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A STOCHASTIC WAVE EQUATION IN TWO SPACE DIMENSION: SMOOTHNESS OF THE LAW

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We prove the existence and uniqueness, for any time, of a real-valued process solving a nonlinear stochastic wave equation driven by a Gaussian noise white in time and correlated in the two-dimensional space variable. We prove that the solution is regular in the sense of the Malliavin calculus. We also give a decay condition on the covariance function of the noise under which the solution has Hölder continuous trajectories and show that, under an additional ellipticity assumption, the law of the solution at any strictly positive time has a smooth density.

0. Introduction. In this paper we study the stochastic wave equation with two-dimensional spatial variable

$$
\begin{aligned}
\left(\frac{\partial^2}{\partial t^2} - \Delta\right)u(t,x) &= \sigma(u(t,x))F(dt,dx) + b(u(t,x)),\\
u(0,x) &= u_0(x),\\
\frac{\partial u}{\partial t}(0,x) &= v_0(x).
\end{aligned} \tag{0.1}
$$

We are interested in solutions which are real-valued stochastic processes and want to establish sufficient conditions ensuring the existence and smoothness of density for the law of the solution $u(t,x)$ for fixed $t>0$, $x\in\mathbf{R}^2$. It is well known (see, for instance, [13]) that, unlike for the one-dimensional spatial variable studied in [3] our requirements on the process u exclude $F(t,x)$ from being a time–space white noise (see [1] for a different approach). This is because the fundamental solution of the wave equation becomes less smooth as the dimension increases. In [8] the noise $F(t,x)$ is assumed to be a generalized Gaussian field with covariance

$$
E(F(t,x)F(s,y)) = \delta(t-s)f(|x-y|), \tag{0.2}
$$

where δ denotes the Dirac delta function and $f\colon \mathbf{R}_+\to\mathbf{R}_+$ is bounded. In their recent paper [4], Dalang and Frangos have weakened the conditions on the covariance function. The noise $F(t,x)$ denotes a martingale measure defined by an extension of a generalized centered Gaussian field $(F(\varphi);\varphi\in$

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$\mathscr{D}(\mathbf{R}_+ \times \mathscr{R}^2))$ with covariance functional J defined by

$$J(\varphi, \psi) = \int_{\mathbf{R}_+} dt \int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dy\, \varphi(t, x) f(|x-y|) \psi(t, y). \tag{0.3}$$

They assume that f is continuous on $]0, \infty[$, satisfies $\int_0^{r_0} rf(r)\,dr < \infty$ for some $r_0 > 0$ and, in addition, that the functional J: $\mathscr{D}(\mathbf{R}_+ \times \mathbf{R}^2) \times \mathscr{D}(\mathbf{R}_+ \times \mathbf{R}^2) \to \mathbf{R}$ is positive definite. These properties assure the existence of a Gaussian process $F(t, x)$ satisfying (0.2). Using Walsh's theory on stochastic integration with respect to martingale measures developed in [13], these authors give a rigorous formulation of equation (0.1) as an evolution equation. Their main concern has been to state results on the existence and uniqueness of solution of equation (0.1). In [8] this has been done for $b \equiv 0$ under two different type of assumptions:

(1) $|\sigma(y)| \le C(|y|+1) \log(|y|+2)^{\alpha}$, $0 < \alpha < \frac{1}{2}$, and σ locally Lipschitz.
(2) σ Lipschitz and bounded.

The approach of [4] allows considering unbounded functions f, such as $f(r) = r^{-\alpha}$, $0 < \alpha < 2$. The main result claims that, assuming σ and b Lipschitz, the condition

$$G(r_0) := \int_0^{r_0} rf(r) \ln\left(\frac{1}{r}\right) dr < +\infty \quad \text{for some } r_0 > 0 \tag{0.4}$$

yields the existence of a unique jointly measurable, L^2-continuous real-valued process $\{u(t, x); t \in [0, t_0], x \in \mathbf{R}^2\}$ solution to (0.1) up to time t_0, where t_0 is some positive and finite real number depending on f. If the coefficient $\sigma(\cdot)$ is a constant function, then (0.4) also provides a necessary condition for the existence and uniqueness of the solution and, in this case, $t_0 = \infty$. Finally, a decay assumption on $G(r_0)$ as $r_0 \to 0$ provides a Hölder continuous version of the solution. Some related work in the case of a d-dimensional spatial variable, for any $d \ge 1$, a constant $\sigma(\cdot)$ and $b \equiv 0$, has been developed independently in [6]. Their formulation uses stochastic equations in infinite dimensions and the main tool is the Fourier transform.

In Section 1 of this paper we improve Dalang's and Frangos's local result, showing that (0.4) is a sufficient condition for the existence and uniqueness of a solution $u(t, x)$ to (0.1), for any $t > 0$, $x \in \mathbf{R}^2$. We also prove a sharper result for the Hölder continuity of $u(\cdot, \cdot)$ under condition (1.19) on f, which is slightly weaker than the integrability hypothesis made in [4]. The cornerstone for avoiding locality is given by Lemma A1, which gives precise lower and upper bounds of the integral

$$\int_0^t ds \int_{|x|<s} dx \int_{|y|<s} dy \frac{1}{\sqrt{s-|x|^2}} f(|x-y|) \frac{1}{\sqrt{s-|y|^2}}, \qquad t > 0. \tag{0.5}$$

If σ is bounded away from zero, the condition (0.4) turns out to be necessary. Sections 2 and 3 are devoted to the proof of the existence and smoothness of the density of the law of $u(t, x)$, $t > 0$, $x \in \mathbf{R}^2$, using the Malliavin calculus. Our framework is as follows.

Let $\mathscr{E}$ denote the inner product space of measurable functions φ: $\mathbf{R}^2 \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dy |\varphi(x)| f(|x-y|) |\varphi(y)| < \infty,$$

endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathscr{E}} := \int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dy \varphi(x) f(|x-y|) \psi(y),$$

and let $\mathscr{H}$ denote the completion of $\mathscr{E}$. Set $\mathscr{H}_T = L^2([0,T]; \mathscr{H})$; notice that $\mathscr{H}$ and $\mathscr{H}_T$ need not be spaces of functions. The space $\mathscr{H}_T$ is a real Hilbert separable space isomorphic to the reproducing kernel Hilbert space of the centered Gaussian noise $(F(\varphi); \varphi \in \mathscr{D}([0,T]\times\mathbf{R}^2))$, which can be identified with a Gaussian process $(W(h), h \in \mathscr{H}_T)$ as follows. Let $(e_j; j \geq 0) \subset \mathscr{E}$ be a CONS of $\mathscr{H}$; then $(W_j(t) = \int_0^t \int_{\mathbf{R}^2} e_j(x) F(ds, dx); j \geq 0)$ is a sequence of independent standard Brownian motions such that

$$F(\varphi) = \sum_{j\geq 0} \int_0^T \langle \varphi(s, *), e_j \rangle_{\mathscr{H}} \, dW_j(s), \qquad \varphi \in \mathscr{D}([0,T] \times \mathbf{R}^2).$$

For $h \in \mathscr{H}_T$, set

$$W(h) = \sum_{j\geq 0} \int_0^T \langle h(s), e_j \rangle_{\mathscr{H}} \, dW_j(s).$$

Therefore, we can use the framework of the Malliavin Calculus described in [10] (see also [9] and [14]). The smoothness of the density requires non-degeneracy conditions for the coefficient σ and for the integral (0.5). Using again Lemma A1, the latter can be formulated in terms of an integrability condition on the function f (see (3.1)). The assumption on σ could possibly be relaxed using a Taylor expansion.

The Appendix contains technical results used in the paper. We usually denote all constants by C, independently of their values. We only make their dependence explicit if it is either important or enlightening for the argument.

1. The solution to the wave equation. Let $F(t,x)$ be a centered Gaussian noise in $\mathbf{R}_+ \times \mathbf{R}^2$ with covariance given by (0.2). As shown in [4], such a noise can be defined on $\mathscr{D}(\mathbf{R}_+ \times \mathbf{R}^2)$ and then extended to bounded measurable subsets of $\mathbf{R}_+ \times \mathbf{R}^2$ to become a martingale measure

$$M_t(A) = F([0,t] \times A),$$

$t \geq 0$, $A \in \mathscr{B}(\mathbf{R}^2)$ (see [13]) for the filtration $\mathscr{F}_t$ defined by

$$\mathscr{F}_t = \sigma(F([0,s] \times A); 0 \leq s \leq t, A \in \mathscr{B}(\mathbf{R}^2)).$$

We assume that the function f: $]0, +\infty[\to \mathbf{R}_+$ is continuous and satisfies

$$\int_0^{r_0} r f(r) \, dr < +\infty, \tag{1.1}$$

for some $r_0 > 0$. In addition we suppose that the functional $J(\cdot, \cdot)$ defined by (0.3) is nonnegative definite. Consider the stochastic wave equation defined in (0.1). We assume that $(\partial u/\partial t)(0, x)$ is a measure with density $v_0(x)$ and solve (0.1) in terms of an evolution equation as follows. Let

$$S(t, x) = \frac{1}{2\pi}(t^2 - |x|^2)^{-1/2} 1_{\{|x|<t\}} \tag{1.2}$$

and consider

$$\begin{aligned} u(t, x) = & \int_{\mathbf{R}^2} S(t, x - y)\, v_0(y)\, dy + \frac{\partial}{\partial t}\left(\int_{\mathbf{R}^2} S(t, x - y)\, u_0(y)\, dy\right) \\ & + \int_0^t \int_{\mathbf{R}^2} S(t - s, x - y) \\ & \qquad \times \left[\sigma(u(s, y))F(ds, dy) + b(u(s, y))\, ds\, dy\right]. \end{aligned} \tag{1.3}$$

The stochastic integral in (1.3) is defined with respect to the $\mathscr{F}_t$-martingale measure M_t (see [13]).

