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ON THE BLOCKWISE BOOTSTRAP FOR EMPIRICAL PROCESSES FOR STATIONARY SEQUENCES

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In this paper, we study the weak convergence to an appropriate Gaussian process of the empirical process of the block-based bootstrap estimator proposed by Künsch for stationary sequences. The classes of processes investigated are weak dependent and associated sequences. We also prove that, differently from the independent situation, the bootstrapped estimator of the mean of certain dependent sequences satisfies the central limit theorem while the mean of the original sequence does not.

0. Introduction.

Efron's bootstrap (1979) provides a very important nonparametric technique to study the sampling distribution of statistics. Bickel and Freedman (1981) proved that the bootstrap also works for an empirical process in the i.i.d. case. In his Remark 2.1, Singh (1981) pointed out that, for dependent random variables, the variance of the bootstrap estimator of the mean does not have the same asymptotic behavior as the variance of the mean itself and therefore the inconsistencies appear. In order to compensate for this deficiency, Künsch (1989) and Liu and Singh (1992) proposed a block bootstrap to estimate the sample distribution of a statistic for general stationary observations. The technique involves the selection of k blocks of l consecutive observations with replacement from the blocks of observations $(X_{j+1}, X_{j+2}, \dots, X_{j+l})$, $j = 0, 1, \dots, n-l$.

Several recent papers studied the consistency of the bootstrapped estimator for the sample mean or for empirical processes under mixing types of dependence. Shao and Yu (1993) established several central limit theorems a.s. conditionally on the observations for the bootstrap estimator of the sample mean for several classes of mixing sequences. In other recent papers, it was shown that, conditionally on the observations, the bootstrapped empirical process of mixing sequences converges weakly to a corresponding Gaussian process, almost surely. In papers by Naik-Nimbalkar and Rajarshi (1994) and Bühlmann (1994), the sequences considered are strongly mixing at a polynomial rate. In Shao and Yu (1992), the problem was solved for ρ-mixing sequences having a logarithmic rate.

With a view toward applications, the aim of this paper is to formulate general theorems for the weak convergence of the bootstrapped estimator of the

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empirical processes to a Gaussian process, almost surely given the observations. Theorem 2.2 gives sufficient conditions for the validity of such a result in terms of moments, easy to verify. By verifying the conditions of Theorem 2.2 for strongly mixing sequences, we obtain the weak convergence to a Brownian bridge of the bootstrapped empirical process under a mixing rate which significantly improves that one used in Naik-Nimbalkar and Rajarshi (1994) and also Bühlmann (1994). We also establish the validity of the blockwise bootstrapped estimator for an empirical process of an associated sequence. The limiting distribution obtained is that expected one, the same Brownian bridge to which the empirical process of strongly mixing sequences or of associated sequences converges, defined in Billingsley [(1968), pages 200–201]. It comes somewhat as a surprise to discover that the sufficient conditions we use for convergence of the blockwise bootstrapped estimator of the empirical process are weaker than those used in establishing the convergence of an empirical process for strong mixing or associated sequences [see Philipp (1986), Shao (1986), Yu (1993)].

However, maybe more surprising is the following result contained also in this paper. In the independent case, Giné and Zinn (1989) proved that the central limit theorem (CLT) for the bootstrapped estimator for the sample mean necessarily implies that the sample mean itself satisfies the central limit theorem. Here we shall prove that the situation is different in the dependent case, and we shall point out a class of examples (Remark 2.1) for which the blockwise bootstrapped estimator of the mean satisfies the central limit theorem, its variance is asymptotically equivalent to the variance of the sample mean, but the sample mean itself fails to satisfy the CLT. For such dependent situations, the bootstrapped estimator for the mean appears to be even more important than for the independent case. On one hand, it allows us to construct confidence intervals for the sample mean, and on the other hand, its variance provides a consistent estimator of the variance of the sample mean, a quantity very difficult to estimate otherwise.

1. Notations. Let $\{X_n\}_{n\in Z}$ be a stationary sequence of random variables with common continuous distribution function $F(t) = P(X_0 \le t)$ on a probability space $(\Omega, \mathscr{F}, P)$. Assume $0 \le X_0 \le 1$, which can always be arranged by a transformation. The empirical process is defined as

$$B_n(t) = \sqrt{n}(F_n(t) - F(t)),$$

where

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le t) \tag{1.1}$$

is the empirical distribution function of $\{X_n\}$. Under certain regularity conditions [see Billingsley (1968), Deo (1973), Philipp (1986), Shao (1986), Yu (1993)], we have that

$$B_n(t) \to_{\mathscr{D}} B(t)$$

in the spaces $D[0,1]$ endowed with Skorohod topology where B is a Gaussian process specified by $EB(t)=0$ for every t, and for every t and s,

$$\operatorname{cov}(B(s), B(t)) = \sum_{k=-\infty}^{\infty} \operatorname{cov} I(X_0 \le s, X_k \le t). \tag{1.2}$$

In addition,

$$\lim_{n\to\infty} \operatorname{cov}(B_n(s), B_n(t)) = \operatorname{cov}(B(s), B(t)). \tag{1.3}$$

One can see from (1.2) and (1.3) that the variance of $B_n(t)$ is a rather complicated quantity and, in order to estimate it, a consistent bootstrapping procedure is desirable.

According to Künsch (1989) and further modifications by Politis and Romano (1992) and independently by Shao and Yu (1993), the block-based bootstrap estimators of the mean and empirical process are defined as follows.

Let k and l be two integers such that $n=kl$. Let $T_{n1}, T_{n2}, \dots, T_{nk}$ be i.i.d. random variables each having uniform distribution on $\{1,2,\dots,n\}$. Define the triangular array $\{X_{ni};\ 1\le i\le n+l\}$ by $X_{ni}=X_i$ for $1\le i\le n$ and $X_{ni}=X_{i-n}$ for $n<i\le n+l$. In other words, we extend our sample of size n by another l observations, namely, $X_1, X_2, \dots, X_l$.

Then the bootstrapped estimator of the mean is defined as

$$\bar{X}_n^* = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{l}\sum_{j=T_{ni}}^{T_{ni}+l-1} X_j, \tag{1.4}$$

and the bootstrapped estimator of the empirical process is

$$F_n^*(t) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{l}\sum_{j=T_{ni}}^{T_{ni}+l-1} I(X_{nj}\le t). \tag{1.5}$$

The bootstrapped empirical process is then defined as

$$B_n^*(t) = n^{1/2}[F_n^*(t) - F_n(t)]. \tag{1.6}$$

In the following text, E^*, var^* and so on will denote the moments under the conditional probability measure P^* induced by the resampling mechanism, that is, P^* is the conditional probability given $(X_1, X_2, \dots, X_n)$.

In this paper, we shall investigate the weak convergence of $B_n^*(t)$ defined by (1.6) to $B(t)$ defined by (1.2) in the Skorohod topology on $D[0,1]$, almost surely. In order to prove these results, we shall investigate first the validity of the CLT for $\bar{X}_n^*$ defined by (1.4) when the variables are bounded. This will be done in Theorem 2.1. Next, we provide sufficient conditions for the weak convergence of $B_n^*(t)$ to $B(t)$ in the sense described above, which are summarized in Proposition 4.1 and Theorem 2.2. Theorem 2.3 deals with the special case of strongly mixing sequences, which is a very important class of dependent random variables. Doukhan's (1994) book and Bradley's (1986) survey paper contain many examples of such sequences including Gaussian processes, time series, Markov chains and so on.

DEFINITION 1.1. Given two sub σ-algebras of $\mathscr{F}$, $\mathscr{A}$ and $\mathscr{B}$, the strong mixing coefficient is defined as

$$\alpha(\mathscr{A}, \mathscr{B}) = \sup(|P(AB) - P(A)P(B)|;\ A \in \mathscr{A},\ B \in \mathscr{B}).$$

A stationary sequence $\{X_n\}_{n \in Z}$ is called strongly mixing if $\alpha_n \to 0$ when $n \to \infty$:

$$\alpha_n = \alpha(\sigma(X_i;\ i \le 0), \sigma(X_k;\ k \ge n)).$$

The last theorem is an application of Theorem 2.2 to associated sequences of random variables, which is a class of dependent random variables which appears in the context of the percolation models, Ising models of statistical mechanics and statistics in general. We refer to Newman (1984) for a survey.

