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SETS AVOIDED BY BROWNIAN MOTION

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A fixed two-dimensional projection of a three-dimensional Brownian motion is almost surely neighborhood recurrent; is this simultaneously true of all the two-dimensional projections with probability 1? Equivalently: three-dimensional Brownian motion hits any infinite cylinder with probability 1; does it hit all cylinders? This papers shows that the answer is no. Brownian motion in three dimensions avoids random cylinders and in fact avoids bodies of revolution that grow almost as fast as cones.

1. Introduction. Let $\{B(t)\colon 0 \le t < \infty\}$ be a Brownian motion started from the origin in three dimensions, with coordinates $(X(t), Y(t), Z(t))$ defined on $(\Omega, \mathcal{F}(t), \mathsf{P})$; we will use P_{v} to denote the law of $B(t)$ translated by v. Any projection of $\{B(t)\}$ onto a plane is a version of a two-dimensional Brownian motion and its almost sure properties are well known: it is neighborhood recurrent, its range is two-dimensional with exact Hausdorff gauge $x^2 \log(1/x) \log\log\log(1/x)$; the list goes on. Some of these properties are known to hold uniformly over all projections, while others fail (necessarily on a set of projections of measure zero, by Fubini's theorem). What about neighborhood recurrence: is this a property inherited simultaneously by all projections of $\{B(t)\}$? An equivalent question is:

Does $\{B(t)\}$ with probability one intersect every infinite cylinder?

In this paper we give a negative answer: with probability 1, there are random cylinders disjoint from the range of a three-dimensional Brownian motion. In fact we show more. Let f be a strictly positive increasing function on $\mathbb{R}^+$ and let $\mathcal{C}_f$ be the set or *thorn*

$$\big\{(x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 + z^2 \ge 1 \text{ and } \sqrt{x^2 + y^2} \le f(|z|)\big\}.$$

Say that Brownian motion *avoids f-thorns* if there is with probability 1 a random set congruent to $\mathcal{C}_f$ avoided by Brownian motion. A zero–one law holds, so the alternative is that with probability 1 Brownian motion intersects all sets congruent to $\mathcal{C}_f$. Our main results are contained in Theorems 2.3 and 2.4 below: under an integral condition on f, satisfied for example when $f(z) = z/\exp(\log^{1/2+\varepsilon} z)$, Brownian motion avoids f-thorns; moreover, in this

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case the set of directions of axes of f-thorns avoided by Brownian motion has Hausdorff dimension 2, with positive probability. On the other hand, if $f(z) = z/\exp(c \log^{1/2} z)$ for sufficiently small c, then Brownian motion does not avoid f-thorns, a.s.

REMARK 1. It would have been equally natural to consider one-sided thorns, $\mathscr{C}_f \cap \{(x, y, z)\colon z \geq 0\}$, but there seems to be little difference since we cannot find an f for which Brownian motion intersects all two-sided f-thorns but misses some one-sided f-thorns.

REMARK 2. One original motivation for this question was to shed some light on the complement of the Wiener sausage $\mathscr{W} := \{B(t) + \mathsf{x}\colon t \in \mathbb{R}^+, |\mathsf{x}| \leq 1\}$. For example, we do not know an elementary proof that $\mathbb{R}^3 \setminus \mathscr{W}$ has an unbounded connected component. This follows from the weakest of our avoidance results. An elementary argument, based on the existence of arbtrarily large values of t for which $\sup_{s \leq t} |B(s)| - \inf_{s \geq t} |B(s)|$ is smaller than any arbitrary fixed positive number (see [1] or [3], Proposition 1), shows that there must be at most one unbounded component.

REMARK 3. The notion of properties holding uniformly over planar projections of higher dimensional Brownian motion is similar to the notion of *quasi-everywhere* properties of the Brownian path; that is, properties that w.p.1 hold simultaneously for every cross section of the Brownian sheet. See, for example, [4] or [8].

We now briefly outline the arguments, setting forth notation that will be used throughout.

NOTATION. For any unit vector $\mathsf{v} \in \mathbb{R}^3$, let $\mathscr{C}_{\mathsf{v}} = \mathscr{C}_{f, \mathsf{v}}$ denote the image of $\mathscr{C}_f$ under any origin-preserving rotation mapping $(0, 0, 1)$ to v. Usually f will be fixed and will be dropped from the notation. Let v_θ denote $(\sin(\theta), 0, \cos(\theta))$ and let $\mathscr{C}_\theta$ denote $\mathscr{C}_{\mathsf{v}_\theta}$. For any set A, let τ_A denote the time Brownian motion first hits the set A. Let $\mathscr{B}(\mathsf{x}, L)$ denote the ball of radius L about the point x, let $\mathscr{B}_L$ denote $\mathscr{B}(0, L)$, and let τ_L be shorthand for $\tau_{\partial \mathscr{B}_L}$. Let

$$q(L) = \mathsf{P}(\tau_L < \tau_{\mathscr{C}})$$

be the probability that Brownian motion reaches modulus L before hitting the f-thorn. Let

$$q(L, \theta) = \mathsf{P}(\tau_L < \tau_{\mathscr{C}} \wedge \tau_{\mathscr{C}_\theta})$$

be the probability that Brownian motion reaches modulus L before hitting either of two f-thorns separated by an angle of θ. Write μ_L for the hitting subprobability measure on $\partial \mathscr{B}_L$ of Brownian motion absorbed by $\mathscr{C}$, so that for $A \subseteq \partial \mathscr{B}_L$, $\mu_L(A) = \mathsf{P}(\tau_L < \tau_{\mathscr{C}},\ B(\tau_L) \in A)$. Let $\mu_{L, \theta}$ be the same for $\mathscr{C} \cup \mathscr{C}_\theta$:

$$\mu_{L, \theta}(A) = \mathsf{P}(\tau_L < \tau_{\mathscr{C}} \wedge \tau_{\mathscr{C}_\theta},\ B(\tau_L) \in A).$$

Theorem 2.4 is proved by the second moment method and the easier Theorem 2.3 is proved by a first moment estimate. We first restrict our attention from all sets congruent to $\mathscr{C}$ to only the rotations, $\mathscr{C}_{\mathrm{v}}$; we will show that Brownian motion avoids f-thorns if and only if with positive probability there is a set $\mathscr{C}_{\mathrm{v}}$ avoided by Brownian motion. Let W_L be the measure of the set of all vectors v in the unit sphere for which $\tau_L < \tau_{\mathscr{C}_{\mathrm{v}}}$. Estimates on $q(L)$ and $q(L,\theta)$ yield estimates on $\mathsf{E}W_L$ and $\mathsf{E}W_L^2$. When $\mathsf{E}W_L^2/(\mathsf{E}W_L)^2$ is bounded, it follows that $\liminf \mathsf{P}(W_L > 0) > 0$ and hence that Brownian motion avoids f-thorns; when $\mathsf{E}W_L = o(f(z)/z)$, it follows that $\mathsf{P}(W_L > 0) \to 0$ and hence that Brownian motion does not avoid f-thorns.

All the work is in obtaining the estimates on $q(L)$ and, particularly, $q(L,\theta)$. The remainder of the paper is organized as follows. Section 2 contains precise statements of the main results and contains rigorous versions of the arguments mentioned above (zero–one laws, the first and second moment methods). It also contains a proof of Theorem 2.3, which requires very little computation. Section 3 contains proofs of the estimates on $q(L)$ and $q(L,\theta)$ in the special case where $f(z) = z^\alpha$. The reason for separating this from the general case is that in the z^α case we have reasonably accurate estimates of both $q(L)$ and $q(L,\theta)$. While the boundedness of $\mathsf{E}W_L^2/(\mathsf{E}W_L)^2$ in this case is subsumed by our later results, these do not contain separate estimates for $q(L)$ and $q(L,\theta)$, and we suspect that the estimate on $q(L)$, Lemma 3.3, will be useful in other contexts. Section 4 begins the proof of Theorem 2.4, breaking it down into a series of lemmas. Section 5 proves those lemmas with soft proofs, Section 6 proves those lemmas involving manipulation of Green's functions and Section 7 gives the proofs that require geometric analysis. The most important ingredient in these last four sections is the integration by parts device, Theorem 6.1, which allows the computation of $U(L,\theta) := q(L,\theta)/q(L)^2$ without exact or even asymptotic knowledge of $q(L)$. In addition to sharpening the dividing line between thorns that are avoided and thorns that are not, Theorem 6.1 should be useful in any situation where one wishes to estimate the probability of simultaneously avoiding two sets.

We will use many notions and results from the classical potential theory and their probabilistic counterparts. A good presentation of different aspects of this theory may be found in [2], [6] and [10].

2. Main results.

Write $\mathscr{R}$ for the range of the Brownian motion $\{B(t)\}$. Throughout, we let $g(z)$ denote the function $z/f(z)$. From the fact that the radial projection of Brownian motion onto the unit sphere is dense, we get the well-known fact that Brownian motion cannot avoid cones.

THEOREM 2.1. *If $f(z) = cz$ for some $c > 0$ then Brownian motion does not avoid f-thorns.*

In the next section, we prove a first result in the other direction.

THEOREM 2.2. *If $f(z) = z^\alpha$ for some $\alpha \in [0,1)$, then Brownian motion avoids f-thorns.*

In particular, when $\alpha = 0$ we recover the result first mentioned in the introduction: some planar projections of three-dimensional Brownian motion are not neighborhood recurrent.

Our sharpest nonavoidance result is the following.

THEOREM 2.3. *If* $f(z) = z/\exp(c\log^{1/2} z)$ *for* $c > 0$ *sufficiently small, then Brownian motion does not avoid* f*-thorns.*

Let $A = \{\mathrm{v}\colon |\mathrm{v}| = 1, \mathscr{R}\cap\mathscr{C}_{\mathrm{v}} = \varnothing\}$ be the set of directions of f-thorns avoided by Brownian motion. Our sharpest avoidance result is the following.

THEOREM 2.4. *Assume the following hypotheses on* f *and on* $g(z) := z/f(z)$*:*

$$f(z) \text{ and } g(z) \text{ are increasing and tend to infinity as } z \to \infty; \tag{2.1}$$

$$g(1) \geq 2; \tag{2.2}$$

$$\text{the circle } A \text{ lies inside the region } |x| \leq f(z), \tag{2.3}$$

where A *is the circle in the* z–x *plane centered on the* z *axis and tangent to the graph* $|x| = f(z)$ *at the points* $(z, \pm f(z))$*. If*

$$\int_1^\infty \frac{1}{z\log^2 g(z)}\,dz < \infty, \tag{2.4}$$

then Brownian motion avoids f*-thorns, and in fact the set* $A = \{\mathrm{v}\colon |\mathrm{v}| = 1, \mathscr{R}\cap\mathscr{C}_{\mathrm{v}} = \varnothing\}$ *of directions of axes of* f*-thorns avoided by Brownian motion has Hasdorff dimension 2, with positive probability.*

Note that the set $A = \{\mathrm{v}\colon |\mathrm{v}| = 1, \mathscr{R}\cap\mathscr{C}_{\mathrm{v}} = \varnothing\}$ of directions of axes of f-thorns avoided by Brownian motion can be empty, with positive probability, for every nontrivial f.

2.1. *Remarks on the hypotheses.* Whether Brownian motion avoids f-thorns is a monotone function of f and does not depend on the values of f on any bounded interval, so the hypothesis (2.2), which is a convenience measure in the proofs, is not really needed. Hypothesis (2.1) is needed to rule out wildly oscillating f, since these require different estimation techniques and Theorem 2.4 probably does not hold for such f. Of course one can prove avoidance for some such f by comparing to an upper envelope function $\tilde{f} \geq f$. To see that (2.3) is not too burdensome, note that it is satisfied in the special cases $f(z) = z^\alpha$ and $f(z) = z/\exp(\log^\alpha z)$, which come up naturally in this paper, and apparently whenever $f''(z)$ behaves in a regular manner. Note that (2.4) is satisfied for $g(z) = \exp(\log^{1/2+\varepsilon} z)$, thus providing a near converse to Theorem 2.3. When there is a gap between first and second moment results, the second moment result is usually sharp. Thus Theorem 2.3 is almost certainly not sharp. But Theorem 2.4 is probably not sharp either, since even if in principle the second moment method yields a sharp condition via Lemma 2.6 below, we do not know whether (2.4) is necessary for this.

LEMMA 2.5. *The probability p of Brownian motion avoiding some set congruent to $\mathscr{C}_f$ is 0 or 1. If the probability of Brownian motion avoiding some $\mathscr{C}_{f,\mathrm{v}}$ is positive, then $p = 1$. If for some fixed $\varepsilon \in (0,1)$, the probability of Brownian motion avoiding some $\mathscr{C}_{(1-\varepsilon)f,\mathrm{v}}$ is 0, then $p = 0$.*

PROOF. The event that some random set congruent to $\mathscr{C}_f$ is avoided after a random finite time is a tail event, so its probability, p_∞, is 0 or 1. Let p_t be the probability of avoiding some random set congruent to $\mathscr{C}_f$ from time t onwards; then $p_t \uparrow p_\infty$. But by the strong Markov property,

$$p_t = \mathsf{E}_{B(s)} p_{t-s} = p_{t-s},$$

and therefore $p = p_0 = p_\infty$, proving a zero–one law for existence of a set congruent to $\mathscr{C}_f$ avoided by Brownian motion. Of course it must be 1 if Brownian motion can avoid a $\mathscr{C}_\mathrm{v}$. Conversely, if the probability of avoiding some random $\mathrm{w} + \mathscr{C}_\mathrm{v}$ is 1, then choosing an arbitrary $\varepsilon \in (0,1)$ and $\mathrm{y} \in (\varepsilon/2)\mathrm{Z}^3$ as close as possible to w, there is a positive probability of avoiding some $\mathrm{y} + \mathscr{C}_{(1-\varepsilon)f,\mathrm{v}}$. By countable additivity this probability is positive for some fixed y, and by coupling, for every fixed y and in particular for $\mathrm{y} = 0$. □

NOTATION. Recall that

$$U(L,\theta) = \frac{q(L,\theta)}{q(L)^2}.$$

We will identify points in R^3 with vectors, in the obvious way. Let $\theta(\mathrm{v},\mathrm{w})$ denote the angle between v and w and let $\theta(\mathrm{v}) = \theta(\mathrm{v},(0,0,1))$.

The second moment method is stated in the following lemma.

LEMMA 2.6 (Second moment method). *Suppose that there is a function $U(\theta) \le \infty$ such that*

$$U(L,\theta) \le U(\theta)$$

for all sufficiently large L. If

$$\int U(\theta(\mathrm{v}))\, dS(\mathrm{v}) < \infty,$$

where dS is surface measure on the unit sphere, then Brownian motion avoids f-thorns. If, furthermore,

$$\int U(\theta(\mathrm{v}))\theta^{-\beta}\, dS(\mathrm{v}) < \infty,$$

then the set $A = \{\mathrm{v}\colon |\mathrm{v}| = 1,\ \mathscr{R} \cap \mathscr{C}_\mathrm{v} = \varnothing\}$ of directions of f-thorns avoided by Brownian motion has dimension at least β, with positive probability.

PROOF. Let W_L be the measure of the set $\{\mathsf{v}\colon |\mathsf{v}| = 1,\ \mathscr{R} \cap \mathscr{C}_{\mathsf{v}} \cap \mathscr{B}(0, L) = \varnothing\}$. By Fubini's theorem, $\mathsf{E} W_L = 4\pi q(L)$. Another application of Fubini's theorem gives

$$\mathsf{E} W_L^2 = \int\int \mathsf{P}(\tau_L < \tau_{\mathscr{C}_{\mathsf{v}}} \wedge \tau_{\mathscr{C}_{\mathsf{w}}})\, dS(\mathsf{v}) dS(\mathsf{w}) = 4\pi \int q(L, \theta(\mathsf{v}))\, dS(\mathsf{v}).$$

By Cauchy–Schwarz, $\mathsf{P}(W_L > 0) \ge (\mathsf{E} W_L)^2 / \mathsf{E} W_L^2$, and this in turn is, up to a factor of 4π, equal to the reciprocal of $\int U(L, \theta(\mathsf{v}))\, dS(\mathsf{v})$. Thus finiteness of $\int U(\theta(\mathsf{v}))\, dS(\mathsf{v})$ implies that $\mathsf{P}(W_L > 0)$ is bounded away from zero for large L, which implies that

$$\mathsf{P}\Big(\lim_{L\to\infty} 1_{W_L > 0} = 1\Big) > 0.$$

This and the previous lemma complete the proof of the first statement.