Our first purpose is to establish the existence and uniqueness of a solution to (1.3). A local existence result ($t \in [0, t_0]$ for some $t_0 > 0$) has been proved by Dalang and Frangos in [4], Theorem 2 when $u_0 = v_0 = 0$. A natural way to give a rigorous meaning to (0.1) is by means of its weak formulation, as follows. Let φ be a $\mathscr{C}^2$ function with compact support included in $[0, T] \times \mathbf{R}^2$. Multiplying the first equation in (0.1) by φ and integrating by parts on $[0, T] \times \mathbf{R}^2$ yields

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi\right)(t, x) u(t, x)\, dt\, dx \\ & \quad + \int_{\mathbf{R}^2} \left(\frac{\partial \varphi}{\partial t}(0, x) u_0(x) - \varphi(0, x) v_0(x)\right) dx \\ & \quad = \int_0^T \int_{\mathbf{R}^2} \varphi(t, x)\left[\sigma(u(t, x))\, F(dt, dx) + b(u(t, x))\, dt\, dx\right]. \end{aligned} \tag{1.4}$$

This leads to the following notion.

DEFINITION 1.1. *A stochastic process* $\{u(t, x), (t, x) \in \mathbf{R}_+ \times \mathbf{R}^2\}$ *is said to be a weak solution of* (0.1) *if it is measurable*, *adapted to* $\{\mathscr{F}_t,\ t \geq 0\}$ *and satisfies* (1.4) *for any* $\mathscr{C}^2$ *function* φ *with compact support included in* $[0, T] \times \mathbf{R}^2$, *for some* $T > 0$.

We prove in this section (see Theorem 1.2) that (1.3) has a unique solution u and give sufficient conditions on f for the trajectories of u to be Hölder continuous. Then, it is easy to check that this is also a weak solution. Indeed, for any $t \in \mathbf{R}_+$, $x \in \mathbf{R}^2$, set

$$X(t, x) = \int_{\mathbf{R}^2} S(t, x - y)\, v_0(y)\, dy + \frac{\partial}{\partial t}\left(\int_{\mathbf{R}^2} S(t, x - y)\, u_0(y)\, dy\right)$$

and let φ be a $\mathscr{C}^2$ function with compact support included in $[0, T] \times \mathbf{R}^2$. Then

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi \right)(t, x) X(t, x)\, dt\, dx \\
(1.5) \qquad &\qquad + \int_{\mathbf{R}^2} \left(\frac{\partial \varphi}{\partial t}(0, x)\, u_0(x) - \varphi(0, x) v_0(x) \right) dx = 0.
\end{aligned}$$

In fact, it is well known (see, for instance, [15]) that $X(t, x)$ is solution of the wave equation

$$\frac{\partial^2 X}{\partial t^2} - \Delta X = 0,$$

with initial condition $X(0, x) = u_0(x)$, $(\partial/\partial t) X(0, x) = v_0(x)$, $x \in \mathbf{R}^2$. An integration by parts yields

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi \right)(t, x) X(t, x)\, dt\, dx \\
&\quad = \int_0^T \int_{\mathbf{R}^2} \left(\frac{\partial^2 X}{\partial t^2} - \Delta X \right)(t, x) \varphi(t, x)\, dt\, dx \\
&\qquad + \int_{\mathbf{R}^2} \left[\varphi(0, x) \frac{\partial X}{\partial t}(0, x) - \frac{\partial \varphi}{\partial t}(0, x)\, X(0, x) \right] dx,
\end{aligned}$$

which proves (1.5). We also have

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi \right)(t, x) \\
&\qquad \times \left[\int_0^t \int_{\mathbf{R}^2} S(t - s, x - y) \right. \\
(1.6) \qquad &\qquad\qquad \left. \times \left\{ \sigma(u(s, y))\, F(ds, dy) + b(u(s, y))\, ds\, dy \right\} \right] dt\, dx \\
&= \int_0^T \int_{\mathbf{R}^2} \varphi(s, y) \left\{ \sigma(u(s, y))\, F(ds, dy) + b(u(s, y))\, ds\, dy \right\}.
\end{aligned}$$

Indeed, by a stochastic Fubini theorem (see, e.g., [13], Theorem 2.6), the left-hand side of (1.6) equals

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^2} \left(\int_s^T \int_{|x-y|<t-s} S(t - s, x - y) \left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi \right)(t, x)\, dt\, dx \right) \\
&\qquad \times \left\{ \sigma(u(s, y))\, F(ds, dy) + b(u(s, y))\, ds\, dy \right\}.
\end{aligned}$$

Since $S(\cdot, \cdot)$ is the Green function of the wave equation, that means

$$\frac{\partial^2 S}{\partial t^2} - \Delta S = \delta,$$

where δ is the Dirac delta distribution at $(0, 0)$ it follows that

$$\int_s^T \int_{|x-y|<t-s} S(t-s, x-y)\left(\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi\right)(t, x)\, dt\, dx = \varphi(s, y).$$

This proves (1.6). The identities (1.5) and (1.6) show that the solution of (1.3) is also a solution in the sense given in Definition 1.1. The next theorem shows the existence and uniqueness of a solution for (1.3). Note that, unlike [4], we obtain a global result.

THEOREM 1.2. *Let* $u_0\colon \mathbf{R}^2 \to \mathbf{R}$ *be of class* $\mathscr{C}^1$ *and bounded,* $v_0\colon \mathbf{R}^2 \to \mathbf{R}$ *such that* $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbf{R}^2)$, *for some* $q_0 \in]2, \infty]$, *and* $\sigma, b\colon \mathbf{R} \to \mathbf{R}$ *be globally Lipschitz functions. We assume that the function* f *associated with the noise* F *satisfies*

$$\int_0^{r_0} r f(r) \ln\left(\frac{1}{r}\right) < +\infty, \tag{1.7}$$

for some $r_0 > 0$. *Then equation* (1.3) *has a unique solution. Moreover, for any* $T > 0$, $p \in [1, \infty)$,

$$\sup_{x\in\mathbf{R}^2} \sup_{0\le t\le T} E(|u(t, x)|^p) < +\infty. \tag{1.8}$$

PROOF. Consider the Picard iteration scheme

$$\begin{aligned} u^0(t, x) &= \int_{\mathbf{R}^2} S(t, x-y) v_0(y)\, dy \\ &\quad + \frac{\partial}{\partial t}\left(\int_{\mathbf{R}^2} S(t, x-y) u_0(y)\, dy\right), \\ u^{n+1}(t, x) &= u^0(t, x) + \int_0^t \int_{\mathbf{R}^2} S(t-s, x-y) \\ &\quad \times \left[\sigma(u^n(s, y)) F(ds, dy) + b(u^n(s, y))\, ds\, dy\right], \qquad n \ge 0. \end{aligned} \tag{1.9}$$

We at first prove that given $t > 0$, $2 \le p < \infty$,

$$\sup_{n\ge 0} \sup_{x\in\mathbf{R}^2} \sup_{0\le s\le t} E(|u^n(s, x)|^p) \le C < +\infty, \tag{1.10}$$

where C is a constant depending on p, t, the L^{q_0}-norm of $|v_0|$ and $|\nabla u_0|$, the supremum norm of u_0 and the Lipschitz constants of σ and b. Indeed, let $1/q_0 + 1/q = 1$ and set $y - x = s\xi$. Then Hölder's inequality yields

$$\begin{aligned} &\left|\int_{\mathbf{R}^2} S(s, x-y)\, v_0(y)\, dy\right| \\ &\quad = \frac{1}{2\pi}\left|\int_{|x-y|<s} (s^2 - |x-y|^2)^{-1/2} v_0(y)\, dy\right| \\ &\quad \le C\|v_0\|_{q_0}\left(\int_{|\xi|<1} s^{2-q}(1-|\xi|^2)^{-q/2}\, d\xi\right)^{1/q} \le C\|v_0\|_{q_0} s^{(2-q)/q}. \end{aligned} \tag{1.11}$$

If $q_0 = +\infty$, $|\int_{\mathbf{R}^2} S(s, x-y)\, v_0(y)\, dy| \le \|v_0\|_\infty s$.

