqrt{v_j - v_i}}\bigg)\\ &\quad \times \mathbb{P}\bigg(W^{\#}(1) \ge \frac{\log t_j}{\sigma\sqrt{v_j - v_i}} - \frac{\log t_{i+1}}{\sigma\sqrt{v_j - v_i}}\bigg)\\ &\le \frac{C_{63}\sigma}{\sqrt{f(t_j)}} \exp\bigg(-\frac{f(t_j)}{\sigma^2}\bigg) + \frac{C_{64}\sigma}{f(t_j)} \exp\bigg(-\frac{f(t_j)}{\sigma^2}\bigg)\\ &\le C_{65}\mathbb{P}(F_j), \end{aligned} \tag{8.17}$$

using Lemma 5.5 and (8.15). For $2 \le j - i < (\log i)^2$,

$$\begin{aligned} I_{11} &\le \mathbb{P}\bigg(W^{\#}(1) \ge \frac{\log t_j - \log t_{i+1}}{\sigma\sqrt{v_j - v_i}}\bigg)\\ &\le \begin{cases} C_{66}\, i^{-C_{67}}, & \text{if } \log i < j - i < (\log i)^2,\\ C_{68} \exp(-C_{69}(j - i)), & \text{if } 2 \le j - i \le \log i, \end{cases} \end{aligned}$$

which, jointly considered with (8.16) and (8.17), implies that

$$\liminf_{n\to\infty} \sum_{i=i_0}^{n} \sum_{j=i_0}^{n} \mathbb{P}(F_i \cap F_j) \Big/ \left(\sum_{i=i_0}^{n} \mathbb{P}(F_i) \right)^2 \le C_{70}.$$

According to the Borel–Cantelli lemma in [23], $\mathbb{P}(F_i \text{ i.o.}) \ge 1/C_{70} > 0$. Applying a 0–1 argument readily yields (8.14), and hence (8.2). □

The proof of (8.3) is very similar to (and a lot easier than) that of (8.2), using Proposition 3.3 instead of Proposition 3.2. We feel free to omit the details.

9. Skorokhod embedding. This brief section is devoted to a Skorokhod-type embedding of Sinai's RWRE. The result (cf. Proposition 9.1 below) is not new, since it was previously stated in [17] (cf. also [29]). The proof is given here in full detail.

Let $\{S_n\}_{n\ge 0}$ be Sinai's random walk in random environment $\Xi = \{\xi_j\}_{j\in\mathbb{N}}$, as in Section 1.1. Assume (1.9). From the environment Ξ, we define $\{W_\Xi(y); y \in \mathbb{R}\}$ as a step function with $W_\Xi(0) = 0$, which is flat in each interval $[n, n+1)$ (for $n \in \mathbb{Z}$), with jumps

$$\mathbb{W}_\Xi(n) - \mathbb{W}_\Xi(n-) = \log\frac{1-\xi_n}{\xi_n} \quad \text{as defined.}$$

Consider the diffusion process $\{\mathbb{X}(t),\ t \ge 0\}$ with random potential $\mathbb{W}_\Xi$, defined via (3.2)–(3.4), with $\mathbb{W}_\Xi$ in the place of $\mathbb{W}$. According to [16], Theorem 3.1, condition (1.9) ensures that $\limsup_{t\to\infty} \mathbb{X}(t) = \infty$ and $\liminf_{t\to\infty} \mathbb{X}(t) = -\infty$ almost surely. Define the sequence of stopping times $\{\mu_n,\ n \ge 0\}$ by $\mu_0 = 0$ and

$$\mu_n = \inf\{t > \mu_{n-1}: |\mathbb{X}(t) - \mathbb{X}(\mu_{n-1})| = 1\}, \qquad n = 1, 2, \dots \quad \text{as defined.}$$

PROPOSITION 9.1. *Under* (1.9), *the two sequences* $\{\mathbb{X}(\mu_n)\}_{n\ge 0}$ *and* $\{S_n\}_{n\ge 0}$ *have the same distributions. Furthermore,* $\{\mu_n - \mu_{n-1}\}_{n\ge 1}$ *are iid variables, distributed as* $\inf\{t > 0: |B(t)| = 1\}$.