For the second statement, let $\Xi = \bigcap_{\varepsilon > 0} \Xi_\varepsilon$ be a random nonempty subset of the unit sphere with the property that if x, y are points of the unit sphere, then

$$\mathsf{P}(\mathsf{x} \in \Xi_\varepsilon) \ge C\varepsilon^\beta$$

and

$$\mathsf{P}(\mathsf{x}, \mathsf{y} \in \Xi_\varepsilon) \le C\varepsilon^{2\beta} |\mathsf{x} - \mathsf{y}|^{-\beta}.$$

It is shown in [9], Lemma 5.1 how to construct such sets using a Cantor-like construction, and that any set A with $\mathsf{P}(\Xi \cap A \ne \varnothing) > 0$ must have dimension at least β.

Construct the sets Ξ_ε independent of the Brownian motion. Recall that $A = \{\mathsf{v}\colon |\mathsf{v}| = 1,\ \mathscr{R} \cap \mathscr{C}_{\mathsf{v}} = \varnothing\}$ and let $A_L = \{\mathsf{v}\colon |\mathsf{v}| = 1,\ \mathscr{R} \cap \mathscr{C}_{\mathsf{v}} \cap \mathscr{B}(0, L) = \varnothing\}$. Let $A'_L = A_L \cap \Xi_{1/L}$. Let W'_L be the measure of A'_L. Then Fubini's theorem gives

$$\mathsf{E} W'_L \ge C q(L) L^{-\beta}$$

and

$$\mathsf{E}(W'_L)^2 \le C \int q(L, \theta(\mathsf{v})) L^{-2\beta} |\theta(\mathsf{v})|^{-\beta}\, dS(\mathsf{v}).$$

Thus by Cauchy–Schwarz again, $\mathsf{P}(W'_L > 0)$ is at least a constant times the reciprocal of $\int U(\theta(\mathsf{v})) |\theta(\mathsf{v})|^{-\beta}\, dS(\mathsf{v})$. If this integral is finite then

$$\mathsf{P}(A \cap \Xi \ne \varnothing) = \mathsf{P}\Big(\bigcap_L A'_L \ne 0\Big) \ge \limsup \mathsf{P}(W'_L > 0) > 0,$$

which shows that A intersects Ξ with positive probability, and hence has dimension at least β, with positive probability. □

As mentioned before, the estimates on $q(L)$ and $q(L, \theta)$ that we plug into this lemma in order to prove Theorems 2.2 and 2.4 are proved in subsequent sections. We end this section with a proof of Theorem 2.3. The version of the first moment method that we need is the following.

LEMMA 2.7. *Fix any $\varepsilon > 0$. Recall that $g(z) = z/f(z)$ and suppose that*

$$\lim_{L\to\infty} q(L)g(L) = 0. \tag{2.5}$$

Then Brownian motion does not avoid $(1+\varepsilon)f$-thorns.

PROOF. By Lemma 2.5, it suffices to show that Brownian motion avoids no $\mathscr{C}_{(1+\varepsilon/2)f,\,\mathrm{v}}$. On the event H that Brownian motion avoids some $\mathscr{C}_{(1+\varepsilon/2)f,\,\mathrm{v}}$, stopping at τ_L we see that $\tau_L < \tau_{\mathscr{C}_{\mathrm{w}}}$ for every w such that $\theta(\mathrm{w},\mathrm{v}) < (\varepsilon/4)f(L)/L = \varepsilon/(4g(L))$. Thus on H there is a $c > 0$ such that $W_L \ge c/g(L)$. Recall that $\mathsf{E}W_L = 4\pi q(L)$. This and (2.5) imply $\mathsf{P}(W_L \ge c/g(L)) \to 0$ for all $c > 0$. Thus, $\mathsf{P}(H) = 0$, which completes the proof. □

PROOF OF THEOREM 2.3. Fix $g(z) = \exp(\alpha \log^{1/2} z)$. Let $L = 2^k$ for some integer k. We compute $q(L)$ as follows. Define

$$A_j = \left\{\mathrm{v} := (x, y, z)\colon 2^j < z \text{ and } |\mathrm{v}| \le 2^{j+1} \text{ and } \sqrt{x^2+y^2} \le f(2^j)\right\}.$$

Then $\mathscr{C}$ contains the disjoint union A of the sets A_j and the event $\{\tau_L < \tau_{\mathscr{C}}\}$ implies the event $\{\tau_L < \tau_A\}$. Conditioning on successive values of $B(\tau_{2^j})$ and using the strong Markov property, we get

$$q(L) \le \prod_{j=1}^{k} \mathsf{P}_{B(\tau_{2^j})}\big(B(s) \notin A_j \text{ for } 0 < s < \tau_{2^{j+1}}\big).$$

The terms of the product may be bounded as follows. Scaling down by a factor of 2^{j+1} transforms $B(\tau_{2^j})$ into a point on the sphere of radius 1/2 and A_j into a superset of a cylinder whose axis is the segment [1/2, 3/4] and whose radius is

$$\frac{f(2^j)}{2^{j+1}} = \frac{1}{2}\exp\big(-\alpha\sqrt{j\log 2}\big).$$

The probability of avoiding a cylinder of length 1/4 and radius r before hitting the boundary of the unit sphere, starting at a point of modulus 1/2, is bounded above by $1 - K/|\log r|$ for some constant K, and since in our case $|\log r| = (\alpha\sqrt{\log 2} + o(1))\sqrt{j}$, we get

$$q(L) \le O(1)\prod_{j=1}^{k}\left(1 - \frac{K_1}{\alpha\sqrt{j}}\right) \le O(1)\exp\left(-\sum_{j=1}^{k}\frac{K_1}{\alpha\sqrt{j}}\right) \le \exp(-\sqrt{k})$$

when α is small. Thus for sufficiently small α, $q(L)g(L) \le \exp((\sqrt{\log 2}\,\alpha - 1)\sqrt{k}) \to 0$, which together with Lemma 2.7 proves that Brownian motion does not avoid $(1+\varepsilon)f$-thorns. It remains to notice that $(1+\varepsilon)z/\exp(\alpha\log^{1/2} z) \le z/\exp((\alpha-\varepsilon)\log^{1/2} z)$ for large z, to complete the proof of Theorem 2.3.

3. Proof of Theorem 2.2. The theorem is proved via the second moment method, following immediately from the following estimate.

LEMMA 3.1. *Suppose $f(z) = z^\gamma$. Then there are constants M and β for which*

$$U(L, \theta) \le M|\log\theta|^{1/(1-\gamma)} \log^\beta |\log\theta|. \tag{3.1}$$

Lemma 3.1 will be proved at the end of Section 3.3.

Recall that $B(t) = (X(t), Y(t), Z(t))$ and let $V(t) = \sqrt{X(t)^2 + Y(t)^2}$. For any process $\{\Lambda(t)\}$, let $T^\Lambda(L) = \inf\{t > 0: \Lambda(t) = L\}$ be the time to hit the value L, so that the notation τ_L is the same as $T^{|B|}(L)$. For the duration of this section, fix a $\gamma \in (0, 1)$ and define $f(z) = z^\gamma$. Since f is fixed, we suppress it from the notation. Define $\mathscr{C}^{L_0} = \mathscr{C} \setminus (\mathbb{R}^2 \times (-L_0, L_0))$ to be the part of $\mathscr{C}$ with z-coordinate at least L_0 in magnitude. Let $\mathscr{C}_\theta = \mathscr{C}_{\mathsf{v}_\theta}$ as before, and define $\mathscr{C}_\theta^{L_0} = \{\mathsf{y} \in \mathscr{C}_\theta: |\mathsf{y} \cdot \mathsf{v}_\theta| \ge L_0\}$ analogously to $\mathscr{C}^{L_0}$. Frequent use is made of the following fact (see [7], proof of Theorem 4.3.8, page 103).

FACT (∗). For any $0 < v_1 < v_2 < v_3$ and any point $\mathsf{v} = (x, y, z)$ such that $x^2 + y^2 = v_2^2$, the hitting probabilities for the radial process $V(t)$ obey

$$\mathsf{P}_\mathsf{v}(T^V(v_1) < T^V(v_3)) = \frac{\log v_3 - \log v_2}{\log v_3 - \log v_1}.$$

Fix parameters $\alpha > \beta > 2$, to be used throughout Section 3 (they are different from α and β in other sections). The proof of (3.1) is based on estimating $q(L)$ and $q(L, \theta)$ separately. To begin, record the following useful bound.

PROPOSITION 3.2. *There exists an absolute constant $c_{\mathsf{cyl}} < 1$ such that if $b \ge 2a > 0$, $\mathsf{v} = (x, y, z)$, $|z| \le a$ and $x^2 + y^2 \le a^2$, then*

$$\mathsf{P}_\mathsf{v}(T^V(a) \ge T^{|Z|}(b)) \le c_{\mathsf{cyl}}^{b/a}.$$

PROOF. It is elementary to see that there is a $c'_{\mathsf{cyl}} < 1$ independent of v such that under the above conditions,

$$\mathsf{P}_\mathsf{v}(T^V(a) \ge T^Z(z + a) \wedge T^Z(z - a)) \le c'_{\mathsf{cyl}}.$$

By applying the strong Markov property at the times when $B(t)$ hits the planes $\{(x', y', z'): z' = z + ja\}$ for integer values of j, we obtain

$$P_\mathsf{v}(T^V(a) \ge T^{|Z|}(b)) \le (c'_{\mathsf{cyl}})^{b/a-1}.$$

Now let $c_{\mathsf{cyl}} = \sqrt{c'_{\mathsf{cyl}}}$ and observe that $(c'_{\mathsf{cyl}})^{b/a-1} \le c_{\mathsf{cyl}}^{b/a}$ as long as $b/a \ge 2$. □

3.1. *Estimating* $q(L)$. Define sequences of constants $m_k = k(\log k)^\alpha$, $r_k = e^{m_k}$ and $q_k = r_k/(\tilde{c}\log k)$, where $\tilde{c}$ is chosen so that $\tilde{c}\log c_{\mathrm{cyl}} \le -2$ and c_{cyl} is the constant from Proposition 3.2.

LEMMA 3.3. *Let* $j(L)$ *be the smallest integer* j *for which* $r_{j-1} \ge L$. *There are constants* k_0 *and* c_q *for which the following estimate holds:*

$$q(L) \ge c_q \exp\left(-\sum_{k=k_0}^{j(L)} \frac{1}{1-\gamma}\left(\frac{1}{k} + \frac{\alpha}{k\log k}\right)\right). \tag{3.2}$$

PROOF. The constant k_0 will be chosen large enough so that certain inequalities hold; we use the usual convention of replacing k_0 by something larger when necessary to satisfy each subsequent inequality. The method of achieving a lower bound on $q(L)$ is to require something stronger, namely that the radial part $V(t)$ reach q_k before the z-component reaches magnitude r_k for each k. With hindsight (i.e., comparing to the upper bound at the end of this section) we can see that this method is sharp up to a constant factor: conditional on avoiding $\mathscr{C}$ up to time τ_L it will be true with probability bounded away from zero that V reaches each q_k before $|Z|$ reaches r_k.

Suppose that $\mathsf{v} = (x, y, z)$ with $|z| \le r_{k-1}$ and $x^2 + y^2 = q_{k-1}^2$. Then

$$\begin{aligned} \mathsf{P}_{\mathsf{v}}(T^V(q_k) < T^{|Z|}(r_k) \wedge T^B(\mathscr{C})) \ge \mathsf{P}_{\mathsf{v}}(T^V(q_k) < T^V(r_k^\gamma)) \\ - \mathsf{P}_{\mathsf{v}}(T^V(q_k) \ge T^{|Z|}(r_k)). \end{aligned} \tag{3.3}$$

We have

$$\mathsf{P}_{\mathsf{v}}(T^V(r_k^\gamma) \le T^V(q_k)) = \frac{\log q_k - \log q_{k-1}}{\log q_k - \log(r_k^\gamma)} \tag{3.4}$$

$$= \frac{k(\log k)^\alpha - \log\log k - (k-1)(\log(k-1))^\alpha + \log\log(k-1)}{(1-\gamma)k(\log k)^\alpha - \log\tilde{c} - \log\log k}. \tag{3.5}$$

Our next goal is to simplify this expression. First we observe that

$$\begin{aligned} k(\log k)^\alpha - (k-1)(\log(k-1))^\alpha \\ = (\log k)^\alpha + (k-1)[(\log k)^\alpha - (\log(k-1))^\alpha]. \end{aligned} \tag{3.6}$$

Next we apply the Taylor series expansion. For $k > k_0$,

$$\begin{aligned} (\log(k-1))^\alpha \ge (\log k)^\alpha - \frac{\alpha}{k}(\log k)^{\alpha-1} \\ + \frac{1}{(k-1)^2}\left[(\alpha-1)\alpha(\log(k-1))^{\alpha-2} - \alpha(\log(k-1))^{\alpha-1}\right] \end{aligned} \tag{3.7}$$

$$\ge (\log k)^\alpha - \frac{\alpha}{k}(\log k)^{\alpha-1} - \frac{1}{k^2}\alpha(\log k)^{\alpha-1}, \tag{3.8}$$

since the difference between k^{-2} and $(k-1)^{-2}$ is $O(k^{-3}) = o((k-1)^{-2}(\log(k-1))^{\alpha-2})$. This combined with (3.6) gives

$$k(\log k)^\alpha - (k-1)(\log(k-1))^\alpha \le (\log k)^\alpha + (k-1)\frac{\alpha}{k}(\log k)^{\alpha-1} + (k-1)\frac{1}{k^2}\alpha(\log k)^{\alpha-1} = (\log k)^\alpha + \alpha(\log k)^{\alpha-1} - \frac{1}{k^2}\alpha(\log k)^{\alpha-1}.$$

Thus for $k > k_0$, throwing out two negative terms,

$$\mathsf{P}_\mathsf{v}(T^V(r_k^\gamma) \le T^V(q_k)) \le \frac{(\log k)^\alpha + \alpha(\log k)^{\alpha-1} + \log\log(k-1)}{(1-\gamma)k(\log k)^\alpha - 2\log\log k}. \tag{3.9}$$

The following inequality is valid for $a, b, c > 0$ such that $b \ge 2c$:

$$\frac{a}{b-c} \le \frac{a}{b} + \frac{2ac}{b^2}.$$

This and (3.9) imply that for large k,

$$\begin{aligned} &\mathsf{P}_\mathsf{v}(T^V(r_k^\gamma) \le T^V(q_k)) \\ &\quad\le \frac{(\log k)^\alpha + \alpha(\log k)^{\alpha-1} + \log\log(k-1)}{(1-\gamma)k(\log k)^\alpha} \\ &\qquad + \frac{4(\log\log k)((\log k)^\alpha + \alpha(\log k)^{\alpha-1} + \log\log(k-1))}{(1-\gamma)^2k^2(\log k)^{2\alpha}} \\ &\quad\le \frac{1}{1-\gamma}\left(\frac{1}{k} + \frac{\alpha}{k\log k} + \frac{2\log\log(k-1)}{k(\log k)^\alpha}\right). \end{aligned} \tag{3.10}$$

Recall that $\tilde{c}\log c_{\mathsf{cyl}} \le -2$ and choose k_0 such that for $k \ge k_0$ we have $r_{k-1} < q_k$. For $k \ge k_0$ we can apply Proposition 3.2 to obtain

$$\mathsf{P}_\mathsf{v}(T^V(q_k) < T^{|Z|}(r_k)) \le c_{\mathsf{cyl}}^{r_k/q_k} = c_{\mathsf{cyl}}^{\tilde{c}\log k} \le k^{-2}.$$

Together with (3.3) and (3.10), this yields for large k,

$$\begin{aligned} &\mathsf{P}_\mathsf{v}(T^V(q_k) < T^{|Z|}(r_k) \wedge T^B(\mathscr{C})) \\ &\quad\ge 1 - \frac{1}{1-\gamma}\left(\frac{1}{k} + \frac{\alpha}{k\log k} + \frac{2\log\log(k-1)}{k(\log k)^\alpha}\right) - \frac{1}{k^2}. \end{aligned}$$

The $1/k^2$ term is small enough to be absorbed into the last error term, so setting

$$p_k := \frac{1}{1-\gamma}\left(\frac{1}{k} + \frac{\alpha}{k\log k} + \frac{4\log\log(k-1)}{k(\log k)^\alpha}\right),$$

we have finally

$$\mathsf{P}_\mathsf{v}(T^V(q_k) < T^{|Z|}(r_k) \wedge T^B(\mathscr{C})) \ge 1 - p_k.$$

It remains to multiply these estimates together qua conditional probabilities.