DEFINITION 1.2. A finite collection of random variables $X_1, \dots, X_m$ is said to be associated if, for any two coordinatewise nondecreasing functions f, g on R^m,

$$\operatorname{cov}(f(X_1, \dots, X_m), g(X_1, \dots, X_m)) \ge 0$$

whenever the covariance exists.

A sequence $\{X_i\}$ is said to be associated if every finite subcollection is associated.

Several notations will be used everywhere in the paper:

$$S_{li} = X_i + \cdots + X_{i+l-1}; \qquad S_{l1} = S_l, \tag{1.7}$$

$$\overline{X}_{li} = \frac{1}{l}(X_i + \cdots + X_{i+l-1}); \qquad \overline{X}_{l1} = \overline{X}_l. \tag{1.8}$$

For a stationary sequence with the distribution function $F(x)$, we denote

$$Y_{li}(s,t) = \sum_{j=i}^{i+l-1} I(s < X_j \le t) - (F(t) - F(s)) \tag{1.9}$$

and

$$Y_{li}(s) = \sum_{j=i}^{i+l-1} I(X_i \ge s) - (1 - F(s)). \tag{1.10}$$

The notation $\ll$ is sometimes used to replace the Vinogradov symbol 0.

The paper is organized in the following way. Section 2 contains the statements of the main results. In Section 3, we prove some preliminary results. Sections 4 and 5 contain the proofs of the theorems.

2. Results. Our first theorem is a step in proving the convergence to the Brownian bridge of $B_n^*(t)$. It provides sufficient conditions for the convergence in distribution to the normal distribution P^*-almost surely for the sample mean and it also has interest in itself.

THEOREM 2.1. *Assume* $\{X_n\}_{n\in Z}$ *is a stationary sequence of random variables such that the variables are bounded almost surely. Denote* $\mu = EX_0$ *and assume* $l = l_n$, $k = k_n$ *are sequences of integers such that*

$$l^2/n \to 0, \qquad l \to \infty \quad \text{as } n \to \infty \quad \text{and} \quad n = lk. \tag{2.1}$$

Assume $\{X_n\}$ *satisfies*

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right)\Big/ n \to \sigma^2 > 0 \quad \text{as } n \to \infty \tag{2.2}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \frac{((X_i - \mu) + \cdots + (X_{i+l-1} - \mu))^2}{l} \to \sigma^2 \quad \text{a.s. as } n \to \infty. \tag{2.3}$$

Then, for $\overline{X}_n^*$ *defined by (1.4), we have*

$$\operatorname{var}^*(\sqrt{n}\ \overline{X}_n^*) \to \sigma^2 \quad \text{a.s. as } n \to \infty \tag{2.4}$$

and

$$\sqrt{n}(\overline{X}_n^* - \overline{X}_n) \to_{\mathscr{D}} N(0, \sigma^2) \quad P^*\text{-a.s. as } n \to \infty. \tag{2.5}$$

The following result provides sufficient conditions for the existence of a blockwise bootstrapped estimator of the mean which satisfies the conclusions of the above theorem. One can easily see that if $\{X_n\}$ is a stationary and ergodic sequence of bounded random variables, one can construct a sequence $l_n(w) \to \infty$ as $n \to \infty$ for which the conclusions (2.4) and (2.5) of Theorem 2.1 hold [with l_n replaced by $l_n(w)$].

If we restrict even more the dependence considering the class of strong mixing random variables, the situation is different. For strong mixing random variables, a similar result holds with l_n deterministic, selected to be the same for all trajectories. In general, the size of l_n which assures the consistency for strong mixing sequences is related to the size of the mixing coefficients. In the next result, we are rather interested in the weakest possible conditions which, imposed to the strong mixing coefficients, will still permit the construction of a blockwise bootstrapped estimator of the mean satisfying the conditions of Theorem 2.1. The condition (2.6) we impose next is optimal in the context of the law of large numbers for bounded strong mixing sequences as proved by Berbee (1987).

PROPOSITION 2.1. *Assume* $\{X_n\}$ *is a strongly mixing stationary sequence of bounded random variables satisfying (2.2). Assume that the strong mixing coefficients satisfy*

$$\sum_n \frac{\alpha_n}{n} < \infty. \tag{2.6}$$

Then there is a sequence $l_n \to \infty$ *such that the conclusions (2.4) and (2.5) of Theorem 2.1 hold.*

REMARK 2.1. Proposition 2.1 shows that, for a large class of dependent random variables, there is a blockwise bootstrapped estimator of the mean whose variance can be used by (2.2) and (2.4) to approximate the limiting variance of the sample mean. Moreover, the bootstrapped estimator satisfies the central limit theorem almost surely. However, the mean itself, $\overline{X}_n$, does not necessarily satisfy the CLT. Bradley (1989) constructed a stationary sequence, bounded, pairwise independent, strongly mixing at a rate $\alpha(n) = O(1/n)$ and such that $S_n/\sqrt{n}$ does not converge in distribution to a nondegenerate normal distribution. This example satisfies, of course, all the conditions of Corollary 2.1 and Proposition 2.1. As a matter of fact, for this example the conclusions (2.4) and (2.5) hold for any sequence $l_n \to \infty$ such that

$$l_n = O\big(n^{1/2}(\log n)^{-3}\big), \qquad l_n = l(2^k) \quad \text{for } 2^k \le n < 2^{k+1}. \tag{2.7}$$

REMARK 2.2. Condition (2.2) is satisfied by a variety of stochastic processes. A large class of examples are provided by processes for which the spectral density exists and is continuous at the origin. In this case, $\sigma^2 = 2\pi f(0)$ [Ibragimov and Linnik (1971), Theorem 18.2.1]. Sufficient conditions for the validity of (2.2) in terms of the covariances of sums of variables in disjoint sets are discussed in recent papers by Bradley (1992) and Bradley and Utev (1994). One aspect of the statistical importance of Theorem 2.1 is that, by its conclusion, (2.4) provides an alternative estimator for σ^2, a quantity hard to estimate. This estimator can also be used to test (2.2) or (2.3), while (2.5) is useful to construct confidence intervals for σ^2.

REMARK 2.3. A result of type (2.5) in probability for strong mixing sequences of random variables satisfying the central limit theorem was obtained independently by Radulovic (1995).

We provide next some general sufficient conditions in terms of moments for convergence of the blockwise bootstrapped estimator of the empirical process of a stochastic process to a corresponding Brownian bridge. The proof is based on a more general result in terms of almost sure convergences given in Section 4.

Without loss of generality, we assume in the next three theorems $0 \le X_0 \le 1$ because this can be always achieved by a transformation.

THEOREM 2.2. *Let $\{X_n\}_{n\in Z}$ be a stationary sequence of random variables. Let l_n, k_n be sequences of natural numbers satisfying*

$$n^h \ll l \ll n^{1/3-a} \quad \textit{for some } 0 < h < \tfrac{1}{3} - a, \quad 0 < a < \tfrac{1}{3}, \tag{2.8}$$

$l_n = l(2^k)$ for $2^k \le n < 2^{k+1}$, $l_n \to \infty$ as $n \to \infty$ and $n = k_n l_n$.

Assume there are two constants C_1 and C_2 such that, for some $\gamma > 0$ and every $0 \le s, t \le 1$,

$$\sup_{n>m}\left|\sum_{i=m}^{n} \operatorname{cov}(I(s < X_0 \le t), I(s < X_i \le t))\right| \le C_1 m^{-\gamma}, \tag{2.9}$$

and for every $1 \le m \le n$,

$$\operatorname{var}\left(\sum_{i=1}^{m} Y_{l_n i}^2(s,t)\right) \le C_2 m l_n^4 \tag{2.10}$$

[where $Y_{li}(s,t)$ *is defined by (1.9)].*

Then

$$B_n^*(t) \to_{\mathscr{D}} B(t) \quad \text{as } n \to \infty \tag{2.11}$$

P^**-almost surely in the Skorohod topology on* $D[0,1]$, *where* B *is a Brownian bridge with the covariance structure given by (1.2). Moreover, if the distribution of* X_0 *is continuous,* $P^*(B(t) \in C[0,1]) = 1$.