REMARK 9.2. Condition (1.9) in the proposition is only to ensure that μ_n ($n \ge 1$) are well defined.

The proof of the proposition is based on the following result which we have learnt from [34].

LEMMA 9.3 (Yor [34]). *Let ϱ be the process of first hitting times of B as in* (4.1). *For all positive a and b,*

$$\int_0^{\varrho_a \wedge \varrho_{-b}} \left(a^{-2}\, \mathbb{1}_{\{B(s)>0\}} + b^{-2}\, \mathbb{1}_{\{B(s)<0\}} \right) ds \overset{\text{law}}{=} \inf\{t < 0: |B(t)| = 1\}.$$

PROOF OF LEMMA 9.3. For $t > 0$, let $B^+(t)$ and $B^-(t)$ be, respectively, the positive and negative parts of $B(t)$. By Tanaka's formula,

$$a^{-1}B^+(t) + b^{-1}B^-(t) = \int_0^t \big(a^{-1}\mathbb{1}_{\{B(s)>0\}} - b^{-1}\mathbb{1}_{\{B(s)<0\}}\big)\,dB(s) + \frac{a^{-1}+b^{-1}}{2}L(t,0),$$

where $L(\cdot,\cdot)$ is as before the local time of B. This is a Skorokhod-type reflection equation. By Lemma VI.2.1 of [27], there exists one-dimensional reflecting Brownian motion, denoted by γ, such that for all $t \ge 0$, $a^{-1}B^+(t) + b^{-1}B^-(t) = \gamma(D(t))$, where

$$D(t) = \int_0^t \big(a^{-2}\mathbb{1}_{\{B(s)>0\}} + b^{-2}\mathbb{1}_{\{B(s)<0\}}\big)\,ds \quad \text{as defined.}$$

Hence

$$D(\varrho_a \wedge \varrho_{-b}) = \inf\{t > 0 : \gamma(t) = 1\},$$

as desired. □

PROOF OF PROPOSITION 9.1. Define $\mathbb{A}_\Xi$ and $\mathbb{T}_\Xi$ as in (3.2)–(3.4), with $\mathbb{W}_\Xi$ in place of $\mathbb{W}$. Since $\mathbb{X}$ is a diffusion process with scale function $\mathbb{A}_\Xi$,

$$\mathbb{P}\big(\mathbb{X}(\mu_n) = i+1 \,|\, \mathbb{X}(\mu_{n-1}) = i; \Xi\big) = \frac{\mathbb{A}_\Xi(i) - \mathbb{A}_\Xi(i-1)}{\mathbb{A}_\Xi(i+1) - \mathbb{A}_\Xi(i-1)} = \xi_i,$$

the last equality following from the definition of $\mathbb{W}_\Xi$. This identifies the distributions of $\{\mathbb{X}(\mu_n),\ n \ge 0\}$ and $\{S_n,\ n \ge 0\}$.

It remains to prove the second part of the proposition. The independence of B and Ξ plays an important role. Conditionally on Ξ and on the event $\{\mathbb{X}(\mu_{n-1}) = i\}$ [hence $B(\mathbb{T}_\Xi^{-1}(\mu_{n-1})) = \mathbb{A}_\Xi(i)$],

$$\mu_n - \mu_{-1} = \inf\big\{s > 0 : B\big(\mathbb{T}_\Xi^{-1}(s + \mu_{n-1})\big) = \mathbb{A}_\Xi(i+1) \text{ or } \mathbb{A}_\Xi(i-1)\big\},$$

which in words means that $\mathbb{T}_\Xi^{-1}(\mu_n)$ is the first exit time of B from the interval $[\mathbb{A}_\Xi(i-1), \mathbb{A}_\Xi(i+1)]$ after time $\mathbb{T}_\Xi^{-1}(\mu_{n-1})$; thus equivalently,

$$\mathbb{T}_\Xi^{-1}(\mu_n) - \mathbb{T}_\Xi^{-1}(\mu_{n-1}) = \inf\big\{t > 0 : B\big(t + \mathbb{T}_\Xi^{-1}(\mu_{n-1})\big) = \mathbb{A}_\Xi(i+1) \text{ or } \mathbb{A}_\Xi(i-1)\big\}.$$