Recall that j is defined so that $r_{j-2} < L \le r_{j-1}$ and that all our estimates are valid for $k \ge k_0$. The strong Markov property applied at each time $T^V(q_k)$ implies that

$$q(L) \ge c'_q \prod_{k=k_0}^{j} (1 - p_k).$$

For small $a > 0$ we have $\log(1-a) \ge -a - a^2$ and so (enlarging k_0 if necessary, and thereby introducing a constant factor),

$$\log\left(\prod_{k=k_0}^{j} (1 - p_k)\right) \ge \hat{c} + \sum_{k=k_0}^{j} (-p_k - p_k^2).$$

Since p_k^2 is summable, as is the lowest order term $\log\log k/k(\log k)^\alpha$ in the definition of p_k, we get

$$\log\left(\prod_{k=k_0}^{j} (1 - p_k)\right) \ge c_{\mathrm{sum}} - \sum_{k=k_0}^{j} \frac{1}{1-\gamma}\left(\frac{1}{k} + \frac{\alpha}{k \log k}\right),$$

which completes the proof of (3.2). □

3.2. *Estimating* $q(L, \theta)$. Now begins the task of estimating the probability $q(L, \theta)$ of avoiding both $\mathscr{C}$ and $\mathscr{C}_\theta$ until τ_L. Let $T_k = \tau_{r_k}$. The argument proceeds by estimating the conditional probability of avoiding both $\mathscr{C}$ and $\mathscr{C}_\theta$ between each T_k and T_{k+1} given $B(T_k)$ and multiplying the supremum of these conditional probabilities to give an upper bound. For values of k greater than some $k_1(\theta)$, this will be close to

$$1 - \frac{2}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)}\right),$$

corresponding to intersections with $\mathscr{C}$ and $\mathscr{C}_\theta$ being roughly independent, while for small k the $2/(1-\gamma)$ is replaced by a $1/(1-\gamma)$. Multiplying these together and identifying the value of k_1 will then give an upper bound on $q(L, \theta)$. This bound loses sharpness where k_1 must be chosen large enough to give a leading term of $2/((1-\gamma)k)$ even in the worst case, that being the case $B(T_k) \in \mathscr{C}$, which is not likely to happen.

Again define sequences of constants: $a_k = r_k k(\log k)^\beta$, $\rho_k = \tilde{c} a_k \log k$, $b_k = r_{k+1}/(\tilde{c}\log k)$ and $d_k = r_k \tilde{c} \log k$, where $\tilde{c}$ still satisfies $\tilde{c}\log c_{\mathrm{cyl}} \le -2$. Set $k_1(\theta) = \exp(|\log\theta|^{1/(\alpha-1)})$. Lemma 3.4 and Corollary 3.6 provide estimates for small k and large k, respectively.

LEMMA 3.4. *Let*

$$t_k := \frac{1}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)}\right) - \frac{c_1}{k(\log k)^{\beta-1}}. \tag{3.11}$$

There are constants k_0 and c_1 such that for any $k \ge k_0$ and any v *with* $|\mathsf{v}| = r_k$,

$$\mathsf{P}_{\mathsf{v}}(B[0, T_{k+1}] \cap \mathscr{C}^{d_k} \ne \varnothing) \ge t_k.$$

LEMMA 3.5. *For* $k \geq k_1(\theta) := \lceil \log(1/\theta) \rceil$ *and* $|\mathrm{v}| = r_k$,

$$\begin{aligned} &\mathsf{P}_{\mathrm{v}}\big(B[0, T_{k+1}] \cap \mathscr{C}^{d_k} \neq \varnothing \text{ and } B[0, T_{k+1}] \cap \mathscr{C}_\theta^{d_k} \neq \varnothing\big) \\ &\quad \leq \frac{9}{(1-\gamma)^2 k (\log k)^\alpha}. \end{aligned} \tag{3.12}$$

COROLLARY 3.6. *For* $k \geq k_1(\theta)$ *and* $|\mathrm{v}| = r_k$,

$$\begin{aligned} &\mathsf{P}_{\mathrm{v}}\big[B[0, T_{k+1}] \cap (\mathscr{C}^{d_k} \cup \mathscr{C}_\theta^{d_k}) \neq \varnothing\big] \\ &\quad \geq \frac{2}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)} - \frac{c_1}{k(\log k)^{\beta-1}}\right) \\ &\qquad - \frac{9}{(1-\gamma)^2 k (\log k)^\alpha}. \end{aligned} \tag{3.13}$$

□

PROOF OF LEMMA 3.4. Let

$$S_k = \inf\{t > 0 \colon \exists\, s \in (0, t) \colon V(s) = a_k \text{ and } V(t) = r_k^\gamma\} \quad \inf \varnothing = \infty,$$

be the first time when $V(t) = r_k^\gamma$ after the first time that a_k is hit by V. Letting E_{y} denote expectation with respect to P_{y}, we have for every y of modulus r_k,

$$\begin{aligned} &\mathsf{P}_{\mathrm{y}}(B[0, T_{k+1}] \cap \mathscr{C}^{d_k} \neq \varnothing) \\ &\quad \geq \mathsf{E}_{\mathrm{y}} \mathsf{P}_{B(T^V(a_k))}(T^V(r_k^\gamma) < T^V(b_k)) - \mathsf{P}_{\mathrm{y}}(T^V(a_k) \geq T^{|Z|}(\rho_k)) \\ &\qquad - \mathsf{E}_{\mathrm{y}} 1_{\{|Z(T^V(a_k))| \leq \rho_k\}} \mathsf{P}_{B(T^V(a_k))}(T^{|Z|}(r_{k+1}) < T^V(b_k)) \\ &\qquad - \mathsf{P}_{\mathrm{y}}(|Z(S_k)| \leq d_k). \end{aligned} \tag{3.14}$$

In words, this says wait until the radial part reaches a_k then see if it comes back to r_k^γ before reaching b_k; if it does, it must hit $\mathscr{C}^{d_k}$ at this time S_k unless the z-coordinate is wrong. This is covered by the union of three events:

1a. $|Z|$ might reach ρ_k before the radial part reaches a_k;
1b. $|Z|$ might reach r_{k+1} before S_k, despite having magnitude at most ρ_k at time $T^V(a_k)$; or
2. $|Z|$ may be smaller than d_k at time S_k.

The point of waiting for the radial part to reach a_k before coming back is to make event 2 unlikely. We give easy estimates on these three probabilities before doing the Taylor series computation for the probability of the radial part coming back to r_k^γ before hitting b_k.

For 1a we use Proposition 3.2. Recalling that $|\mathrm{y}| = r_k$ gives

$$\mathsf{P}_{\mathrm{y}}(T^V(a_k) < T^{|Z|}(\rho_k)) \geq 1 - c_{\mathrm{cyl}}^{\rho_k/a_k} = 1 - c_{\mathrm{cyl}}^{\tilde{c}\log k} \geq 1 - k^{-2}. \tag{3.15}$$

For 1b, condition on $B(T^V(a_k))$ to get

$$\begin{aligned} &\mathsf{E}_{\mathrm{y}} 1_{\{|Z(T^V(a_k))| \leq \rho_k\}} \mathsf{P}_{B(T^V(a_k))}(T^{|Z|}(r_{k+1}) < T^V(b_k)) \\ &\quad \leq \sup\{\mathsf{P}_{(x,y,z)}(T^{|Z|}(r_{k+1}) < T^V(b_k)) \colon x^2 + y^2 = a_k^2, |z| \leq \rho_k\}. \end{aligned}$$

Since a_k and ρ_k are less than b_k for large k, Proposition 3.2 gives

$$\mathsf{E}_{\mathsf{y}} 1_{\{|Z(T^V(a_k))| \le \rho_k\}} \mathsf{P}_{B(T^V(a_k))}(T^{|Z|}(r_{k+1}) < T^V(b_k)) \le c_{\mathsf{cyl}}^{r_{k+1}/b_k} = c_{\mathsf{cyl}}^{\tilde{c}\log k} \le k^{-2}. \tag{3.16}$$

For 2, let A_k denote the event that $|Z(S_k)| \le d_k$. Since $a_k/r_k = k(\log k)^\beta$, the P_{y} distribution of $T^V(a_k)$ is stochastically greater than $(r_k k(\log k)^\beta)^2$ times some fixed distribution. The P_{y} distribution of S_k is even greater. Since $|Z|$ is independent of V, the P_{y} density of $|Z(S_k)|$ is bounded by $c_{\mathsf{density}}/(r_k k(\log k)^\beta)$ for some constant c_{density}. Hence there exist constants c'_{density} and c''_{density} for which

$$\mathsf{P}_{\mathsf{y}}(A_k) \le \frac{c'_{\mathsf{density}} d_k}{r_k k(\log k)^\beta} \le \frac{c''_{\mathsf{density}}}{k(\log k)^{\beta-1}}. \tag{3.17}$$

For the final estimate, we condition on $B(T^V(a_k))$ and use the fact that the event in question depends only on V to get

$$\mathsf{E}_{\mathsf{y}} \mathsf{P}_{B(T^V(a_k))}(T^V(r_k^\gamma) < T^V(b_k)) = \frac{\log b_k - \log a_k}{\log b_k - \log(r_k^\gamma)}.$$

Expanding the right-hand side according to the definitions gives

$$\begin{aligned}&\frac{(k+1)(\log(k+1))^\alpha - \log\tilde{c} - \log\log k - k(\log k)^\alpha - \log k - \beta\log\log k}{(k+1)(\log(k+1))^\alpha - \log\tilde{c} - \log\log k - \gamma k(\log k)^\alpha}\\ &\quad\ge \frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha - 2\log k}{(1-\gamma)(k+1)(\log(k+1))^\alpha + \gamma((k+1)(\log(k+1))^\alpha - k(\log k)^\alpha)}.\end{aligned}$$

For positive a, b and c we always have

$$\frac{a}{b+c} \ge \frac{a}{b} - \frac{ac}{b^2}$$

and so this is at least

$$\begin{aligned}&\frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha - 2\log k}{(1-\gamma)(k+1)(\log(k+1))^\alpha} - \gamma((k+1)(\log(k+1))^\alpha\\ &\quad - k(\log k)^\alpha)\frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha - 2\log k}{(1-\gamma)^2(k+1)^2(\log(k+1))^{2\alpha}}\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\ge \frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha - 2\log k}{(1-\gamma)(k+1)(\log(k+1))^\alpha}\\ &\quad - \frac{\gamma[(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha]^2}{(1-\gamma)^2(k+1)^2(\log(k+1))^{2\alpha}}.\end{aligned} \tag{3.19}$$

For large k, the Taylor series expansion gives

$$(\log k)^\alpha \le (\log(k+1))^\alpha - \frac{\alpha}{k+1}(\log(k+1))^{\alpha-1}.$$

Hence

$$(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha \tag{3.20}$$
$$\geq (\log(k+1))^\alpha + k\frac{\alpha}{k+1}(\log(k+1))^{\alpha-1}$$
$$= (\log(k+1))^\alpha + \alpha(\log(k+1))^{\alpha-1} - \frac{\alpha}{k+1}(\log(k+1))^{\alpha-1}. \tag{3.21}$$

On the other hand, (3.8) with k replaced by $k+1$ gives

$$\begin{aligned}(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha &= (\log(k+1))^\alpha + k[(\log(k+1))^\alpha - (\log k)^\alpha] \\ &\leq (\log(k+1))^\alpha + \alpha\left(\frac{k}{k+1} + \frac{k}{(k+1)^2}\right)\log(k+1))^{\alpha-1} \\ &\leq 2(\log(k+1))^\alpha.\end{aligned} \tag{3.22}$$

Plugging (3.21) and (3.22) into (3.19) gives, for large k,

$$\mathsf{E}_y \mathsf{P}_{B(T^V(a_k))}(T^V(r_k^\gamma) < T^V(b_k))$$
$$\geq \frac{(\log(k+1))^\alpha + \alpha(\log(k+1))^{\alpha-1} - (\alpha/k+1)(\log(k+1))^{\alpha-1} - 2\log k}{(1-\gamma)(k+1)(\log(k+1))^\alpha}$$
$$- \frac{\gamma(2(\log(k+1))^\alpha)^2}{(1-\gamma)^2(k+1)^2(\log(k+1))^{2\alpha}} \tag{3.23}$$
$$\geq \frac{1}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)} - \frac{4\log(k+1)}{(k+1)(\log(k+1))^\alpha}\right)$$
$$- \frac{4\gamma}{(1-\gamma)^2(k+1)^2}$$
$$\geq \frac{1}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)} - \frac{8}{(k+1)(\log(k+1))^{\alpha-1}}\right). \tag{3.24}$$

All the parts of inequality (3.14) have now been estimated. Plugging in (3.24), (3.15), (3.16) and (3.17) gives

$$\mathsf{P}_y(B[0, T_{k+1}] \cap \mathscr{C}^{d_k} \neq \varnothing)$$
$$\geq \frac{1}{1-\gamma}\left(\frac{1}{k+1} + \frac{\alpha}{(k+1)\log(k+1)} - \frac{8}{(k+1)(\log(k+1))^{\alpha-1}}\right)$$
$$- \frac{2}{k^2} - \frac{c''_{\text{density}}}{k(\log k)^{\beta-1}},$$

which may be written in the form (3.11) thus proving Lemma 3.4. □

PROOF OF LEMMA 3.5. If the event in (3.12) occurs, then it occurs as follows: $B(t)$ hits one of the two sets $\mathscr{C}^{d_k}$ or $\mathscr{C}_\theta^{d_k}$, and then the other. By symmetry, the probability of this is at most twice the supremum of the probability of hitting $\mathscr{C}_\theta^{d_k}$ and then hitting $\mathscr{C}^{d_k}$, where the supremum is taken over all starting points y with $|\mathrm{y}| = r_k$. Conditioning on $T^V(r_k)$ and on the first point z where $B(t)$ hits $\mathscr{C}_\theta$ and using the strong Markov property shows that the probability in (3.12) is at most $2(p_1 + p_2 p_3)$ where

$$p_1 = \sup\{\mathsf{P}_\mathrm{y}(T^V(r_k) > T^{|Z|}(d_k)) \colon |\mathrm{y}| = r_k\};$$

$$p_2 = \sup\{\mathsf{P}_{(x,y,z)}(T^V(r_{k+1}^\gamma) < T^V(r_{k+1})) \colon x^2 + y^2 = r_k\};$$

$$\begin{aligned} p_3 &= \sup\{\mathsf{P}_\mathrm{z}(B[0, T_{k+1}] \cap \mathscr{C}_\theta^{d_k} \neq \varnothing) \colon \mathrm{z} \in \mathscr{C}_\theta\} \\ &= \sup\{\mathsf{P}_\mathrm{z}(B[0, T_{k+1}] \cap \mathscr{C}_\theta \neq \varnothing) \colon \mathrm{z} \in \mathscr{C}_\theta^{d_k}\} \\ &\leq \sup\{\mathsf{P}_\mathrm{z}(T^V(r_{k+1}^\gamma) < T^V(r_{k+1})) \colon \mathrm{z} \in \mathscr{C}_\theta^{d_k}\}. \end{aligned}$$

