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ASYMPTOTIC DISTRIBUTION OF QUADRATIC FORMS

BY F. GÖTZE[1] AND A. N. TIKHOMIROV[1,2]

University of Bielefeld and Syktyvkar State University and Russian Academy of Sciences

We consider quadratic forms

$$Q_n = \sum_{1 \le j \ne k \le n} a_{jk} X_j X_k,$$

where X_j are i.i.d. random variables with finite third moment. We obtain optimal bounds for the Kolmogorov distance between the distribution of Q_n and the distribution of the same quadratic forms with X_j replaced by corresponding Gaussian random variables. These bounds are applied to Toeplitz and random matrices as well as to nonstationary AR(1) processes.

1. Introduction and results. Let $X_1, X_2, \ldots$ denote independent random variables (r.v.) such that $\mathbf{E}X_j = 0$ and $\mathbf{E}X_j^2 = 1$, $j = 1, 2, \ldots$. Let $A = A^{(n)} = \{a_{njk}\}_{j,k=1}^n$ denote a symmetric matrix with eigenvalues $\lambda_{n1}, \ldots, \lambda_{nn}$ ordered to be nonincreasing in absolute value. Throughout we shall assume that

$$\sum_{j,k=1}^n a_{njk}^2 = \sum_{j=1}^n \lambda_{nj}^2 = 1. \tag{1.1}$$

Consider the quadratic forms (q.f.)

$$Q_n = \sum_{j,k=1}^n a_{njk} X_j X_k \quad \text{and} \quad G_n = \sum_{j,k=1}^n a_{njk} Y_j Y_k,$$

where $Y_1, Y_2, \ldots$ are orthonormal Gaussian r.v.'s.

There exists an extensive literature on different probability problems for quadratic forms. One of the most important problems is the investigation of the asymptotic of distributions of quadratic forms G_n and Q_n. Sevastyanov (1961) described the class of distributions which are limits of G_n as $n \to \infty$. Whittle (1960, 1964) obtained inequalities for moments of Q_n similar to the Rosenthal inequalities for sums of independent r.v.'s and proved a central limit theorem (CLT) under the additional condition that the matrices $A^{(n)}$ have a special quasi-diagonal structure. Varberg (1966) obtained conditions of convergence of Q_n in quadratic mean and almost surely and investigated

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weak convergence of G_n, too. There are a number of papers on the convergence of quadratic forms Q_n and especially of G_n to various limit distributions. See, for example, de Jong (1987) (CLT for quadratic forms), Fox and Taqqu (1987) (limit theorems for quadratic forms G_n of Gaussian r.v.'s with long-range dependence), Mikosch (1990, 1991) (functional CLT for quadratic forms, law of iterated logarithm). Venter and de Wet (1973) gave conditions under which the distribution of a quadratic form Q_n converges to that of $\sum_m \alpha_m (Y_m^2 - 1)$, where $\{\alpha_m\}$ is a sequence of real numbers.

Rotar' (1973) proved that under sufficiently weak conditions on the matrix A and for large n, the distribution of Q_n is close to the distribution of G_n. Gamkrelidze and Rotar' (1977) obtained bounds for the error of this approximation, which were improved by Rotar' and Shervashidze (1985). Below we formulate their result. We shall use the following notation:

$$F_j(x) = \mathbf{P}\{X_j \le x\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-u^2/2\}\, du,$$

$$\nu_j = 3\int_{-\infty}^{\infty} x^2 |F_j(x) - \Phi(x)|\, dx, \qquad s_j^2 = \sum_{k=1}^{n} a_{njk}^2,$$

$$L = \sum_{j=1}^{n} \nu_j s_j^3 + \sum_{j,k=1}^{n} \nu_j \nu_k |a_{njk}|^3, \qquad \Delta = \sum_{j=1}^{n} \lambda_{nj}^4.$$

Here and in what follows we shall consider matrices A with zero diagonal only; that is, $a_{njj} = 0$ for $1 \le j \le n$.

THEOREM [Rotar' and Shervashidze (1985)]. *Let* $\Delta < 1/2$. *Then*

$$\sup_x |\mathbf{P}\{Q_n \le x\} - \mathbf{P}\{G_n \le x\}| \le C(1 - \log(1 - 2\Delta))^{3/4} L^{1/4},$$

where C is an absolute constant.

The aim of our paper is to improve this bound for independent identically distributed (i.i.d.) r.v.'s $X_1, X_2, \dots, X_n$ under various assumptions on the spectra of the matrices A. We apply these results in particular to investigate the rate of convergence of the distributions of Q_n to Gaussian distributions.

Write

$$\delta_n(A, F) = \sup_x |\mathbf{P}\{Q_n \le x\} - \mathbf{P}\{G_n \le x\}|,$$

and

$$\Delta_n(A, F) = \sup_x \left| \mathbf{P}\left\{ Q_n / \sqrt{\operatorname{Var}(Q_n)} \le x \right\} - \Phi(x) \right|,$$

where F is the distribution function (d.f.) $F(x) = \mathbf{P}\{X_1 \le x\}$, $\operatorname{Var}(Q_n)$ denotes the variance of Q_n, and A is the symmetric matrix of coefficients of Q_n. We shall assume that the matrices $A = A^{(n)}$ satisfy the same conditions as in the paper of Rotar' and Shervashidze (1985). That is, we assume (1.1) and

$$a_{njj} = 0, \qquad j = 1, \dots, n. \tag{1.2}$$

Then we have $\mathrm{Var}(Q_n) = 2$. Throughout the paper we shall suppose that $X, X_1, X_2, \dots$ are i.i.d. and

$$\mathbf{E}X = 0, \qquad \mathbf{E}X^2 = 1, \qquad \beta_3 = \mathbf{E}|X|^3 < \infty. \tag{1.3}$$

By C (with an index or without it), we shall denote absolute constants, whereas $C(\cdot,\cdot)$ will denote positive constants depending on arguments.

THEOREM 1. *Assume the conditions* (1.1)–(1.3) *hold. Then there exists an absolute constant C such that*

$$\max\{\delta_n(A,F), \Delta_n(A,F)\} \le C\beta_3^2|\lambda_{n1}|.$$

Write

$$\mathscr{L}_n^2 = \max_{1\le j\le n} \sum_{k=1}^n a_{njk}^2 \quad \text{and} \quad \Gamma_n = \max_{1\le j\le n} \sum_{k=1}^n |a_{njk}|.$$

It is well known that

$$\mathscr{L}_n \le |\lambda_{n1}| \le \Gamma_n$$

[see, for example, Lancaster (1969), pages 208–215]. Theorem 1 immediately implies the following result.

COROLLARY 2. *Assume that the conditions* (1.1)–(1.3) *hold. Then there exist absolute positive constants C_1 such that*

$$\max\{\delta_n(A,F), \Delta_n(A,F)\} \le C_1\beta_3^2\Gamma_n.$$

For $x > 0$ define the function $\log^+ x = \max\{\frac{1}{2}, |\log x|\}$.

THEOREM 3. *Assume that the conditions* (1.1)–(1.3) *hold and that there exists a positive constant b_0 such that for some $q \ge 2$,*

$$|\lambda_{nq}| \ge b_0. \tag{1.4}$$

Then there exists a constant $C(b_0)$ such that

$$\delta_n(A,F) \le C(b_0)\beta_3^2 \begin{cases} \mathscr{L}_n^{4/(16-q)}, & \text{if } 2 \le q \le 11, \\ \mathscr{L}_n \log^+ \mathscr{L}_n, & \text{if } q = 12, \\ \mathscr{L}_n, & \text{if } q = 13. \end{cases}$$

Note that, if $\mathscr{L}_n \ge e^{-1/2}$, the results of Theorem 3 are trivial. We shall assume that $\mathscr{L}_n \le e^{-1/2}$, and therefore, $\log^+ \mathscr{L}_n = -\log \mathscr{L}_n$.

REMARK 1. Note that the bounds in Theorem 1 have an optimal dependence on λ_{n1} and on $\mathscr{L}_n$, respectively. Indeed, consider the following q.f.:

$$Q_n = n^{-1/2} \sum_{k=1}^n X_{2k-1}X_{2k}.$$

Here $|\lambda_{n1}| = 1/\sqrt{n}$, and if $X_j = \pm 1$ are i.i.d. Rademacher r.v., then

$$\min\{\delta_n(A, F), \Delta_n(A, F)\} \geq C/\sqrt{n} = C|\lambda_{n1}|.$$

REMARK 2. Note for comparison that the best available result so far (by Rotar') yields

$$\Delta_n(A, F) \leq C\mathscr{L}^{1/4}.$$

The remaining part of the paper is divided into Sections 2–6. Section 2 presents the applications of Theorems 1 and 3 for Toeplitz matrices, random matrices and nearly nonstationary autoregressive processes of order one. Sections 3–6 are devoted to the proofs of Theorems 1 and 3. Section 3 provides estimates of characteristic functions using a symmetrization inequality due to Götze (1979). In Section 4 we investigate spectral properties of some special type of submatrices of the matrix A, which we need for the proof of Theorems 1 and 3 in Section 6. Section 5 presents bounds for distributions of randomized linear forms appearing in Lemma 3.2. Finally, the proofs of Theorems 1 and 3 are given in Section 6.

2. Applications.

Toeplitz matrices. Let $b_0, b_1, \dots$ be a sequence of real numbers such that $b_0 = 0$. Assume

$$B = \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{2.1}$$

Introduce the quantity $D_n^2 = \sum_{j=1}^{n}(n-j)b_j^2$, and define the matrix A with elements

$$a_{jk} = \frac{1}{D_n} b_{|j-k|}. \tag{2.2}$$

Whittle (1964) proved that $\delta_n(A, F) \to 0$, as $n \to \infty$. We prove the following sharpening of his result.

PROPOSITION 2.1. *Assume* (2.1) *holds. Then there exists an absolute constant C such that*

$$\delta_n(A, F) \leq C\beta_3 n^{-1/2} \sum_{j=1}^{n} |b_j|.$$

PROOF. From Corollary 2 it follows that

$$\delta(A, F) \leq C\beta_3^2 \Gamma_n. \tag{2.3}$$

By (2.2)

$$\Gamma_n \leq \frac{2}{D_n} \sum_{j=1}^{n} |b_j|. \tag{2.4}$$

Without loss of generality we may assume that $\sum_{j \ge n/2} b_j^2 \le B/2$. Then it is easy to check that

$$D_n^2 \ge Bn/4. \tag{2.5}$$

The inequalities (2.3)–(2.5) together prove Proposition 2.1. □

Random matrices. Let $B = (b_{jk})$ denote a symmetric $n \times n$ matrix such that the entries for $k > j$ are independent r.v.'s which are independent of $X, X_1, \dots$. Assume that there exists a constant C_0 such that

$$|b_{jk}| \le C_0, \qquad \mathbf{E} b_{jk} = 0, \quad \text{and} \quad \mathbf{E} b_{jk}^2 = \sigma^2 \ \text{for } j > k. \tag{2.6}$$

Denote by D_n the quantity $D_n^2 = 2\sum_{1 \le j < k \le n} b_{jk}^2$. Let $A = B/D_n$. Then we have the following result.