Accordingly,

$$\begin{aligned}
\mu_n - \mu_{n-1} &= \int_{\mathbb{T}_\Xi^{-1}(\mu_{n-1})}^{\mathbb{T}_\Xi^{-1}(\mu_n)} \exp\big[-2\mathbb{W}_\Xi\big(\mathbb{A}_\Xi^{-1}(B(s))\big)\big]\,ds \\
&= \int_{\mathbb{T}_\Xi^{-1}(\mu_{n-1})}^{\mathbb{T}_\Xi^{-1}(\mu_n)} \big(\exp(-2\mathbb{W}_\Xi(i))\mathbb{1}_{\{B(s)>\mathbb{A}_\Xi(i)\}} \\
&\qquad + \exp(-2\mathbb{W}_\Xi(i-1))\mathbb{1}_{\{B(s)<\mathbb{A}_\Xi(i)\}}\big)\,ds \\
&= \int_0^{\hat{\varrho}} \big(\exp(-2\mathbb{W}_\Xi(i))\mathbb{1}_{\{\hat{B}(s)>0\}} + \exp(-2\mathbb{W}_\Xi(i-1))\mathbb{1}_{\{\hat{B}(s)<0\}}\big)\,ds,
\end{aligned}$$

where $\hat{B}(t) = B(t + \mathbb{T}_{\Xi}^{-1}(\mu_{n-1})) - B(\mathbb{T}_{\Xi}^{-1}(\mu_{n-1}))$ (for $t \geq 0$) as defined and $\hat{\varrho} = \inf\{t > 0\colon \hat{B}(t) = \exp(\mathbb{W}_{\Xi}(i))$ or $-\exp(\mathbb{W}_{\Xi}(i-1))\}$ as defined. By Lemma 9.3, for any n, conditionally on Ξ, $\{\mathbb{W}_{\Xi}(\mu_{n-1}) = i\}$ and $(\mu_1, \dots, \mu_{n-1})$, the variable $\mu_n - \mu_{n-1}$ is distributed as the first hitting time at 1 of standard reflecting Brownian motion. This yields the second part of the proposition. □

10. Proofs of Theorems 1.3–1.5. We say a few words about the proofs of Theorems 1.3–1.5, which are now consequences of Proposition 9.1 and Theorem 8.1. Let us, for example, prove the first half of Theorem 1.3 (the rest of Theorems 1.3–1.5 can be verified exactly in the same spirit). As was pointed out in Section 8, we only have to characterize the upper functions of $\max_{0 \leq k \leq n} S_k$ [where $\{S_n\}_{n \geq 0}$ is of course Sinai's simple RWRE whose environment Ξ satisfies (1.9)].

We use the embedding described in Section 8, and continue adopting the same notation. By the second part of Proposition 9.1, $\{\mu_n\}_{n \geq 1}$ is the partial sum process of a sequence of iid variables with common mean 1 and finite variance, say c^2. According to the usual law of the iterated logarithm,

$$\limsup_{n \to \infty} \frac{|\mu_n - n|}{c(2n \log\log n)^{1/2}} = 1 \quad \text{a.s.}$$

In particular, almost surely for all large n,

$$n - n^{2/3} \leq \mu_n \leq n + n^{2/3}.$$

To verify the convergence part of Theorem 1.3, let $\{a_n\}_{n \geq 1}$ be a sequence of positive nondecreasing numbers, such that

$$\sum_n \frac{a_n}{n \log n} \exp\left(-\frac{\pi^2 \sigma^2}{8} a_n\right) < \infty. \tag{10.1}$$

As in (8.4) in the continuous-time setting, we assume without loss of generality that

$$\frac{4}{\pi^2 \sigma^2} \log\log\log n \leq a_n \leq \frac{16}{\pi^2 \sigma^2} \log\log\log n.$$