We have $|(\partial/\partial s)(\int_{\mathbf{R}^2} S(s, x-y)u_0(y)\,dy)| \leq C(A_1 + A_2)$, with

$$A_1 = \left|\int_{|\xi|<1} (1-|\xi|^2)^{-1/2} u_0(x+s\xi)\,d\xi\right| \leq C\|u_0\|_\infty$$

and

$$A_2 = \left|\int_{|\xi|<1} s(1-|\xi|^2)^{-1/2} \frac{\partial}{\partial s}\big(u_0(x+s\xi)\big)\,d\xi\right|.$$

Then, as before,

$$\begin{aligned} A_2 &\leq s\int_{|\xi|<1} (1-|\xi|^2)^{-1/2} \big|\nabla u_0(x+s\xi)\big|\,d\xi \\ &= \int_{|x-y|<s} \big|\nabla u_0(y)\big| \big(s^2 - |x-y|^2\big)^{-1/2}\,dy \qquad (1.12) \\ &\leq C\|\nabla u_0\|_{q_0} s^{(2-q)/q}, \end{aligned}$$

for finite q_0. If $q_0 = +\infty$, the right-hand side of (1.12) is replaced by $\|\nabla u_0\|_{q_0} s$. Therefore, (1.11) and (1.12) yield $\sup_{x\in\mathbf{R}^2}\sup_{0\leq s\leq t}|u^0(s,x)| \leq C$ for some positive constant C depending only on t, the supremum norm of u_0 and the L^{q_0}-norm of v_0, ∇u_0. Let $n \geq 0$. Then

$$E\big(|u^{n+1}(t,x)|^p\big) \leq C\big(1 + A_p^n(t,x) + B_p^n(t,x)\big),$$

where

$$A_p^n(t,x) = E\left(\left|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\sigma(u^n(s,y))F(ds,dy)\right|^p\right),$$

$$B_p^n(t,x) = E\left(\left|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)b(u^n(s,y))\,ds\,dy\right|^p\right).$$

Let $J(s)$ and μ_t be defined by (A.1) and (A.2), respectively. Then, Burkholder's inequality and Hölder's inequality applied to integrals with respect to the measure $S(t-s, x-y)f(|y-y'|)S(t-s, x-y')\,ds\,dy\,dy'$ yield

$$\begin{aligned} A_p^n(t,x) \leq CE\Bigg(\Bigg|\int_0^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy'\, & S(t-s, x-y)\sigma(u^n(s,y))f(|y-y'|) \\ & \times \sigma(u^n(s,y'))\,S(t-s, x-y')\Bigg|^{p/2}\Bigg) \qquad (1.13) \\ \leq C(\mu_t)^{(p/2)-1} \int_0^t \Bigg(1 + \sup_{x\in\mathbf{R}^2} & \sup_{0\leq r\leq s} E(|u^n(r,x)|^p)\Bigg) J(t-s)\,ds. \end{aligned}$$

Let ν_t be defined by (A.3). Hölder's inequality applied to integrals with respect to the measure $S(t-s, x-y)\,ds\,dy$ implies

$$B_p^n(t,x) \leq C(\nu_t)^{p-1} \int_0^t \left(1 + \sup_{x\in\mathbf{R}^2}\sup_{0\leq r\leq s} E\big(|u^n(r,x)|^p\big)\right) ds. \qquad (1.14)$$

We prove in the Appendix (see Remark A2) that assumption (1.7) implies $\sup_{0\le t\le T}|J(t)| \le C_T$, for any $T > 0$, for some finite constant C_T depending only on T. Thus, (1.13) and (1.14) show (1.10), by Gronwall's lemma. Using the Lipschitz property of σ and b, a similar computation implies for $0 \le t \le T$, $n \ge 1$,

$$\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t} E\big(|u^{n+1}(s,x)-u^n(s,x)|^p\big)$$
$$\le C\big\{(\mu_T)^{(p/2)-1}+(\nu_T)^{p-1}\big\}\int_0^t \sup_{x\in\mathbf{R}^2}\sup_{0\le r\le s} E\big(|u^n(r,x)-u^{n-1}(r,x)|^p\big)\,ds.$$

Moreover, (1.9) and (1.10) yield

$$\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t} E\big(|u^1(s,x)-u^0(s,x)|^p\big)\le C.$$

Therefore, the sequence $\{u^n(t,x), n\ge 0\}$ converges in L^p uniformly in $x\in\mathbf{R}^2$, $0\le t\le T$. Set $u(t,x) = L^p - \lim_{n\to\infty} u^n(t,x)$. It is easy to see that $u(t,x)$ satisfies (1.3), (1.8) and

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t} E\big(|u^n(s,x)-u(s,x)|^p\big)=0, \tag{1.15}$$

which proves the existence of a solution. The existence of a jointly measurable version of u continuous in L^p has been proved in [4] in the case $u_0 = v_0 = 0$, but it can be easily extended to our setting. Uniqueness is checked by standard arguments. □

REMARK 1.3. Let u_0, v_0: $\mathbf{R}^2\to\mathbf{R}$ be as in Theorem 1.2. Let σ, b: $\mathbf{R}\to\mathbf{R}$ be such that

$$|b(x)|\le C_1(1+|x|),\qquad |\sigma(x)|\ge C_2>0,\ \forall\, x\in\mathbf{R}.$$

Assume that there exists a measurable process $(u(t,x); t\in[0,T], x\in\mathbf{R}^2)$ solution of (1.3) [hence for which the stochastic integral in (1.3) is defined] and such that $\sup_{x\in\mathbf{R}^2}\sup_{0\le t\le T} E(|u(t,x)|^2)<\infty$; then (1.7) holds. Indeed, if u^0 is defined as in the proof of Theorem 1.2, $\sup_{x\in\mathbf{R}^2}\sup_{0\le t\le T} u^0(t,x)<\infty$. The growth assumption on b and (A.3) imply that if

$$A(t,x)=E\bigg(\bigg|\int_0^t\int_{\mathbf{R}^2} S(t-s,x-y)\sigma(u(s,y))\,F(ds,dy)\bigg|^2\bigg),$$

$\sup_{x\in\mathbf{R}^2}\sup_{0\le t\le T} A(t,x)<\infty$. The isometry property of the stochastic integral and the lower estimate on σ imply

$$A(t,x)\ge C_2^2\int_0^t ds\int_{\mathbf{R}^2}dy\int_{\mathbf{R}^2}dz\,S(t-s,x-y)f(|y-z|)S(t-s,x-z)$$
$$=C_2^2\mu_t,$$

with μ_t defined by (A.2). Thus, (1.7) is a consequence of the lower bound of μ_t proved in Lemma A1(b).

We conclude this section by analyzing the path regularity of $u(t,x)$. The next result, which is crucial in Section 3, extends and improves similar estimates proved by Dalang and Frangos [4] when $\sigma = 1$, $u_0 = v_0 \equiv 0$.

PROPOSITION 1.4. *Suppose that there exists* $\beta \in (0,1)$ *and* $r_0 > 0$ *such that*

$$\int_0^{r_0} r^{1-\beta} f(r)\, dr < \infty. \tag{1.16}$$

Let $u_0\colon \mathbf{R}^2 \to \mathbf{R}$ *be of class* $\mathscr{C}^1$ *and bounded, with* $\beta/2(1+\beta)$*-Hölder continuous partial derivatives,* $v_0\colon \mathbf{R}^2 \to \mathbf{R}$ *be such that* $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbf{R}^2)$ *for some* $q_0 \in]4, +\infty]$, $\sigma, b\colon \mathbf{R} \to \mathbf{R}$ *be globally Lipschitz functions. Then for any compact set* $K \subset \mathbf{R}^2$, *for every* $T > 0$, $p \in [2,\infty)$, $0 \le t' \le t \le T$, $x, x' \in K$, $0 < \gamma < \beta/2(1+\beta)$,

$$E\big(|u(t,x) - u(t',x)|^p\big) \le C|t-t'|^{\gamma p}, \tag{1.17}$$

$$E\big(|u(t,x) - u(t,x')|^p\big) \le C|x-x'|^{\gamma p}. \tag{1.18}$$

Therefore, the trajectories of u *are* γ*-Hölder continuous in* $(t,x) \in [0,T] \times K$ *for* $\gamma \in [0, \beta/2(1+\beta)[$.

REMARK 1.5. (i) The integrability condition (1.16) implies (1.7). Hence the assumptions on f, u_0 and v_0 are stronger than in Theorem 1.2.

(ii) The proof of the Proposition shows that (1.17) and (1.18) hold if one replaces (1.16) by the following weaker technical assumption: There exists $\beta \in (0,1)$, $r_0 > 0$ such that for $0 < t < r_0$,

$$\int_0^t r f(r) \ln\left(1 + \frac{t}{r}\right) dr \le C t^{\beta}. \tag{1.19}$$

This will be used in the next section.