We shall verify the conditions of Theorem 2.2 for classes of strong mixing or associated random variables and we shall establish the following two theorems. The first one improves on the mixing rates used in the papers by Naik-Nimbalkar and Rajarshi (1994) or Bühlmann (1994). As a special feature, the conditions imposed to the strongly mixing coefficients in Theorem 2.3 or to the covariances of the associated sequence in Theorem 2.4 are both weaker than those used in the best known results so far, which establish the convergence of $B_n(t) \to_{\mathscr{D}} B(t)$ for these classes of processes [Shao (1986), Yu (1993)].

THEOREM 2.3. *Assume* $\{X_n\}$ *is a strongly mixing stationary sequence of random variables. Assume that* l_n *satisfies (2.8) and*

$$\sum_{m>n} \alpha_m = O(n^{-\gamma}) \quad \text{for some } \gamma > 0. \tag{2.12}$$

Then the series in (1.2) is convergent and the conclusion (2.11) of Theorem 2.2 holds.

THEOREM 2.4. *Assume* $\{X_n\}$ *is a stationary associated sequence of random variables and* X_0 *has a continuous bounded density. Assume that* l_n *satisfies (2.8) and*

$$\sum_{i=m}^{\infty} \operatorname{cov}^{1/3}(X_0, X_i) = O(m^{-\gamma}) \quad \text{for some } \gamma > 0. \tag{2.13}$$

Then the series in (1.2) is convergent and the conclusion (2.11) of Theorem 2.2 holds with $P^*(B(t) \in C[0,1]) = 1$.

3. Preliminary results. In this section, we fix a realization of the stochastic process, $\{x_i\}$. Therefore, the randomness is due only to resampling procedure. By using the notations (1.4) and (1.8), we have the following.

PROPOSITION 3.1. *Let* $\{x_i\}_{i\ge 1}$ *be a bounded sequence of real numbers. Let* k *and* l *be integers such that* $n = kl$ *and*

$$l^2/n \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

For each n, *let* $\{T_{n1}, T_{n2}, \dots, T_{nk}\}$ *be i.i.d. uniform on* $\{1, 2, \dots, n\}$ *and* $(x_{n1}, x_{n2}, \dots, x_{n,\,l+n}) = (x_1, x_2, \dots, x_n, x_1, x_2, \dots, x_l)$. *Assume*

$$V_n = \frac{1}{k}\sum_{i=1}^{n}(\bar{x}_{li} - \bar{x}_n)^2 \to \sigma^2 > 0 \quad \textit{as } n \to \infty. \tag{3.2}$$

Then

$$\sqrt{n}(\bar{x}_n^* - \bar{x}_n) \to_{\mathscr{D}} N(0, \sigma^2) \quad \textit{as } n \to \infty.$$

PROOF. It is obvious that we have the representation

$$\bar{x}_n^* = \frac{1}{k}\sum_{j=1}^{k}\left(\sum_{i=1}^{n} I(T_{nj} = i)\bar{x}_{li}\right). \tag{3.3}$$

By (3.3) and the definition of x_{ni}, $1 \le i \le n + l$, we have

$$E\bar{x}_n^* = \bar{x}_n.$$

Using now the independence of $\{T_{nj}\}_j$ and the fact that the sets $\{I(T_{nj} = i);\ 1 \le i \le n\}$ are disjoint, we obtain

$$\begin{aligned}
\mathrm{var}(\bar{x}_n^*) &= \frac{1}{k^2}\sum_{j=1}^{k} E\left(\sum_{i=1}^{n} I(T_{nj} = i)(\bar{x}_{li} - \bar{x}_n)\right)^2 \\
&= \frac{1}{k^2}\sum_{j=1}^{k}\sum_{i=1}^{n} EI(T_{nj} = i)(\bar{x}_{li} - \bar{x}_n)^2 \\
&= \frac{1}{kn}\sum_{i=1}^{n}(\bar{x}_{li} - \bar{x}_n)^2.
\end{aligned}$$

Therefore,

$$\mathrm{var}(\sqrt{n}\bar{x}_n^*) = \frac{1}{k}\sum_{i=1}^{n}(\bar{x}_{li} - \bar{x}_n)^2,$$

and by condition (3.2), we have

$$\lim_{n\to\infty} \mathrm{var}(\sqrt{n}\bar{x}_n^*) = \sigma^2 > 0. \tag{3.4}$$

Denote $U_{nj} = (\sqrt{n}/k)\sum_{i=1}^{n} I(T_{nj} = i)(\bar{x}_{li} - \bar{x}_n)$ and notice that $\{U_{nj}\}_{1\le j\le k}$ are independent and $\sqrt{n}(\bar{x}_n^* - \bar{x}_n) = \sum_{j=1}^{k} U_{nj}$.

By (3.4), in order to establish the CLT we have only to check the Lindeberg conditions, which is equivalent in this setting to proving that

$$E \max_{1\le j\le k} U_{nj}^2 \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Once again, by the fact that $\{I(T_{nj} = i);\ 1 \le i \le n\}$ are disjoint and x_i's are bounded, we have, for every $1 \le j \le k$,

$$|U_{nj}| \le \frac{\sqrt{n}}{k}\max_{1\le i\le n}|\bar{x}_{li} - \bar{x}_n| = O\left(\frac{\sqrt{n}}{k}\right),$$

whence

$$U_{nj}^2 = O\left(\frac{l^2}{n}\right),$$

which proves (3.5) by (3.1). □

In order to state the next propositions, we denote by $f_n^*(t)$ the expression in (1.5) with X_{ni} replaced by x_{ni} and also

$$Z_n^*(t) = \sqrt{n}(f_n^*(t) - f_n(t)). \tag{3.6}$$

We shall also use the notations

$$f_{li}(s) = \frac{1}{l}\sum_{j=i}^{i+l-1} I(x_j \le s), \qquad f_n(s) = f_{n1}(s),$$

$$f_{li}(s,t) = f_{li}(s) - f_{li}(t), \qquad f_n(s,t) = f_{n1}(s,t),$$

$$V_n(s,t) = \mathrm{var}(Z_n^*(t) - Z_n^*(s)).$$

PROPOSITION 3.2. *Let $\{x_i; 1 \le i \le n\}$, $\{x_{ni};\ 1 \le i \le n+l\}$ and $\{T_{ni}; 1 \le i \le k\}$ as in Proposition 3.1, $0 \le x_i \le 1$ for every $1 \le i \le n$. Assume*

$$\text{for some } 0 < a < \tfrac{1}{2}, \qquad l = O(n^{1/2-a}), \quad n = kl, \tag{3.7}$$

and there are constants $K > 0$, $0 < b < 1$, $c > 0$, such that, for every s and t in $[0,1]$ and every $n \ge 1$, we have

$$V_n(s,t) = \frac{1}{k}\sum_{i=1}^{n}(f_{li}(s,t) - f_n(s,t))^2 \le K(|s-t|^b + n^{-c}). \tag{3.8}$$

Then $Z_n^*(t)$ defined by (3.6) is tight, that is, for every $\varepsilon, \eta > 0$ there is a δ, $0 < \delta < 1$ and N_0 such that, for every $n \ge N_0$,

$$P\Big(\sup_{|t-s|<\delta} |Z_n^*(t) - Z_n^*(s)| \ge \varepsilon\Big) \le \eta,$$

and as a consequence, if Y is taken as a limiting distribution on a subsequence, $P(Y \in C[0,1]) = 1$.

PROOF. The proof of tightness is based on the approach used by Naik-Nimbalkar and Rajarshi (1994).