To estimate these three probabilities, we use Proposition 3.2 and Fact $(*)$ twice. First, by Proposition 3.2, when $|\mathrm{y}| = r_k$, we have

$$\mathsf{P}_\mathrm{y}(T^V(r_k) > T^{|Z|}(d_k)) \leq c_{\mathrm{cyl}}^{d_k/r_k} \leq k^{-2}. \tag{3.25}$$

Second, for $x^2 + y^2 = r_k^2$, Fact $(*)$ gives

$$\begin{aligned} \mathsf{P}_{(x,y,z)}(T^V(r_{k+1}^\gamma) < T^V(r_{k+1})) &= \frac{\log r_{k+1} - \log r_k}{\log r_{k+1} - \log(r_{k+1}^\gamma)} \\ &= \frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha}{(1-\gamma)(k+1)(\log(k+1))^\alpha}. \end{aligned}$$

Recalling from (3.22) that

$$(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha \leq 2(\log(k+1))^\alpha$$

and assuming $k \geq k_0$ then yields

$$\mathsf{P}_{(x,y,z)}(T^V(r_{k+1}^\gamma) < T^V(r_{k+1})) \leq \frac{2}{(1-\gamma)(k+1)}. \tag{3.26}$$

Finally, assume $k \geq k_1(\theta) = \lceil \log(1/\theta) \rceil$. Then the distance Δ between $\mathscr{C}^{d_k}$ and $\mathscr{C}_\theta^{d_k}$ is at least

$$\theta r_k - 2r_k^\gamma \geq r_k(e^{-k} - 2r_k^{\gamma-1}) \geq \frac{r_k}{e^{k+1}}$$

for $k \geq k_0$ where k_0 is independent of θ. If $k \geq k_1(\theta)$ and z is any point on $\mathscr{C}_\theta^{d_k}$, we have

$$
\begin{aligned}
\mathrm{P}_{\mathrm{z}}(T^V(r_{k+1}^\gamma) < T^V(r_{k+1})) &\leq \frac{\log r_{k+1} - \log \Delta}{\log r_{k+1} - \log(r_{k+1}^\gamma)} \\
&\leq \frac{(k+1)(\log(k+1))^\alpha - k(\log k)^\alpha + k + 1}{(1-\gamma)(k+1)(\log(k+1))^\alpha} \\
&\leq \frac{2(\log(k+1))^\alpha}{(1-\gamma)(k+1)(\log(k+1))^\alpha} + \frac{1}{(1-\gamma)(\log k)^\alpha} \qquad (3.27)\\
&\leq \frac{2}{(1-\gamma)(k+1)} + \frac{1}{(1-\gamma)(\log k)^\alpha} \\
&\leq \frac{2}{(1-\gamma)(\log k)^\alpha}.
\end{aligned}
$$

Putting together (3.25), (3.26) and (3.27) gives

$$2(p_1 + p_2 p_3) \leq 2\left(k^{-2} + \frac{4}{(1-\gamma)^2 k(\log k)^\alpha}\right)$$

which proves Lemma 3.5. □

Proofs of Lemma 3.1 and Theorem 2.2. It remains to multiply all the conditional probabilities. Recall the definition of t_k as the right-hand side of (3.11) and let $s_k = 2t_k - 9/((1-\gamma)^2 k(\log k)^\alpha)$ be the right-hand side of (3.13). When $|\mathrm{y}| = r_k$, the P_{y}-probability of the event

$$\{B[0, T_{k+1}] \cap \mathscr{C}^{d_k} \neq \varnothing\} \cup \{B[0, T_{k+1}] \cap \mathscr{C}_\theta^{d_k} \neq \varnothing\}$$

is bounded below by t_k for any $k \geq k_0$ and by s_k in the case that $k \geq k_1(\theta)$. Let m be such that $r_{m+1} < L \leq r_{m+2}$, that is, $m = j - 3$ where j is defined in Lemma 3.3. A repeated application of the strong Markov property at times T_k then gives

$$q(L, \theta) \leq c_{\mathrm{upper}} \prod_{k=k_0}^{k_1} (1 - t_k) \prod_{k=k_1+1}^{m} (1 - s_k).$$

For small $a > 0$ we have $\log(1-a) \leq -a - a^2$ and so

$$
\begin{aligned}
&\log\left(c_{\mathrm{upper}} \prod_{k=k_0}^{k_1} (1-t_k) \times \prod_{k=k_1+1}^{m} (1-s_k)\right) \\
&\qquad \leq c'_{\mathrm{upper}} + \sum_{k=k_0}^{k_1} (-t_k - 2t_k^2) + \sum_{k=k_1+1}^{m} (-s_k - 2s_k^2).
\end{aligned}
$$

The series s_k^2 and t_k^2 are summable, being $O(k^{-2})$, and the series

$$\sum_{k=1}^{\infty} \frac{3c_1}{k(\log k)^{\beta-1}} + \frac{9}{(1-\gamma)^2 k(\log k)^\alpha}$$

is summable as well, which implies that there is a c''_{upper} for which

$$\log\Big(c_{\mathrm{upper}}\prod_{k=k_0}^{k_1}(1-t_k)\times\prod_{k=k_1+1}^{m}(1-s_k)\Big)$$
$$\le c''_{\mathrm{upper}}-\sum_{k=k_0}^{k_1}\frac{1}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)$$
$$-\sum_{k=k_1+1}^{m}\frac{2}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big).$$

Thus, using the estimate of $q(L)$ in Lemma 3.3 in the last step, we have

$$\begin{aligned}q(L,\theta)&\le\exp\Big(c''_{\mathrm{upper}}-\sum_{k=k_0}^{k_1}\frac{1}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)\\&\qquad-\sum_{k=k_1+1}^{m}\frac{2}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)\Big)\\&=\exp\Big(c''_{\mathrm{upper}}+\sum_{k=k_0}^{k_1}\frac{1}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)\\&\qquad-\sum_{k=k_0}^{m}\frac{2}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)\Big)\\&\le c^{*}_{\mathrm{upper}}\exp\Big(\frac{1}{1-\gamma}\log k_1+\frac{\alpha}{1-\gamma}\log\log k_1\\&\qquad-\sum_{k=k_0-1}^{j-1}\frac{2}{1-\gamma}\Big(\frac{1}{k+1}+\frac{\alpha}{(k+1)\log(k+1)}\Big)\Big)\\&\le M|\log\theta|^{1/(1-\gamma)}(\log|\log\theta|)^{\zeta}q(L)^2.\end{aligned}$$

This proves Lemma 3.1. Since the right-hand side of (3.1) is integrable over the unit sphere, Theorem 2.2 then follows from Lemma 2.6. □

4. Avoidance of thorns passing an integral test. Let f and g be fixed functions satisfying the hypotheses (2.1)–(2.3) of of Theorem 2.4, and satisfying the integral test (2.4). Recall $U(L,\theta)$ from Section 2. The remainder of the paper is devoted to proving

$$U(L,\theta)\le C_f\theta^{-\xi} \tag{4.1}$$

for some constant C_f and arbitrarily small $\xi>0$. This, together with the second moment lemma, proves Theorem 2.4. In this section we outline the proof of (4.1).

The idea of the proof is that if $h(\mathrm{x})=\mathrm{P}_{\mathrm{x}}(\tau_L<\tau_{\mathscr{C}})$ solves a Dirichlet problem for $\mathscr{B}_L\backslash\mathscr{C}$ and h_θ is the analogous function when $\mathscr{C}$ is replaced by $\mathscr{C}_\theta$, then $h\cdot h_\theta$

"almost" solves the Dirichlet problem on $\mathscr{B}_L \setminus (\mathscr{C} \cup \mathscr{C}_\theta)$; evaluating at the origin, $q(L, \theta)$ is almost equal to $q(L)^2$. The correction term is the integral of $\nabla h \cdot \nabla h_\theta$ against Green's function for Brownian motion absorbed by $\mathscr{C} \cup \mathscr{C}_\theta$, as stated with some obfuscation in Lemma 4.3 below. Thus to prove (4.1), it suffices to get good bounds on $|\nabla h|$ and on Green's function $G^\theta(0, \mathrm{x})$. The bounds on $|\nabla h|$ are somewhat tedious to derive, being based on geometric arguments that involve first getting bounds on $|h|$, but are reasonably straightforward.

Bounds on G, however, are not straightforward, since if we knew G we could solve the problem directly. One approach is to use the bound $G^\theta(\mathrm{x}, \mathrm{y})$ by the unrestricted Green's function $|\mathrm{x} - \mathrm{y}|^{-1}$. Not only does this give reasonable results [under a stronger hypothesis than (2.4)], but it may be bootstrapped to give better and better bounds on G^θ. The (transfinite) limit of such bootstrapping is to get an implicit inequality obeyed by G^θ and $q(\cdot, \theta)$ in the form of Lemma 4.4 below. This together with Lemma 4.3 gives an integral inequality satisfied by $U(\cdot, \theta)$, Lemma 4.7 below, which leads directly to (4.1).

That being the conceptual outline, we now state a sequence of lemmas, including those mentioned above, which form the technical breakdown of the necessary steps. The first two are merely useful and intuitively obvious principles which are used repeatedly in the remaining proofs.

We start with a few technical changes to our set-up. First of all, we will give a new meaning to the symbol $\mathscr{C}^L$, different from that in Section 3. The change is small and will not confuse a reader who forgets, so we risk the duplication of notation. We start with a set $\mathscr{C} \cap \mathscr{B}_L$ and smooth it in an appriopriate way so that the resulting set has a C^2-boundary. Recall that $\mathscr{C}$ is defined by a twice differentiable function f but it is truncated near the origin so that the origin is outside $\mathscr{C}$. The boundary of each of the two components of $\mathscr{C} \cap \mathscr{B}_L$ is smooth except for a circle at each end of this truncated set. We modify the set $\mathscr{C} \cap \mathscr{B}_L$ to obtain $\mathscr{C}^L$ so that (i) the sets $\mathscr{C} \cap \mathscr{B}_L$ and $\mathscr{C}^L$ may differ only in a neighborhood of radius one around each of the circles mentioned above; (ii) the boundary of $\mathscr{C}^L$ is C^2-smooth; (iii) for large $L < L'$, the sets $\mathscr{C}^L \cap \mathscr{B}_{L/2}$ and $\mathscr{C}^{L'} \cap \mathscr{B}_{L/2}$ are identical.

We will also need a new definition similar to that of $q(L)$. Define $\tilde{q}(L)$ to be the probability of hitting $\partial\mathscr{B}_L$ before hitting $\mathscr{C}^{L/2}$ for Brownian motion starting from the origin. The meaning of $\tilde{q}(L, \theta)$ is derived in an analogous way: it is the probability of avoiding $\mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2}$ until the hitting time of $\partial\mathscr{B}_L$. We change the meaning of $U(L, \theta)$, again so that now

$$U(L, \theta) := \frac{\tilde{q}(L, \theta)}{\tilde{q}(L)^2};$$

it is elementary to check that the second moment method applies equally well when U is defined in terms of $\tilde{q}$ as when U is defined in terms of q, so the substitution is not dangerous and saves us from a page full of tildes or an unfamiliar letter.

We say that L is a *regular value* for f and θ if $U(L, \theta) \geq U(L/4, \theta)$. Probably all values are regular, but in lieu of a proof of that we must consider both alternatives.

LEMMA 4.1. (i) *Suppose μ_r is the subprobability hitting measure on $\partial\mathscr{B}_r$ of Brownian motion on the domain $\mathscr{B}_r \setminus \mathscr{C}^\rho$ for some $\rho > 0$,*

$$\mu_r(A) = \mathsf{P}_0(B(\tau_{\mathscr{C}^\rho \cup \partial\mathscr{B}_r}) \in A).$$

Then the density

$$\frac{d\mu_L}{dS}(\mathsf{x})$$

of μ_L with respect to area dS on the L-sphere is an increasing function of the angle between x and the z-axis.

(ii) *Suppose $r_1 < r$ and $\rho > 0$. For x satisfying $|\mathsf{x}| = r_1$, the probability $P_{\mathsf{x}}(\tau_r < \tau_{C^\rho})$ is an increasing function of the angle between x and the z-axis.*

COROLLARY 4.2. *Assume that $g(z) \to \infty$ as $z \to \infty$. Then $\widetilde{q}(2L)/\widetilde{q}(L) \to 1$ as $L \to \infty$.*

LEMMA 4.3. *Fix f, L and θ. Let $h_1(\mathsf{x}) = \mathsf{P}_{\mathsf{x}}(\tau_{\partial\mathscr{B}_L} < \tau_{\mathscr{C}^{L/2}})$ be the probability of hitting the L-sphere before $\mathscr{C}^{L/2}$ starting at x. Similarly, let $h_2(\mathsf{x}) = \mathsf{P}_{\mathsf{x}}(\tau_{\partial\mathscr{B}_L} < \tau_{\mathscr{C}_\theta^{L/2}})$ be the probability of hitting the L-sphere before hitting the rotated cylinder $\mathscr{C}_\theta^{L/2}$. Let G^θ denote the Green's function for the region $\mathscr{B}_L \setminus (\mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2})$. Then there is a constant r_f such that for all regular values of $L \geq r_f$,*

$$\widetilde{q}(L,\theta) \leq 2\left[\widetilde{q}(L)^2 + \int_{\mathscr{B}_{L/4}\setminus(\mathscr{C}\cup\mathscr{C}_\theta)} (\triangledown h_1(\mathsf{x}) \triangledown h_2(\mathsf{x})) G^\theta(0,\mathsf{x})\, d\mathsf{x}\right].$$

LEMMA 4.4. *There exist an absolute constant K and a constant R_f depending on f, such that for any θ, any $L \geq R_f$ and any x with $|\mathsf{x}| \geq R_f$,*

$$G^\theta(0,\mathsf{x}) \leq K\widetilde{q}(|\mathsf{x}|/2,\theta)|\mathsf{x}|^{-1}.$$

For any values of the parameters, one has the weaker bound

$$G^\theta(0,\mathsf{x}) \leq |\mathsf{x}|^{-1}.$$

LEMMA 4.5. *Let $h(u,r) = h_1(x,y,u)$ for any point (x,y,u) such that $x^2 + y^2 = r^2$. Suppose $r \leq z \leq L/3$. There is a constant r_f and a $c_* > 0$ such that for all $z \geq r_f$,*

$$h(z,r) \leq c_* \frac{\widetilde{q}(L)}{\widetilde{q}(z)} \frac{\log(r/f(z))}{\log g(z)}. \tag{4.2}$$

If $r \geq z$ but the other hypotheses remain the same, then

$$h(z,r) \leq c_* \frac{\widetilde{q}(L)}{\widetilde{q}(r)}.$$

LEMMA 4.6. *Recall that* $h(u, r) = h_1(x, y, u)$ *for* (x, y, u) *such that* $x^2 + y^2 = r^2$. *Assume that* $(x, y, z) \in \mathscr{B}_{L/4}$. *If* $r \le z$ *and* $r_f \le z \le L/2$ *with* r_f *as in the previous lemma, then*

$$|\bigtriangledown h(z, r)| \le K_f \frac{\widetilde{q}(L)}{\widetilde{q}(z)} \frac{1}{r \log g(z)}, \tag{4.3}$$

where the constant K_f *depends on* f *but not on* L. *If* $L/2 \ge r > z$, $r > r_f$, *and if* ρ *denotes* $\sqrt{z^2 + r^2}$, *then we have as well*

$$|\bigtriangledown h(z, r)| \le K_f \frac{\widetilde{q}(L)}{\rho \widetilde{q}(\rho) \log g(\rho)}. \tag{4.4}$$

Finally, if r *and* z *are both at most* $2r_f$, *then* $|\bigtriangledown h(z, r)| \le c\widetilde{q}(L)$ *where* c *depends on* f.