PROOF OF PROPOSITION 1.4. We at first prove (1.17). For $0 \le t' \le t \le T$ with $t - t' \le \frac{1}{2}$, $x \in K$, we write

$$E\big|u(t,x) - u(t',x)\big|^p \le C \sum_{i=1}^4 R_i, \tag{1.20}$$

with

$$R_1 = \left| \int_{\mathbf{R}^2} (S(t, x-y) - S(t', x-y)) v_0(y)\, dy \right|^p,$$

$$R_2 = \left| \frac{\partial}{\partial s}\left(\int_{\mathbf{R}^2} S(s, x-y)\, u_0(y)\, dy \right)_{s=t} - \frac{\partial}{\partial s}\left(\int_{\mathbf{R}^2} S(s, x-y) u_0(y)\, dy \right)_{s=t'} \right|^p,$$

$$\begin{aligned} R_3 = E\bigg| &\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\sigma(u(s,y)) F(ds,dy) \\ &- \int_0^{t'} \int_{\mathbf{R}^2} S(t'-s, x-y)\sigma(u(s,y)) F(ds,dy) \bigg|^p, \end{aligned}$$

$$R_4 = E\left|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y) b(u(s,y))\, ds\, dy \right.$$
$$\left. - \int_0^{t'} \int_{\mathbf{R}^2} S(t'-s, x-y) b(u(s,y))\, ds\, dy \right|^p.$$

We have $R_1 \le C(R_{11}^p + R_{12}^p)$, where

$$R_{11} = \left| \int_{|x-y|<t'} (S(t, x-y) - S(t', x-y)) v_0(y)\, dy \right|,$$
$$R_{12} = \left| \int_{t' \le |x-y| < t} S(t, x-y) v_0(y)\, dy \right|.$$

Let q be the conjugate exponent of q_0. Since $q_0 > 4$, $1 < q < \frac{4}{3}$ and for $0 \le t' < t \le T$, using (A.22), we have

$$\begin{aligned} R_{11} &\le C \int_{|z|<t'} |v_0(x+z)| \left(\frac{1}{\sqrt{t'^2 - |z|^2}} - \frac{1}{\sqrt{t^2 - |z|^2}} \right) dz \\ &\le C \|v_0\|_{q_0} \left(\int_{|z|<t'} \left| \frac{1}{\sqrt{t'^2 - |z|^2}} - \frac{1}{\sqrt{t^2 - |z|^2}} \right|^q dz \right)^{1/q} \\ &\le C \|v_0\|_{q_0} |t-t'|^{1/q - 1/2} \\ &\le C \|v_0\|_{q_0} |t-t'|^{\beta/(2(1+\beta))}, \end{aligned}$$

because $\beta/2(1+\beta) < 1/4 < 1/q - 1/2$. Furthermore,

$$\begin{aligned} R_{12} &\le C \|v_0\|_{q_0} \left(\int_{t'}^t \frac{\rho\, d\rho}{(t^2 - \rho^2)^{q/2}} \right)^{1/q} \\ &\le C \|v_0\|_{q_0} |t^2 - t'^2|^{1/q - 1/2} \le C \|v_0\|_{q_0} |t-t'|^{\beta/(2(1+\beta))} \end{aligned}$$

and hence

$$R_1 \le C \|v_0\|_{q_0}^p |t-t'|^{p(\beta/2(1+\beta))}. \tag{1.21}$$

An obvious change of variables implies $R_2 \le C(R_{21}^p + R_{22}^p + R_{23}^p)$, where

$$R_{21} = \left| \int_{|\xi|<1} \frac{t'}{\sqrt{1-|\xi|^2}} \left(\frac{\partial}{\partial s}(u_0(x+s\xi))_{s=t} - \frac{\partial}{\partial s}(u_0(x+s\xi))_{s=t'} \right) d\xi \right|,$$
$$R_{22} = \left| \int_{|\xi|<1} \frac{1}{\sqrt{1-|\xi|^2}} \Big(u_0(x+t\xi) - u_0(x+t'\xi) \Big) d\xi \right|,$$
$$R_{23} = \left| \int_{|\xi|<1} \frac{1}{\sqrt{1-|\xi|^2}} (t-t') \frac{\partial}{\partial s} \big(u_0(x+s\xi) \big)_{s=t} d\xi \right|.$$

Clearly,

$$
\begin{aligned}
R_{21} &\le \int_{|\xi|<1} \frac{1}{\sqrt{1-|\xi|^2}} \big|\nabla u_0(x+t\xi) - \nabla u_0(x+t'\xi)\big|\,|\xi|\,d\xi \\
&\le C|t-t'|^{\beta/(2(1+\beta))} \int_{|\xi|<1} \frac{|\xi|^{1+\beta/(2(1+\beta))}}{\sqrt{1-|\xi|^2}}\,d\xi \qquad (1.22)\\
&\le C|t-t'|^{\beta/(2(1+\beta))}.
\end{aligned}
$$

Moreover, using the estimate (1.12) we obtain

$$R_{22} + R_{23} \le C|t-t'|.$$

Thus,

$$R_2 \le C|t-t'|^{p(\beta/2(1+\beta))}. \qquad (1.23)$$

Clearly, $R_3 \le C(R_{31} + R_{32} + R_{33})$, where

$$
\begin{aligned}
R_{31} &= E\left|\int_{t'}^{t}\int_{\mathbf{R}^2} S(t-s, x-y)\,\sigma(u(s,y))\,F(ds,dy)\right|^p, \\
R_{32} &= E\left|\int_0^{t'}\int_{|x-y|<t'-s} \big(S(t'-s, x-y) - S(t-s, x-y)\big)\right. \\
&\qquad\qquad \left. \times\, \sigma(u(s,y))F(ds,dy)\right|^p, \\
R_{33} &= E\left|\int_0^{t'}\int_{t'-s<|x-y|<t-s} S(t-s,x-y)\sigma(u(s,y))F(ds,dy)\right|^p.
\end{aligned}
$$

Burkholder's and Hölder's inequalities, (1.8) and the property $\sup_{0\le s\le T} J(s) = C_T < \infty$ (see Remark A2) yield

$$
\begin{aligned}
R_{31} &= CE\left(\left|\int_{t'}^{t} ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy'\, S(t-s,x-y)\,\sigma(u(s,y))\,f(|y-y'|)\right.\right. \\
&\qquad\qquad \left.\left. \times\, \sigma(u(s,y'))\,S(t-s,x-y')\right|^{p/2}\right) \\
&\le C(\mu_{t-t'})^{p/2}\Big(1 + \sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t} E\big|u(s,x)\big|^p\Big) \qquad (1.24)\\
&\le C\left(\int_{t'}^{t} J(s)\,ds\right)^{p/2} \\
&\le C(t-t')^{p/2}.
\end{aligned}
$$

For $t > 0$, $h \geq 0$ set

$$\mu_{t,h} = \int_0^t ds \int_{|y|<s} dy \int_{|y'|<s} dy' \big[S(s,y) - S(s+h,y)\big] \times f\big(|y-y'|\big)\big[S(s,y') - S(s+h,y')\big],$$

$$\tilde{\mu}_{t,h} = \int_0^t ds \int_{s<|y|<s+h} dy \int_{s<|y'|<s+h} dy'\, S(s+h,y) f\big(|y-y'|\big) S(s+h,y').$$

Given $a < \beta$, (1.16) yields that $\int_0^t r f(r) \ln(1+t/r)\, dr \leq C t^a$ for $t \leq r_0$. Hence, Lemma A5 yields that for $\gamma \in [0, \beta/2(1+\beta)[$, $0 \leq t \leq r_0$, $0 \leq h \leq 1/2$,

$$\mu_{t,h} + \tilde{\mu}_{t,h} \leq C h^{2\gamma}.$$

Burkholder's and Hölder's inequalities imply

$$R_{32} \leq C\big(\mu_{t',t-t'}\big)^{p/2},$$
$$R_{33} \leq C\big(\tilde{\mu}_{t',t-t'}\big)^{p/2},$$

and hence for $\gamma \in [0, \beta/2(1+\beta)[$, $|t-t'| \leq 1/2$,

$$R_3 \leq C\big(|t-t'|^{p/2} + |t-t'|^{p\gamma}\big) \leq C|t-t'|^{p\gamma}. \tag{1.25}$$

Finally, $R_4 \leq C \sum_{j=1}^3 R_{4j}$ where R_{4j} is the analogue of R_{3j}, $j = 1,2,3$, with the coefficient σ replaced by b and the integrator $F(ds,dy)$ replaced by $ds\,dy$. Hölder's inequality implies

$$R_{41} \leq C(\nu_t - \nu_{t'})^p \leq C(t-t')^{2p},$$
$$R_{42} \leq C(\nu_{t',t-t'})^p,$$
$$R_{43} \leq C(\tilde{\nu}_{t',t-t'})^p,$$

where, for any t, $h \geq 0$, ν_t has been defined in (A.3) and

$$\nu_{t,h} = \int_0^t ds \int_{|y|<s} dy \big(S(s,y) - S(s+h,y)\big),$$

$$\tilde{\nu}_{t,h} = \int_0^t ds \int_{s<|y|<s+h} dy\, S(s+h,y).$$

An explicit integration yields $\nu_{t,h} + \tilde{\nu}_{t,h} \leq C h^{1/2}$. Consequently,

$$R_4 \leq C(t-t')^{p/2}. \tag{1.26}$$

The inequalities (1.21)–(1.26) yield (1.17). We now prove (1.18). For $t > 0$, $x, x' \in K$ with $|x-x'| \leq \frac{1}{2}$, let

$$E\big|u(t,x) - u(t,x')\big|^p \leq C \sum_{i=1}^4 U_i,$$

with

$$U_1 = \left| \int_{\mathbf{R}^2} \big(S(t, x-y) - S(t, x'-y)\big) v_0(y)\, dy \right|^p,$$

$$U_2 = \left| \frac{\partial}{\partial t} \left[\int_{\mathbf{R}^2} \big(S(t, x-y) - S(t, x'-y)\big) u_0(y)\, dy \right] \right|^p,$$

$$U_3 = E\left| \int_0^t \int_{\mathbf{R}^2} (S(t-s, x-y) - S(t-s, x'-y)) \sigma(u(s, y)) F(ds, dy) \right|^p,$$

$$U_4 = E\left| \int_0^t \int_{\mathbf{R}^2} \big(S(t-s, x-y) - S(t-s, x'-y)\big)\, b\big(u(s, y)\big)\, ds\, dy \right|^p.$$