Notice that, similarly to (3.3), we have the following representation:

$$Z_n^*(t) - Z_n^*(s) = \sum_{j=1}^{k} U_{nj}(s,t),$$

where, for each $1 \le j \le k$,

$$U_{nj}(s,t) = \frac{\sqrt{n}}{k}\sum_{i=1}^{n} I(T_{nj} = i)(f_{li}(s,t) - f_n(s,t))$$

is a triangular array of independent random variables and, after a simple computation, we obtain by (3.8),

$$\begin{aligned} V_n(s,t) &= \mathrm{var}(Z_n^*(t) - Z_n^*(s)) \\ &= \frac{1}{k}\sum_{i=1}^{n}(f_{li}(s,t) - f_n(s,t))^2 \qquad (3.9)\\ &= O(|s-t|^b + n^{-c}), \end{aligned}$$

and also by (3.7),

$$|U_{nj}(s,t)| = O\left(\frac{\sqrt{n}}{k}\right) = O\left(\frac{l}{\sqrt{n}}\right) = O(n^{-a}). \qquad (3.10)$$

Therefore, by Bennett's inequality [Pollard (1984), page 192], for every $\eta > 0$,

$$P(|Z_n^*(t) - Z_n^*(s)| > \eta) \le 2\exp\left(\frac{-\eta^2}{C_1(|s-t|^b + n^{-c})} B\left(\frac{C_2 n^{-a}}{C_1(|s-t|^b + n^{-c})}\right)\right),$$

where $B(x)$ is continuous and decreasing and $B(0+) = 1$. This is the key step in applying now a restricted chaining argument given in Theorem 2.6 in Pollard (1984) applied with the semimetric $d(s,t) = C_3|t-s|^{b/2}$, $0 < b < 1$, and the covering number $N(\delta, d, T) = 1 + [(\delta/C_3)^{-2/b}]$, where $[x]$ denotes the integer part of x and $T = [0,1]$. We shall not give here the detail of the proof. We just notice that (3.9) and (3.10) are the only ingredients needed to make the arguments in Naik-Nimbalkar and Rajarshi (1994), clearly expressed at pages 990–992, work in our context. □

PROPOSITION 3.3. *Assume all the conditions of Proposition 3.2 are satisfied and in addition, for every s, t in $[0,1]$,*

$$\lim_{n\to\infty} V_n(s,t) \quad \text{exists}. \qquad (3.11)$$

Then, for every s, t in $[0,1]$, we have that

$$\lim_{n\to\infty} \frac{1}{k}\sum_{i=1}^{n}(f_{li}(s) - f_n(s))(f_{li}(t) - f_n(t)) = g(s,t) \qquad (3.12)$$

exists and

$$Z_n^*(t) \to_{\mathscr{D}} B(t) \quad \text{on } D[0,1], \qquad (3.13)$$

where $B(t)$ is a Brownian bridge with the covariance structure $g(s,t)$ given by the limit in (3.12) and $P(B(t) \in C[0,1]) = 1$.

PROOF. In order to prove Proposition 3.3, we mention that Proposition 3.2 implies that $Z_n^*(t)$ is tight, and if Y is a limiting process on a subsequence, then $P(Y \in C[0,1]) = 1$. According to Theorems 15.4 and 15.5 in Billingsley (1968), we have only to prove that the finite-dimensional distributions of $Z_n^*(t)$

converge weakly to the corresponding ones of the Brownian bridge $B(t)$. We show first that (3.11) implies (3.12).

By the fact that

$$\operatorname{var}(Z_n^*(t) - Z_n^*(s)) = \operatorname{var} Z_n^*(s) + \operatorname{var} Z_n^*(t) - 2\operatorname{cov}(Z_n^*(t), Z_n^*(s)),$$

it is easy to see by (3.9) that

$$V_n(s,t) = V_n(0,t) + V_n(0,s) - \frac{2}{k}\sum_{i=1}^{n}(f_{li}(s) - f_n(s))(f_{li}(t) - f_n(t)).$$

By (3.11), the limits $\lim_{n\to\infty} V_n(s,t)$, $\lim_{n\to\infty} V_n(0,s)$ and $\lim_{n\to\infty} V_n(0,t)$ all exist, and as a consequence, the limit in (3.12) exists, too. Moreover, we have

$$\operatorname{cov}(Z_n^*(t), Z_n^*(s)) = \frac{1}{k}\sum_{i=1}^{n}(f_{li}(s) - f_n(s))(f_{li}(t) - f_n(t)). \tag{3.14}$$

Now let $0 \le t_1 < t_2 < \cdots < t_m \le 1$ and let $\alpha_1, \alpha_2, \dots, \alpha_m$ be real numbers. We have to prove only that

$$\sum_{u=1}^{m} \alpha_u Z_n^*(t_u) \to_{\mathscr{D}} \sum_{u=1}^{m} \alpha_u B(t_u) \quad \text{as } n \to \infty.$$

Remark that

$$\sum_{u=1}^{m} \alpha_u Z_n^*(t_u) = \frac{\sqrt{n}}{k}\sum_{j=1}^{k}\sum_{i=1}^{n} I(T_{nj} = i)\sum_{u=1}^{m}\alpha_u(f_{li}(t_u) - f_n(t_u)).$$

Denote $y_i = \sum_{u=1}^{m} \alpha_u I(x_i \le t_u)$ and apply now Proposition 3.1 with x_i replaced by y_i. We obtain

$$\sum_{u=1}^{m} \alpha_u Z_n^*(t_u) \to_{\mathscr{D}} N(0,\sigma^2) \quad \text{as } n \to \infty,$$

provided the following limit exists:

$$\sigma^2 = \lim_{n\to\infty} \operatorname{var} \sum_{u=1}^{m} \alpha_u Z_n^*(t_u). \tag{3.15}$$

To prove (3.15), we use the trivial computation

$$\begin{aligned}\operatorname{var}\left(\sum_{u=1}^{m} \alpha_u Z_n^*(t_u)\right) &= \sum_{u=1}^{m} \alpha_u^2 \operatorname{var} Z_n^*(t_u)\\ &\quad + 2\sum_{u=1}^{m-1}\sum_{v=u+1}^{m} \alpha_u\alpha_v \operatorname{cov}(Z_n^*(t_u), Z_n^*(t_v)).\end{aligned}$$

By (3.12), (3.14) and (3.4), it follows that

$$\begin{aligned}\sigma^2 &= \lim_{n\to\infty} \operatorname{var}\biggl(\sum_{u=1}^{m} \alpha_u Z_n^*(t_u)\biggr)\\ &= \sum_{u=1}^{m} \alpha_u^2 g(t_u, t_u) + 2\sum_{u=1}^{m-1}\sum_{v=u+1}^{m} \alpha_u\alpha_v g(t_u, t_v)\\ &= \operatorname{var}\biggl(\sum_{u=1}^{m} \alpha_u B(t_u)\biggr),\end{aligned}$$

where $B(u)$ is the desired Brownian bridge. This convergence proves (3.15) and completes the proof of this proposition. $\square$

4. Proofs of Theorems 2.1 and 2.2. In this section, we consider, instead of the sequences of numbers $\{x_i\}$ from the preceding section, the sequences of random variables $\{X_i\}$. If we fix w and denote $x_i = X_i(w)$, all the propositions of Section 3 can be viewed as a behavior of a fixed trajectory $X_1(w), X_2(w), \dots$. Fixing the variables $\{X_k\}$, recall that P^*, E^*, var^*, refer to the probability, expected value, variance, conditioned on $\{X_k\}$. In other words, if we replace in Propositions 3.1–3.3 P with P^*, $x_i = X_i(w)$ and assume, according to the case, that the conditions (3.2), (3.8) or (3.12) hold almost surely, then the results in all these three propositions also hold almost surely under P^*.

In this section, we shall analyze the conditions (3.2), (3.8) and (3.12) which require now to prove almost sure results for some functions of the random variables $\{X_i\}$. Our goal is to give general sufficient conditions in terms of moments in order for the required almost sure results to hold.

For the proof of Theorem 2.1, we need the following well-known lemma which is a consequence of Theorem 3.7.6 of Stout (1974). This lemma was also used by Shao and Yu (1993).