LEMMA 4.7. *There exist constants* $c_f > 0$ *and* $R_f > 1$ *and a function* $b(r)$ *such that for any* θ *and for any* $L \ge R/4 \ge R_f$,

$$U(L, \theta) \le b(R) + c_f(1 + |\log \theta|) \int_R^L \frac{U(s, \theta)}{s \log^2 g(s)}\, ds. \tag{4.5}$$

Lemma 4.1 and Corollary 4.2, are proved in the next section; these require little computation. Lemma 4.3 is proved in the section following. No computation is required, but the fact that the estimate (4.3) for $|\bigtriangledown h|$ only holds away from $\partial\mathscr{B}_L$ forces us to restrict the integral to a smaller ball and results in some extra estimates. Lemma 4.4 is also proved in the same section. Lemmas 4.5, 4.6 and 4.7 are proved in the subsequent, final section.

We conclude this section with a proof of Theorem 2.4 from the above results.

PROOF OF THEOREM 2.4. By assumption, $\int^{\infty^-} (z \log^2 g(z))^{-1}\, dz < \infty$, so we may choose R large enough so that $R/4 \ge R_f$ and

$$c_f \int_R^{\infty} (z \log^2 g(z))^{-1}\, dz < \xi$$

for ξ arbitrarily small. By Lemma 4.7, for any $L \ge R$, $U(L, \theta)$ is bounded above by the value $u_\theta(L)$, where u_θ solves the integral equation

$$u_\theta(x) = b(R) + c_f(1 + |\log \theta|) \int_R^x \frac{u_\theta(s)}{s \log^2 g(s)}\, ds.$$

Differentiating, one sees that

$$u'_\theta(x) = c_f(1 + |\log \theta|) \frac{u_\theta(x)}{x \log^2 g(x)}$$

and hence that

$$u_\theta(x) = b(R) \exp\left(c_f(1 + |\log \theta|) \int_R^x \frac{1}{s \log^2 g(s)}\, ds \right).$$

By the choice of R, the integral, ξ, may be made arbitrarily small, and so

$$U(L,\theta) \le u_\theta(L) \le C_f\theta^{-\xi},$$

proving (4.1). The function $\theta(\mathrm{v})^{-\xi}$ is integrable over the unit sphere, so the second moment method completes the proof that Brownian motion avoides f-thorns. In fact $\theta(\mathrm{v})^{-\xi-\beta}$ is integrable for β arbitrarily close to 2 (by picking $\xi < 2-\beta$), so the second moment method shows that the dimension of the set of directions of axes of f-thorns avoided by Brownian motion is greater than β for any $\beta < 2$, proving the dimension result. □

5. Noncomputational proofs.

PROOF OF LEMMA 4.1. We prove only part (i) as (ii) has a similar proof. We will use a skew-product decomposition. This is a standard technique, so we will limit ourselves to the description of the decomposition. See [5], Section 7.15, for more information. Let B_t^* be B_t reversed at time τ_r, the time when it hits the sphere of radius r. In other words, $B_t^* = B(\tau_r - t)$ for $t \in (0,\tau_r)$. Let $R_t = |B_t^*|$ denote the modulus and $A_t = \theta(B_t^*)$ denote the angle with the z-axis. Then R_t is the time-reversal of a stopped three-dimensional Bessel process and A_t is a diffusion ψ_t on the interval $[0,\pi]$ time-changed according to a clock determined by R_t but otherwise independent of R_t. The processes are related by $A_t = \psi_{\beta(t)}$ where $\beta(t) = \int_0^t R_u^{-2}\,du$. We have $\beta(t) \to \infty$ as $t \to \tau_r$.

Note that $B_t \in \mathscr{C}^\rho$ for some $t < \tau_r$ if and only if $\psi_s < D_s$ for some $s < \infty$, where D_s is the maximal angle with the z-axis of any vector in $\mathscr{C}^\rho$ of length $R_{\beta^{-1}(s)}$.

The hitting distribution on a sphere is uniform for Brownian motion starting from its center. In order to prove the lemma, it will suffice to show that the probability of hitting $\mathscr{C}^\rho$ before hitting $\partial\mathscr{B}_r$ for a Brownian motion starting from the center and conditioned to exit the sphere at $\mathrm{x} \in \partial\mathscr{B}_r$ is a decreasing function of the angle $\theta(\mathrm{x})$ that x makes with the z-axis. This is equivalent to proving that the probability of $\{\psi_s < D_s\}$ is a decreasing function of $\theta(\psi_0)$.

To show this, we use a coupling argument. We consider a process $(\tilde{R},\psi^1,\psi^2)$ such that $\tilde{R}$ has the same distribution as R and such that ψ^1 and ψ^2 have the same transition probabilities as ψ given R. We let $\psi_0^1 > \psi_0^2$ and require that if $\psi_t^1 = \psi_t^2$ then $\psi_s^1 = \psi_s^2$ for all $s > t$ (in other words the processes stay coupled if they meet). Then clearly $\psi_t^1 \ge \psi_t^2$ for all t, so the probability that $\psi_s^1 < D_s$ for some s is smaller than the probability that $\psi_s^2 < D_s$ for some s. □

PROOF OF COROLLARY 4.2. Consider an arbitrary $a < 1$. Let dS denote the normalized surface area measure on $\mathscr{B}(0,L)$ and let μ_L be defined for $\Lambda \subset \partial\mathscr{B}_L$ by $\mu_L(\Lambda) = \mathrm{P}_0(B(\tau_L \wedge \tau_{C^{L/2}}) \in \Lambda)$. Choose a small $\delta > 0$ such that the S-measure of $A = \{\mathrm{v} \in \mathscr{B}(0,L)\colon r(\mathrm{v}) > \delta L\}$ is greater than $\sqrt{a}$. Then Lemma 4.1 implies that $\mu_L(A) > \sqrt{a}\mu_L(\mathscr{B}(0,L))$. Note that $\mathscr{C} \cap \mathscr{B}(0,2L)$ is a subset of $\{\mathsf{v}: r(\mathsf{v}) \le f(2L)\}$. This and Fact $(*)$ imply that for $\mathsf{x} \in A$,

$$\mathsf{P}_{\mathsf{x}}(\tau_{2L} < \tau_{\mathscr{C}^L}) \ge \mathsf{P}_{\mathsf{x}}(\tau_{2L} < \tau_{\mathscr{C}}) \ge \frac{\log r(\mathsf{x}) - \log f(2L)}{\log(2L) - \log f(2L)} \ge \frac{\log \delta + \log g(2L)}{\log 2 + \log g(2L)}.$$

Since $g(2L) \to \infty$ as $L \to \infty$, we have

$$\mathsf{P}_{\mathsf{x}}(\tau_{2L} < \tau_{\mathscr{C}^L}) \ge \sqrt{a}$$

for $\mathsf{x} \in A$ and large L. By the strong Markov property and the definition of μ_L, for large L,

$$\begin{aligned}\widetilde{q}(2L) &= \int_{\mathscr{B}(0,L)} \mathsf{P}_{\mathsf{x}}(\tau_{2L} < \tau_{\mathscr{C}^L})\, d\mu_L(\mathsf{x}) \\ &\ge \int_A \mathsf{P}_{\mathsf{x}}(\tau_{2L} < \tau_{\mathscr{C}^L})\, d\mu_L(\mathsf{x}) \\ &\ge \sqrt{a}\mu_L(A) \ge a\mu_L(\mathscr{B}(0, L)) = a\widetilde{q}(L).\end{aligned}$$

Since a can be chosen arbitrarily close to 1, the proof is complete. □

6. Green's function methods.

We begin this section with a theorem that is not sufficient for our purposes, but is a cleaner version of the one we will use.

THEOREM 6.1. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be any two closed regions contained in a ball $\mathscr{B}_L$ of radius L centered at the origin 0. Assume the origin is in neither. Suppose h_i is harmonic on $\mathscr{B}_L \setminus \mathscr{C}_i$. Let $G(\cdot,\cdot)$ denote Green's function for the interior of $\mathscr{B}_L \setminus (\mathscr{C}_1 \cup \mathscr{C}_2)$; in other words, if τ is the exit time from $\mathscr{B}_L \setminus (\mathscr{C}_1 \cup \mathscr{C}_2)$, then the expected occupation of a set A up to time τ is*

$$\mathsf{E}_{\mathsf{y}} \int_0^\tau 1_A(B(t))\, dt = \int_A G(\mathsf{y}, \mathsf{x})\, d\mathsf{x}.$$

Then

$$\mathsf{P}_0(\tau_L < \tau_{\mathscr{C}_1 \cup \mathscr{C}_2}) = h_1(0)h_2(0) + \int_{\mathscr{B}_L \setminus (\mathscr{C}_1 \cup \mathscr{C}_2)} [\triangledown h_1(\mathsf{x}) \triangledown h_2(\mathsf{x})] G(0, \mathsf{x})\, d\mathsf{x}, \tag{6.1}$$

provided the integral is absolutely convergent.

PROOF. We will write $\triangledown^2$ for the Laplacian. The function

$$\phi(\mathsf{y}) = \mathsf{E}_{\mathsf{y}} \int_0^\tau f(B(t))\, dt$$

satisfies $\triangledown^2 \phi = -2f$ for any continuous f for which $\mathsf{E} \int_0^\tau |f(B(t))|\, dt$ is finite. Applying the dominated convergence theorem to the defining equation for G we see that if $f(\mathsf{x}) = \triangledown h_1(\mathsf{x}) \triangledown h_2(\mathsf{x})$ then

$$\mathsf{E}_{\mathsf{y}} \int_0^\tau f(B(t))\, dt = \int f(\mathsf{x})\, G(\mathsf{y}, \mathsf{x})\, d\mathsf{x},$$

the right-hand side (and hence the left-hand side) being absolutely integrable by assumption. Since f is bounded and continuous, we see that the Laplacian in y of $\int f(\mathrm{x})G(\mathrm{y},\mathrm{x})\,d\mathrm{x}$ is $-2\bigtriangledown h_1(\mathrm{y})\bigtriangledown h_2(\mathrm{y})$. By the product rule for C^2 functions,

$$\bigtriangledown^2(h_1h_2) = h_1\bigtriangledown^2 h_2 + h_2\bigtriangledown^2 h_1 + 2\bigtriangledown h_1\bigtriangledown h_2;$$

adding this to the equation

$$\bigtriangledown^2\Big(\int f(\mathrm{x})G(\mathrm{y},\mathrm{x})\,d\mathrm{x}\Big) = -2f(\mathrm{y})$$

and remarking that $\bigtriangledown^2 h_1 = \bigtriangledown^2 h_2 = 0$ shows that $\bigtriangledown^2\Psi = 0$, where

$$\Psi(\mathrm{y}) = h_1(\mathrm{y})h_2(\mathrm{y}) + \int_{\mathscr{B}_L\setminus(\mathscr{C}_1\cup\mathscr{C}_2)}[\bigtriangledown h_1(\mathrm{x})\bigtriangledown h_2(\mathrm{x})]G(\mathrm{y},\mathrm{x})\,d\mathrm{x}.$$

The function h_1h_2 has boundary values 1 on $\partial\mathscr{B}_L$ and 0 on $\mathscr{C}_1\cup\mathscr{C}_2$. Since $G(\mathrm{y},\mathrm{x}) \to 0$ as $\mathrm{y} \to \partial(\mathscr{B}_L\setminus(\mathscr{C}_1\cup\mathscr{C}_2))$, these are the boundary values of Ψ as well. This forces $\Psi(\mathrm{y}) = \mathrm{P}_\mathrm{y}(\tau_L < \tau_{\mathscr{C}_1\cup\mathscr{C}_2})$, by the maximum principle (see [2], Theorem II.1.8), since both sides are harmonic with the same boundary conditions. Setting $\mathrm{y} = 0$ proves the theorem. □

We wish to apply this to the case where $\mathscr{C}_1 = \mathscr{C}^{L/2}$ and $\mathscr{C}_2 = \mathscr{C}_\theta^{L/2}$, plugging in the bounds on $|\bigtriangledown h_i|$ from Lemma 4.6. The gradient of $h_i(\mathrm{x})$ is difficult to control near the boundary of $\mathscr{B}_L$. The following lemma allows us to restrict attention to $\mathscr{B}_{L/4}$.

LEMMA 6.2. *Let $\mu_{L/4,\,\theta}$ be the hitting subprobability measure on $\partial\mathscr{B}_{L/4}$ defined by*

$$\mu_{L/4,\,\theta}(A) = \mathrm{P}_0(B(\tau_{\mathscr{C}\cup\mathscr{C}_\theta\cup\partial\mathscr{B}_{L/4}}) \in A).$$

Let $h_1(\mathrm{x})$ and $h_2(\mathrm{x})$ be the probabilities from x *of hitting $\partial\mathscr{B}_L$ before hitting $\mathscr{C}^{L/2}$ and $\mathscr{C}_\theta^{L/2}$, respectively. There is a constant r_f such that for any θ and any regular $L \ge r_f$,*

$$\widetilde{q}(L,\theta) \le 2\int h_1(\mathrm{x})h_2(\mathrm{x})\,d\mu_{L/4,\,\theta}(\mathrm{x}).$$

PROOF. By Corollary 4.2 we may choose r_f great enough so that $\widetilde{q}(L) \ge 0.9\widetilde{q}(L/4)$ for all $L \ge r_f$. When L is regular for θ, it follows that

$$\widetilde{q}(L,\theta) \ge \widetilde{q}(L/4,\theta)\frac{\widetilde{q}(L)^2}{\widetilde{q}(L/4)^2} \ge 0.81\widetilde{q}(L/4,\theta).$$

An upper bound for $\widetilde{q}(L,\theta)$ is the probability of avoiding both $\mathscr{C}^{L/2}$ and $\mathscr{C}_\theta^{L/2}$ until $\tau_{L/4}$ and then avoiding $\mathscr{C}^{L/2}$ until time τ_L. By the Markov property this upper bound is $\int h_1(\mathsf{x})\,d\mu_{L/4,\,\theta}(\mathsf{x})$. A similar bound holds for h_2. Thus we have

$$\tilde{q}(L,\theta) \le \int h_1(\mathsf{x})\,d\mu_{L/4,\,\theta}(\mathsf{x}),$$
$$\tilde{q}(L,\theta) \le \int h_2(\mathsf{x})\,d\mu_{L/4,\,\theta}(\mathsf{x}),$$
$$\tilde{q}(L/4,\theta) = \int 1\,d\mu_{L/4,\,\theta}(\mathsf{x}).$$

Now use the fact that $x+y-1 \le xy$ for x and y in $[0,1]$ to get that when $L \ge r_f$,

$$\begin{aligned}\tilde{q}(L,\theta) &\le 4\tilde{q}(L,\theta) - 2\tilde{q}(L/4,\theta)\\ &\le 2\int (h_1(\mathsf{x}) + h_2(\mathsf{x}) - 1)\,d\mu_{L/4,\,\theta}(\mathsf{x})\\ &\le 2\int h_1(\mathsf{x})h_2(\mathsf{x})\,d\mu_{L/4,\,\theta}(\mathsf{x}).\end{aligned}$$ □

PROOF OF LEMMA 4.3. Let τ be the hitting time for the set $\partial\mathscr{B}_{L/4} \cup \mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2}$. The function $\Psi(\mathsf{x}) := \mathsf{E}_\mathsf{x} h_1(B_\tau)h_2(B_\tau)$ is harmonic in the interior of $\mathscr{B}_{L/4} \setminus (\mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2})$ with boundary conditions h_1h_2, so by the same argument as in the proof of Theorem 6.1,

$$\Psi(\mathsf{y}) = h_1(\mathsf{y})h_2(\mathsf{y}) + \int_{\mathscr{B}_{L/4}\setminus(\mathscr{C}^{L/2}\cup\mathscr{C}_\theta^{L/2})} [\nabla h_1(\mathsf{x}) \triangledown h_2(\mathsf{x})]\tilde{G}^\theta(\mathsf{y},\mathsf{x})\,d\mathsf{x},$$

where $\tilde{G}^\theta(\cdot,\cdot)$ is the Green's function for $\mathscr{B}_{L/4} \setminus (\mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2})$. Since $\tilde{G}^\theta(\cdot,\cdot)$ is less than $G^\theta(\cdot,\cdot)$, the Green's function for $\mathscr{B}_L \setminus (\mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2})$, we obtain an upper bound for $\Psi(\mathsf{y})$ by replacing $\tilde{G}^\theta$ with G^θ in the last formula. It can be shown just as in the last part of Lemma 4.6 that the gradients are bounded on $\mathscr{B}_{L/4}\setminus(\mathscr{C}^{L/2}\cup\mathscr{C}_\theta^{L/2})$ so that there is no problem with convergence of the integral (we do not make any assertion about the size of the bound at this point). By the previous lemma, $\tilde{q}(L,\theta) \le 2\Psi(0)$, which, together with the formula for Ψ and the fact $h_1(0) = h_2(0) = \tilde{q}(L)$ proves the lemma. □

PROOF OF LEMMA 4.4. The weaker bound comes from bounding G^θ by Green's function $G(\cdot,\cdot)$ for all of $\mathbb{R}^3$.