PROPOSITION 2.2. *Assume* (2.6) *holds. Then*

$$\mathbf{P}\Big\{ \sup_x \Big| \mathbf{P}\big\{ Q_n \le x \,\big|\, a_{jk}, j > k \big\} - \mathbf{P}\big\{ G_n \le x \,\big|\, a_{jk}, j > k \big\} \Big| \le C\beta_3^2 n^{-1/2} \Big\} \to 1,$$

as $n \to \infty$.

PROOF. By Theorem 1 we have

$$\sup_x \Big| \mathbf{P}\big\{ Q_n \le x \,\big|\, a_{jk}, j > k \big\} - \mathbf{P}\big\{ G_n \le x \,\big|\, a_{jk}, j > k \big\} \Big| \le C\beta_3^2 |\lambda_1|. \tag{2.7}$$

By Theorem 2 of Füredi and Komlos (1980),

$$\mathbf{P}\big\{ |\lambda_1| \le C n^{1/2} / D_n \big\} \to 1, \tag{2.8}$$

as $n \to \infty$. It is easy to check that

$$\mathbf{E} D_n^2 = n(n-1)\sigma^2, \qquad \mathbf{E}\big(D_n^2 - \mathbf{E} D_n^2 \big)^2 \le n(n-1) C_0^4. \tag{2.9}$$

By (2.9),

$$\mathbf{P}\{ D_n \ge Cn \} \le C n^{-2}. \tag{2.10}$$

The inequalities (2.7), (2.8), and (2.10) together complete the proof. □

Nearly nonstationary AR(1) *processes.* Assume that $Z_j(n)$, $j = 1, \dots, n$, $n = 1, 2, \dots$ is a first order autoregressive process with unknown parameter $\rho_n = 1 - \alpha_n / n$, where $\alpha_n > 0$,

$$Z_j(n) = \rho_n Z_{j-1}(n) + X_j, \qquad Z_0(n) = 0 \quad \text{for all } n. \tag{2.11}$$

Here $X_1, X_2, \dots$ are i.i.d. r.v.'s satisfying condition (1.3). The least-squares estimate of the parameter ρ_n based on the observations $Z_1(n), \dots, Z_n(n)$ is given by

$$\hat{\rho}_n = \sum_{j=1}^n Z_j(n) Z_{j-1}(n) \left(\sum_{k=1}^n Z_{k-1}^2(n) \right)^{-1}. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$W_n = (\hat{\rho}_n - \rho_n) \sum_{k=1}^{n} Z_{k-1}^2(n) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} \rho_n^{j-k} X_j X_k.$$

Such process are for instance described in Chan and Wei (1987) and Rachkauskas (1996). It is well known that, if $\alpha_n \to \infty$ for $n \to \infty$,

$$\delta_n = \sup_x \left| \mathbf{P}\left\{ W_n / \sqrt{\operatorname{Var}(W_n)} \le x \right\} - \Phi(x) \right| \to 0,$$

as $n \to \infty$. Our results yield the following rates of convergence.

PROPOSITION 2.3. *Assume that* (1.3) *holds. Then there exists an absolute constant C such that*

$$\delta_n = \sup_x \left| \mathbf{P}\left\{ W_n / \sqrt{\operatorname{Var}\{W_n\}} \le x \right\} - \Phi(x) \right| \le C \beta_3^2 \alpha_n^{-1/2}.$$

PROOF. We can represent the statistic $W_n / \sqrt{\operatorname{Var}(W_n)}$ in the form of a q.f. Q_n with matrix A defined by

$$a_{jk} = \frac{\rho_n^{|j-k|}}{2\sqrt{\operatorname{Var}(W_n)}}. \tag{2.13}$$

By Theorem 1 we have $\delta_n \le C\beta_3 |\lambda_1|$. It is easy to see that

$$\operatorname{Var}(W_n) = \sum_{j=1}^{n} (n-j) \rho_n^{2j} = \frac{n \rho_n^2}{1 - \rho_n^2} - \frac{\rho_n^2 (1 - \rho_n^{2n})}{(1 - \rho_n^2)^2}.$$

A simple calculation yields

$$C_1 n^2 \alpha_n^{-1} \le \operatorname{Var}(W_n) \le C_2 n^2 \alpha_n^{-1}. \tag{2.14}$$

From (2.14) we derive

$$\Gamma_n \le C n^{-1} \alpha_n^{1/2} \sum_{j=1}^{n} \rho_n^j \le C \alpha_n^{-1/2}. \tag{2.15}$$

Corollary 2 and inequality (2.15) together complete the proof. □

Consider now the case

$$\alpha_n \to \alpha > 0, \qquad n \to \infty. \tag{2.16}$$

In this case we obtain the following result.

PROPOSITION 2.4. *Assume that* (1.3) *and* (2.16) *hold. Then there exists a constant $C(\alpha)$ such that*

$$\sup_{x \in \mathbb{R}^1} \left| \mathbf{P}\{W_n \le x\} - \mathbf{P}\{G_n \le x\} \right| \le C(\alpha) \beta_3^2 n^{-1/2},$$

where G_n is the q.f. of Gaussian r.v.'s and the matrix A is defined in (2.13).

PROOF. Using (2.16) we obtain that

$$\mathscr{L}_n^2 \le \frac{2}{\operatorname{Var}(W_n)} \sum_{k=1}^{n} \rho_n^{2k} \le Cn^{-1}. \tag{2.17}$$

It finally remains to check the condition (1.4) and to apply Theorem 3. It is not difficult to prove that the spectrum of the matrix A satisfies, for any $q \ge 1$,

$$v_\alpha \frac{2\alpha}{\alpha^2 + \pi^2 (q + 1/2)^2} \le \lim_{n\to\infty} |\lambda_q(n)| \le v_\alpha \frac{2\alpha}{\alpha^2 + \pi^2 q^2}, \tag{2.18}$$

where $v_\alpha = (4\alpha^2/(\exp\{-2\alpha\} - 1 + 2\alpha))^{1/2}$. Combining Theorem 3, relations (2.17) and (2.18), we obtain the result of Proposition 2.4. □

3. The method of symmetrization. Let $\overline{X}$ denote a copy of the r.v. X which is independent of all other r.v.'s. Throughout, $\tilde{X}$ denote a symmetrization of X, that is, $\tilde{X} = X - \overline{X}$. By $\mathbf{N}$ we denote the set of integer numbers $\{1, \dots, n\}$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ be a random vector with zero–one coordinates, which is independent of $X_1, \dots, X_n$. Write $\mathscr{X} = (X_1, \dots, X_n)$ and $\xi_j = (0, \dots, 0, X_j, 0, \dots, 0)$, $j \in \mathbf{N}$. By $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ we shall denote the Euclidean scalar product and the Euclidean norm, respectively. For simplicity of notation, the index n will be omitted. In this notation we have $Q = \langle A\mathscr{X}, \mathscr{X} \rangle$ and $f(t) = \mathbf{E}\exp\{itQ\}$.

There is an extensive literature on the distribution of quadratic forms of the type $R = \|Z_1 + \cdots + Z_n\|^2$ where Z_j, $J \in \mathbf{N}$, denotes a sequence of i.i.d. random vectors with values in $\mathbb{R}^q$, assuming nondegenerate covariance operators and finite third absolute moments. Here the error of approximation in replacing Z_j by Gaussian random vectors is of order $n^{-1/2}$ and depends on the smallest eigenvalue of the covariance of Z_1. The first result of this type is due to Götze (1979). The crucial step of the proofs is an estimate of the characteristic function (ch.f.) of R using a symmetrization inequality reducing the quadratic form to a conditional linear form [see Götze (1979)]. In our case this procedure yields the following result.

LEMMA 3.1. *For all $t \in \mathbb{R}$, we have*

$$|f(t)|^4 \le \mathbf{E}\exp\Big\{2it \sum_{j,k \in \mathbf{N}} a_{jk}\varepsilon_j(1 - \varepsilon_k)\tilde{X}_j\tilde{X}_k\Big\}. \tag{3.1}$$

PROOF. Assume that $U, V, V_1, V_2 \in \mathbb{R}^n$ are independent random vectors such that V, V_1, V_2 are i.i.d. Let $h(u, v)$ denote an arbitrary measurable real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$. Then

$$|\mathbf{E}\exp\{ith(U, V)\}| \le \mathbf{E}\exp\{it\Delta_2(V_1 - V_2)h(U, V_2)\}, \tag{3.2}$$

where the difference operator $\Delta_2(s)h(u, v) = h(u, v + s) - h(u, v)$ acts on the second variable of h. Indeed, applying Hölder's inequality and using that

V, V_1, V_2 are i.i.d., we have

$$\begin{aligned}|\mathbf{E}\exp\{ith(U,V)\}| &\le \mathbf{E}|\mathbf{E}(\exp\{ith(U,V)\}|U)|^2\\ &= \mathbf{E}\exp\{ith(U,V_1)\}\exp\{-ith(U,V_2)\},\end{aligned}$$

which coincides with (3.2) [cf. Götze (1984)]. For the proof of (3.1) we fix $\boldsymbol{\varepsilon}$ and apply the inequality (3.2) twice with

$$U = (\varepsilon_1 X_1, \dots, \varepsilon_n X_n), \qquad V = ((1-\varepsilon_1)X_1, \dots, (1-\varepsilon_n)X_n) \tag{3.3}$$

and $h(U,V) = \langle A(U+V), (I+V)\rangle$. Note that

$$\Delta_1(U_1 - U_2)\Delta_2(V_1 - V_2)h(U_2, V_2) = 2\langle A(U_1 - U_2), (V_1 - V_2)\rangle,$$

where the difference operator Δ_1 acts on the first arguments of h. This obviously yields the result. □

Denote by $\tilde{X}^{(H)}$ the truncation of $\tilde{X}$, that is,

$$\tilde{X}^{(H)} = \tilde{X}\mathbf{I}\{|\tilde{X}| \le H\},$$

where $\mathbf{I}\{B\}$ denotes the indicator of an event B. Introduce the notation

$$\sigma^2(H) = \mathbf{E}(\tilde{X}^{(H)})^2, \qquad \mu_4(H) = \mathbf{E}(\tilde{X}^{(H)})^4.$$

LEMMA 3.2. *Let $\varepsilon_1, \dots, \varepsilon_n$ denote i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then, for all $t \ge 0, K > 0$

$$\begin{aligned}|f(t)|^4 \le{}& \mathbf{E}\exp\Big\{\tfrac{1}{4}i(t\wedge T_0)\sigma(H)\sum_{j,k\in\mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j)Y_jY_k\Big\}\\ &+ \mathbf{I}\{|t| > \gamma\mathscr{L}^{-1}\}\\ &+ \mathbf{P}\Big\{\max_{k\in\mathbf{N}}\Big|\sum_{j\in\mathbf{N}} a_{jk}(1-\varepsilon_j)Y_j\Big| \ge K\mathscr{L}\log^+\mathscr{L}\Big\}\\ &+ \mathbf{P}\Big\{\max_{j\in\mathbf{N}}\Big|\sum_{k\in\mathbf{N}} a_{jk}\varepsilon_k\tilde{X}_k^{(H)}\Big| > K\mathscr{L}\Big\},\end{aligned} \tag{3.4}$$

where

$$T_0 = \gamma\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1},$$
$$t\wedge T_0 = \min\{t, T_0\} \quad \text{and} \quad \gamma = \sigma(H)/\Big(2K\sqrt{\mu_4(H)}\Big).$$

The corresponding estimates for negative t is similar since $|f(t)| = |f(-t)|$.