LEMMA 4.1. *Let $\{\xi_n\}$ be a sequence of random variables with $E\xi_n = 0$ for every $n \ge 1$ and $\sup_{n\ge1} E\xi_n^2 < \infty$. Assume there is a constant $C > 0$ such that, for any $n \ge 1$,*

$$\sup_{k>0} E\biggl(\sum_{i=k+1}^{k+n} \xi_i\biggr)^2 \le Cn. \tag{4.1}$$

Then

$$\lim_{n\to\infty} \frac{|\sum_{i=1}^{n} \xi_i|}{n^{1/2}\log^2 n} = 0 \quad \textit{a.s.}$$

4.1. *Proof of Theorem* 2.1. In order to prove Theorem 2.1, we shall verify the conditions of Proposition 3.1. By using the notation (1.8), we have only to prove the almost sure variant of (3.2), namely,

$$V_n = \frac{1}{k}\sum_{i=1}^{n}(\overline{X}_{li} - \overline{X}_n)^2 \to \sigma^2 \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{4.2}$$

Without any loss of generality, we assume that X_i's are all centered ($\mu = 0$). Simple computations show that

$$\begin{aligned} V_n &= \frac{1}{k}\sum_{i=1}^{n}(\overline{X}_{li})^2 - \frac{n}{k}(\overline{X}_n)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{S_{li}^2 - ES_{li}^2}{l} + \frac{\operatorname{var} S_l}{l} - l\overline{X}_n^2. \end{aligned} \tag{4.3}$$

One can see now that by (2.3), the first term in (4.3) converges to 0 almost surely. By Lemma 4.1 applied to $\{X_n\}$ and by the size of l given in (2.1), the last term in (4.3) is approaching 0 almost surely as $n \to \infty$. Condition (2.2) shows that the limit in (4.3) is σ^2 almost surely. Therefore, Proposition 3.1 applies and the result follows. □

We shall base the proof of Proposition 2.1 on the following lemma, which can be found in Rio [(1995), inequality (5.1) on page 936] or in Peligrad [(1994), Theorem 2.2], with $\delta = \infty$.

LEMMA 4.2. *Assume $\{X_n\}_{n\in Z}$ is a strongly mixing sequence of centered random variables such that, for every i, $|X_i| < C$ a.s. Then there is a universal constant K such that, for every $x > 0$ and every n,*

$$P\Big(\max_{1\le i\le n}|S_i| > x\Big) \le Kx^{-2}\bigg(\sum_{i=1}^{n}EX_i^2 + C^2 n\sum_{i=1}^{n}\alpha_i\bigg).$$

4.2. *Proof of Proposition* 2.1. As before, we assume without loss of generality $\mu = 0$. By Theorem 2.1, we have only to prove that there is a sequence $l_n \to \infty$ satisfying (2.1) and such that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{S_{li}^2 - ES_{li}^2}{l} \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{4.4}$$

We remark that $\{S_{li}\}_i$ is a strong mixing sequence of random variables satisfying $|S_{li}| < Cl$ a.s. for every i, where C is such that $|X_i| < C$ a.s. The strong mixing coefficients associated to the sequence $\{S_{li}\}_i$ will be denoted by $\overline{\alpha}_i$ and obviously,

$$\overline{\alpha}_i \begin{cases} \le \alpha_{i-l}, & \text{for } i \ge l, \\ \le 1, & \text{for } 1 \le i < l. \end{cases} \tag{4.5}$$

By Lemma 4.2, stationarity and (4.5), we get, for every $n \geq 1$, $l = l_n$, for some numerical constants K_1 and K_2,

$$
\begin{aligned}
P\biggl(\max_{1\le i\le n}\biggl|\sum_{j=1}^{i} S_{lj}^2 - ES_{lj}^2\biggr| > x\biggr) &\le K_1 x^{-2}\biggl(nES_l^4 + nl^4\sum_{i=1}^{n}\overline{\alpha}_i\biggr) \\
&\le K_2 x^{-2} n l^5 \sum_{i=1}^{n}\alpha_i.
\end{aligned}
\tag{4.6}
$$

In order to establish (4.4) by the Borel–Cantelli lemma, it is sufficient to show that, for every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} P\biggl(\max_{2^k\le n<2^{k+1}} \frac{1}{n}\biggl|\sum_{i=1}^{n}\frac{S_{li}^2 - ES_{li}^2}{l}\biggr| \ge \varepsilon\biggr) < \infty.$$

Without restricting the generality, we shall consider that $l_n = l(2^k)$ for $2^k \leq n < 2^{k+1}$.

Relation (4.6) implies

$$
\begin{aligned}
&P\biggl(\max_{2^k\le n<2^{k+1}} \frac{1}{n}\biggl|\sum_{i=1}^{n}\frac{S_{li}^2 - ES_{li}^2}{l}\biggr| \ge \varepsilon\biggr)\\
&\quad\le P\biggl(\max_{1\le n\le 2^{k+1}}\biggl|\sum_{i=1}^{n}\bigl(S_{l(2^k),\,i}^2 - ES_{l(2^k),\,i}^2\bigr)\biggr| \ge \varepsilon 2^k l(2^k)\biggr)\\
&\quad= 0\biggl(2^{-k} l^3(2^k)\sum_{i=1}^{2^k}\alpha_i\biggr),
\end{aligned}
$$

and the result follows if l_n can be constructed to satisfy

$$\sum_n \frac{l_n^3 \sum_{i=1}^n \alpha_i}{n^2} < \infty.$$

By an elementary argument, we see that the existence of such a sequence l_n is guaranteed under the condition (2.6).

We shall base the proof of Theorem 2.2 on the following result.

PROPOSITION 4.1. *Let $\{X_n\}_{n\in Z}$ be a stationary stochastic process, $0 \leq X_0 \leq 1$, with a continuous distribution function $F(t)$. Assume l satisfies (2.8) with $\frac{1}{3}$ replaced by $\frac{1}{2}$ and the series (1.2) is convergent for every s and t. Assume also the following conditions are satisfied: there is $K_1 > 0$, $d > 0$, $0 < b < 1$, such that, for every $0 \leq s, t \leq 1$ and every n,*

$$\sigma_n(s,t) = \frac{\operatorname{var}(\sum_{i=1}^n I(s < X_i \le t))}{n} \le K_1(|s-t|^b + n^{-d}), \tag{4.7}$$

$$\sup_{s,\,t\in[0,\,1]} \frac{1}{n}\biggl|\sum_{i=1}^{n}\frac{Y_{li}^2(s,t) - EY_{li}^2(s,t)}{l}\biggr| = O(n^{-g}) \quad \text{a.s.} \quad \text{as } n\to\infty, \tag{4.8}$$

for some $g > 0$.

Then (2.11) holds, that is, $B_n^*(t) \to_{\mathscr{D}} B(t)$ as $n \to \infty$, P^*-a.s. where $B(t)$ is defined by (1.2) and $P^*(B(t) \in C[0,1]) = 1$, where $C[0,1]$ are the continuous functions on $[0,1]$.

PROOF. By a standard transformation, we may assume without loss of generality that X_0 is uniformly distributed on $[0,1]$.

According to Proposition 3.3, we have only to verify (3.8) and (3.11) with x_k replaced by $X_k(w)$. In this context, $V_n(s,t) = \mathrm{var}(B_n^*(t) - B_n^*(s))$ and the conditions (3.8) and (3.11) translate to

$$\lim_{n\to\infty} V_n(s,t) \quad \text{exist a.s.,} \tag{4.9}$$

and for some $K > 0$, depending only on trajectory, $0 < b < 1$ and $c > 0$, we have

$$V_n(s,t) \le K(|s-t|^b + n^{-c}) \quad \text{a.s.} \tag{4.10}$$

for each s and t and $n \to \infty$.