Define the nondecreasing function $f(t) = a_{[t/2]} - 1$ as defined. Condition (10.1) guarantees $J_1(f) < \infty$, where J_1 is defined in Theorem 8.1. By (8.1), with probability 1, for all large t, $\sup_{0 \leq s \leq t} \mathbb{X}(s) \leq (\log t)^2 f(t)$. Consequently, for all large n,

$$\begin{aligned}
\max_{0 \leq k \leq n} \mathbb{X}(\mu_k) &\leq \sup_{0 \leq s \leq \mu_n} \mathbb{X}(s) + 1 \\
&\leq (\log \mu_n)^2 f(\mu_n) + 1 \\
&\leq \left(\log(n + n^{2/3})\right)^2 (a_n - 1) + 1 \\
&< (\log n)^2 a_n,
\end{aligned}$$

which, in view of Proposition 9.1, implies the convergence part of Theorem 1.3.

11. Weak convergence and limit distribution. Although the main object of the paper is to study almost sure asymptotics, our approach allows us to recover, as a by-product, the weak convergence of $\max_{0\le k\le n} S_k$ (as well as its explicit limit distribution) stated in Theorem A (cf. Section 1.1). In view of the embedding in Proposition 9.1, we only have to treat the continuous-time case: the diffusion process $\mathbb{X}$ with random potential $\mathbb{W}$ [cf. (3.2) for definition]. Recall $\mathbb{T}$, $\mathbb{A}$, H_- from (3.3)–(3.6), ϱ from (4.1), and $\overline{\mathbb{X}}$ and $W^\#$ from (2.1)–(2.4). Write for all $v > 0$,

$$\overline{\mathbb{X}}^{-1}(v) = \inf\{t > 0: \overline{\mathbb{X}}(t) > v\} \quad \text{as defined.} \tag{11.1}$$

Note that $\overline{\mathbb{X}}^{-1}(v)$ is nothing else by $\mathbb{T}(\varrho_{\mathbb{A}(v)})$.

PROPOSITION 11.1. *Let* $\{\mathbb{W}(x);\ x \in \mathbb{R}\}$ *be a cadlag process satisfying* (3.1), *and* $\{\mathbb{X}(t);\ t \ge 0\}$ *a diffusion process with potential* $\mathbb{W}$ [*cf*. (3.2)]. *As v goes to infinity,*

$$\frac{1}{\sqrt{v}}\left[\frac{\log \overline{\mathbb{X}}^{-1}(v)}{\sigma} - W^\#(v) \vee U_-\big(\overline{W}(v)\big)\right] \to 0, \tag{11.2}$$

in probability, where U_- *is defined in* (4.6).

COROLLARY 11.2. *We have*

$$\frac{\sigma^2}{(\log t)^2}\overline{\mathbb{X}}(t) \overset{\text{law}}{\to} \Lambda = \big(W^\#(1) \vee U_-\big(\overline{W}(1)\big)\big)^{-2}, \qquad t \to \infty \ (\textit{as defined}).$$

Moreover, the limit law is characterized either by distribution function or by Laplace transform:

$$\mathbb{P}(\Lambda < x) = 1 - \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2\pi^2}\exp\left(-\frac{(2k+1)^2\pi^2 x}{8}\right), \qquad x > 0, \tag{11.3}$$

$$\mathbb{E}\exp(-\lambda\Lambda) = \frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}}, \qquad \lambda > 0. \tag{11.4}$$

REMARK 11.3. From (11.3) or (11.4), it is noted that Λ is distributed as the last zero of B before exiting from $[-1, 1]$.

PROOF OF COROLLARY 11.2. The weak convergence follows from (11.1) and (11.2) and from the scaling property. It remains to verify (11.3)–(11.4). Observe that $\mathbb{P}(U_-(r) < x) = (x - r)/x$ for all $x > r$. Since U_- is independent of $(W^{\#}, \overline{W})$,