PROOF. By Lemma 3.1

$$|f(t)|^4 \le \mathbf{E}\exp\Big\{2it\sum_{j,\,k\in\mathbf{N}} a_{jk}\,\varepsilon_k(1-\varepsilon_j)\tilde{X}_j\tilde{X}_k\Big\}.$$

Since $\tilde{X}$ has a symmetric distribution and

$$\mathbf{E}\cos(a\tilde{X})\mathbf{I}\{|\tilde{X}|\ge H\} \le \mathbf{E}\cos(a\cdot 0)\mathbf{I}\{|\tilde{X}|\ge H\},$$

it is easily seen that

$$|f(t)|^4 \le \mathbf{E}\exp\Big\{2it\sum_{j,\,k\in\mathbf{N}} a_{jk}\,\varepsilon_j(1-\varepsilon_k)\tilde{X}_j^{(H)}\tilde{X}_k^{(H)}\Big\}. \tag{3.5}$$

With U, V defined by (3.3) write

$$\tilde{U} = \big(\varepsilon_1\tilde{X}_1,\dots,\varepsilon_n\tilde{X}_n\big), \qquad \tilde{V} = \big((1-\varepsilon_1)\tilde{X}_1,\dots,(1-\varepsilon_n)\tilde{X}_n\big).$$

Write

$$S_j(\boldsymbol{\varepsilon}) = \sum_{k\in\mathbf{N}} a_{jk}\,\varepsilon_k\tilde{X}_k^{(H)}.$$

Since the coordinates of $\tilde{V}$ are conditionally independent given $\varepsilon_1,\dots,\varepsilon_n$ and $\tilde{U}$, we have

$$|f(t)|^4 \le \mathbf{E}\prod_{j\in\mathbf{N}}\mathbf{E}\Big(\exp\Big\{2it(1-\varepsilon_j)\tilde{X}_j^{(H)}S_j(\boldsymbol{\varepsilon})\Big\}\Big|S_j(\boldsymbol{\varepsilon})\Big). \tag{3.6}$$

Note that

$$\max_{j\in\mathbf{N}}\big|S_j(\boldsymbol{\varepsilon})\big| \le K\mathscr{L}, \tag{3.7}$$

implies

$$\begin{aligned}
&\Big|\mathbf{E}\Big(\exp\Big\{2it(1-\varepsilon_j)\tilde{X}_j^{(H)}S_j(\boldsymbol{\varepsilon})\Big\}\Big|S_j(\boldsymbol{\varepsilon})\Big)\Big|\\
&\quad\le 1-\frac{t^2}{2}(1-\varepsilon_j)\sigma^2(H)S_j^2(\boldsymbol{\varepsilon}) + t^4(1-\varepsilon_j)\mu_4(H)S_j^4(\boldsymbol{\varepsilon})\\
&\quad\le \max\Big\{\mathbf{I}\{t>\gamma\mathscr{L}^{-1}\},\ \exp\Big\{-\frac{1}{4}t^2\sigma^2(H)S_j^2(\boldsymbol{\varepsilon})(1-\varepsilon_j)\Big\}\Big\}.
\end{aligned} \tag{3.8}$$

The inequalities (3.6)–(3.8) together imply

$$\begin{aligned}
|f(t)|^4 &\le \mathbf{E}\exp\Big\{\tfrac{1}{2}it\sigma(H)\sum_{j,\,k\in\mathbf{N}} a_{jk}\,\varepsilon_k(1-\varepsilon_j)Y_j\tilde{X}_k^{(H)}\Big\}\\
&\qquad + \mathbf{I}\{t>\gamma\mathscr{L}^{-1}\}\\
&\qquad + \mathbf{P}\Big\{\max_{j\in\mathbf{N}}\big|S_j(\boldsymbol{\varepsilon})\big| > K\mathscr{L}\Big\}.
\end{aligned} \tag{3.9}$$

Repeating the arguments in (3.6)–(3.9) and replacing $\mathscr{L}$ in (3.7) by $(\mathscr{L}\log^+\mathscr{L})$, we obtain

$$
\begin{aligned}
|f(t)|^4 \le{}& \mathbf{E}\exp\Big\{\tfrac{1}{4}i(t\wedge T_0)\sigma(H)\sum_{j,k\in\mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j)Y_jY_k\Big\}\\
&+\mathbf{I}\{t>\gamma\mathscr{L}^{-1}\}\\
&+\mathbf{P}\Big\{\max_{k\in\mathbf{N}}\Big|\sum_{j\in\mathbf{N}} a_{jk}(1-\varepsilon_j)Y_j\Big|\ge K\mathscr{L}\log^+\mathscr{L}\Big\}\\
&+\mathbf{P}\Big\{\max_{j\in\mathbf{N}}|S_j(\boldsymbol{\varepsilon})|>K\mathscr{L}\Big\}.
\end{aligned}\tag{3.10}
$$

□

To investigate the ch.f. in a neighborhood of zero, we shall use Lemma 3.3 below, which is a generalization of Lemma 3.1 in Bentkus and Götze (1996). Let J_0, J_1 denote an arbitrary partition of $\mathbf{N}$ and let J_{il} denote an arbitrary partition of J_i, $i=0,1$, respectively. Introduce the corresponding vectors

$$
U_l=\sum_{j\in J_l}\xi_j,\qquad U_{lm}=\sum_{j\in J_{lm}}\xi_j,\ l=0,1,\ m=1,\dots,5.\tag{3.11}
$$

Note that

$$
\mathscr{X}=U_0+U_1,\qquad U_i=\sum_{m=1}^{5}U_{im},\ i=0,1.
$$

LEMMA 3.3. *For any $\eta\in\mathbb{R}^n$, $t\in\mathbb{R}$, we have*

$$
\begin{aligned}
&\big|\mathbf{E}\langle A\eta,\mathscr{X}\rangle^3\exp\{it\langle A\eta,\mathscr{X}\rangle+itQ\}\big|^2\\
&\quad\le C\max_{\substack{l=0,1\\1\le m\le 5}}\mathbf{E}^2|\langle A\eta,U_{l,m}\rangle|^3\\
&\quad\quad\times\Big(\max_{1\le m\le 5}\mathbf{E}\exp\big\{2it\langle\tilde U_0,A\tilde U_{1,m}\rangle\big\}\\
&\quad\quad\quad+\max_{1\le k,m\le 5}\mathbf{E}\exp\big\{2it\langle A\tilde U_{1k},\tilde U_{0m}\rangle\big\}\Big).
\end{aligned}\tag{3.12}
$$

PROOF. Write

$$
\begin{aligned}
&\mathbf{E}\langle A\eta,\mathscr{X}\rangle^3\exp\{it\langle A\eta,\mathscr{X}\rangle+itQ_n\}\\
&\quad=\sum_{\alpha_0+\alpha_{1,1}+\alpha_{1,2}+\cdots+\alpha_{1,5}=3}H(\alpha_0,\alpha_{1,1},\dots,\alpha_{1,5}),
\end{aligned}\tag{3.13}
$$

where

$$
\begin{aligned}
&H(\alpha_0,\alpha_{1,1},\dots,\alpha_{1,5})\\
&\quad=\mathbf{E}\langle A\eta,U_0\rangle^{\alpha_0}\prod_{m=1}^{5}\langle A\eta,U_{1,m}\rangle^{\alpha_{1,m}}\exp\{it\langle A\eta,\mathscr{X}\rangle+itQ\}.
\end{aligned}
$$

If $\alpha_0 = 0$, there are at least two r.v.'s, U_{1,j_1}, U_{1,j_2}, which are not involved in the product used in the definition of the function $H(\alpha_0, \dots, \alpha_{1,5})$. Arguing in the same way as in the proof of Lemma 3.1 in Bentkus and Götze (1996), we obtain the estimate

$$|H(0, \alpha_{1,1}, \dots, \alpha_{1,5})| \le \max_{\substack{l=0,1\\ 1\le m\le 5}} \mathbf{E}|\langle A\eta, U_{lm}\rangle|^3 \times \max_{1\le m\le 5} \mathbf{E}^{1/2} \exp\Bigl\{2it\langle A\tilde U_0, \tilde U_{1m}\rangle\Bigr\}. \tag{3.14}$$

In the case $\alpha_0 \ne 0$, we fix a set J_{1,m_0} which is not contained in the product used in the definition of the function $H(\alpha_0, \alpha_{1,1}, \dots, \alpha_{1,5})$ (in this case we have at least three similar sets). We represent U_0 as $\sum_{m=1}^5 U_{0,m}$. Raising $\langle A\eta, U_0\rangle$ to the power α_0, we get a similar sum as in equality (3.13). For each term in this equality there exist at least two sets J_{0,m_1}, J_{0,m_2} which are not involved in the product used in the definition of this term. In the same way as in (3.14) for $k = 1,2,3$, we obtain the estimate

$$|H(k, \alpha_{1,1}, \dots, \alpha_{1,5})| \le C \max_{\substack{l=0,1;\\ 1\le m\le 5}} \mathbf{E}|\langle A\xi_k, U_{l,m}\rangle|^3 \times \max_{1\le l,m\le 5} \mathbf{E}^{1/2} \exp\Bigl\{2it\langle \tilde U_{0,l}, A\tilde U_{1,m}\rangle\Bigr\}. \tag{3.15}$$

The conclusion of Lemma 3.3 follows immediately from the inequalities (3.14) and (3.15). □

REMARK. The conclusions of Lemma 3.3 are still valid, if the partition J_{0j}, $j = 1, \dots, 5$, depends on J_{1l} for any fixed $l = 1, \dots, 5$.

Denote by $g_{l,m}(t)$, $l, m = 1, \dots, 5$ the functions

$$g_{l,m}(t) = \mathbf{E}\exp\Bigl\{2it\langle A\tilde U_{0l}, \tilde U_{1m}\rangle\Bigr\}, \tag{3.16}$$

where U_{0l} and U_{1m} were defined in (3.11). Then

LEMMA 3.4. *For all $l, m = 1, \dots, 5$ and for all $t \ge 0$, $K > 0$, we have*

$$\begin{aligned} g_{l,m}(t) \le{}& \mathbf{E}\exp\Biggl\{\tfrac14 it\sigma(H) \sum_{j\in J_{0l},\, k\in J_{1m}} a_{jk} Y_j Y_k\Biggr\} \\ &+ \mathbf{I}\bigl\{t > \gamma\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1}\bigr\}, \\ &+ \mathbf{P}\Biggl\{\max_{j\in J_{0l}} \Biggl|\sum_{k\in J_{1m}} a_{jk} X_k\Biggr| \ge K\mathscr{L}\log^+\mathscr{L}\Biggr\} \\ &+ \mathbf{P}\Biggl\{\max_{k\in J_{1m}} \Biggl|\sum_{j\in J_{0l}} a_{jk} \tilde Y_j^{(H)}\Biggr| \ge K\mathscr{L}\log^+\mathscr{L}\Biggr\}, \end{aligned} \tag{3.17}$$

where $\gamma = \sigma(H)/(2K\sqrt{\mu_4(H)})$.