By the same kind of computations leading to (4.3), we get the following expansion:

$$\begin{aligned} V_n(s,t) &= \frac{1}{n}\sum_{i=1}^{n} \frac{Y_{li}^2(s,t) - EY_{li}^2(s,t)}{l} + \frac{\mathrm{var}\sum_{i=1}^{l} I(s < X_i \le t)}{l} \\ &\quad - l\left(\frac{(\sum_{i=1}^{n} I(s < X_i \le t) - (t-s))}{n}\right)^2 \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{4.11}$$

Because of (4.7), by stationarity and Lemma 4.1, the last term in (4.11) is convergent to 0 a.s., and by (4.8), the first term of $V_n(s,t)$ is convergent to 0 a.s. For the middle term, we use the fact that the convergence of the series in (1.2) implies that

$$\sigma(s,t) = \sum_{i=-\infty}^{\infty} \mathrm{cov}(I(s < X_0 \le t), I(s < X_{|i|} \le t))$$

is convergent for every s, t.

By stationarity,

$$\lim_{l\to\infty} \frac{\mathrm{var}\sum_{i=1}^{l} I(s < X_i \le t)}{l} = \sigma(s,t)$$

and (4.9) follows.

In order to verify (4.10), we use the expansion (4.11). As a consequence of (4.8), we obtain

$$\sup_{s,\,t\in[0,1]} \mathrm{I} = O(n^{-g}) \quad \text{a.s.} \quad \text{for some } g > 0 \quad \text{as } n \to \infty.$$

By (4.7) and (2.8),

$$\mathrm{II} \le K_1(|s-t|^b + l^{-d}) \le K_1(|s-t|^b + n^{-hd}) \quad \text{a.s.}$$

In order to evaluate III, let $M_n = l_n^{1/2+\varepsilon}$ for some $0 < \varepsilon < 2a/(1-2a)$, $0 < a < 1/2$. For s and t fixed between 0 and 1, let u be such that $(u-1)/M_n < s \le u/M_n$ and let v be such that $(v-1)/M_n < t \le v/M_n$. If we denote

$$G_n(s,t) = F_n(s,t) - (t-s) = \frac{1}{n}\sum_{i=1}^{n} I(s < X_i \le t) - (t-s),$$

we can easily verify that

$$|G_n(s,t) - G_n(u,v)| \le \max_{1\le u\le M_n} |G_n(u-1,u)| + \frac{2}{M_n},$$

which implies

$$\sup_{s,t} \text{III} = \sup_{s,t} lG_n^2(s,t) \le 4 \max_{1\le u,\, v\le M_n} lG_n^2(u,v) + \frac{4l}{M_n^2} = A + B.$$

By the selection of M_n and (2.8),

$$B = \frac{4l}{M_n^2} \ll l^{-2\varepsilon} \ll n^{-2\varepsilon h}.$$

It remains to prove that $A = 0(n^{-g})$ for some $g > 0$ as $n \to \infty$. Denote $M_k = M_{2^{k+1}}$. In order to prove this, we shall apply the Borel–Cantelli lemma, according to which it is enough to estimate

$$\begin{aligned}&\sum_{k=0}^{\infty} P\Big(\max_{2^k\le n<2^{k+1}} \max_{1\le u,\, v\le M_k} n^g l_n G_n^2(u,v) > \varepsilon\Big)\\ &\qquad \le \sum_{k=0}^{\infty}\sum_{u=1}^{M_k}\sum_{v=1}^{M_k} P\Big(\max_{1\le n\le 2^{k+1}} l_{2^k} G_n^2(u,v) > \varepsilon 2^{-kg}\Big).\end{aligned} \tag{4.12}$$

By stationarity, condition (4.7) assures the applicability of Corollary 3.1 in Moricz, Serfling and Stout (1982); therefore, applying first Chebyshev's inequality and after the above-mentioned theorem, we get

$$P\Big(\max_{1\le n\le 2^{k+1}} l_{2^k} G_n^2(u,v) > \varepsilon 2^{-kg}\Big) \ll \frac{l_{2^k} 2^{k(1+g)} (\log 2^k)^2}{2^{2k}} \ll \frac{l_{2^k} 2^{kg}}{2^k} k^2$$

for every u, v. Therefore, by the selection of M and l, the sum in (4.12) does not exceed

$$\sum_{k=0}^{\infty} \frac{M_k^2 l_{2^k} 2^{kg}}{2^k} k^2 \ll \sum_{k=0}^{\infty} \frac{(2^k)^{(1+\varepsilon)(1-2a)} 2^{kg}}{2^k} k^2. \tag{4.13}$$

One can easily see from the definition of a and ε that there is a $g > 0$ such that the series in (4.13) is convergent.

These three estimates show that (4.10) is verified and we have the desired result by Proposition 3.3. □

For proving Theorem 2.2, we need the following lemma.

LEMMA 4.3. *Assume $\{X_i\}_{i\geq 1}$ is a stationary sequence of random variables uniformly distributed on $[0,1]$, satisfying (2.9). Then, there is a positive constant K_1 such that, for every $0 \leq s \leq t \leq 1$, we have*

$$\left|\sum_{i=-\infty}^{\infty} \operatorname{cov}(I(s < X_0 \leq t), I(s < X_{|i|} \leq t))\right| \leq K_1 |t-s|^{\gamma/(1+\gamma)}, \tag{4.14}$$

and there is a positive constant K_2 such that, for every $0 \leq s \leq t \leq 1$ and every $n \geq 1$,

$$\frac{\operatorname{var}(\sum_{i=1}^n I(s < X_i \leq t))}{n} \leq K_2 \big(|t-s|^{\gamma/(1+\gamma)} + n^{-\gamma}\big). \tag{4.15}$$

PROOF. We divide the sum (4.14) in a sum with i running between $-u$ and u, and the sums of the other variables where u is a natural number. We get, by (2.9) and the fact that X_0 is uniformly distributed,

$$\begin{aligned}
&\left|\sum_{i=-\infty}^{\infty} \operatorname{cov}(I(s < X_0 \leq t), I(s < X_{|i|} \leq t))\right| \\
&\quad \leq 2u \sup_{i\geq 0} \big|\operatorname{cov}(I(s < X_0 \leq t), I(s < X_i \leq t))\big| \\
&\qquad + 2\left|\sum_{i=u+1}^{\infty} \operatorname{cov}(I(s < X_0 \leq t), I(s < X_i \leq t))\right| \\
&\quad \ll u|t-s| + (u+1)^{-\gamma}.
\end{aligned}$$

The relation (4.14) follows by selecting u to be the integer part of $(t-s)^{-[1/(\gamma+1)]}$. Relation (4.15) follows easily from (4.14) and (2.9). □

4.3. *Proof of Theorem* 2.2. We shall assume first X_0 has a uniform distribution on $[0,1]$, and we shall check the conditions of Proposition 4.1, (4.7) and (4.8). Lemma 4.3 indicates that condition (4.7) is satisfied.

We shall verify now (4.8). Let us recall the notation $Y_{li}(s,t) = \sum_{j=i}^{i+l-1} I(s < X_j \leq t) - (t-s)$. Let $M = M_n = l_n^{1/2+\varepsilon}$, where $\varepsilon < 9a/2(1-3a)$, and a is as in (2.8). For s, t fixed, let u, v be two integers such that $(u-1)/M < s \leq u/M$ and $(v-1)/M < t \leq v/M$. Denote

$$D_i(s,t) = Y_{li}(s,t) - Y_{li}\left(\frac{u}{M}, \frac{v}{M}\right)$$

and

$$V_n(s,t) = \frac{1}{n}\sum_{i=1}^n \frac{Y_{li}^2(s,t) - EY_{li}^2(s,t)}{l}.$$

Simple arithmetic involving the use of the trivial inequality $2|xy| \leq ax^2 + (1/a)y^2$ for every x, y and $a > 0$ shows that, for every $0 < d < 1$, we have

$$Y_{li}^2(s,t) \leq (1+d) Y_{li}^2\left(\frac{u}{M}, \frac{v}{M}\right) + \left(1 + \frac{1}{d}\right) D_i^2(s,t) \tag{4.16}$$

and

$$Y_{li}^2(s,t) \geq (1-d)Y_{li}^2\left(\frac{u}{M}, \frac{v}{M}\right) + \left(1 - \frac{1}{d}\right)D_i^2(s,t). \tag{4.17}$$

By subtracting from (4.16) the expected value in (4.17), and then by subtracting from (4.17) the expected value in (4.16), we obtain two other relations. By adding these new relations for $1 \leq i \leq n$ and dividing by $1/nl$, we obtain

$$\begin{aligned} |V_n(s,t)| \leq 2\Bigg(\left|V_n\left(\frac{u}{M}, \frac{v}{M}\right)\right| &+ d\frac{\operatorname{var} S_l(u/M, v/M)}{l} \\ &+ d^{-1}\left[\frac{1}{nl}\sum_{i=1}^n (D_i^2(s,t) + ED_i^2(s,t))\right]\Bigg). \end{aligned} \tag{4.18}$$

By (4.15), we can find a positive constant K_3 such that

$$\frac{\operatorname{var} S_l(u/M, v/M)}{l} \leq K_3. \tag{4.19}$$

In order to evaluate the other terms from the right-hand side of (4.18), we shall prove that, for some $\delta > 0$, we have

$$\max_{1 \leq u,\, v \leq M} \left|V_n\left(\frac{u}{M}, \frac{v}{M}\right)\right| \leq K(w) \cdot n^{-\delta} \quad \text{a.s.} \tag{4.20}$$

for a certain constant $K(w)$ depending only on the trajectory.