$$\begin{aligned}
&\mathbb{P}\big(W^{\#}(1) \vee U_-\big(\overline{W}(1)\big) < t\big)\\
&\quad = \mathbb{P}\big(W^{\#}(1) < t;\, U_-\big(\overline{W}(1)\big) < t\big)\\
&\quad = \mathbb{E}\bigg(\frac{t - \overline{W}(1)}{t} 1_{\{W^{\#}(1) \le t\}}\bigg)\\
&\quad = \int_0^t \frac{2}{t} \sum_{k=0}^{\infty} \frac{t-a}{t} \exp\bigg(-\frac{(2k+1)^2\pi^2}{8t^2}\bigg)\cos\bigg(\bigg(k+\frac{1}{2}\bigg)\frac{\pi a}{t}\bigg)\, da,
\end{aligned}$$

the last equality due to Theorem 2.1. This yields (11.3). The Laplace transform (11.4) can be obtained using (11.3) and the well-known relation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 + x^2} = \frac{\pi}{4x}\tanh\bigg(\frac{\pi x}{2}\bigg), \qquad x > 0.$$

This completes the proof of Corollary 11.2. □

PROOF OF PROPOSITION 11.1. By (4.5), $\mathbb{X}^{-1}(v) = I_1(v) + I_2(v)$ for all $v > 0$, which implies $\log(I_1(v) \vee I_2(v)) \le \log \mathbb{X}^{-1}(v) \le \log(I_1(v) \vee I_2(v)) + \log 2$. The proof of the proposition is thus reduced to showing the following estimate: for any $a > 0$ and $\varepsilon > 0$, there exists $v_0 > 0$ such that for all $v \ge v_0$,

$$\mathbb{P}\Big(\Big|\log\big(I_1(v) \vee I_2(v)\big) - \sigma W^{\#}(v) \vee \sigma U_-\big(\overline{W}(v)\big)\Big| \ge a\sqrt{v}\Big) \le 4\varepsilon. \tag{11.5}$$

According to (4.7) and (4.11), on $E_7 \cap E_9$,

$$\log\big(I_1(v) \vee I_2(v)\big) \le \big(\sigma W^{\#}(v) + \log^4 v\big) \vee \sigma U_-\big(\overline{W}(v) + \log^4 v\big).$$

Let $E_{26} = \{U_-(\overline{W}(v) + \log^4 v) \le U_-(\overline{W}(v)) + v^{1/3}\}$ as defined. Then $\mathbb{P}(E_{26}^c) \le \mathbb{P}(U_-(\log^4 v) \ge v^{1/3}) = (\log^4 v)/v^{1/3} < \varepsilon$, for large v. On the other hand, it follows from Lemmas 4.1 and 4.2 that $\mathbb{P}(E_7^c \cup E_9^c) \le (C_9 + C_{10})\exp(-\log^2 v) < \varepsilon$. Now, on $E_7 \cap E_9 \cap E_{26}$,

$$\begin{aligned}
\log\big(I_1(v) \vee I_2(v)\big) &\le \sigma v^{1/3} + \sigma W^{\#}(v) \vee \sigma U_-\big(\overline{W}(v)\big)\\
&< a\sqrt{v} + \sigma W^{\#}(v) \vee \sigma U_-\big(\overline{W}(v)\big).
\end{aligned}$$

Consequently, for all sufficiently large v,

$$\mathbb{P}\Big[\log\big(I_1(v) \vee I_2(v)\big) \ge a\sqrt{v} + \sigma W^{\#}(v) \vee \sigma U_-\big(\overline{W}(v)\big)\Big] < 2\varepsilon. \tag{11.6}$$

A similar argument based again on Lemmas 4.1 and 4.2 leads to the following lower bound for $\log(I_1(v) \vee I_2(v))$:

$$\mathbb{P}\Big[\log\big(I_1(v) \vee I_2(v)\big) \le -a\sqrt{v} + \sigma W^{\#}(v) \vee \sigma U_-\big(\overline{W}(v)\big)\Big] \le 2\varepsilon. \tag{11.7}$$

Combining (11.6) with (11.7) yields (11.5), hence Proposition 11.1. □

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[34] YOR, M. (1998). Personal communication.

LABORATOIRE DE PROBABILITÉS
UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
F-75252 PARIS CEDEX 05
FRANCE
E-MAIL: hu@proba.jussieu.fr

L.S.T.A.
UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
F-75252 PARIS CEDEX 05
FRANCE
E-MAIL: shi@ccr.jussieu.fr