PROOF. Similarly to (3.6) we get

$$g_{lm}(t) \le \mathbf{E} \prod_{j \in J_{0l}} \mathbf{E}\left(\exp\left\{ 2it(1-\varepsilon_j)\tilde{X}_j^{(H)} \sum_{k \in J_m} a_{jk}\varepsilon_k \tilde{X}_k^{(H)} \right\} | \tilde{U}_m \right).$$

Using instead of (3.7) the inequality,

$$\max_{j \in \mathbf{N}} \left| \sum_{k \in \mathbf{N}} a_{jk}\varepsilon_k \tilde{X}_k^{(H)} \right| \le K\mathscr{L} \log^+ \mathscr{L},$$

we obtain

$$\begin{aligned} \mathbf{E}\left(\exp\left\{ 2it(1-\varepsilon_j)\tilde{X}_j^{(H)} \sum_{k \in J_m} a_{jk}\varepsilon_k \tilde{X}_k^{(H)} \right\} \middle| \tilde{U}_m \right) \\ \le \mathbf{I}\left\{ t > \gamma \mathscr{L}^{-1} (\log^+ \mathscr{L})^{-1} \right\} \\ + \exp\left\{ -\tfrac{1}{4} t^2 \sigma^2(H)(1-\varepsilon_j) \left(\sum_{k \in \mathbf{N}} a_{jk}\varepsilon_k \tilde{X}_k^{(H)} \right)^2 \right\}. \end{aligned}$$

Repeating the arguments for Gaussian r.v.'s, we obtain the result of the lemma. □

4. Spectral properties of submatrices. In order to estimate the right-hand side of (3.4), we need lower bounds for sufficiently many eigenvalues of the matrix

$$A_{\boldsymbol{\varepsilon}} = (a_{\boldsymbol{\varepsilon} jk})_{j,k \in \mathbf{N}} \quad \text{with } a_{\boldsymbol{\varepsilon} jk} = \big(\varepsilon_j(1-\varepsilon_k) + \varepsilon_k(1-\varepsilon_j)\big) a_{jk},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is a random vector with zero–one coordinates, which is independent of $X_1, \dots, X_n$. Introduce the diagonal matrix $\mathscr{E}$ with $\mathscr{E}(j,j) = \varepsilon_j$. Then we have

$$A_{\boldsymbol{\varepsilon}} = \mathscr{E}A(I-\mathscr{E}) + (I-\mathscr{E})A\mathscr{E}. \tag{4.1}$$

Define for any $J \subset \mathbf{N}$ the diagonal matrix $\mathscr{E}_J$ with $\mathscr{E}_J(j,j) = 1$, if $j \in J$ and 0 otherwise. Let

$$A_J = \mathscr{E}_J A(I-\mathscr{E}_J) + (I-\mathscr{E}_J)A\mathscr{E}_J. \tag{4.2}$$

By $\lambda_k(J)$, $k \in \mathbf{N}$, we denote the eigenvalues of the matrix A_J, ordered to be nonincreasing in absolute value.

Unfortunately, our assumptions are expressed in terms of spectral properties of A. Hence we need to find a partition $\boldsymbol{\varepsilon}$ (respectively, a subset J) such that $A_{\boldsymbol{\varepsilon}}$ (respectively, A_J) satisfies all the necessary spectral conditions. To this end we will investigate the spectral properties of $A_{\boldsymbol{\varepsilon}}$ and A_J in Lemmas 3.1–3.3.

Let $\Lambda^q(\mathbb{R}^n)$ denote the qth exterior product of $\mathbb{R}^n$ endowed with the Euclidean norm,

$$\|x\|^2 = \sum_{1 \le i_1 < \cdots < i_q \le n} x_{i_1 \cdots i_q}^2, \qquad x = \sum_{1 \le i_1 < \cdots < i_q \le n} x_{i_1 \cdots i_q} e_{i_1} \wedge \cdots \wedge e_{i_q},$$

where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

In what follows let $A \wedge B$ denote the exterior product of the matrices A and B and let $A^{\wedge q} = A \wedge \cdots \wedge A$ (q times) denote the exterior power of A, which maps $\Lambda^q(\mathbb{R}^n)$ into itself. Let $j^r = (j_1, j_2, \dots, j_r)$ denote a multiindex such that $1 \le j_1 < \cdots < j_r \le n$, $r = 1, 2, \dots, q$. The symbol $A_{j^r k^r}$ will denote the minor of the matrix A with rows $1 \le j_1 < \cdots < j_r \le n$ and columns $1 \le k_1 < \cdots < k_r \le n$. It is well known that the j^q, k^q entry of $A^{\wedge q}$, say $A^{\wedge q}(j^q, k^q)$ is given by $A_{j^q k^q}$. Furthermore, the eigenvalues of the matrix $A^{\wedge q}$ are $\lambda^*_{j^q} = \prod_{\nu=1}^q \lambda_{j_\nu}$, and the eigenvalue with largest absolute value is given by

$$\lambda_{*q} = \prod_{\nu=1}^{q} \lambda_\nu. \tag{4.3}$$

By $\lambda^*_{j^q}(J)$, $\lambda_{*q}(J)$, $\lambda^*_{\varepsilon j^q}$, $\lambda_{\varepsilon q}$, $q \in \mathbf{N}$, we shall denote the corresponding eigenvalues of the qth exterior power of the matrices A_J and A_ε defined in (4.1) and (4.2), respectively. Note that $A^{\wedge 1} = A$ and $A^{\wedge 0} = I$.

Below we shall prove some results about the spectrum of the matrix A_J.

LEMMA 4.1. *Assume that the conditions* (1.1) *and* (1.2) *hold. Then there exist sets* J, J_1 *such that, for* $q \in \mathbf{N}$,

$$\left|\lambda_q(J)\right| \ge \frac{1}{2^q}|\lambda_{*q}| - 16\mathscr{L}\mathbf{I}\{q \ne 1\}, \tag{4.4}$$

and

$$\sum_{k\in\mathbf{N}} \lambda_k^2(J_1) \ge \tfrac{1}{2}, \qquad \left|\lambda_1(J_1)\right| \le |\lambda_1|. \tag{4.5}$$

Let $J_1 \subset J \subset \mathbf{N}$. Denote by A_{JJ_1} and by A'_{JJ_1} the matrices defined by

$$A_{JJ_1} = \mathscr{E}_{J_1} A_J + A_J \mathscr{E}_{J_1}, \qquad A'_{JJ_1} = \left(\mathscr{E} - \mathscr{E}_{J_1}\right) A_J + A_J\left(\mathscr{E} - \mathscr{E}_{J_1}\right).$$

By $\lambda_k(JJ_1)$, respectively, $\lambda'_k(JJ_1)$, $k \in \mathbf{N}$, we denote the eigenvalues of these matrices. Note that, in general, $A_{JJ_1} \ne A_{J_1}$.

LEMMA 4.2. *Assume that* (1.1) *and* (1.2) *are fulfilled. Then, for any* $J \subset \mathbf{N}$ *there exists a set* J_1 *such that* $J_1 \subset J$ *and*

$$\min\left\{\left|\lambda_q(JJ_1)\right|, \left|\lambda_q\left(JJ_1^{(c)}\right)\right|\right\} \ge \frac{1}{2^{q+1}}\left|\lambda_{*q}(J)\right| - 19\mathscr{L}\,\mathbf{I}\{q \ne 1\}, \tag{4.6}$$

and

$$\min\left\{\sum_{k\in\mathbf{N}} \lambda_k^2(JJ_1), \sum_{k\in\mathbf{N}} \lambda_k^2\left(JJ_1^{(c)}\right)\right\} \ge \tfrac{1}{4}\sum_{k\in\mathbf{N}} \lambda_k^2(J) - \sqrt{2}\mathscr{L}, \tag{4.7}$$

where $J_1^{(c)} = J \setminus J_1$.

REMARK. The inequalities (4.6) and (4.7) are still valid if $\lambda_q(JJ_1)$ is replaced by $\lambda'_q(JJ_1)$.

LEMMA 4.3. *Let* $\varepsilon_1, \dots, \varepsilon_n$ *be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then, for any $0 \le \alpha < 2 - 2p$,

$$\mathbf{P}\bigg\{\sum_{j\in\mathbf{N}} |\lambda_{\varepsilon j}|^2 \le \alpha p\bigg\} \le C(2-2p-\alpha)^{-6}\mathscr{L}^6 p^{-6}. \tag{4.8}$$

Let $\mathscr{L} \le 2^{-5}B(q,p)|\lambda_{*q}|^2$, $B(q,p) = 2^{2q-1}(1-p)^{2q}$. *Then, for any* $q \ge 1$, *there exists an absolute constant* C *such that, for* $0 \le \alpha^2 \le B(q,p)$, *we have*

$$\mathbf{P}\big\{|\lambda_{*\varepsilon q}| \le \alpha p^q |\lambda_{*q}|\big\} \le C\big(B(q,p)-\alpha^2\big)^{-6}|\lambda_{*q}|^{-12q}\mathscr{L}^6 p^{-12q}. \tag{4.9}$$

PROOF OF LEMMA 4.1. Introduce a random vector $\boldsymbol{\varepsilon}$ with i.i.d. Bernoulli coordinates ε_j, $j \in \mathbf{N}$ such that $\mathbf{P}\{\varepsilon_j = 1\} = p$. Denote by A_{*q} and $A_{*\varepsilon q}$ the qth exterior powers of the matrices A and $A_{\boldsymbol{\varepsilon}}$, respectively. The proof is based on the following relations:

$$\mathbf{E}|\lambda_{*\varepsilon q}| \ge \big\|(\mathbf{E}A_{*\varepsilon q})x\big\| \quad \text{for any } x \in \Lambda^q(\mathbb{R}^n), \text{ such that } \|x\| = 1, \tag{4.10}$$

$$\mathbf{E}A_{\varepsilon j^q k^q} = \big(2p(1-p)\big)^q A_{j^q k^q} \quad \text{for any } j^q, k^q \text{ such that } j^q \cap k^q = \varnothing, \tag{4.11}$$

and

$$\sum_{j^q, k^q : j^q \cap k^q \ne \varnothing} |A_{j^q k^q}|^2 \le 256\mathscr{L}^2 \mathbf{I}\{q \ne 1\}. \tag{4.12}$$

Let $q \ge 2$. Then, by (4.11) and (4.12),

$$\Big\|\mathbf{E}A_{*\varepsilon q} - \big(2p(1-p)\big)^q A_{*q}\Big\|_2 \le 16\mathscr{L}. \tag{4.13}$$

Here $\|\cdot\|$ denotes the Frobenius-norm of matrix, that is, $\|A\|_2^2 = \sum_{j,k\in\mathbf{N}} a_{jk}^2$. Since $\|A\| \le \|A\|_2$, we obtain by (4.13),

$$\|\mathbf{E}A_{*\varepsilon q}x\| \ge \big(2p(1-p)\big)^q \|A_{*q}x\| - 16\mathscr{L}. \tag{4.14}$$

The inequalities (4.10) and (4.14) imply

$$\mathbf{E}|\lambda_{*\varepsilon q}| \ge \big(2p(1-p)\big)^q \|A_{*q}x\| - 16\mathscr{L}. \tag{4.15}$$

Choosing here x as the eigenvector of A_{*q} corresponding to the eigenvalue λ_{*q}, $\|A_{*q}x\| = |\lambda_{*q}|$, we obtain

$$\mathbf{E}|\lambda_{*\varepsilon q}| \ge \big(2p(1-p)\big)^q |\lambda_{*q}| - 16\mathscr{L}. \tag{4.16}$$

Let $\lambda_{\varepsilon k}$ denote the kth eigenvalue of $A_{\boldsymbol{\varepsilon}}$ (ordered in absolute value). Since $\lambda_{\varepsilon k} \le 1$ for all $k \in \mathbf{N}$, the equality (4.3) implies that

$$|\lambda_{\varepsilon q}| \ge \prod_{\nu=1}^{q} |\lambda_{\varepsilon\nu}| = |\lambda_{*\varepsilon q}|. \tag{4.17}$$

Inequality (4.4) for $q \ge 2$ follows from (4.16) and (4.17) with $p = \frac{1}{2}$. In the case $q = 1$ we have $\mathbf{E}a_{\varepsilon jk} = p(1-p)a_{jk}$, and (4.10) implies (4.4).