According to the Borel–Cantelli lemma, we have to show that

$$\sum_k P\left(\max_{2^k < n \leq 2^{k+1}} \max_{1 \leq u,\, v \leq M} n^\delta \left|V_n\left(\frac{u}{M}, \frac{v}{M}\right)\right| > \varepsilon\right) < \infty. \tag{4.21}$$

Each term of this sum is majorated by

$$\sum_{v=1}^M \sum_{u=1}^M P\left(\max_{1 \leq n \leq 2^k} \frac{1}{2^{k(1-\delta)}} \frac{1}{l_{2^k}} \left|\sum_{i=1}^n \left(Y_{li}^2\left(\frac{u}{M}, \frac{v}{M}\right) - EY_{li}^2\left(\frac{u}{M}, \frac{v}{M}\right)\right)\right| > \varepsilon\right).$$

We remark that, by Corollary 3.1 in Moricz, Serfling and Stout (1982) and by stationarity, the condition (2.10) implies that, for every $x > 0$ and every s and t, we have

$$P\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j \left(Y_{li}^2(s,t) - EY_{li}^2(s,t)\right)\right| > x\right) \ll x^{-2} l_n^4 n (\log n)^2.$$

This estimate, together with the definition of $M = M_n$, proves that the sum in (4.21) is bounded by

$$\sum_k M_{2^k}^2 \frac{l_{2^k}^4 \cdot 2^k \cdot k^2}{l_{2^k}^2 2^{2k(1-\delta)}} \ll \sum_k \frac{l_{2^k}^{3+2\varepsilon} k^2}{2^{k-2\delta}}.$$

Now, by the selection of ε and the condition (2.8) imposed to l, one can easily see that the selection of $\delta > 0$ is possible such that the sum in (4.21) is convergent.

Now we show that an $\eta > 0$ can be found such that, for a constant $C(w)$, we have

$$\sup_{s,t} \frac{1}{nl} \sum_{i=1}^{n} D_i^2(s,t) \le C(w) n^{-\eta} \quad \text{a.s.} \tag{4.22}$$

Since for every $1 \le j \le n$,

$$\begin{aligned} -I\left(\frac{v-1}{M} < X_j \le \frac{v}{M}\right) &\le I(s < X_j \le t) - I\left(\frac{u}{M} < X_j \le \frac{v}{M}\right) \\ &\le I\left(\frac{u-1}{M} < X_j \le \frac{u}{M}\right), \end{aligned}$$

we note that, for every s, t,

$$|D_i(s,t)| \le \max_{1 \le u \le M} \left| Y_{li}\left(\frac{u-1}{M}, \frac{u}{M}\right)\right| + \frac{2l}{M}.$$

By taking into account that $(a+b)^2 \le 2(a^2+b^2)$ and the definition of $V_n(s,t)$, we deduce that

$$\begin{aligned} \frac{1}{nl} \sum_{i=1}^{n} D_i^2(s,t) &\ll \max_{1 \le u \le M} V_n\left(\frac{u-1}{M}, \frac{u}{M}\right) \\ &\quad + \max_{1 \le u \le n} \frac{EY_{l1}^2((u-1)/M, u/M)}{l} + \frac{l}{M^2}. \end{aligned} \tag{4.23}$$

The relation (4.22) follows from (4.23) after using (4.20), (4.15) and the definition of M and l.

From (4.23), conditions (2.8), (2.10) and the definition of M, we also have, for some $\gamma > 0$ and $C > 0$,

$$\frac{1}{nl} \sum_{i=1}^{n} ED_i^2(s,t) \le Cn^{-\gamma}. \tag{4.24}$$

To complete the proof, we just notice that, by (4.18), (4.19), (4.20), (4.22) and (4.24), we are able to select $d = d_n = n^{-\beta}$ for $0 < \beta < \min(\eta, \gamma)$ such that (4.8) is verified for some $g < \min(\delta, \beta, (\eta - \beta), (\gamma - \beta))$. $\square$

Suppose now that X_0 has an arbitrary continuous distribution $F(x)$. In a routine manner as in Billingsley [(1968), page 197], or in Naik-Nimbalkar and Rajarshi [(1994), page 993], $\xi_i = F(X_i)$ has a uniform distribution and satisfies (2.9) and (2.10). Therefore, the conclusion of Theorem 2.2 holds, and if $\bar{B}_n^*(t)$ is the bootstrapped empirical process for ξ_i, then $\bar{B}_n^*(F(t)) = B_n^*(t)$ and the arguments continue exactly as in Billingsley [(1968), page 197], and we obtain the result of Theorem 2.2 with $P^*(B(t) \in C) = 1$.

If the distribution of X is not continuous, we define [Billingsley (1968), page 142] a generalized inverse $\varphi(s) = \inf\{t\colon s \le F(t)\}$. The mapping theorem (5.1) from Billingsley (1968) still applies, but the limiting distribution will not be in C with probability 1 if F has discontinuities.

5. Proofs of Theorems 2.3 and 2.4. Both of these theorems will be proved by applying Theorem 2.2. They require several lemmas which lead to the validity of (2.9) and (2.10).

LEMMA 5.1 [Ibragimov and Linnik (1971), Theorem 17.2.1]. *Assume X and Y are two random variables such that $|X| < M$ a.s. and $|Y| < N$ a.s. Then*

$$|\operatorname{cov}(X, Y)| \le 4MN\alpha(\sigma(X), \sigma(Y)).$$

LEMMA 5.2 [Bagai and Prakasa Rao (1991), Roussas (1991)]. *Assume $\{X, Y\}$ is an associated vector of random variables each having a continuous bounded density. Then, for some $C \ge 0$,*

$$\sup_{u,v}(P(X \le u, Y \le v) - P(X \le u)P(Y \le v)) \le C\operatorname{cov}^{1/3}(X, Y).$$

LEMMA 5.3. *Let $\{Y_1, Y_2, Y_3, Y_4\}$ be an associated vector of bounded random variables $|Y_i| < l$ for $i = 1, 2, 3, 4$. Then*

$$\operatorname{cov}(Y_1Y_2, Y_3Y_4) \le 4l^2 \sum_{i\in\{1,2\}} \sum_{j\in\{3,4\}} \operatorname{cov}(Y_i, Y_j).$$

PROOF. By Lemma (3.1)(ii) in Birkel (1988), for every function $h\colon R \to R$ bounded differentiable with bounded derivative,

$$\operatorname{cov}(h(Y_1)h(Y_2), h(Y_3)h(Y_4)) \le \|h\|_\infty^2 \cdot \|h'\|_\infty^2 \sum_{i\in\{1,2\}} \sum_{j\in\{1,2\}} \operatorname{cov}(Y_i, Y_j).$$

We select now $h(x)$ a differentiable increasing function, $-2l \le h(x) \le 2l$ for every x such that $h(x) = x$ for $-l \le x \le l$, $h(x)$ is concave upwards on $(-\infty, -l)$ and concave downwards on $(l, +\infty)$. For this selection, because $h(Y_i) = Y_i$, $\|h'\|_\infty \le 1$ and $\|h\|_\infty \le 2l$, we obtain this result. $\square$

Next we shall prove the following lemma for strong mixing sequences.