Then, $U_1 \le C(U_{11}^p + U_{12}^p + U_{13}^p)$ where, for $\xi = x' - x$,

$$U_{11} = \left| \int_{|z|<t,\, |z-\xi|<t} \left(\frac{1}{\sqrt{t^2 - |z|^2}} - \frac{1}{\sqrt{t^2 - |z-\xi|^2}} \right) v_0(x+z)\, dz \right|,$$

$$U_{12} = \left| \int_{|z|<t<|z-\xi|} \frac{v_0(x+z)}{\sqrt{t^2 - |z|^2}}\, dz \right|,$$

$$U_{13} = \left| \int_{|z-\xi|<t<|z|} \frac{v_0(x'+z-\xi)}{\sqrt{t^2 - |z-\xi|^2}}\, dz \right|.$$

Clearly U_{12} and U_{13} are analogous. Let $1/q_0 + 1/q = 1$; Hölder's inequality implies

$$U_{12} \le C \|v_0\|_{q_0} \left(\int_{|z|<t<|z-\xi|} \frac{dz}{(t^2 - |z|^2)^{q/2}} \right)^{1/q}.$$

We have

$$\int_{|z|<t<|z-\xi|} \frac{dz}{(t^2 - |z|^2)^{q/2}} = 2\pi \int_{(t-|\xi|)^+}^{t} \frac{\rho\, d\rho}{(t^2 - \rho^2)^{q/2}} \le C|\xi|^{1-q/2}.$$

Thus, the choice of q yields

$$U_{12} \le C \|v_0\|_{q_0} |x' - x|^{\beta/2(1+\beta)}. \tag{1.27}$$

Analogously,

$$U_{11} \le C \|v_0\|_{q_0} \left(\int_{|z-\xi|<|z|<t} \left| \frac{1}{\sqrt{t^2 - |z|^2}} - \frac{1}{\sqrt{t^2 - |z-\xi|^2}} \right|^q dz \right)^{1/q}.$$

Let $\zeta \in \mathbf{R}_+$ be such that $\beta/2(1+\beta) < \zeta/2q < 1/q - 1/2$. Notice that $q \in (1, 4/3)$ and $\zeta < 2 - q$. Therefore the inequality (A.23) of Lemma A4 yields

$$U_{11} \le C \|v_0\|_{q_0} |x' - x|^{\beta/2(1+\beta)}. \tag{1.28}$$

Let $y - x = tz$ $(y - x' = tz)$. Then $U_2 \le C(U_{21} + U_{22})$, with

$$U_{21} = \left| \int_{|z|<1} \frac{1}{\sqrt{1-|z|^2}} \left(\frac{\partial}{\partial t} \big(u_0(x+tz) - u_0(x'+tz)\big)\right) dz \right|^p,$$

$$U_{22} = \left| \int_{|z|<1} \frac{1}{\sqrt{1-|z|^2}} \big(u_0(x+tz) - u_0(x'+tz)\big)\, dz \right|^p.$$

The Hölder continuity of ∇u_0 yields $U_{21} \le C|x-x'|^{p(\beta/2(1+\beta))}$. Clearly, $U_{22} \le C|x-x'|^p$. Thus

$$U_2 \le C|x-x'|^{p(\beta/2(1+\beta))}. \tag{1.29}$$

Set $U_3 \le C\sum_{i=1}^{3} U_{3i}$, where

$$U_{31} = E\left| \int_0^{t-|x-x'|/2} \int_{|x-y|<t-s, |x'-y|<t-s} \big(S(t-s, x-y) - S(t-s, x'-y)\big) \times \sigma(u(s,y))\, F(ds, dy) \right|^p,$$

$$U_{32} = E\left| \int_0^t \int_{|x'-y|<t-s, |x-y|>t-s} S(t-s, x'-y)\, \sigma(u(s,y))\, F(ds,dy) \right|^p,$$

$$U_{33} = E\left| \int_0^t \int_{|x-y|<t-s,\, |x'-y|>t-s} S(t-s, x-y) \sigma(u(s,y))\, F(ds,dy) \right|^p.$$

Clearly $U_{32} = U_{33}$. For any $t > 0$, $\xi \in \mathbf{R}^2$, set

$$M_{t,\xi} = \int_0^t ds \int_{|z|<s,\, |z-\xi|>s} dz \int_{|z'|<s,\, |z'-\xi|>s} dz' S(s,z)\, f(|z-z'|)\, S(s,z'),$$

$$N_{t,\xi} = \int_{|\xi|/2}^t ds \int_{|z|<s, |z-\xi|<s} dz \int_{|z'|<s, |z'-\xi|<s} dz' \, \big|S(s,z) - S(s,z-\xi)\big| f\big(|z-z'|\big) \times \big|S(s,z') - S(s,z'-\xi)\big|.$$

Burkholder's, Hölder's inequality and (1.8) imply

$$U_3 \le C\big(M_{t,x'-x}^{p/2} + N_{t,x'-x}^{p/2}\big).$$

Lemma A.5 yields that for $0 < \gamma < \beta/2(1+\beta)$, $|\xi| \le 1/2$, $M_{t,\xi} + N_{t,\xi} \le C|\xi|^{2\gamma}$. Hence

$$U_3 \le C\,|x'-x|^{\gamma p} \quad \text{for } |x'-x| \le \tfrac{1}{2}. \tag{1.30}$$

Finally, we study U_4. It is decomposed as U_3 into $\sum_{j=1}^{3} U_{4,j}$, where $U_{4,j}$ is defined as $U_{3,j}$ with b instead of σ and $ds\,dy$ instead of the noise $F(ds,dy)$. Again $U_{4,2} = U_{4,3}$ and by Hölder's inequality and an explicit integration,

$$\begin{aligned} U_{4,2} &\le C\left| \int_0^t ds \int_{|x-y|<t-s,\, |x'-y|>t-s} dy\, S(t-s, x-y) \right|^p \\ &\le C|x-x'|^{p/2}. \end{aligned} \tag{1.31}$$

Using (A.24) in Lemma A4, similar computations imply

$$U_{4,1} \le C\bigg(\int_0^{t-(|x-x'|/2)} ds \int_{|x-y|<t-s,\,|x'-y|<t-s} \big|S(t-s,x-y)-S(t-s,x'-y)\big|\,dy\bigg)^p \le C|x'-x|^{p/2}. \tag{1.32}$$

Inequalities (1.27)–(1.32) show (1.18) and conclude the proof of the proposition. □

2. Regularity of the solution. This section is devoted to establishing that the solution of (1.3), for fixed $t > 0$, $x \in \mathbf{R}^2$, belongs to the space $\mathbf{D}^\infty = \bigcap_{N\in\mathbf{N}}\bigcap_{p\in[1,\infty)}\mathbf{D}^{N,p}$ of the Malliavin calculus developed in the framework that has been described in the introduction. We recall that the Sobolev spaces $\mathbf{D}^{N,p}$ are defined by means of iterations of the Malliavin derivative operator D (see [10], Section 1.1) and that, for a random variable X and a positive integer N, $D^N X$ defines, whenever it exists, a random variable with values in $\mathscr{H}_T^{\otimes N}$. For $h \in \mathscr{H}_T$ set $D_h X = \langle DX, h\rangle_{\mathscr{H}_T}$. Since $\mathscr{H}_T = L^2([0,T];\mathscr{H})$, for $r \in [0,T]$, $DX(r)$ defines an element in $\mathscr{H}$, which will be denoted by $D_{r,*}X$. Then, clearly, for any $h \in \mathscr{H}_T$,

$$D_h X = \int_0^T \big\langle D_{r,*}X, h(r)\big\rangle_{\mathscr{H}}\,dr.$$

We will also use the following notation: for $r_i \in [0,T]$, $\varphi_i \in \mathscr{H}$, $i = 1,\dots,N$, set

$$D^N_{((r_1,\varphi_1),\dots,(r_N,\varphi_N))}X = \big\langle D^N_{(r_1,\dots,r_N),*}X, \varphi_1\otimes\cdots\otimes\varphi_N\big\rangle_{\mathscr{H}^{\otimes N}}.$$

For $N = 1$ we write $D_{r,\varphi}X = \langle D_{r,*}X, \varphi\rangle_{\mathscr{H}}$, $r \in [0,T]$, $\varphi \in \mathscr{H}$. By convention, $\mathbf{D}^{0,p} = L^p(\Omega)$. The regularity result of this section is proved using an induction argument described in the following lemma.