It is easy to see that for any $J \subset \mathbf{N}$,

$$|\lambda_1(J)| \le |\lambda_1|.$$

Furthermore,

$$\sum_{k \in \mathbf{N}} \lambda_{\boldsymbol{\varepsilon} k}^2 = \|A_{\boldsymbol{\varepsilon}}\|_2^2. \tag{4.18}$$

The relations (4.11), (4.12) with $q = 1$ and (4.18) together imply

$$\mathbf{E} \sum_{k \in \mathbf{N}} \lambda_{\boldsymbol{\varepsilon} k}^2 = 2p(1-p)\|A\|_2^2. \tag{4.19}$$

This implies for $p = \frac{1}{2}$ inequality (4.5). It remains to prove the assertions (4.10)–(4.12). Since the matrix $A_{\boldsymbol{\varepsilon}}$ is symmetric, we have

$$|\lambda_{*\boldsymbol{\varepsilon} q}| = \sup_{x \in \Lambda^q(\mathbb{R}^n):\, \|x\| = 1} \|A_{*\boldsymbol{\varepsilon} q} x\|.$$

This equality implies (4.9). Let $j^q \cap k^q = \varnothing$. Note that in this case $\varepsilon_{j_1}, \dots, \varepsilon_{j_q}$ and $\varepsilon_{k_1}, \dots, \varepsilon_{k_q}$ are independent. It is easy to see that

$$\varepsilon_j(1 - \varepsilon_k) + \varepsilon_k(1 - \varepsilon_j) = (\varepsilon_j - \varepsilon_k)^2. \tag{4.20}$$

Hence, for any permutation π of $\{1, \dots, q\}$,

$$\begin{aligned} \mathbf{E} \prod_{\nu=1}^{q} (\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 &= \mathbf{E} \prod_{\nu=1}^{q} \mathbf{E}\Big((\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 | \varepsilon_{j_\nu} \Big) \\ &= \prod_{\nu=1}^{q} \mathbf{E} p^{(1-\varepsilon_{j_\nu})} (1-p)^{\varepsilon_{j_\nu}} = (2p(1-p))^q. \end{aligned} \tag{4.21}$$

Expanding the minor $A_{\boldsymbol{\varepsilon} j^q k^q}$, we get

$$A_{\boldsymbol{\varepsilon} j^q k^q} = \sum_{\pi} (-1)^{\pi} \prod_{\nu=1}^{q} a_{\boldsymbol{\varepsilon} j_\nu k_{\pi(\nu)}} = \sum_{\pi} (-1)^{\pi} \prod_{\nu=1}^{q} (\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 \prod_{\nu=1}^{q} a_{j_\nu k_{\pi(\nu)}}. \tag{4.22}$$

Averaging (4.22) over all $\boldsymbol{\varepsilon}$ and applying (4.21), we obtain

$$\mathbf{E} A_{\boldsymbol{\varepsilon} j^q k^q} = (2p(1-p))^q \sum_{\pi} (-1)^{\pi} \prod_{\nu=1}^{q} a_{j_\nu k_{\pi(\nu)}} = (2p(1-p))^q A_{j^q k^q}.$$

This proves (4.11).

Finally let $j^q \cap k^q \ne \varnothing$. The assumption (1.2) yields (4.12) for $q = 1$. If $q = 2$, we note that $|A_{\boldsymbol{\varepsilon} j^q k^q}| = |a_{\boldsymbol{\varepsilon} j k_1} a_{\boldsymbol{\varepsilon} j_2 j}|$ for $j \in j^q \cap k^q$. This implies for $q = 2$,

$$\sum_{j^q, k^q:\, j^q \cap k^q \ne \varnothing} |A_{\boldsymbol{\varepsilon} j^q k^q}|^2 \le \sum_{j, k, l \in \mathbf{N}} |a_{jk}|^2 |a_{lj}|^2 \le \mathscr{L}^2 \sum_{j, k \in \mathbf{N}} a_{jk}^2 = \mathscr{L}^2, \tag{4.23}$$

thus proving the inequality (4.12) for $q = 2$.

Let $q \geq 3$ and $j \in j^q \cap k^q$. Expanding $A_{\varepsilon j^q k^q}$ first in the jth row and then in the jth column, we obtain

$$\sum_{j^q, k^q: j^q \cap k^q \neq \varnothing} |A_{\varepsilon j^q k^q}|^2 \leq 2q^4 \Bigl(\sum_{j,k,l \in \mathbf{N}} |a_{jk}|^2 |a_{lj}|^2 \Bigr) \times \Bigl(\sum_{j^{(q-2)}, k^{(q-2)}} |A_{\varepsilon j^{(q-2)}, k^{(q-2)}}|^2 \Bigr). \tag{4.24}$$

Note that, for any $r \in \mathbf{N}$,

$$\begin{aligned} \sum_{j^r, k^r} |A_{\varepsilon j^r k^r}|^2 &= \sum_{j^r} |\lambda^*_{\varepsilon j^r}|^2 = \sum_{1 \leq j_1 < \cdots < j_r \leq n} \prod_{\nu=1}^{r} |\lambda_{\varepsilon j_\nu}|^2 \\ &\leq \frac{1}{r!} \Bigl(\sum_{k \in N} |\lambda_{\varepsilon k}|^2 \Bigr)^r \leq \frac{1}{r!} \Bigl(\sum_{j,k \in \mathbf{N}} a_{jk}^2 \Bigr)^4 = \frac{1}{r!}. \end{aligned} \tag{4.25}$$

It is easy to show that $2q^4/(q-2)! \leq 256$. The inequalities (4.24) and (4.25) together imply (4.12) in the general case. □

PROOF OF LEMMA 4.2. Consider the sets L_k, $k = 0, 1, \dots, K = |J|$, consisting of the first k elements of J, that is, $L_0 = \varnothing$ and $L_K = J$. Let $\mathscr{E}'_k$ be the diagonal matrix with $\mathscr{E}'_k(j,j) = \varepsilon_j^{(k)}$, where $\varepsilon_j^{(k)} = 1$, if $i \in L_k$, and 0, otherwise. Let

$$\begin{aligned} \mathscr{C}_k &= \mathscr{E}'_k A(I - \mathscr{E}) + (I - \mathscr{E}) A \mathscr{E}'_k, \\ \mathscr{C}'_k &= (\mathscr{E} - \mathscr{E}'_k) A (I - \mathscr{E}) + (I - \mathscr{E}) A (\mathscr{E} - \mathscr{E}'_k), \end{aligned}$$

and let $\lambda_{*q}(k)$ and $\lambda'_{*q}(k)$ denote the eigenvalues of the qth exterior power of the matrices $\mathscr{C}_k$ and $\mathscr{C}'_k$, respectively, of largest absolute values. Note that $|\lambda_{*q}(A)|$ is of the operator-norm in the space of qth exterior powers of symmetric matrices, which is continuous with respect to the Frobenius-norm of matrices. It is well known that

$$\bigl| |\lambda_{*q}(A)| - |\lambda_{*q}(B)| \bigr| \leq |\lambda_{*q}(A - B)| \leq \|A_{*q} - B_{*q}\|^2$$

for any symmetric matrices A and B [see Lancaster (1969), Chapter 6]. Assume that, for any $q \in \mathbf{N}$,

$$\|\mathscr{C}_k^{\wedge q} - \mathscr{C}_{k+1}^{\wedge q}\|_2^2 \leq 2 \frac{q^2}{(q-1)!} \mathscr{L}^2. \tag{4.26}$$

Then

$$\max_{1 \leq k \leq K-1} \bigl| |\lambda_1(k+1)| - |\lambda_1(k)| \bigr| \leq \sqrt{2 \frac{q^2}{(q-1)!}} \mathscr{L}. \tag{4.27}$$

We note that $2q^2/(q-1)! \le 9$ for $q \ge 1$. Since $\lambda_{*q}(0) = 0$, (4.27) implies that one can find a number k_0 such that

$$\tfrac{1}{2}\left|\lambda_{*q}(K)\right| - 3\mathscr{L} \le \left|\lambda_{*q}(k_0)\right| \le \tfrac{1}{2}\left|\lambda_{*q}(K)\right| + 3\mathscr{L}. \tag{4.28}$$

Put $J_1 = L_{k_0}$. For $J_{k_0}^{(c)} = J \setminus J_{k_0}$, we have

$$\left|\lambda'_{*q}(k_0)\right| \ge \left|\lambda_{*q}(K)\right| - \left|\lambda_{*q}(k_0)\right| \ge \tfrac{1}{2}\left|\lambda_{*q}(K)\right| - 3\mathscr{L}. \tag{4.29}$$

The inequalities (4.28) and (4.29) together imply (4.6). The same proof yields (4.7). Thus it remains to prove (4.26) only. If $q = 1$, we have

$$\|\mathscr{C}_{k+1} - \mathscr{C}_k\|_2^2 \le 2\mathscr{L}^2$$

and (4.26) holds. The proof of (4.26) in the case $q \ge 2$ is similar to the proof of (4.12). Indeed, let l_k be the kth element of J. Then

$$\|\mathscr{C}_k^{\wedge q} - \mathscr{C}_{k+1}^{\wedge q}\|_2^2 \le 2 \sum_{l^q,\, j^q:\, l_{k+1} \in l^q} |A_{\varepsilon l^q, j^q}|^2.$$

Expanding the minor $A_{\varepsilon l^q j^q}$ in the l_{k+1}th row, we obtain

$$\sum_{l^q,\, j^q:\, l_{k+1} \in l^q} |A_{\varepsilon l^q j^q}|^2 \le q^2 \left(\sum_{j \in \mathbf{N}} a_{l_{k+1} j}^2 \right) \sum_{l^{q-1},\, j^{q-1}} |A_{*\, l^{q-1}, j^{q-1}}|^2. \tag{4.30}$$

The inequalities (4.30) and (4.25) together imply (4.26). □

PROOF OF LEMMA 4.3. We shall prove the second inequality only, because the first one can be proved similarly. First note that

$$|\lambda_{*\varepsilon q}|^2 = \sup_{x \in \Lambda^q(\mathbb{R}^n):\, \|x\|^2 = 1} \|A_{*\varepsilon q} x\|^2.$$

Hence, for any $x \in \Lambda^q(\mathbb{R}^n)$ such that $\|x\|^2 = 1$, we have

$$\mathbf{P}\left\{|\lambda_{*\varepsilon q}|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\right\} \le \mathbf{P}\left\{\|A_{*\varepsilon q} x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\right\}.$$

Let x be the eigenvector of matrix $A^{\wedge q}$ corresponding to the eigenvalue λ_{*q}. For the proof of the lemma it is sufficient to estimate the quantity

$$D_n = \mathbf{P}\left\{\|A_{*\varepsilon q} x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\right\}.$$