Let S be the normalized surface area measure on $\partial\mathscr{B}_L$. To prove the stronger bound, we first claim that for any fixed θ and some absolute constant K,

$$\frac{d\mu_{L,\,\theta}}{dS} \le K\tilde{q}(L/2,\theta),$$

where $\mu_{L,\,\theta}$ is the hitting (subprobability) measure on $\partial\mathscr{B}_L$ of Brownian motion started at 0 and killed at $\partial\mathscr{B}_L \cup \mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2}$. Let ν_x denote the P_x law of $B(\tau_L)$, that is, the hitting distribution on $\partial\mathscr{B}_L$ of an unkilled Brownian motion started at x. The Harnack principle shows that the densities $d\nu_{\mathsf{x}}/dS$ are bounded for $\mathsf{x} \in \mathscr{B}_{L/2}$ by an absolute constant K. Thus

$$\mu_{L,\,\theta}(A) \le \int \nu_{\mathsf{x}}(A)\, d\mu_{L/2,\,\theta}(\mathsf{x}) \le K\|\mu_{L/2,\,\theta}\| \int_A dS = K\widetilde{q}(L/2, \theta)S(A),$$

proving the claim.

Now let A be any set disjoint from the ball $\mathscr{B}_r \subseteq \mathscr{B}_L$. Letting τ be the hitting time of $\partial\mathscr{B}_L \cup \mathscr{C}^{L/2} \cup \mathscr{C}_\theta^{L/2}$ and using the strong Markov property at time τ_r gives

$$\begin{aligned}\int_A G^\theta(0, \mathsf{y})\, d\mathsf{y} &= \int\left(\int\int 1_{B_t\in A}1_{\tau>t}\, dt\, d\mathsf{P}_{\mathsf{y}}\right) d\mu_{r,\,\theta}(\mathsf{y})\\ &\le \int\left(\int\int 1_{B_t\in A}\, dt\, d\mathsf{P}_{\mathsf{y}}\right) d\mu_{r,\,\theta}(\mathsf{y})\\ &\le K\widetilde{q}(r/2, \theta)\int\left(\int\int 1_{B_t\in A}\, dt\, d\mathsf{P}_{\mathsf{y}}\right) dS_r(\mathsf{y}),\end{aligned}$$

by the above claim for $L = r$, where S_r is normalized surface measure on $\partial\mathscr{B}_r$. But this last quantity is just

$$K\int_A \widetilde{q}(r/2, \theta)G(0, \mathsf{y})\, d\mathsf{y} \le K\int_A \widetilde{q}(r/2, \theta)|\mathsf{y}|^{-1}\, d\mathsf{y}.$$

Letting A shrink around x and leting $r \uparrow |\mathsf{x}|$ then proves the lemma. □

7. Geometric bounds. The following lemma is needed in the proof of Lemma 4.5. It is a version of the boundary Harnack principle but we could not find a version of that theorem that would apply directly in our case.

LEMMA 7.1. *Suppose that for some* z_0,

$$\begin{aligned}A_1 &= \{(x, y, z)\colon x^2 + y^2 < c_1^2 z_0^2,\, c_2 z_0 < z < c_3 z_0\},\\ A_2 &= \{(x, y, z)\colon x^2 + y^2 < c_4^2 z_0^2,\, c_2 z_0 < z < c_3 z_0\},\\ W &= \{(x, y, z)\colon x^2 + y^2 = c_1^2 z_0^2,\, c_2 z_0 < z < c_3 z_0\},\end{aligned}$$

and $\mathsf{v} = (x_1, y_1, z_1)$ *is a point with* $c_5 < z_1 < c_6$, $c_4^2 z_0^2 < x_1^2 + y_1^2 < c_1^2 z_0^2$. *Assume that* $c_4 < c_1$ *and* $c_2 < c_5 < c_6 < c_3$. *Then there exists* $c_7 > 0$ *which depends on* c_1, c_2, c_3, c_5 *and* c_6 *but does not depend on* c_4 *or* z_0, *and such that*

$$\mathsf{P}_{\mathsf{v}}(B(\tau_{A_1^c}) \in W \mid \tau_{A_1^c} < \tau_{A_2}) > c_7.$$

PROOF. We will prove the lemma for $z_0 = 1$. The general case follows by scaling. We will also assume that $c_4 < c_1/4$. The other case requires minor modifications.

Let $c_8 = \max(c_3 - c_6, c_5 - c_2)$ and choose c_9 so that $\sum_{k=1}^{\infty} c_9 k 2^{-k} < c_8/2$. Let $m_k = \sum_{j=k}^{\infty} c_9 j 2^{-j}$,

$$D_k = \left\{(x, y, z)\colon \sqrt{x^2 + y^2} < c_1 2^{-k}, c_5 - m_k < z < c_6 + m_k\right\},$$
$$W_k = \left\{(x, y, z) \in \partial D_k\colon \sqrt{x^2 + y^2} = c_1 2^{-k}\right\},$$
$$U_k = \partial D_k \setminus W_k.$$

Note that for any $\mathrm{w} \in W_k$, the distance from w to W_{k-1} is $c_1 2^{-k}$ but the distance to U_{k-1} is not less than $c_9 k 2^{-k}$. It easily follows from Proposition 3.2 that if $\mathrm{w} \in W_k$, then

$$\mathrm{P}_{\mathrm{w}}(\tau_{U_{k-1}} < \tau_{W_{k-1}}) < \exp(-c_{10} k).$$

If $c_1 2^{-k} \geq 2c_4$, then it is easy to see that for any $\mathrm{w} \in W_k$,

$$\mathrm{P}_{\mathrm{w}}(\tau_{A_2} < \tau_{\partial D_k}) < c_{11} < 1.$$

Hence, for $\mathrm{w} \in W_k$, assuming $c_1 2^{-k} \geq 2c_4$,

$$\mathrm{P}_{\mathrm{w}}(\tau_{U_{k-1}} < \tau_{W_{k-1}} \mid \tau_{\partial D_k} < \tau_{A_2}) < \exp(-c_{12} k).$$

Now suppose that $\mathrm{v} \in W_n$ where n is the smallest number such that $c_1 2^{-n} \geq 2c_4$. If we condition Brownian motion not to hit A_2 between the first hitting times of ∂D_k and ∂D_{k-1} for $k \leq n$, then, using the strong Markov property, we see that for such conditioned process we may have $B(\tau_{\partial D_k}) \in W_k$ for all $k = n, n-1, \ldots, 2$, with probability not less than

$$\prod_{k=2}^{n} (1 - \exp(-c_{12} k)) \geq \prod_{k=2}^{\infty} (1 - \exp(-c_{12} k)) = c_{13} > 0.$$

Hence, Brownian motion conditioned to avoid A_2 before exiting D_1 can hit W_1 with probability greater than c_{13}. Brownian motion starting from a point of W_1 can hit W before hitting any other part of the boundary of A_1 or A_2 with probability greater than c_{14}, independent of c_4. An application of the strong Markov property at the hitting time of W_1 shows that

$$\mathrm{P}_{\mathrm{v}}(B(\tau_{A_1^c}) \in W \mid \tau_{A_1^c} < \tau_{A_2}) > c_{13} c_{14}.$$

The same proof applies to $\mathrm{v} \in W_k$ for $k < n$. The result can be extended to all points $\mathrm{v} = (x_1, y_1, z_1)$ with $x_1^2 + y_1^2 > c_1 2^{-n}$ and $c_5 < z_1 < c_6$ using the Harnack inequality. Finally, it extends to v with $c_4 < \sqrt{x_1^2 + y_1^2} < c_1 2^{-n}$ by the boundary Harnack principle. See [2] for the Harnack inequality and the boundary Harnack principle. □

PROOF OF LEMMA 4.5. The idea is that escaping from (z, r) to $\partial \mathscr{B}_L$ takes two steps. First, one has to escape to $\{r \approx z\}$. Approximating $\mathscr{C}$ by a cylinder of radius $f(z)$ about the z-axis, we see that this probability is roughly $\log(r/f(z))/\log(z/f(z))$. Second, one must escape from radius roughly z to radius L. This probability is the conditional probability of escaping to radius

L given having escaped to radius z, and is thus roughly $\widetilde{q}(L)/\widetilde{q}(z)$. When $r < 2f(z)$, the cylinder approximation is too course and we need a third step, namely first escaping to $\{r \approx 2f(z)\}$. We now rigorize this.

Let v_0 be the point (x_0, y_0, z_0), where all coordinates are assumed w.l.o.g. to be positive. Let $r_0 = \sqrt{x_0^2 + y_0^2}$. We use the unsubscripted symbols x, y, z, r to refer to the x, y, z and $\sqrt{x^2 + y^2}$ coordinate functions, respectively. Assume first that $r_0 < 2f(z_0)$. Let A_0 be the circle in the x, z-plane, centered on the z-axis and tangent to the curve $x = f(z)$ at the point $(z_0, f(z_0))$. By assumption (2.3), A_0 lies completely inside the region $x \le f(z)$. Rotating this circle around the z-axis gives a sphere, A_1, tangent to $\partial\mathscr{C}$ at all points with $r = f(z_0)$ and $z = z_0$, and lying inside $\mathscr{C}$. Let A_2 be the sphere with 4 times the radius and the same center. Clearly we may write

$$h_1(\mathsf{v}_0) \le \mathsf{P}_{\mathsf{v}_0}(\tau_{A_2} < \tau_{A_1}) \tag{7.1}$$

$$\times \sup_{\mathsf{v}\in A_2} \mathsf{P}_{\mathsf{v}}(\tau_{(5\sqrt{2}/4)z_0} < \tau_{\mathscr{C}}) \tag{7.2}$$

$$\times \sup_{\mathsf{v}\in\partial\mathscr{B}_{(5\sqrt{2}/4)z_0}} \mathsf{P}_{\mathsf{v}}(\tau_L < \tau_{\mathscr{C}}). \tag{7.3}$$

To estimate the term (7.1), use the facts that the radius R_1 of A_1 is at least $f(z_0)$ and that the distance $d(\mathsf{v}_0, A_1)$ from v_0 to A_1 is at most $r_0 - f(z_0)$, to get

$$\begin{aligned}\mathsf{P}_{\mathsf{v}_0}(\tau_{A_2} < \tau_{A_1}) &= \frac{4}{3}\left(1 - \frac{R_1}{d(\mathsf{v}_0, A_1)}\right) \le \frac{4}{3}\left(1 - \frac{R_1}{R_1 + r_0 - f(z_0)}\right)\\ &\le \frac{4}{3}\frac{r_0 - f(z_0)}{f(z_0)} \le \frac{4}{3}c_1 \log\left(1 + \frac{r_0 - f(z_0)}{f(z_0)}\right)\\ &= \frac{4}{3}c_1 \log(r_0/f(z_0)).\end{aligned}$$

Let A_3 be the cylinder $\{r \le f(z_0)/2\}$. Let A_4 be the cylinder with radius $5z_0/4$ whose axis is the subinterval $[3z_0/4, 5z_0/4]$ of the z-axis. Observe that A_4 lies inside $\mathscr{B}_{5\sqrt{2}z_0/4}$, and that $A_3 \cap A_4$ lies inside $\mathscr{C} \cap A_4$ [since $f(z_0)/2 \le f(z_0/2) \le f(3z_0/4)$]. Thus (7.2) may be bounded above by $\mathsf{P}_{\mathsf{v}}(\tau_{A_4} < \tau_{A_3})$. When z_0 is sufficiently large, $4f(z_0) \le z_0/10$, and thus the z-coordinate of v is in $[0.9z_0, 1.1z_0]$ for every $\mathsf{v} \in A_4$. By scaling, there is a uniform lower bound $\varepsilon_1 > 0$ for the probability

$$\mathsf{P}_{\mathsf{v}}(B(\tau_{A_4}) \in \mathbb{R}^2 \times (3z_0/4, 5z_0/4))$$

that a Brownian motion started at v exits A_4 along the curved boundary $W := \{(x, y, z)\colon x^2 + y^2 = 25z_0^2/16, 3z_0/4 < z < 5z_0/4\}$. By Lemma 7.1, it follows that

$$\mathsf{P}_{\mathsf{v}}(B(\tau_{A_4}) \in W \mid \tau_{A_4} < \tau_{A_3}) \ge \varepsilon.$$

Thus the term (7.2) is at most

$$\varepsilon^{-1} \sup_{\mathsf{v}\in A_2} \mathsf{P}_{\mathsf{v}}(\tau_W < \tau_{A_3}).$$

Using Fact $(*)$, this gives an upper bound of

$$\varepsilon^{-1} \frac{\log(4f(z_0)) - \log(f(z_0)/2)}{\log(5z_0/4) - \log(f(z_0)/2)} = \varepsilon^{-1} \frac{\log 8}{\log(5g(z_0)/2)}.$$

To estimate (7.3), note that by Lemma 4.1(ii) the supremum is achieved at points $(x, y, 0)$ such that $x^2 + y^2 = 25z_0^2/8$. Let v be such a point. The sphere of radius $z_0/2$ around v is disjoint from $\mathscr{C}$, so applying the Harnack principle to points w in the set Λ of points on $\partial\mathscr{B}_{5z_0/4}$ within distance $z_0/4$ from the x, y-plane, we see that there is a universal constant C such that

$$\mathsf{P}_{\mathsf{v}}(\tau_L < \tau_{\mathscr{C}^{L/2}}) \le C\mathsf{P}_{\mathsf{w}}(\tau_L < \tau_{\mathscr{C}^{L/2}}).$$

Lemma 4.1(i) implies that the μ_L-measure of the set of points on $\mathscr{B}_L$ which form an angle greater than ψ with the z-axis is greater than the normalized surface measure of the same set. Hence,

$$\mathsf{P}_0(B(\tau_{5\sqrt{2}z_0/4}) \in \Lambda \,|\, \tau_{5\sqrt{2}z_0/4} \le \tau_{\mathscr{C}^{L/2}}) \ge \frac{|\Lambda|}{4\pi(5\sqrt{2}z_0/4)^2} = \widetilde{c}.$$

Thus

$$\begin{aligned}
\frac{\widetilde{q}(L)}{\widetilde{q}(5\sqrt{2}z_0/4)} &= \mathsf{P}_0(\tau_L < \tau_{\mathscr{C}^{L/2}} \,|\, \tau_{5\sqrt{2}z_0/4} < \tau_{\mathscr{C}^{L/2}}) \\
&\ge \widetilde{c}\,\mathsf{P}_0(\tau_L < \tau_{\mathscr{C}^{L/2}} \,|\, \tau_\Lambda < \tau_{\mathscr{C}^{L/2}}) \\
&= \widetilde{c}\,\mathsf{E}[\mathsf{P}_{\tau_\Lambda}(\tau_L < \tau_{\mathscr{C}^{L/2}}) \mid \tau_\Lambda < \tau_{\mathscr{C}^{L/2}}] \\
&\ge \frac{\widetilde{c}}{C}\mathsf{P}_{\mathsf{v}}(\tau_L < \tau_{\mathscr{C}^{L/2}}).
\end{aligned}$$

Thus (7.3) is bounded above by

$$(C/\widetilde{c})\frac{\widetilde{q}(L)}{\widetilde{q}(5\sqrt{2}z_0/4)} \le 2(C/\widetilde{c})\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}$$

for z_0 sufficiently large.