LEMMA 2.1 (Lemma 3.2 [11]). *Let $\{X_n : n \ge 1\}$ be a sequence of random variables in $\mathbf{D}^{N,p}$, $N \in \mathbf{N}$, $p \in [2,\infty)$. Assume there exists $X \in \mathbf{D}^{N-1,p}$ such that $\{D^{N-1}X_n, n \ge 1\}$ converges to $D^{N-1}X$ in $L^p(\Omega, \mathscr{H}_T^{\otimes(N-1)})$ as $n \to \infty$ and moreover, the sequence $\{D^N X_n, n \ge 1\}$ is bounded in $L^p(\Omega; \mathscr{H}_T^{\otimes N})$. Then $X \in \mathbf{D}^{N,p}$.*

THEOREM 2.2. *Let σ, b be of class $\mathscr{C}^\infty$ with bounded derivatives of any order $k \ge 1$ and assume that v_0, u_0 and f satisfy the hypotheses of Theorem 1.2. Then, for any $T > 0$, $t \in [0,T]$, $x \in \mathbf{R}^2$, the solution $u(t,x)$ of (1.3) belongs to $\mathbf{D}^\infty$. Furthermore, given $p \in [1,\infty[$, there exists a constant $C_p(T)$ depending* *on p and T such that, for $0 \le s < t \le T$,*

$$\sup_{s\le\tau\le t}\sup_{x\in\mathbf{R}^2} E\left(\left|\int_s^t \left\| D_{r,*}u(\tau,x)\right\|_{\mathscr{H}}^2 dr\right|^p\right) \le C_p(T)\mu_{t-s}^p, \tag{2.1}$$

with μ_t defined in (A.2).

PROOF. Consider the sequence of Picard's approximations $\{u^n(t,x), n \ge 0\}$, $t \ge 0$, $x \in \mathbf{R}^2$, defined by (1.9). For any integer $N \ge 1$, set

(H_N): for any $t \in [0,T]$, $x \in \mathbf{R}^2$, $p \in [2,\infty)$:

(i) $\{u^n(t,x), n \ge 0\} \subset \mathbf{D}^{N,p}$,
(ii) $\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\|D^{N-1}(u^n(s,x)-u(s,x))\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes(N-1)})} \to 0$ as $n\to\infty$;
(iii) $\sup_{n\ge 0}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\|D^N u^n(s,x)\|_{L^p(\Omega;\mathscr{H}_T^{\otimes N})} = C_N < \infty$.

We prove by induction that the assumptions of the theorem imply that (H_N) holds for every $N \ge 1$. Then Lemma 2.1 concludes the proof. Let $N = 1$, $t \ge 0$, $x \in \mathbf{R}^2$ be fixed. Since $u^0(t,x)$ is deterministic, $u^0(t,x) \in \mathbf{D}^{1,p}$ and $Du^0(t,x) = 0$. Assume $u^j(t,x) \in \mathbf{D}^{1,p}$, for any $j = 0,1,\ldots,n$, $n \ge 0$. By the rules of Malliavin's calculus the right-hand side of (1.9) belongs to $\mathbf{D}^{1,p}$ and, in addition, for any $\varphi \in \mathscr{H}$,

$$\begin{aligned} D_{r,\varphi}u^{n+1}(t,x) &= \langle S(t-r,x-*)\sigma(u^n(r,*)),\varphi\rangle_{\mathscr{H}} \\ &\quad + \int_r^t\int_{\mathbf{R}^2} S(t-s,x-y)D_{r,\varphi}u^n(s,y) \\ &\qquad \times \left[\sigma'\big(u^n(s,y)\big)F(ds,dy) + b'\big(u^n(s,y)\big)\,ds\,dy\right] \end{aligned}$$

if $r \in [0,t]$ and $D_{r,\varphi}u^{n+1}(t,x) = 0$ if $r > t$. In the proof of Theorem 1.2 [see (1.15)], we have shown that for $p \in [2,\infty[$,

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t} E\big(|u^n(s,x)-u(s,x)|^p\big) = 0.$$

Next we prove

$$\sup_{n\ge 0}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\left\|Du^n(s,x)\right\|_{L^p(\Omega;\mathscr{H}_T)} = C_1 < \infty, \tag{2.2}$$

for some constant C_1 . Then (H_1) will be established. For fixed $t \in \mathbf{R}_+$ and $x \in \mathbf{R}^2$, consider the decomposition

$$\left\|Du^{n+1}(t,x)\right\|^p_{L^p(\Omega;\mathscr{H}_T)} \le C_p\sum_{i=1}^3 T_i^n, \qquad n \ge 0, \tag{2.3}$$

where

$$T_1^n = \|S(t-\cdot, x-*)\,\sigma(u^n(\cdot,*))\|^p_{L^p(\Omega;\mathscr{H}_T)},$$

$$T_2^n = \left\|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\sigma'(u^n(s,y))Du^n(s,y)F(ds,dy)\right\|^p_{L^p(\Omega;\mathscr{H}_T)},$$

$$T_3^n = \left\|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)b'(u^n(s,y))\,Du^n(s,y)\,ds\,dy\right\|^p_{L^p(\Omega;\mathscr{H}_T)}.$$

Let μ_t be defined by (A.2); Hölder's inequality and (1.10) imply

$$\begin{aligned} T_1^n &= E\biggl(\biggl|\int_0^t dr \int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} dz'\, S(t-r, x-z)\sigma(u^n(r,z))f(|z-z'|) \\ &\qquad \times \sigma(u^n(r,z'))\,S(t-r, x-z')\biggr|^{p/2}\biggr) \\ &\le C(\mu_t)^{p/2}\Bigl(1+\sup_{x\in\mathbf{R}}\sup_{0\le s\le t} E|u^n(s,x)|^p\Bigr) \le C(\mu_t)^{p/2}. \end{aligned} \tag{2.4}$$

Fix $t \in [0,T]$ and consider the continuous $\mathscr{H}_T$-valued $\mathscr{F}_\tau$-martingale

$$Y_\tau = \int_0^\tau \int_{\mathbf{R}^2} S(t-s, x-y)\sigma'(u^n(s,y))\,Du^n(s,y)F(ds,dy), \qquad \tau\in[0,T].$$

Denote by $\langle Y\rangle_\tau$ the unique $\mathscr{F}_\tau$-predictable, increasing process such that $\langle Y\rangle_0 = 0$ and $\|Y_\tau\|^2_{\mathscr{H}_T} - \langle Y\rangle_\tau$ is a real $\mathscr{F}_\tau$-martingale. Using the Itô formula, one easily checks that

$$\langle Y\rangle_\tau = \sum_{j\ge 0} \|1_{[0,\tau]}(\cdot)S(t-\cdot, x-*)\sigma'(u^n(\cdot,*))D_{e_j}u^n(\cdot,*)\|^2_{\mathscr{H}_T},$$

where $\{e_j, j\ge 0\}$ is a CONS of $\mathscr{H}_T$. Therefore, Burkholder's inequality for Hilbert-valued martingales (see, e.g., [7], page 212) and Parseval's identity yield

$$\begin{aligned} T_2^n &= E\biggl(\biggl\|\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\sigma'(u^n(s,y))\,Du^n(s,y)F(ds,dy)\biggr\|^p_{\mathscr{H}_T}\biggr) \\ &\le C_p E\biggl(\biggl|\sum_{j\ge 0}\int_0^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy' S(t-s, x-y)\sigma'(u^n(s,y))D_{e_j}u^n(s,y) \\ &\qquad \times f(|y-y'|)S(t-s, x-y')\sigma'(u^n(s,y'))D_{e_j}u^n(s,y')\biggr|^{p/2}\biggr) \\ &= C_p E\biggl(\biggl|\int_0^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy' S(t-s, x-y)\sigma'(u^n(s,y))f(|y-y'|) \\ &\qquad \times S(t-s, x-y')\bigl\langle Du^n(s,y), Du^n(s,y')\bigr\rangle_{\mathscr{H}_T}\sigma'(u^n(s,y'))\biggr|^{p/2}\biggr). \end{aligned} \tag{2.5}$$

Apply Schwarz's inequality to the scalar product in $\mathscr{H}_T$, then Hölder's and Schwarz's inequalities. The right-hand side of (2.5) is bounded by

$$C_p\|\sigma'\|_\infty^p \mu_t^{p/2-1}\int_0^t J(t-s)\sup_{x\in\mathbf{R}^2}\sup_{0\le\tau\le s} E\big(\|D\,u^n(\tau,x)\|_{\mathscr{H}_T}^p\big)\,ds. \tag{2.6}$$

Let ν_t be defined by (A.3); Hölder's inequality yields

$$\begin{aligned} T_3^n &\le \|b'\|_\infty^p E\left(\left|\int_0^t ds\int_{\mathbf{R}^2} dy\, S(t-s,x-y)\|D\,u^n(s,y)\|_{\mathscr{H}_T}\right|^p\right)\\ &\le C\|b'\|_\infty^p \nu_t^{p-1}\int_0^t \sup_{y\in\mathbf{R}^2}\sup_{0\le\tau\le s} E\big(\|D\,u^n(\tau,y)\|_{\mathscr{H}_T}^p\big)\,ds. \end{aligned}\tag{2.7}$$

Therefore, (2.3), (2.4), (2.6), (2.7), (A.3) and (A.6) yield

$$\begin{aligned} &\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\big\|Du^{n+1}(s,x)\big\|_{L^p(\Omega,\mathscr{H}_T)}^p\\ &\quad\le C\left(\mu_t^{p/2}+\int_0^t\sup_{x\in\mathbf{R}^2}\sup_{0\le\tau\le s}\big\|D\,u^n(\tau,x)\big\|_{L^p(\Omega,\mathscr{H}_T)}^p\,ds\right). \end{aligned}$$