Note that, for $0 \le u \le 1$,

$$|u - v|^2 \ge u^2 - 2v.$$

Lyapunov's inequality, (4.14), and the last inequality together imply

$$\begin{aligned} \mathbf{E}\|A_{*\varepsilon q} x\|^2 &\ge 2^{2q} p^{2q} (1-p)^{2q} \|A_{*q} x\|^2 - 2^5 \mathscr{L} \\ &= 2^{2q} p^{2q} (1-p)^{2q} |\lambda_{*q}|^2 - 2^5 \mathscr{L} \ge B(q,p) p^{2q} |\lambda_{*q}|^2. \end{aligned} \tag{4.31}$$

Write $\zeta = \|A_{*\varepsilon q} x\|^2$. If we show

$$\mu_6 = \mathbf{E}|\zeta - \mathbf{E}\zeta|^6 \le C\mathscr{L}^6, \tag{4.32}$$

the lemma follows. Indeed, (4.31) implies

$$\begin{aligned} D_n &= \mathbf{P}\Big\{\|A_{*\varepsilon q} x\|^2 - \mathbf{E}\|A_{*\varepsilon q} x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2 - \mathbf{E}\|A_{*\varepsilon q} x\|^2\Big\} \\ &\le \mathbf{P}\Big\{\|A_{*\varepsilon q} x\|^2 - \mathbf{E}\|A_{*\varepsilon q} x\|^2 \le -\big(B(p,q) - \alpha^2\big) p^{2q} |\lambda_{*q}|^2\Big\} \\ &\le P\Big\{|\zeta - \mathbf{E}\zeta| \ge \big(B(p,q) - \alpha^2\big) p^{2q} |\lambda_{*q}|^2\Big\} \end{aligned}$$

Applying Chebyshev's inequality and inequality (4.32) we obtain (4.8). It remains to prove (4.32) only. Write μ_6 in the form

$$\mu_6 = \sum_{i_1^q} \cdots \sum_{i_6^q} \mathbf{E} \prod_{k=1}^{6} \big(\zeta(i_k^q) - \mathbf{E}\zeta(i_k^q)\big) \quad \text{where } \zeta(i_k^q) = \Big(\sum_{l^q} A_{\varepsilon i_k^q, l^q} x_{l^q}\Big)^2.$$

Introducing for $k = 1, 2, \dots, 6$ the notation

$$\zeta(i_k^q, l_1^q, l_2^q) = \big(A_{\varepsilon i_k^q, l_1^q} A_{\varepsilon i_k^q, l_2^q} - \mathbf{E} A_{\varepsilon i_k^q, l_1^q} A_{\varepsilon i_k^q, l_2^q}\big) x_{l_1^q} x_{l_2^q},$$

we can rewrite the equality (4.31) in the form

$$\mu_6 = \sum_{i_1^q, l_1^q, j_1^q} \cdots \sum_{i_6^q, l_6^q, j_6^q} \mathbf{E}\zeta(i_1^q, l_1^q, j_1^q) \cdots \zeta(i_6^q, l_6^q, j_6^q). \tag{4.33}$$

Without loss of generality we may assume that, for $k = 1, \dots, 6$, the multiindices i_k^q are disjoint to the multiindices l_k^q and j_k^q. Write

$$J_k = \{i_k^{(1)}, \dots, i_k^{(q)}\} \cup \{l_k^{(1)}, \dots, l_k^{(q)}\} \cup \{j_k^{(1)}, \dots, j_k^{(q)}\}.$$

If for some fixed k_0 and a multiindex i_k^q, l_k^q, j_k^q, $k = 1, 2, \dots, 6$,

$$J_{k_0} \cap \Big(\bigcup_{k \ne k_0} J_k\Big) = \varnothing,$$

then

$$\mathbf{E}\zeta(i_1^q, l_1^q, j_1^q)\zeta(i_1^q, l_2^1, j_2^q) \cdots \zeta(i_6^q, l_6^1, j_6^q) = 0.$$

This implies that the summands in (4.33) are nonzero in three cases only.

(i) The set of indices $J_1, \dots, J_6$ can be decomposed in three nonintersecting pairs of subsets (J_{i_1}, J_{i_2}), (J_{i_3}, J_{i_4}), (J_{i_5}, J_{i_6}), such that any pair has at least one common index.

(ii) The set of indices $J_1, \dots, J_6$ can be decomposed in two nonintersecting triples of subsets $(J_{i_1}, J_{i_2}, J_{i_3})$, $(J_{i_4}, J_{i_5}, J_{i_6})$, such that any triple has at least one common index.

(iii) The set of indices $J_1, \dots, J_6$ can be decomposed in two nonintersecting subsets: a quadruple $(J_{i_1}, J_{i_2}, J_{i_3}, J_{i_4})$ and a pair (J_{i_5}, J_{i_6}), such that any both systems have at least one common index. We show that in all these cases the following estimate holds:

$$\Big|\sum_{i_1^q, l_1^q, j_1^q} \cdots \sum_{i_6^q, l_6^q, j_6^q} \mathbf{E}\zeta(i_1^q, l_1^q, j_1^q) \cdots \zeta(i_6^q, l_6^q, j_6^q)\Big| \le C\mathscr{L}^6. \tag{4.34}$$

Note, for instance, that in the first case the sum on the right-hand side of (4.33) can be decomposed in a product of the following sums:

$$\mathscr{S}_{\nu,\mu} = \sum_{i^q_\nu, l^q_\nu, j^q_\nu} \sum_{i^q_\mu, l^q_\mu, j^q_\mu} \zeta(i^q_\nu, l^q_\nu, j^q_\nu)\zeta(i^q_\mu, l^q_\mu, j^q_\mu),$$

where the sets J_ν and J_μ have nonempty intersection, $1 \le \nu \ne \mu \le 6$.

Let i_0 be the common index of J_ν and J_μ. For simplicity we can assume that i_0 is a common index for l^q_ν and j^q_μ. Expanding the determinants $A_{\boldsymbol{\varepsilon} i^q_\nu, l^q_\nu}$ and $(A(\varepsilon))^{\wedge q}(i^q_\mu, j^q_\mu)$ in the common column, and the determinants $A_{\boldsymbol{\varepsilon} i^q_\nu, j^q_\nu}$, and $A_{\boldsymbol{\varepsilon} i^q_\mu, l^q_\mu}$ in the common row and using $a_{nii} = 0$, we can obtain the following estimate:

$$\begin{aligned}\mathscr{S}_{\nu,\mu} \le q^2 &\sum_{i_0, j_1, l_1, j_2, l_2 \in \mathbf{N}} |a_{nj_1,i_0}|\,|a_{nl_1,i_0}|\,|a_{nl_1,l_2}|\,|a_{nj_1,j_2}|\,|x_{i_0}|^2 |x_{l_2}|\,|x_{j_2}| \\ &\times \sum_{i^{q-2}, j^{q-2}} |A_{i^{q-2} j^{q-2}}|\,|x_{i^{q-2}}|\,|x_{j^{q-2}}|.\end{aligned}$$

Hölder's inequality, (4.23) and $\sum_i x_i^2 = 1$ together imply

$$\begin{aligned}\mathscr{S}_{\nu,\mu} &\le q^2/(q-2)! \sum_{i_0, l_1, j_1 \in \mathbf{N}} \Bigl(\sum_{l_2 \in \mathbf{N}} a^2_{l_1,l_2}\Bigr)^{1/2} \Bigl(\sum_{j_2 \in \mathbf{N}} a^2_{l_1,l_2}\Bigr)^{1/2} |a_{nj_1,i_0}|\,|a_{nl_1,i_0}| x^2_{i_0} \\ &\le q^2/(q-2)! \sum_{i_0 \in \mathbf{N}} \Bigl(\sum_{l_1 \in \mathbf{N}} a^2_{l_0,i_0}\Bigr)^{1/2} \Bigl(\sum_{j_1 \in \mathbf{N}} a^2_{j_1,i_0}\Bigr)^{1/2} x^2_{i_0} \le C\mathscr{L}^2.\end{aligned}$$

The last inequality proves (4.34) for case (i). The proof of inequality (4.34) in cases (ii) and (iii) is similar. Inequality (4.34) finally implies (4.32). □

5. Some bounds for distributions of randomized linear forms. In this section we obtain the bounds for the last two terms in (3.4). Note that the results of this section can be extended to the case of non-i.i.d. r.v.'s $X_1, \dots, X_n$. Let

$$\mathscr{L}_j^2 = \sum_{k \in \mathbf{N}} a^2_{jk}.$$

Without loss of generality we shall assume that $\mathscr{L}_j \ne 0$. Let $\rho_j = \mathscr{L}\mathscr{L}_j^{-1}(\log^+ \mathscr{L}_j)^{-1}$ and $\varkappa = (e/H^2)\log^+ \mathscr{L}$.

LEMMA 5.1. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 =\} = p.$$

Then for $\mathscr{L} \le e^{-1/2}$

$$\mathbf{P}\Biggl\{\max_{j \in \mathbf{N}} \Biggl| \sum_{\substack{k \in \mathbf{N}: \\ |a_{jk}| \le \rho_j \mathscr{L}_j}} a_{jk}\varepsilon_k \tilde{X}_k^{(H)} \Biggr| > K\mathscr{L}\Biggr\} \le \mathscr{L}^{K/H - 2 - p\varkappa}. \tag{5.1}$$

PROOF. If $p\varkappa \geq K/H - 2$, the inequality (5.1) is trivial. Assume that $p\varkappa < K/H - 2$. Denote by Σ_j the quantity

$$\Sigma_j = \left| \sum_{\substack{k \in \mathbf{N}: \\ |a_{jk}| \leq \rho_j \mathscr{L}_j}} a_{kj} \varepsilon_k \tilde{X}_k^{(H)} \right|, \tag{5.2}$$

and by Σ^*, $\prod^*$ denote the sum and the product over $k \in \mathbf{N}$: $|a_{jk}| \leq \rho_j \mathscr{L}_j$. In this notation we have

$$\mathbf{P}\Big\{ \max_{j \in \mathbf{N}} \Sigma_j > K\mathscr{L} \Big\} \leq \sum_{j \in \mathbf{N}} \mathbf{P}\{\Sigma_j > K\mathscr{L}\}.$$

By Chebyshev's inequality

$$\begin{aligned} \mathbf{P}\{\Sigma_j > K\mathscr{L}\} &\leq 2\exp\{-h_j K\mathscr{L}\} \mathbf{E}\exp\{h_j \Sigma_j\} \\ &= 2\exp\{-h_j K\mathscr{L}\} \prod{}^* \Big(1 + p\Big(\mathbf{E}\exp\Big\{h_j a_{jk} \tilde{X}_k^{(H)}\Big\} - 1\Big)\Big) \\ &\leq 2\exp\{-h_j K\mathscr{L}\} \exp\Big\{ p \sum{}^* \Big(\mathbf{E}\exp\Big\{h_j a_{jk} \tilde{X}_k^{(H)}\Big\} - 1\Big)\Big\}. \end{aligned} \tag{5.3}$$

Furthermore, we have

$$\begin{aligned} \mathbf{E}\exp\Big\{h_j a_{jk} \tilde{X}_k^{(H)}\Big\} - 1 &\leq \sigma^2(H) h_j^2 |a_{jk}|^2 \exp\{h_j H |a_{jk}|\} \\ &\leq h_j^2 |a_{jk}|^2 \exp\{h_j H |a_{jk}|\}. \end{aligned} \tag{5.4}$$