Combine the three pieces (7.1)–(7.3) to yield the bound in the lemma.

In the case where $z_0 \ge r_0 \ge 2f(z_0)$, skip the first step, writing $h_1(\mathsf{v}_0)$ as at most

$$\varepsilon^{-1}\mathsf{P}_{\mathsf{v}_0}(\tau_W < \tau_{A_3})$$

times (7.3). Fact $(*)$ then gives an upper bound of

$$h_1(\mathsf{v}_0) \le \varepsilon^{-1} \frac{\log r_0 - \log(f(z_0)/2)}{\log(5z_0/4) - \log(f(z_0)/2)} \sup_{\mathsf{v}\in\partial\mathscr{B}_{(5\sqrt{2}/4)z_0}} \mathsf{P}_{\mathsf{v}}(\tau_L < \tau_{\mathscr{C}^{L/2}}),$$

which is at most a constant multiple of

$$\frac{\log(r_0/f(z_0))}{\log g(z_0)} \sup_{\mathsf{v}\in\partial\mathscr{B}_{(5\sqrt{2}/4)z_0}} \mathsf{P}_{\mathsf{v}}(\tau_L < \tau_{C^{L/2}}) \le \widehat{c}\,\frac{\log(r_0/f(z_0))}{\log g(z_0)} \frac{\widetilde{q}(L)}{\widetilde{q}(z_0)},$$

since $r_0/f(z_0) \ge 2$.

Finally, in the case where $r_0 \ge z_0$, we have from Lemma 4.1(ii) that $h(z_0, r_0) \le h(\lambda, \lambda)$, where $\lambda = \sqrt{(z_0^2 + r_0^2)/2}$. Now apply (4.2) and observe that by Corollary 4.2, $\widetilde{q}(r_0) = (1 + o(1))\widetilde{q}(\lambda)$ for large r_f. $\square$

PROOF OF LEMMA 4.6. We start with a proof of (4.3). It will suffice to show that

$$|h(\mathsf{v}_0) - h(\mathsf{v}_1)| \le K_f \delta \frac{\widetilde{q}(L)}{\widetilde{q}(z_0)} \frac{1}{r_0 \log g(z_0)}$$

where $\mathsf{v}_i = (x_i, y_i, z_i)$, $r_i = \sqrt{x_i^2 + y_i^2}$ and $\delta := |\mathsf{v}_0 - \mathsf{v}_1|$ is small. We let $\mathscr{K}$ be the plane such that v_0 and v_1 are symmetric with respect to $\mathscr{K}$.

Consider first the case when $r_0 > 2f(z_0)$. Assume $\delta < r_0/100$. For $k \ge 1$, define the following regions.

Let $S_k = \partial\mathscr{B}(\mathsf{v}_0, 2^k r_0/8)$.

Let A_0 be the closure of $\mathscr{B}(\mathsf{v}_0, 2r_0/8)$.

Let A_k be the closure of the spherical shell between S_k and S_{k+1} for $k \ge 1$.

Let $\widetilde{\mathscr{C}}$ be the set symmetric to $\mathscr{C}$ with respect to $\mathscr{K}$.

Let $D_k = (\mathscr{C} \cup \widetilde{\mathscr{C}}) \cap A_k$.

Suppose $\widetilde{\tau}$ is a stopping time such that $\widetilde{\tau} \le \tau_{\mathscr{C}^{L/2}} \wedge \tau_L$ a.s. Since h is harmonic on $\mathscr{B}_L \setminus \mathscr{C}^{L/2}$, we have $h(\mathsf{x}) = \mathsf{E}_{\mathsf{x}} h(B(\widetilde{\tau}))$ for $\mathsf{x} \in \mathscr{B}(\mathsf{v}_0, r_0/4)$. Couple Brownian motions B_0 and B_1 started from points v_0 and v_1 so that they are mirror images in $\mathscr{K}$. We will use superscripts to denote hitting times for B_i. Let

$$\widetilde{\tau} = \tau^0_{\mathscr{K}} \wedge \tau^0_L \wedge \tau^0_{\mathscr{C}^{L/2}} \wedge \tau^1_{\mathscr{K}} \wedge \tau^1_L \wedge \tau^1_{\mathscr{C}^{L/2}}.$$

Note that $h(B_0(\widetilde{\tau})) = h(B_0(\widetilde{\tau}))$ if $\widetilde{\tau} = \tau^0_{\mathscr{K}} \wedge \tau^1_{\mathscr{K}}$. Using E for the law of the coupling we then have

$$\begin{aligned} h(\mathsf{v}_0) - h(\mathsf{v}_1) &= \mathsf{E}[h(B_0(\widetilde{\tau})) - h(B_1(\widetilde{\tau}))] \\ &= \mathsf{E}\big[1_{\tau^0_L \wedge \tau^1_L = \widetilde{\tau}}[h(B_0(\widetilde{\tau})) - h(B_1(\widetilde{\tau}))]\big] \qquad (7.4)\\ &\quad + \mathsf{E}\big[h(B_0(\widetilde{\tau}))1_{\tau^1_{\mathscr{C}^{L/2}} = \widetilde{\tau}} - h(B_1(\widetilde{\tau}))1_{\tau^0_{\mathscr{C}^{L/2}} = \widetilde{\tau}}\big]. \qquad (7.5) \end{aligned}$$

We take care of the term (7.4) first. We may bound it above by

$$\mathsf{E}1_{\tau^0_L = \widetilde{\tau}}(1 - h(B_1(\widetilde{\tau}))) = \int (1 - \mathsf{P}_{\mathsf{x}}(\tau_L < \tau_{\mathscr{C}^{L/2}}))\, d\pi(\mathsf{x}),$$

where π is the subprobability measure corresponding to the location of $B_1(\tau^0_L)$ restricted to the event $\{\tau^0_L = \widetilde{\tau}\}$. This event is contained in $\{\tau^0_L < \tau^0_{\mathscr{K}}\}$ and so it is clear that the total mass $||\pi||$ of π is at most a constant multiple of δ/L, since $r_0, z_0 \le L/4$. Comparing $\mathscr{C}^{L/2}$ to the infinite cylinder of radius $f(L)$, and the ball $\mathscr{B}_L$ to the analogous cylinder of radius L one sees from Fact $(*)$ that for $\mathsf{x} \in \mathscr{B}_{5L/8}$ with $z(\mathsf{x}) = 0$,

$$\mathsf{P}_{\mathsf{x}}(\tau_L > \tau_{\mathscr{C}^{L/2}}) \leq \frac{\log L - \log(L/2)}{\log L - \log f(L)} = \frac{\log 2}{\log g(L)}.$$

By the Harnack principle applied in the shell between $\partial \mathscr{B}_{9L/16}$ and $\partial \mathscr{B}_L$,

$$\mathsf{P}_{\mathsf{x}}(\tau_L > \tau_{\mathscr{C}^{L/2}}) \leq \frac{c_2}{\log g(L)},$$

for all $\mathsf{x} \in \partial B_{5L/8}$. By the maximum principle, the same inequality holds for all x with $|\mathsf{x}| \geq 5L/8$. Since $\mathsf{v}_i \in \mathscr{B}_{5L/16}$, we have $|B_1(\tau_L^0)| \geq 5L/8$ and thus the term (7.4) is at most

$$\int [1 - \mathsf{P}_{\mathsf{x}}(\tau_L < \tau_{\mathscr{C}^{L/2}})]\, d\pi(\mathsf{x}) \leq \frac{c_2}{\log g(L)} ||\pi|| \leq \frac{c_3 \delta}{L \log g(L)}.$$

Recalling from Lemma 4.1 that $\widetilde{q}(2x)/\widetilde{q}(x) \to 1$, it follows easily that $L\widetilde{q}(L) \geq c z_0 \widetilde{q}(z_0)$ and hence that

$$c_3 \frac{\delta}{L \log g(L)} \leq c' \delta \frac{\widetilde{q}(L)}{\widetilde{q}(z_0) z_0 \log(g(z_0))}.$$

Since $z_0 \geq r_0$, this shows that the term (7.4) is bounded by an expression of the form

$$c\delta \frac{\widetilde{q}(L)}{\widetilde{q}(z_0) r_0 \log(g(z_0))}.$$

We now turn to the term (7.5). The event $\{\tau^1_{\mathscr{C}^{L/2}} = \widetilde{\tau}\}$ is contained in the union of events $\{\tau^1_{D_k} \leq \tau^1_{\mathscr{K}}\}$ for $k \geq 0$. Hence, (7.5) is bounded above by

$$\begin{aligned} \mathsf{E}\big[h(B_0(\widetilde{\tau})) 1_{\tau^1_{\mathscr{C}^{L/2}} = \widetilde{\tau}}\big] &= \sum_{k \geq 0} \mathsf{P}_{\mathsf{v}_0}(\tau_{D_k} \leq \tau_{\mathscr{K}}) \sup_{x \in D_k} h(x) \\ &\leq \sum_{k \geq 0} \mathsf{P}_{\mathsf{v}_0}(\tau_{D_k} \leq \tau_{\mathscr{K}}) \sup_{x \in A_k} h(x). \end{aligned} \tag{7.6}$$

We will need the following lemma. Its proof is given at the end of this section.

LEMMA 7.2. *Let p_1 be the probability that Brownian motion started from v_0 will hit $\mathscr{B}(\mathsf{v}_0, r_0/4)$ before hitting $\mathscr{K}$. Let p_2 be an upper bound for the probability that a Brownian motion starting from a point $\mathsf{y} \in S_k$ will hit S_{k+1} before $\mathscr{K}$. Let p_3 be the probability that a Brownian motion starting from a point $\mathsf{y} \in S_k$ will hit D_{k+1} before $\mathscr{K}$. Then $p_2 < 1$ and there are constants $c_i > 0$ and $\alpha < p_2^{-1}$ depending only on f and such that for $2f(z_0) \leq r_0 \leq z_0 \leq L$ and $z_0 \geq r_f$,*

$$\sup_{\mathsf{x} \in A_k} h(\mathsf{x}) \leq c_4 \alpha^k \frac{\widetilde{q}(L)}{\widetilde{q}(z_0)} \frac{\log(r_0/f(z_0))}{\log(g(z_0))}, \tag{7.7}$$

(7.8) $$p_1 \le \frac{c_5\delta}{r_0},$$

(7.9) $$p_3 \le \frac{c_6}{\log(r_0/f(z_0))}.$$

In the case when $r_f \le z_0 \le r_0 \le L$ the estimates (7.7) and (7.9) are replaced by

(7.10) $$\sup_{x\in A_k} h(x) \le c_7\alpha^k \frac{r_0}{\rho}\frac{\widetilde{q}(L)}{\widetilde{q}(\rho)}\frac{\log g(r_0)}{\log(g(\rho))},$$

(7.11) $$p_3 \le \frac{c_8}{\log g(r_0)},$$

where $\rho = \sqrt{r_0^2+z_0^2}$.

Note that (7.9) provides also an upper bound for the probablity of hitting $D_0 \cup D_1$ before hitting $\mathscr{K}$ for Brownian motion starting from v_0 (see the proof of Lemma 7.2 at the end of this section). Assuming this lemma for the moment, use the strong Markov property to see that for $k \ge 0$ and $2f(z_0) \le r_0 \le z_0$,

$$P_{v_0}(\tau_{D_k} \le \tau_{\mathscr{K}}) \le p_1 p_3 p_2^k.$$

Combining this with (7.6) and (7.7) gives

$$E\big[h(B_0(\widetilde{\tau}))1_{\tau^1_{\mathscr{C}L/2}=\widetilde{\tau}}\big] \le \sum_{k\ge 0} c p_1 p_2^k p_3 \alpha^k \frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}\frac{\log(r_0/f(z_0))}{\log(g(z_0))},$$

which reduces to

$$c\frac{\delta}{r_0}\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)\log(g(z_0))},$$

and completes the proof in the case $2f(z_0) \le r_0 \le z_0$.

The proof of (4.4) is completely analogous—estimates (7.10) and (7.11) have to be used in place of (7.7) and (7.9).

Next we consider the case $r_0 \le 2f(z_0)$. Since h is positive harmonic inside the ball $\mathscr{B}(v_0, (r_0 - f(z_0))/2)$, it is a mixture of Poisson kernels which have bounded derivatives inside $\mathscr{B}(v_0, (r_0 - f(z_0))/4)$. Thus the maximum of $|\nabla h|$ inside $\mathscr{B}(v_0, (r_0 - f(z_0))/4)$ is bounded by a constant times the maximum value of h on $\mathscr{B}(v_0, (r_0 - f(z_0))/2)$ divided by the radius of the ball. From Lemma 4.5 we obtain

$$|\nabla h(v_0)| \le c\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}\frac{\log(r_0/f(z_0))}{\log(g(z_0))}\frac{1}{r_0 - f(z_0)}.$$

Let b denote $r_0/f(z_0) - 1$. Then

$$|\nabla h(v_0)| \le c\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}\frac{\log(1+b)}{\log(g(z_0))}\frac{1}{bf(z_0)} \le c'\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}\frac{1}{r_0\log(g(z_0))},$$

completing the proof in this case.

It remains to consider the case when both r_0 and z_0 are at most $2r_f$. Let

$$\chi_1(r, z) = \max\left(\frac{1}{\widetilde{q}(z)} \frac{\log(r/f(z))}{\log g(z)}, \frac{1}{\widetilde{q}(r)}\right)$$

and let $\chi(\mathsf{x})$ be the harmonic function in $\mathscr{B}_{4r_f} \setminus \mathscr{C}$ which is equal to χ_1 on $\partial\mathscr{B}_{4r_f}$ and 0 on $\partial\mathscr{C}$. Recall that we have assumed at the beginning of the section that the boundary of $\mathscr{C}$ is C^2-smooth. It is a standard result that $\chi(\mathsf{x})$ is bounded by a constant (depending on f) times the distance of x from $\partial\mathscr{C}$, for $\mathsf{x} \in \mathscr{B}_{2r_f} \setminus \mathscr{C}$. By Lemma 4.5 we have $h(\mathsf{x}) \le \widetilde{c}\widetilde{q}(L)\chi(\mathsf{x}) \le c\widetilde{q}(L)\mathrm{dist}(\mathsf{x}, \partial\mathscr{C})$. We now apply the same argument as in the previous paragraph to obtain the bound $|\nabla h(\mathsf{v}_0)| \le c'\widetilde{q}(L)\mathrm{dist}(\mathsf{x}, \partial\mathscr{C})/\mathrm{dist}(\mathsf{x}, \partial\mathscr{C}) = c'\widetilde{q}(L)$. □

To prove Lemma 4.7, we will need the following spherical integral.

LEMMA 7.3. *Let A_θ be the region on the s-sphere defined by*

$$A_\theta = \partial\mathscr{B}_s \setminus (\mathscr{C} \cup \mathscr{C}_\theta).$$

Let $r(\mathsf{x})$ [respectively, $r'(\mathsf{x})$] denote the distance between x *and the z-axis (respectively, the axis of $\mathscr{C}_\theta$). Then there are constants κ_1, κ_2 independent of f such that for any s,*

$$\int_{\mathscr{B}_s} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\, dS \le \kappa_1 + \kappa_2 |\log\theta|, \tag{7.12}$$

where dS is (nonnormalized) area measure on $\partial\mathscr{B}_s$. Alternatively,

$$\int_{A_\theta} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\, dS \le 16\pi\log(\pi g(s)) \tag{7.13}$$

independently of θ.