This estimate, together with (A.6) and Gronwall's lemma, show (2.2). Furthermore, similar computations yield, for every $t\in[0,T]$, $n\ge 0$,

$$\begin{aligned} &\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\|D\,u^{n+2}(s,x)-Du^{n+1}(s,x)\big\|_{L^p(\Omega;\mathscr{H}_T)}^p\\ &\quad\le C\Bigg\{\mu_t^{p/2}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\big\|u^{n+1}(s,x)-u^n(s,x)\big\|_{L^p(\Omega)}^p\\ &\qquad+\mu_t^{(p/2)-1}\int_0^t J(t-s)\sup_{x\in\mathbf{R}^2}\sup_{0\le\tau\le s}\big\|D\,u^{n+1}(\tau,x)-D\,u^n(\tau,x)\big\|_{L^p(\Omega;\mathscr{H}_T)}^p\,ds\\ &\qquad\qquad+\nu_t^{p-1}\int_0^t\sup_{x\in\mathbf{R}^2}\sup_{0\le\tau\le s}\big\|Du^{n+1}(\tau,x)-D\,u^n(\tau,x)\big\|_{L^p(\Omega;\mathscr{H}_T)}^p\,ds\Bigg\}. \end{aligned}$$

By Lemma A1 (see Remark A2) and (A.3) this can be bounded by

$$\begin{aligned} &C\Bigg[\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\big\|u^{n+1}(s,x)-u^n(s,x)\big\|_{L^p(\Omega)}^p\\ &\qquad+\int_0^t\sup_{x\in\mathbf{R}^2}\sup_{0\le\tau\le s}\big\|D\,u^{n+1}(\tau,x)-D\,u^n(\tau,x)\big\|_{L^p(\Omega;\mathscr{H}_T)}^p\,ds\Bigg]. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\big\|D\,u^n(s,x)-D\,u(s,x)\big\|_{L^p(\Omega,\mathscr{H}_T)}=0$$

and

$$\sup_{x\in\mathbf{R}^2}\sup_{0\le s\le t}\|D\,u(s,x)\|_{L^p(\Omega;\mathscr{H}_T)}<+\infty. \tag{2.8}$$

The derivative $Du(t, x)$ satisfies the following equation: for any $\varphi \in \mathscr{H}$,

$$D_{r,\varphi}u(t,x) = \langle S(t-r, x-*)\sigma(u(r,*)), \varphi\rangle_{\mathscr{H}} + \int_0^t \int_{\mathbf{R}^2} S(t-s, x-y) \times D_{r,\varphi}u(s,y)\big[\sigma'(u(s,y))\, F(ds,dy) + b'(u(s,y))\, ds\, dy\big], \tag{2.9}$$

if $r \in [0, t]$ and $D_{r,\varphi}u(t, x) = 0$ if $r > t$. Let $0 \le s < t \le T$ and set $\mathscr{H}_{s,t} = L^2([s, t], \mathscr{H})$. Notice that $\mathscr{H}_{s,t} \subset \mathscr{H}_T$ and $\mathscr{H}_{0,T} = \mathscr{H}_T$. Since $D_{r,*}u(\tau, x) = 0$ for $r \in [s, t]$, $\tau < s$ and $x \in \mathbf{R}^2$, computations similar to the previous ones yield, for any $\tau \in [s, t]$, $p \ge 2$,

$$\begin{aligned}\|D u(\tau,x)\|^p_{L^p(\Omega,\mathscr{H}_{s,t})} &\le \bigg(\mu^{p/2}_{\tau-s} + \mu^{(p/2)-1}_{\tau-s}\int_s^\tau J(\tau-r)\sup_{x\in\mathbf{R}^2}\sup_{s\le\rho\le r} E\big(\|Du(\rho,x)\|^p_{\mathscr{H}_{s,t}}\big)\,dr \\ &\qquad + \nu^{p-1}_{\tau-s}\int_s^\tau \sup_{x\in\mathbf{R}^2}\sup_{s\le\rho\le r} E\big(\|Du(\rho,x)\|^p_{\mathscr{H}_{s,t}}\big)\,dr\bigg).\end{aligned}$$

Then, using (2.8) and Gronwall's lemma, we obtain (2.1).

In order to deal with the Malliavin derivatives of any order, we introduce some notation. Let $N \in \mathbf{N}$, fix a set $A_N = \{\alpha_i = (r_i, \varphi_i) \in \mathbf{R}_+ \times \mathscr{H}, i = 1, \dots, N\}$ and set $\bigvee_i r_i = \max(r_1, \dots, r_N)$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\widehat{\alpha}_i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \alpha_N)$. Denote by $\mathscr{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1, \dots, p_m$, $m = 1, \dots, N$ and let $|p_i|$ denote the cardinal of p_i. Let X be any random variable belonging to $\mathbf{D}^{N,2}$, $N \ge 1$, g a real $\mathscr{C}^N$ function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives yields

$$D^N_\alpha(g(X)) = \Gamma_\alpha(g, X) := \sum_{m=1}^N \sum_{\mathscr{P}_m} c_m g^{(m)}(X) \prod_{j=1}^m D^{|p_i|}_{p_i} X,$$

with some positive coefficients c_m, $m = 1, \dots, N$, $c_1 = 1$. Let

$$\Delta_\alpha(g, X) = \Gamma_\alpha(g, X) - g'(X) D^N_\alpha X.$$

From (1.9), using induction on $n \ge 0$, it is easy to check

$$\{u^n(t,x), n \ge 0\} \subset \mathbf{D}^{N,p} \quad \text{for any } t \ge 0,\ x \in \mathbf{R}^2,\ N \in \mathbf{N}$$

and

$$\begin{aligned} D^N_\alpha u^{n+1}(t,x) &= \sum_{i=1}^N \langle \Gamma_{\hat{\alpha}_i}(\sigma, u^n(r_i,*)) S(t-r_i, x-*), \varphi_i\rangle_{\mathscr{H}} \\ &\quad + \int_{\bigvee_i r_i}^t \int_{\mathbf{R}^2} S(t-s,x-y) \\ &\qquad \times \big[\Delta_\alpha(\sigma, u^n(s,y))\, F(ds,dy) + \Delta_\alpha(b, u^n(s,y))\, ds\, dy\big] \\ &\quad + \int_{\bigvee_i r_i}^t \int_{\mathbf{R}^2} S(t-s,x-y)\, D^N_\alpha u^n(s,y) \\ &\qquad \times \big[\sigma'(u^n(s,y))\, F(ds,dy) + b'(u^n(s,y))\, ds\, dy\big]. \end{aligned} \tag{2.10}$$

Assume now (H_N) holds for $1 \le N \le M$, $M \ge 1$. We want to check (H_{M+1}). The preceding argument shows condition (i). For the proof of (ii), we notice that $D^M_\alpha u(t,x)$ satisfies an evolution equation as (2.10) with u^{n+1} and u^n replaced by u and N by M. The terms in this equation containing $\Gamma_{\hat{\alpha}_i}$ and Δ_α involve Malliavin's derivatives of u up to the order $M-1$. For these terms we use the induction hypothesis. For the remaining ones we use the technique based on inequalities developed in the first part of the proof and we conclude by Gronwall's inequality applied to $\varphi(t) := \sup_{x\in\mathbf{R}^2} \sup_{0\le s\le t} E\|D^M(u^n(s,x) - u(s,x))\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes M})}$. Finally, we prove (H_{M+1})(iii). The induction assumption (H_N)(iii), $1 \le N \le M$, yields, for some positive constant C, $g = \sigma, b$,

$$\sup_{n\ge 0} \sup_{i=1,\dots,M+1} \sup_{x\in\mathbf{R}^2} \sup_{0\le s\le t} \big\|\Gamma_{\hat{\alpha}_i}(\sigma, u^n(\alpha_i))\big\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes M})} \le C,$$

$$\sup_{n\ge 0} \sup_{x\in\mathbf{R}^2} \sup_{0\le s\le t} \big\|\Delta_\alpha(g, u^n(s,x)\big\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes M})} \le C.$$

Thus, using the same method as for the proof of (2.2) and the preceding estimates, we obtain

$$\begin{aligned} &\sup_{x\in\mathbf{R}^2} \sup_{0\le s\le t} \big\|D^{M+1}u^n(s,x)\big\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes(M+1)})} \\ &\quad\le C\left(1 + \int_0^t \sup_{x\in\mathbf{R}^2} \sup_{0\le r\le s} \big\|D^{M+1}u^{n-1}(r,x)\big\|^p_{L^p(\Omega;\mathscr{H}_T^{\otimes(M+1)})}\, ds\right). \end{aligned}$$

We conclude by Gronwall's lemma. □

3. Existence and regularity of the density of the solution. Fix $t > 0$ and pairwise distinct points $x_1, \dots, x_d$ of $\mathbf{R}^2$. Let u denote the solution of (1.3) and set

$$u(t,\underline{x}) = (u(t,x_1), \dots, u(t,x_d)).$$

In this section we give sufficient conditions for the existence and smoothness of the density of the law of $u(t,\underline{x})$, using the classical approach provided by the Malliavin calculus. The main result is the following theorem.