By (5.3) and (5.4) it follows that

$$\sum_{j \in \mathbf{N}} \mathbf{P}\{\Sigma_j > K\mathscr{L}\} \leq 2 \sum_{j \in \mathbf{N}} \exp\Big\{-Kh_j\mathscr{L} + p h_j^2 \mathscr{L}_j^2 \exp\{h_j H \rho_j \mathscr{L}_j\}\Big\}. \tag{5.5}$$

Choosing $h_j = H^{-1}\mathscr{L}_j^{-1}\rho_j^{-1} = H^{-1}\mathscr{L}^{-1}\log^+ \mathscr{L}_j$, we get

$$\begin{aligned} &\sum_{j \in \mathbf{N}} \mathbf{P}\{|\Sigma_j| > K\mathscr{L}\} \\ &\quad \leq 2 \sum_{j \in \mathbf{N}} \exp\Big\{-KH^{-1}|\log^+ \mathscr{L}_j| + pH^{-2}\big(\log^+ \mathscr{L}_j\big)^2 \mathscr{L}_j^2 \mathscr{L}^{-2} e\Big\}. \end{aligned} \tag{5.6}$$

Note that

$$p\big(\log^+ \mathscr{L}_j\big)\mathscr{L}_j^2 \mathscr{L}^{-2} \leq p \log^+ \mathscr{L}, \tag{5.7}$$

since the function $x^2 \log^+ x$ is increasing, for $0 \leq x \leq e^{-1/2}$. Inequalities (5.6) and (5.7) together finally imply

$$\mathbf{P}\Big\{|\Sigma_j| > K\mathscr{L}\Big\} \leq 2 \sum_{j \in \mathbf{N}} \mathscr{L}_j^{KH^{-1} - p\varkappa} \leq \mathscr{L}^{KH^{-1} - 2 - p\varkappa}. \tag{5.8}$$

□

LEMMA 5.2. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then, for any $K, H > 0$ *such that* $K/H \geq 3$,

$$\mathbf{P}\left\{\max_{j\in\mathbf{N}}\left|\sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|>\rho_j\mathscr{L}_j}} a_{jk}\,\varepsilon_k\,\tilde{X}_k\right| > K\mathscr{L}\right\} \leq p^{K/H}\mathscr{L}^{-2}(\log^+\mathscr{L})^{K/H}.$$

PROOF. Denote by Σ_j' the quantity

$$\Sigma_j' = \sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|>\rho_j\mathscr{L}_j}} a_{jk}\,\varepsilon_k\,\tilde{X}_k,$$

and let Σ', Π' denote the sum and the product over $k \in \mathbf{N}$: $|a_{jk}| > \rho_j\mathscr{L}_j$. The inequality

$$|\Sigma_j'| \leq H\max_{l\in\mathbf{N}}|a_{jl}|\sum{}'\varepsilon_k \leq H\mathscr{L}_j\sum{}'\varepsilon_k,$$

implies

$$\mathbf{P}\left\{\max_{1\leq j\leq n}|\Sigma_j'| > K\mathscr{L}\right\} \leq \sum_{j\in\mathbf{N}}\mathbf{P}\{|\Sigma_j'| > K\mathscr{L}_j\} \leq \sum_{j\in\mathbf{N}}\mathbf{P}\left\{\sum{}'\varepsilon_k \geq \frac{K}{H}\right\}. \tag{5.9}$$

Let n_j^* the number of elements a_{lj} such that $k \in \mathbf{N}$: $|a_{kj}| > \rho_j\mathscr{L}_j$. Note that $n_j^* \leq 1/\rho_j^2$, $j = 1, 2, \ldots, n$. Let $\mu_n = \Sigma_1^n\varepsilon_j$. Then μ_n has binomial distribution with parameters n and p. Assuming

$$\mathbf{P}\{\mu_n > C\} \leq (np)^C, \qquad C \geq 3, \tag{5.10}$$

we obtain

$$\mathbf{P}\left\{\sum{}'\varepsilon_k > \frac{K}{H}\right\} \leq (n_j^* p)^{K/H} \leq \left(\rho_j^{-2}p\right)^{K/H}. \tag{5.11}$$

Note that

$$\sum_{j\in\mathbf{N}}\left(\rho_j^{-2}p\right)^{K/H} \leq p^{K/H}\mathscr{L}^{-2}(\log^+\mathscr{L})^{K/H}.$$

This implies the result of Lemma 5.2. It remains to prove (5.10). If $np \geq 1$, the inequality (5.10) is trivial. If $np < 1$, assume for simplicity that C is an integer. Then,

$$\begin{aligned}\mathbf{P}\{\mu_n > C\} &= \sum_{j>C} C_n^k p^k (1-p)^k \leq (np)^C \frac{1}{C!}\sum_{k\geq 0}\frac{(np)^k}{k!}\\ &\leq (np)^C\frac{e}{C!} \leq (np)^C.\end{aligned}$$

□

REMARK 5.3. For $\mathscr{L} \leq e^{-1/2}$ we have

$$\begin{aligned}&\mathbf{P}\left\{\max_{1\leq j\leq n}\left|\sum_{k\in\mathbf{N}} a_{kj}\varepsilon_k\tilde{X}_k^{(H)}\right| > 2K\mathscr{L}\right\}\\ &\qquad\leq \mathscr{L}^{KH^{-1}-2-p\varkappa} + p^{K/H}\mathscr{L}^{-2}(\log^+\mathscr{L})^{-2K/H}.\end{aligned} \tag{5.12}$$

LEMMA 5.4. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = \tfrac{1}{2}.$$

Then

$$\mathbf{P}\left\{\max_{1\le j\le n}\left|\sum_{k\in\mathbf{N}} a_{kj}\varepsilon_k \tilde{X}_k^{(H)}\right| > 2K\mathscr{L}|\log^+\mathscr{L}|\right\} \le \exp\{e/2H\}\mathscr{L}^{K/H}. \tag{5.13}$$

PROOF. Note that $\mathscr{L}_j \log^+ \mathscr{L}_j \le \mathscr{L} \log^+ \mathscr{L}$. Therefore,

$$\begin{aligned}\mathbf{P}&\left\{\max_{1\le j\le n}\left|\sum_{k\in\mathbf{N}} a_{kj}\varepsilon_k \tilde{X}_k^{(H)}\right| > 2K\mathscr{L}\log^+\mathscr{L}\right\}\\ &\le \sum_{j\in\mathbf{N}}\mathbf{P}\left\{\left|\sum_{k\in\mathbf{N}} a_{kj}\varepsilon_k \tilde{X}_k^{(H)}\right| > 2K\mathscr{L}_j\log^+\mathscr{L}_j\right\}.\end{aligned}$$

Similarly to (5.5) we obtain that

$$\begin{aligned}\mathbf{P}&\left\{\max_{1\le j\le n}\left|\sum_{k\in\mathbf{N}} a_{kj}\varepsilon_k \tilde{X}_k^{(H)}\right| > 2K\mathscr{L}\log^+\mathscr{L}\right\}\\ &\le \sum_{j\in\mathbf{N}}\exp\left\{-Kh_j\mathscr{L}_j\log^+\mathscr{L}_j + \tfrac{1}{2}h_j^2\mathscr{L}_j^2\exp\{h_jH\rho_j\mathscr{L}_j\}\right\}.\end{aligned} \tag{5.14}$$

By (5.14) with $h_j = H^{-1}\mathscr{L}_j^{-1}$, (5.13) holds.

LEMMA 5.5. *The inequality*

$$\mathbf{P}\left\{\max_{1\le j\le n}\left|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\right| > K\mathscr{L}(\log^+\mathscr{L})^{1/2}\right\} \le \mathscr{L}^{K^2/2-2}$$

holds.

PROOF. Note that $\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k$ has normal distribution with variance $\mathscr{L}_j^2(\varepsilon) = \sum_{k\in\mathbf{N}} a_{jk}^2(1-\varepsilon_k)$. Thus

$$\begin{aligned}\mathbf{P}&\left\{\max_{j\in\mathbf{N}}\left|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\right| > K\mathscr{L}(\log^+\mathscr{L})^{1/2}\right\}\\ &\le \sum_{j\in\mathbf{N}}\mathbf{P}\left\{\left|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\right| > K\mathscr{L}(\log^+\mathscr{L})^{1/2}\right\}\\ &\le \sum_{j\in\mathbf{N}}\exp\left\{-\frac{K^2\mathscr{L}^2\log^+\mathscr{L}_j}{\mathscr{L}_j^2}\right\} \le \sum_{j\in\mathbf{N}}\exp\left\{-\frac{1}{2}K^2\log^+\mathscr{L}_j\right\} \le \mathscr{L}^{K^2/2-2}. \quad \square\end{aligned}$$

6. The proofs of Theorems 1 and 3. Let $f(t)$ and $g(t)$ denote the ch.f.s of Q and G, respectively. Under the conditions of Theorems 1 and 3, the distribution of G has a bounded density. Therefore, by Esseen's inequality, we have, for any $T > 0$,

$$\Delta(A, F) \le C_1 \int_0^T \frac{|f(t) - g(t)|}{t}\, dt + \frac{C_2}{T}. \tag{6.1}$$

See, for instance, Petrov (1975). To bound the right-hand side of (6.1) we need more notation. Denote by ζ_j, $j = 1, 2, \dots, n$, the i.i.d. Gaussian r.v.'s with covariance operator coinciding with that of ξ_k. Introduce the random variables $\mathscr{X}_k = \sum_{j=1}^{k} \xi_j + \sum_{j=k+1}^{n} \zeta_j$, $k = 1, \dots, n-1$ and $\mathscr{X}'_k = \sum_{j=1}^{k-1} \xi_j + \sum_{j=k+1}^{n} \zeta_j$. For $k = n$ we define $\mathscr{X}_n = \mathscr{X}$ and for $k = 0$, respectively, $\mathscr{X}_0 = \sum_{j \in \mathbf{N}} \zeta_j$. Let $Q'_k = \langle A\mathscr{X}'_k, \mathscr{X}'_k \rangle$.