PROOF. The axis of $\mathscr{C}$ intersects $\partial\mathscr{B}_s$ in two points; call them p and $\widetilde{\mathsf{p}}$. There is an arc in $\partial\mathscr{B}_s$ of length $s\theta$ connecting p to one of the intersection points of $\mathscr{C}_\theta$ with $\partial\mathscr{B}_s$; call this point p′. Let w denote the midpoint of the arc pp′. By symmetry through the origin, we may integrate over the set of points making angle at most $\pi/2$ with w, and then double the result. Let $\gamma_\mathsf{u}(\mathsf{v})$ denote the arclength along $\partial\mathscr{B}_s$ between the points v and u. Break the integral in (7.12) into two pieces:

$$\int_{\mathscr{B}_s} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\, dS = 2\left[\int_{\mathsf{x}:\,\gamma_\mathsf{w}(\mathsf{x})\le\theta s} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\, dS + \int_{\mathsf{x}:\,\theta s\le\gamma_\mathsf{w}(\mathsf{x})\le\pi/2} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\, dS\right].$$

When $\gamma_\mathsf{w}(\mathsf{x}) \ge \theta s$, then each of $\gamma_\mathsf{p}(\mathsf{x})$ and $\gamma_{\mathsf{p}'}(\mathsf{x})$ is at least $\gamma_\mathsf{w}(\mathsf{x})/2$, and so $r(\mathsf{x})$ and $r'(\mathsf{x})$ are at least $\gamma_\mathsf{w}(\mathsf{x})/4$. The integrand in the second integral is therefore at most $16/\gamma_\mathsf{w}(\mathsf{x})^2$. Since the area of $\{\mathsf{x}\colon a \le \gamma_\mathsf{w}(\mathsf{x}) \le a + da\}$ is at most $2\pi a\, da$, we may integrate over the parameter $r = \gamma_\mathsf{w}(\mathsf{x})$ to see that the second integral is at most

$$\int_{\theta s}^{\pi s/2} \frac{32\pi}{r}\, dr \le 32\pi(\log(\pi/2) + |\log\theta|).$$

To evaluate the first integral, we may integrate over the region where $\gamma_{\mathsf{p}}(\mathsf{x}) \le \gamma_{\mathsf{p}'}(\mathsf{x})$ and then double. On this region $\gamma_{\mathsf{p}'}(\mathsf{x}) \ge \theta s/2$. Integrating over the parameter $r = \gamma_{\mathsf{p}}(\mathsf{x})$, the first integral is at most

$$2\int_0^{\theta s} \frac{2}{\theta s r} 2\pi r\,dr \le 8\pi.$$

Putting these two pieces together proves (7.12).

To prove (7.13), use Cauchy–Schwarz to see that

$$\int_{A_\theta} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\,dS \le \left(\int_{A_\theta} \frac{1}{r(\mathsf{x})^2}\,dS\right)^{1/2}\left(\int_{A_\theta} \frac{1}{r'(\mathsf{x})^2}\,dS\right)^{1/2} = \int_{A_\theta} \frac{1}{r(\mathsf{x})^2}\,dS.$$

An upper bound for this is

$$2\int_{\mathsf{x}:\, f(s)/2 \le \gamma_{\mathsf{p}}(\mathsf{x}) \le \pi s/2} \frac{4}{\gamma_{\mathsf{p}}(\mathsf{x})^2}\,dS,$$

which is at most

$$2\int_{f(s)/2}^{\pi s/2} 8\pi/r\,dr \le 16\pi \log(\pi g(s)). \qquad \square$$

Proof of Lemma 4.7. We operate by induction on L. First, note that for any R and any $L \le R$, $U(L,\theta) \le \tilde{q}(L)^{-2}$. Thus if we choose $b(r) \ge \tilde{q}(4r)^{-2}$, then the result holds for any $L \in [R/4, R]$. The induction step assumes the result for $L/4$ and proves the result for L. If L is not regular for θ then $U(L,\theta) \le U(L/4,\theta)$ so the induction is trivial. Thus we may assume L is regular. Applying Lemmas 4.3 and 4.4 shows that for any $L \ge R \ge 2r_f$,

$$\tilde{q}(L,\theta) \le 2\bigg[\tilde{q}(L)^2 + 10\int_{\mathscr{B}_{L/4}\setminus(\mathscr{B}_R\cup\mathscr{C}\cup\mathscr{C}_\theta)} |\nabla h_1||\nabla h_2|\tilde{q}(|\mathsf{x}|/2,\theta)|\mathsf{x}|^{-1}\,d\mathsf{x} + \int_{\mathscr{B}_R\setminus(\mathscr{C}\cup\mathscr{C}_\theta)} |\nabla h_1||\nabla h_2||\mathsf{x}|^{-1}\,d\mathsf{x}\bigg].$$

Write this as an iterated integral, over spherical shells; apply the bounds on $|\nabla h_i|$ from Lemma 4.6, replacing z by ρ in (4.3) at a cost of a factor of at most some function $\beta(r_f)$, to get

$$\begin{aligned}\tilde{q}(L,\theta) \le 2\beta(r_f)^2\bigg[\tilde{q}(L)^2 &+ \int_R^{L/4} K_f^2 \frac{\tilde{q}(L)^2}{\tilde{q}(s)^2\log^2 g(s)} \frac{\tilde{q}(s/2,\theta)}{s} \times \left(\int_{\mathscr{B}_s\setminus(\mathscr{C}\cup\mathscr{C}_\theta)} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\,dS\right)ds \\ &+ \int_{\sqrt{2}r_f}^{R} K_f^2 \frac{\tilde{q}(L)^2}{\tilde{q}(s)^2\log^2 g(s)} \frac{1}{s}\left(\int_{\mathscr{B}_s\setminus(\mathscr{C}\cup\mathscr{C}_\theta)} \frac{1}{r(\mathsf{x})r'(\mathsf{x})}\,dS\right)ds\bigg] \\ &+ \int_{\mathscr{B}_{\sqrt{2}r_f}\setminus(\mathscr{C}\cup\mathscr{C}_\theta)} |\nabla h_1||\nabla h_2||\mathsf{x}|^{-1}\,d\mathsf{x}.\end{aligned}$$

The last integral is bounded by $\Xi(r_f)\widetilde{q}(L)^2$, by Lemma 4.6. Let R_f be large enough so that $\widetilde{q}(s) \geq \widetilde{q}(s/4)/2$ for $s \geq R_f/4$. Change variables in the first line to $t = s/4$ and regroup the part where $t < R$ with the second line to get

$$\begin{aligned}\widetilde{q}(L,\theta) \leq 2\beta(r_f)^2\bigg[\widetilde{q}(L)^2 &+ \int_R^{L/16} K_f^2 \frac{\widetilde{q}(L)^2}{(1/4)\widetilde{q}(t)^2\log^2 g(t)}\frac{\widetilde{q}(t,\theta)}{2t}\\ &\times\left(\int_{\mathscr{B}_s\setminus(\mathscr{C}\cup\mathscr{C}_\theta)}\frac{1}{r(\mathrm{x})r'(\mathrm{x})}\,dS\right)(2dt)\\ &+\int_{\sqrt{2}r_f}^R K_f^2\frac{\widetilde{q}(L)^2}{\widetilde{q}(s)^2\log^2 g(s)}\frac{5}{s}\left(\int_{\mathscr{B}_s\setminus(\mathscr{C}\cup\mathscr{C}_\theta)}\frac{1}{r(\mathrm{x})r'(\mathrm{x})}\,dS\right)ds\bigg]\\ &+\Xi(r_f)\widetilde{q}(L)^2,\end{aligned}$$

where the 5 comes from bounding $\widetilde{q}(t,\theta)$ above by one, and adding the regrouped part, which has a total factor of 4. Use the first bound from Lemma 7.3 for the inner integral in the first line and the second bound from the lemma in the inner integral in the second line and divide by $\widetilde{q}(L)^2$ to get

$$\begin{aligned}U(L,\theta) \leq{}& 8\beta(r_f)^2(\kappa_1+\kappa_2|\log\theta|)K_f^2\int_R^{L/16}\frac{U(t,\theta)}{t\log^2 g(t)}\,dt\\ &+2\beta(r_f)^2+10K_f^2\beta(r_f)^2\int_{\sqrt{2}r_f}^R\frac{16\pi\log(\pi g(s))}{s\widetilde{q}(s)^2\log^2 g(s)}\,ds+\Xi(r_f).\end{aligned}$$

Setting

$$c_f = 8\beta(r_f)^2K_f^2\max\{\kappa_1,\kappa_2\}$$

and

$$b(R) = \beta(r_f)^2\left[2+\Xi(r_f)+10K_f^2\int_1^R\frac{16\pi\log(\pi g(s))}{s\widetilde{q}(s)^2\log^2 g(s)}\,ds\right]$$

proves the lemma. $\square$

Proof of Lemma 7.2. The bounds $p_2 < 1$ and $p_1 \leq c_{31}\delta/r_0$ are obvious.

To prove (7.7), consider three cases. First suppose that $2^k r_0/4 \geq L/24$. Then $z_0 \geq r_0 \geq 2^{-k}L/6$, so $\widetilde{q}(z_0)/\widetilde{q}(L) \leq (1+\varepsilon)^{k+3}$, where $1+\varepsilon$ is an upper bound on $\widetilde{q}(x)/\widetilde{q}(2x)$ for $x \geq r_f$. Also, since $z_0 \leq L \leq 6\cdot 2^k r_0$ and $r_0/2 \geq f(z_0)$,

$$\begin{aligned}\frac{\log g(z_0)}{\log(r_0/f(z_0))} &= \frac{\log(z_0/f(z_0))}{\log(r_0/f(z_0))} \leq \frac{\log(z_0/(r_0/2))}{\log(r_0/(r_0/2))}\\ &= \frac{\log(z_0/r_0)+\log 2}{\log 2} \leq 2(k+4).\end{aligned}$$

Thus we obtain

$$\sup_{\mathrm{x}\in A_k} h(\mathrm{x}) \leq 1 \leq c(k+4)(1+\varepsilon)^{k+3}\frac{\widetilde{q}(L)}{\widetilde{q}(z_0)}\frac{\log(r_0/f(z_0))}{\log(g(z_0))}. \tag{7.14}$$

Now take an arbitrarily small $\alpha > 1$. Then choose small $\varepsilon > 0$ (this requires choosing large r_f) and c_* sufficiently large so that $c(k+4)(1+\varepsilon)^{k+3}$ is bounded by $c_*\alpha^k$.

The second case is if $z_0/2 \le 2^k r_0/4 \le L/24$. This ensures that $A_k \subseteq \mathscr{B}_{L/3}$ and thus by Lemma 4.5, $h(\mathrm{x}) \le \widetilde{q}(L)/\widetilde{q}(|\mathrm{x}|)$ for any $\mathrm{x} \in A_k$. If a point $\mathrm{x} \in A_k$ has cylindrical coordinates z_1 and r_1, then

$$z_1 \le z_0 + 2^k r_0/4 \le 2^k r_0/2 + 2^k r_0/4 \le 2^k r_0 \le 2^k z_0$$

and so $\widetilde{q}(z_0)/\widetilde{q}(z_1) \le (1+\varepsilon)^k$ as in the previous case. In view of $z_0 \le 2^k r_0/2$,

$$\frac{\log g(z_0)}{\log(r_0/f(z_0))} \le \frac{\log(z_0/r_0) + \log 2}{\log 2} \le 2(k+1).$$

Thus

$$\begin{aligned}\sup_{\mathrm{x}\in A_k} h(\mathrm{x}) &\le \frac{\widetilde{q}(L)}{\widetilde{q}(|\mathrm{x}|)}(1+\varepsilon)^k \frac{\widetilde{q}(z_1)}{\widetilde{q}(z_0)} \frac{\log(r_0/f(z_0))}{\log(g(z_0))} 2(k+1) \\ &\le c(k+1)(1+\varepsilon)^k \frac{\widetilde{q}(L)}{\widetilde{q}(z_0)} \frac{\log(r_0/f(z_0))}{\log(g(z_0))},\end{aligned}$$

which is analogous to (7.14).

Finally, in the case where $2^k r_0/4 \le z_0/2 \wedge L/24$, let a point $\mathrm{x} \in A_k$ again have cylindrical coordinates (z_1, r_1). Since $z_0/2 \le z_1 \le 3z_0/2$ and $r_1 \le 2\cdot 2^k r_0$, it follows that $\widetilde{q}(z_0)/\widetilde{q}(z_1) \le (1+\varepsilon)^{k+1}$, that $\log g(z_0)/\log g(z_1) \le 2$, and that $\log(r_1/f(z_1))/\log(r_0/f(z_0)) \le 1+(k+1)\log 2$. Lemma 4.5 is again applicable, yielding

$$\begin{aligned}\sup_{\mathrm{x}\in A_k} h(\mathrm{x}) &\le c\frac{\widetilde{q}(L)}{\widetilde{q}(z_1)} \frac{\log(r_1/f(z_1))}{\log g(z_1)}(1+\varepsilon)^k \frac{\widetilde{q}(z_1)}{\widetilde{q}(z_0)} \frac{2\log g(z_1)}{\log g(z_0)} \\ &\quad \times (1+(k+1)\log 2)\frac{\log(r_0/f(z_0))}{\log(r_1/f(z_1))}.\end{aligned}$$

This simplifies again to (7.14).

Recall that α can be chosen arbitrarily close to 1 by choosing c_* sufficiently large in each of the three cases. Choosing $\alpha < p_2^{-1}$ and c_* to be the maximum of the three values proves (7.7).

Next we prove (7.10). Assume that $z_0 \le r_0$ and find a point $\widetilde{\mathrm{v}}_0$ with the same ρ as for v_0 and such that $\widetilde{z}_0 = \widetilde{r}_0$ and $|\mathrm{v}_0 - \widetilde{\mathrm{v}}_0| < r_0$. Then $A_k \subset \widetilde{A}_{k+4}$ and we obtain from (7.7),

$$\sup_{\mathrm{x}\in A_k} h(\mathrm{x}) \le \sup_{\mathrm{x}\in \widetilde{A}_{k+4}} h(\mathrm{x}) \le c_4\alpha^{k+4} \frac{\widetilde{q}(L)}{\widetilde{q}(\widetilde{z}_0)} \frac{\log(\widetilde{r}_0/f(\widetilde{z}_0))}{\log(g(\widetilde{z}_0))}.$$

Since $\rho/2 \le \widetilde{z}_0 = \widetilde{r}_0 \le r_0 \le \rho$, we have $r_0/\rho \ge c$, $\widetilde{q}(\widetilde{z}_0) \ge \widetilde{q}(\rho)$, $\log g(\widetilde{z}_0) \ge c\log g(\rho)$, and

$$\log(\widetilde{r}_0/f(\widetilde{z}_0)) \le c\log(r_0/f(r_0)) = c\log g(r_0),$$

for some absolute constant c. Hence,

$$\sup_{x\in A_k} h(x) \le c'\alpha^k \frac{r_0}{\rho}\frac{\widetilde{q}(L)}{\widetilde{q}(\rho)}\frac{\log g(r_0)}{\log(g(\rho))},$$

which is (7.10).

It remains to prove (7.9) and (7.11). When $r_0 \le z_0$, scaling down by a factor of $2^k r_0$ turns D_{k+1} into a set contained in the union of two cylinders with axes at most 1 and radii at most $1/g(z_0)$, so the capacity of the rescaled set D_{k+1} is at most a constant multiple of $1/\log g(z_0)$. The rescaled point y is at distance at least 1/16 from the rescaled D_{k+1} and at distance at most 1 from the rescaled $\mathscr{K}$, so the probability of hitting D_{k+1} before $\mathscr{K}$ starting from y is at most $c/\log g(z_0) \le c/\log(r_0/f(z_0))$. In the case $z_0 \le r_0$ we use the bound $f(r_0)/r_0 = 1/g(r_0)$ for the cylinder radius. □

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