LEMMA 5.4. *Assume $\{X_n\}$ is a stationary strong mixing sequence of bounded, centered random variables. Then*

$$\operatorname{var}\left(\sum_{j=1}^{n}(X_j + \cdots + X_{j+l-1})^2\right) \ll nl^4 \sum_{i=1}^{n} \alpha_i.$$

PROOF. We recall the notation $S_{li} = X_i + \cdots + X_{i+l-1}$. Obviously, we have

$$|S_{li}| = O(l). \tag{5.1}$$

We have the following estimate:

$$\begin{aligned}\operatorname{var}\left(\sum_{j=1}^{n} S_{lj}^2\right) &\le \sum_{j=1}^{n} ES_{lj}^4 + 2\sum_{j=1}^{n-1}\sum_{i=j+1}^{n} \operatorname{cov}(S_{lj}^2, S_{li}^2)\\ &= \sum_{j=1}^{n} ES_{lj}^4 + 2\sum_{j=1}^{n-1}\sum_{i=j+1}^{j+l} \operatorname{cov}(S_{lj}^2, S_{li}^2)\\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=j+l+1}^{n} \operatorname{cov}(Y_j^2, Y_i^2)\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{5.2}$$

By (5.1) and Lemma 5.1, we have

$$I \ll nl^2 ES_l^2 \ll nl^3 \sum_{i=1}^{n} \alpha_i.$$

We also have, by Lemma 5.1,

$$\operatorname{cov}(S_{lj}^2, S_{li}^2) \le ES_{lj}^2 S_{li}^2 \ll l^2 ES_{li}^2 \ll l^3 \sum_{i=1}^{n} \alpha_i,$$

whence

$$\mathrm{II} \ll nl^4 \sum_{i=1}^{n} \alpha_i.$$

Observe now that S_{lj}^2 and S_{li}^2 which appear in III are $i - j + l$ steps apart, and by Definition 2.1 and Lemma 5.1, we get, for $i \ge j + l$,

$$|\operatorname{cov}(S_{lj}^2, S_{li}^2)| \ll \alpha_{i-j+l} l^4.$$

Therefore,

$$\mathrm{III} \ll l^4 n \sum_{i=1}^{n} \alpha_i,$$

and we have the desired estimate by adding the estimates for I, II and III in (5.2). □

LEMMA 5.5. *Let $\{X_n\}$ be a stationary sequence of associated random variables with continuous bounded density such that*

$$\sum_n \operatorname{cov}^{1/3}(X_0, X_n) < \infty. \tag{5.3}$$

Then, for every $1 \le m \le n$,

$$\operatorname{var}\left(\sum_{j=1}^{m} Y_{lj}^2(s,t)\right) \ll ml^4.$$

PROOF. Observe that $Y_{li}(s)$ defined by (1.10) is an associated sequence bounded by l and

$$Y_{lj}(s,t) = Y_{lj}(s) - Y_{lj}(t).$$

Also,

$$\begin{aligned} Y_{lj}^2(s,t) - EY_{lj}^2(s,t) = (Y_{lj}^2(s) - EY_{lj}^2(s)) + (Y_{lj}^2(t) - EY_{lj}^2(t)) \\ - 2(Y_{lj}(t)Y_{lj}(s) - EY_{lj}(t)Y_{lj}(s)), \end{aligned}$$

whence

$$\begin{aligned} \operatorname{var}\left(\sum_{j=1}^m Y_{lj}^2(s,t)\right) \ll \operatorname{var}\left(\sum_{j=1}^m Y_{lj}^2(s)\right) + \operatorname{var}\left(\sum_{j=1}^m Y_{lj}^2(t)\right) \\ + \operatorname{var}\left(\sum_{j=1}^m Y_{lj}(s)Y_{lj}(t)\right). \end{aligned}$$

For every $s \le t$, we have the estimate

$$\begin{aligned} \operatorname{var}\left(\sum_{j=1}^m Y_{lj}(s)Y_{lj}(t)\right) &= \sum_{j=1}^m \operatorname{var}(Y_{lj}(s)Y_{lj}(t)) \\ &\quad + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^m \operatorname{cov}(Y_{li}(s)Y_{li}(t), Y_{lj}(s)Y_{lj}(t)). \end{aligned} \tag{5.4}$$

By the boundedness of $Y_{li}(s)$, we get

$$\sum_{j=1}^m \operatorname{var}(Y_{lj}(s)Y_{lj}(t)) \ll ml^4. \tag{5.5}$$

According to Lemma 5.3, we also have

$$\operatorname{cov}(Y_{li}(s)Y_{li}(t), Y_{lj}(s)Y_{lj}(t)) \ll l^2 \sum_{v\in\{s,t\}}\sum_{w\in\{s,t\}} \operatorname{cov}(Y_{li}(v)Y_{lj}(w)). \tag{5.6}$$

Now, as a consequence of Lemma 5.2, we obtain uniformly in v, w,

$$\sum_{j=i+1}^m \operatorname{cov}(Y_{li}(v), Y_{lj}(w)) \ll l^2 \sum_{n=1}^\infty \operatorname{cov}^{1/3}(X_0, X_n). \tag{5.7}$$

By introducing the results of the relations (5.5), (5.6) and (5.7) in (5.4), we obtain the conclusion of this lemma. □

5.1. *Proof of Theorem* 2.3. Lemma 5.1 and (2.12) give the following estimate:

$$\sum_{i\ge m}^\infty |\operatorname{cov}(I(s < X_0 \le t), I(s < X_i \le t))| \le \sum_{i\ge m}^\infty \alpha_i \ll m^{-\gamma},$$

which verifies the condition (2.9) of Theorem 2.2, while Lemma 5.4 applied to the sequence $I(s < X_i \leq t) - (F(t) - F(s))$ establishes the validity of (2.10). Theorem 2.3 follows then as a consequence of Theorem 2.2. $\square$

5.2. *Proof of Theorem* 2.4. Similarly as above, we see that Lemma 5.5 guarantees the validity of (2.10) in Theorem 2.2, while Lemma 5.2 and condition (2.13) show that (2.9) holds. $\square$

5.3. *Proof of Remark* 2.1. We have to prove that, for the example mentioned in Remark 2.1, the conclusion of Theorem 2.1 holds for every l_n satisfying (2.7). By Lemma 5.4, for every $1 \leq m \leq n$,

$$\mathrm{var}\left(\sum_{j=1}^{m} S_{l_n j}^2\right) \ll m l_n^4 \sum_{i=1}^{n} \alpha_i. \tag{5.8}$$

By Moricz, Serfling and Stout (1982) and stationarity, for every $x \geq 0$, (5.8) implies

$$P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(S_{l_n i}^2 - ES_{l_n i}^2)\right| \geq x\right) \ll x^{-2} n (\log n)^2 l_n^4 \sum_{i=1}^{n} \alpha_i. \tag{5.9}$$

Now we repeat the arguments from the proof of Proposition 2.1 with the maximal inequality (4.6) replaced by (5.9) and obtain

$$\lim_{n\to\infty} \frac{|\sum_{j=1}^{n}(S_{l_n j}^2 - ES_{l_n j}^2)|}{n^{1/2} l_n^2 (\sum_{i=1}^{n}\alpha_i)^{1/2} \log^2 n} = 0 \quad \text{a.s.} \tag{5.10}$$

The example from Remark 2.1 satisfies the relation $\alpha_i \ll 1/i$. Therefore,

$$n^{1/2} l_n^2 \left(\sum_{i=1}^{n}\alpha_i\right)^{1/2} \log^2 n \ll n^{1/2} l_n^2 \log^3 n,$$

and a selection of l_n as in (2.7) guarantees that the condition (2.3) in Theorem 2.1 is satisfied. Therefore, the conclusion of Theorem 2.1 follows for this particular example under (2.7). $\square$

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