Introduce the quadratic forms $Q_k = \langle A\mathscr{X}_k, \mathscr{X}_k \rangle$ with characteristic functions $f_k(t) = \mathbf{E}\exp\{itQ_k\}$. We have

$$g(t) - f(t) = \sum_{k \in \mathbf{N}} (f_{k-1}(t) - f_k(t)). \tag{6.2}$$

Consider now the difference $h_k(t) = (f_{k-1}(t) - f_k(t))$. It is easy to see that

$$\begin{aligned} Q_k - Q_{k-1} &= \langle A\mathscr{X}_k, \mathscr{X}_k \rangle - \langle A\mathscr{X}_{k-1}, \mathscr{X}_{k-1} \rangle \\ &= \langle A\xi_k, \mathscr{X}'_{k-1} \rangle - \langle A\zeta_k, \mathscr{X}'_{k-1} \rangle. \end{aligned} \tag{6.3}$$

Equality (6.3) implies that

$$\begin{aligned} h_k(t) = {} & \mathbf{E}\big(\exp\{2it\langle A\xi_k, \mathscr{X}'_{k-1}\rangle\} - 1\big)\exp\{itQ'_{k-1}\} \\ & - \mathbf{E}\big(\exp\{2it\langle A\zeta_k, \mathscr{X}'_{k-1}\rangle\} - 1\big)\exp\{itQ'_{k-1}\}. \end{aligned} \tag{6.4}$$

Since ξ_k and ζ_k are independent of $\mathscr{X}'_{k-1}$ and, hence of Q'_{k-1} as well, we can rewrite the equality (6.4) in the form

$$h_k(t) = h_k(t, \xi_k) - h_k(t, \zeta_k), \tag{6.5}$$

where

$$\begin{aligned} h_k(t, \eta) = {} & \mathbf{E}\Big(\exp\{2it\langle A\eta, \mathscr{X}'_{k-1}\rangle\} - 1 - 2it\langle A\eta, \mathscr{X}'_{k-1}\rangle - \tfrac{1}{2}\big(2it\langle A\eta, \mathscr{X}'_{k-1}\rangle\big)^2\Big) \\ & \times \exp\{itQ'_{k-1}\}. \end{aligned}$$

In the last formula we have used $\mathbf{E}\zeta_k^2 = \mathbf{E}\xi_k^2$. Now we shall derive a bound for $h_k(t, \eta)$. A bound for the second term can be proved in the same way. Let τ be a uniformly distributed r.v., on the unit interval, which is independent of all other r.v.'s. By Taylor expansion with integral remainder we have

$$\exp\{iu\} - 1 - iu - \tfrac{1}{2}(iu)^2 = \tfrac{1}{6}(iu)^3\mathbf{E}(1-\tau)^2\exp\{i\tau u\}.$$

Using this formula for $h_k(t, \eta)$, we obtain that

$$h_k(t, \eta) = \tfrac{1}{6}(it)^3\mathbf{E}(1-\tau)^2\langle A\eta, \mathscr{X}'_{k-1}\rangle^3\exp\{i\tau t\langle A\eta, \mathscr{X}_{k-1}\rangle + itQ'_{k-1}\}. \tag{6.6}$$

From Rosenthal's inequality it follows that, for $\eta = \xi_k$, or $\eta = \zeta_k$,

$$\max_{\substack{i=0,1\\ l=1,\dots,5}} \mathbf{E}\left|\langle A\eta, U_{il}\rangle\right|^3 \le C\beta_3^2\left(\sum_{j\in\mathbf{N}} a_{jk}^2\right)^{3/2} = C\beta_3^2\mathscr{L}_k^3. \tag{6.7}$$

Proof of Theorem 1. By Lemmas 4.1 and 4.2 we can choose a subset $J_0 \subset \mathbf{N}$ and a partition J_{1l}, $l = 1,\dots,5$, of $J_1 = \mathbf{N}\setminus J_0$ and

$$\min_{1\le l\le 5}\left\{\sum_{k\in\mathbf{N}} \lambda_k^2(J_0 J_{1l})\right\} \ge \tfrac{1}{16} - 5\sqrt{2}\mathscr{L}. \tag{6.8}$$

For any fixed J_{1l} we can choose a partition J_{0_j}, $j = 1,\dots,5$, such that

$$\min_{1\le m\le 5}\left\{\sum_{k\in\mathbf{N}} \lambda_k^2(J_{1l} J_{0m})\right\} \ge \tfrac{1}{128} - 5\sqrt{2}\mathscr{L}. \tag{6.9}$$

Let $B = (b_{jk})_{j,k\in\mathbf{N}}$ denote a symmetric matrix with eigenvalues μ_k, $k = 1,\dots,n$, which are nondecreasing in absolute value. Consider the quadratic form $G_b = \sum_{j,k\in\mathbf{N}} b_{jk}Y_jY_k$. For its ch.f. $\phi_b(t) = \mathbf{E}\exp\{itG_b\}$, we have the representation

$$\phi_b(t) = \exp\left\{i\int_0^t\left(\sum_{k\in\mathbf{N}}\frac{\mu_k}{1-2iu\mu_k}\right)du\right\}.$$

From this formula, it immediately follows that

$$|\phi_b(t)| \le \exp\left\{-\tfrac{3}{4}t^2\sum_{k\in\mathbf{N}}\mu_k^2\right\} + \mathbf{I}\left\{t > 2|\mu_1|^{-1}\right\}. \tag{6.10}$$

Relations (6.7)–(6.9), Lemmas 3.3, 3.4, 5.4, 5.5 and the inequality (6.10) together imply that

$$|h_k(t,\eta)| \le Ct^3 \max_{\substack{i=0,1\\ l=1,\dots,5}} \mathbf{E}\left|\langle A\eta, U_{il}\rangle\right|^3 r(t), \tag{6.11}$$

where

$$\begin{aligned} r(t) &= \exp\{-1/256\sigma^2(H)t^2\} + \mathbf{I}\left\{t > \gamma\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1}\right\}\\ &\quad + \theta_1\exp\left\{\frac{e}{4H}\right\}\mathscr{L}^{K/2H} + \theta_2\mathscr{L}^{K^2/4-1}. \end{aligned}$$

Note that

$$\sigma^2(H) \ge 1 - \beta_3/H, \qquad \mu_4(H) \le H\beta_3. \tag{6.12}$$

Choose $H = 2\beta_3$ and $K = 16H$. Then, $\gamma \ge (32\beta_3^2)^{-1}$, we obtain from (6.7) and (6.12),

$$\begin{aligned} |h_k(t)| &\le Ct^3\beta_3^2\mathscr{L}_k^3\Big(\exp\{-1/512t^2\} + \mathscr{L}^6\\ &\qquad + \mathbf{I}\left\{t \ge (32\beta_3\mathscr{L})^{-1}(\log^+\mathscr{L})^{-1}\right\}\Big). \end{aligned} \tag{6.13}$$

Applying Lemmas 3.2, 3.3, 5.1, 5.2 and inequalities (6.8) and (4.12) we obtain for $t \ge 0$,

$$|f(t)| \le \exp\left\{-\tfrac{1}{8}\alpha p(t \wedge T_0)^2\right\} + \mathbf{I}\{t > \gamma\mathscr{L}^{-1}\} + C(2-2p-\alpha)^{-3/2}p^{-3/2}\mathscr{L}^{3/2} + \mathscr{L}^{K/4H-1/2-p\varkappa/4} + p^{K/4H}\mathscr{L}^{-1/2}(\log^+\mathscr{L})^{-K/2H}.$$

Let $p = \mathscr{L}^{1/4}$ and $\alpha = 1$. Then,

$$|f(t)| \le \exp\left\{-\tfrac{1}{8}(t \wedge T_0)^2\mathscr{L}^{1/4}\right\} + \mathbf{I}\left\{t > \tfrac{1}{32}\beta_3^{-2}\mathscr{L}^{-1}\right\} + C\mathscr{L}^{9/8} + \mathbf{I}\left\{t > |\lambda_1|^{-1}\right\}, \tag{6.14}$$

and

$$|g(t)| \le \exp\left\{-\tfrac{1}{2}t^2\right\} + \mathbf{I}\left\{t > 2|\lambda_1|^{-1}\right\}. \tag{6.15}$$

Applying inequality (6.13) for $0 \le t \le (32\beta_3^2\mathscr{L})^{-1}(\log^+\mathscr{L})^{-1}$, inequality (6.15) for $(2\beta_3\mathscr{L})^{-1}(\log^+\mathscr{L})^{-1} \le t \le (32\beta_3^2\mathscr{L})^{-1}$, inequality (6.15) and (6.1) together complete the proof of Theorem 1. □

PROOF OF THEOREM 3. Similarly to (6.8) and (6.9), by Lemmas 3.1 and 3.2 we can choose a subset $J_0 \subset \mathbf{N}$ and a partition J_{1l}, $l = 1, \dots, 5$, of $J_1 = \mathbf{N} \setminus J_0$ such that

$$\min_{1 \le l \le 5}\left\{\left|\lambda_q(J_0 J_{1l})\right|\right\} \ge \frac{|\lambda_{*q}|}{2^{2(q+1)}} - 21\mathscr{L}. \tag{6.16}$$

For any fixed J_{1l} we can choose a partition J_{0j}, $j = 1, \dots, 5$, such that

$$\min_{1 \le m \le 5}\left\{\left|\lambda_q(J_{1l} J_{0m})\right|\right\} \ge \frac{|\lambda_{*q}|}{2^{3(q+1)}} - 21\mathscr{L}. \tag{6.17}$$

It is well known that

$$|\phi_b(t)| \le \min\left\{\frac{1}{\left(1 + 2\mu_q^2 t^2\right)^{q/4}}, \frac{1}{|\mu_{*q}| t^{q/2}}\right\}, \tag{6.18}$$

where $\mu_{*q} = \prod_{l=1}^{q}\mu_l$; for instance, Götze (1984), Lemma 5.48. Applying Lemmas 3.3, 3.4, 5.1, 5.4 and inequalities (6.16)–(6.18), we obtain from (6.2) that

$$|h_k(t)| \le C\beta_3^2\mathscr{L}_k^3 t^3\left(\frac{1}{\left(1 + (2\delta b_0 t)^2\right)^{q/8}} + \mathscr{L}^6\right) + \mathbf{I}\left\{t > C\beta_3^{-2}\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1}\right\}, \tag{6.19}$$

where δ is some small positive constant depending on q only. If $2 \leq q < 12$, inequality (6.19) implies that

$$\int_0^T \frac{dt}{t} |f(t)| \leq C\beta_3^2 \mathscr{L} T^{3-q/4}.$$

The last inequality and (6.1) with $T = C\beta_3^{-2}\mathscr{L}^{-1}(\log^+ \mathscr{L})^{-4/(4-q/4)}$ imply the required results. If $q = 12$, we get from (6.19)

$$\int_0^T \frac{dt}{t} |f(t)| \leq C\beta_3^2 \mathscr{L} \log^+ T.$$

Choosing $T = C\beta_3^{-2}\mathscr{L}^{-1}(\log^+ \mathscr{L})^{-1}$ and applying (6.1) we obtain the required result. In the case $q = 13$ we shall apply the following inequality for $C\beta_3^{-2}\mathscr{L}^{-1}(\log^+ \mathscr{L})^{-1} \leq t \leq C\beta_3^{-2}\mathscr{L}^{-1}$. Choose $K = 16qH$ and $p = \mathscr{L}^{1/4q}$. Then, Lemmas 3.2, 4.3, 5.1, 5.2 and 5.5 and inequalities (6.17) and (6.18) together imply that

$$|f(t)| \leq \frac{C}{\left(b_0 t \mathscr{L}^{1/4q}\right)^{q/8}} + \mathscr{L}^3 \leq C\mathscr{L}^{51/32}. \tag{6.20}$$

Note that

$$\int_0^t u^2 (1 + \alpha u)^{-13/4}\, du \leq C, \tag{6.21}$$

and

$$\sum_{k \in \mathbf{N}} \mathscr{L}_k^3 \leq \mathscr{L}. \tag{6.22}$$

The inequalities (6.19) and (6.20) complete the proof. □

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FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
33501 BIELEFELD 1
GERMANY
E-MAIL: goetze@mathematik.uni-bielefeld.de

FACULTY OF MATHEMATICS
RUSSIAN ACADEMY OF SCIENCES
OKTJABRSKYI PROSPEKT
167001, SYKTYVKAR
RUSSIA
E-MAIL tikhomir@ssu.edu.komi.ru