THEOREM 3.1. *Assume that*:

(i) *There exist* $a_1 \ge a_2 > 0$ *such that* $2(1+a_2)(a_1 - a_2) < a_2 \le a_1 < 2$, *positive constants* C_1 *and* C_2 *such that for* $t \in [0,T]$,

$$C_1 t^{a_1} \le \int_0^t y f(y) \ln\left(1 + \frac{t}{y}\right) dy \le C_2 t^{a_2}. \tag{3.1}$$

(ii) u_0: $\mathbf{R}^2 \to \mathbf{R}$ *is of class* $\mathscr{C}^1$, *bounded, with* $a_2/2(1+a_2)$-*Hölder continuous partial derivatives*, v_0: $\mathbf{R}^2 \to \mathbf{R}$ *and there exists* $q_0 \in\,]4, +\infty]$ *such that* $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbf{R}^2)$.

(iii) σ *and* b *are* $\mathscr{C}^\infty$ *with bounded derivatives of any order* $i \ge 1$.

(iv) *There exists* $a > 0$ *such that*

$$\left|\sigma(u(t, x_j))\right| \geq a \quad \text{for any } j = 1, \dots, d.$$

Then the law of the random vector $u(t, \underline{x})$ *has a* $\mathscr{C}^\infty$ *density with respect to the Lebesgue measure on* $\mathbf{R}^d$.

REMARKS. (i) Let $f(x) = x^{-\alpha}$, $0 < \alpha < 2$. Then (3.1) holds with $a_1 = a_2 = 2 - \alpha$.

(ii) Using a localization procedure developed in [2], we can prove a version of Theorem 3.1 without assuming (iv). In this case, we conclude that the law of $u(t, \underline{x})$ has a smooth density ρ on $\{\sigma \neq 0\}^d$, that is, $\rho \in \mathscr{C}^\infty(\{\sigma \neq 0\}^d; \mathbf{R})$ and for any $\varphi \in \mathscr{C}(\mathbf{R}^d, \mathbf{R})$ with compact support included in $\{\sigma \neq 0\}^d$, $E[\varphi(u(t, \underline{x})] = \int_{\mathbf{R}^d} \rho(y)\varphi(y)\, dy$.

Let $\Gamma(t, \underline{x})$ denote the Malliavin covariance matrix $(\langle Du(t, x_i), Du(t, x_j)\rangle_{\mathscr{H}_T}; 1 \leq i,\ j \leq d)$. According to Theorem 2.2, we only need to check

$$\left(\det \Gamma(t, \underline{x})\right)^{-1} \in \bigcap_{1 \leq p < \infty} L^p(\Omega),$$

(see, e.g., [9], Corollary 2.1.2). We recall that, given $\varphi \in \mathscr{H}$, the Malliavin derivative $D_{r,\varphi} u(t, x)$ satisfies the equation

$$\begin{aligned} D_{r,\varphi} u(t, x_i) = \left\langle S(t - r, x_i - *)\sigma(u(r, *)), \varphi\right\rangle_{\mathscr{H}} + \int_r^t \int_{\mathbf{R}^2} S(t - s, x_i - y) \\ \times D_{r,\varphi} u(s, y)\left[\sigma'(u(s, y))F(ds, dy) + b'(u(s, y))\, ds\, dy\right] \end{aligned}$$

for $r \leq t$, $D_{r,z} u(t, x_i) = 0$, if $r > t$, $i = 1, \dots, d$, [see (2.9)].

PROOF OF THEOREM 3.1. The proof consists of checking that, for any $p \geq 2$, there exists $\varepsilon_0(p) > 0$ and for any $0 < \varepsilon \leq \varepsilon_0(p)$, if $P_\varepsilon(v) = P(v^t \Gamma(t, \underline{x}) v \leq \varepsilon)$,

$$\sup_{v \in \mathbf{R}^d,\, |v| = 1} P_\varepsilon(v) \leq C\varepsilon^p. \tag{3.2}$$

(see, e.g., [9], Lemma 2.3.1). We have, for any $\varepsilon, \delta > 0$ with $t - \varepsilon^\delta > 0$,

$$\begin{aligned} v^t \Gamma(t, \underline{x}) v &= \int_0^t dr \left\| \sum_{1 \leq i \leq d} v_i D_{r,*} u(t, x_i) \right\|_{\mathscr{H}}^2 \\ &\geq \sum_{i,\, j=1}^d v_i v_j \int_{t - \varepsilon^\delta}^t dr \left\langle D_{r,*} u(t, x_i), D_{r,*} u(t, x_j)\right\rangle_{\mathscr{H}}. \end{aligned}$$

Set

$$\begin{aligned} &\left\langle D_{r,*} u(t, x_i), D_{r,*} u(t, x_j)\right\rangle_{\mathscr{H}} \\ &\quad = \left\langle S(t - r, x_i - *)\sigma(u(r, *)), S(t - r, x_j - *)\sigma(u(r, *))\right\rangle_{\mathscr{H}} \\ &\quad\quad + U(t, r, x_i, x_j). \end{aligned}$$

The triangle inequality yields $P_\varepsilon(v) \le P^1_\varepsilon(v) + P^2_\varepsilon(v)$, where

$$P^1_\varepsilon(v) = P\bigg(\sum_{i,\,j=1}^d v_i v_j \int_{t-\varepsilon^\delta}^t dr \big\langle S(t-r, x_i - *)\sigma(u(r,*)),\\ S(t-r, x_j - *)\sigma(u(\cdot,*))\big\rangle_{\mathscr{H}} < 2\varepsilon\bigg),$$

$$P^2_\varepsilon(v) = P\bigg(\sum_{i,\,j=1}^d v_i v_j \int_{t-\varepsilon^\delta}^t dr\, U(t, r, x_i, x_j) \ge \varepsilon\bigg).$$

First we study $P^2_\varepsilon(v)$. By Chebyshev's inequality, for any $q \in [1,\infty)$, since $|v| \le 1$,

$$P^2_\varepsilon(v) \le C\varepsilon^{-q} \sum_{i,\,j=1}^d E\bigg(\bigg|\int_{t-\varepsilon^\delta}^t dr\, U(t, r, x_i, x_j)\bigg|^q\bigg).$$

To simplify the notation, let $\mathscr{H}(\varepsilon,\delta) = \mathscr{H}_{t-\varepsilon^\delta,\,t}$, that is, for functions $\varphi, \psi \in \mathscr{H}_T$, set

$$\langle \varphi, \psi\rangle_{\mathscr{H}(\varepsilon,\,\delta)} = \int_{t-\varepsilon^\delta}^t dr \langle \varphi(r), \psi(r)\rangle_{\mathscr{H}}.$$

Then, the definition of $U(t, r, x_i, x_j)$ and the fact that $D_{r,*}u(s, y) = 0$ if $s < r$ yield

$$P^2_\varepsilon(v) \le C\varepsilon^{-q} \sum_{i,\,j=1}^d \sum_{k=1}^5 T_k(i, j), \tag{3.3}$$

where

$$T_1(i, j) = E\bigg(\bigg|\bigg\langle S(t-\cdot, x_i - *)\sigma(u(\cdot,*)),\\ \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)\sigma'(u(s, y))F(ds, dy)\bigg\rangle_{\mathscr{H}(\varepsilon,\,\delta)}\bigg|^q\bigg),$$

$$T_2(i, j) = E\bigg(\bigg|\bigg\langle S(t-\cdot, x_i - *)\sigma(u(\cdot,*)),\\ \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)b'(u(s, y))\, ds\, dy\bigg\rangle_{\mathscr{H}(\varepsilon,\,\delta)}\bigg|^q\bigg),$$

$$T_3(i, j) = E\bigg(\bigg|\bigg\langle \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_i - y)\, Du(s, y)\sigma'(u(s, y))F(ds, dy),\\ \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)\sigma'(u(s, y))\, F(ds, dy)\bigg\rangle_{\mathscr{H}_{\varepsilon,\,\delta}}\bigg|^q\bigg),$$

$$T_4(i, j) = E\bigg(\bigg|\bigg\langle \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_i - y)\, Du(s, y)\sigma'(u(s, y))F(ds, dy),\\ \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)b'(u(s, y))\, ds\, dy\bigg\rangle_{\mathscr{H}_{\varepsilon,\,\delta}}\bigg|^q\bigg),$$

$$T_5(i, j) = E\left(\left|\left\langle \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_i - y)\, Du(s, y) b'(u(s, y))\, ds\, dy, \right.\right.\right.$$
$$\left.\left.\left. \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y) b'(u(s, y))\, ds\, dy \right\rangle_{\mathscr{H}_{\varepsilon,\delta}}\right|^q\right).$$

Schwarz's inequality implies, for $i, j = 1, \dots, d$,

$$T_1(i, j) \leq T_{11}(i)^{1/2} T_{12}(j)^{1/2},$$

where

$$T_{11}(i) = E\left(\left\| S(t-\cdot, x_i - *)\sigma(u(\cdot, *))\right\|^{2q}_{\mathscr{H}(\varepsilon,\delta)}\right),$$
$$T_{12}(j) = E\left(\left\| \int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)\sigma'(u(s, y)) F(ds, dy)\right\|^{2q}_{\mathscr{H}(\varepsilon,\delta)}\right).$$

Hölder's inequality applied to the measure μ_{ε^δ} defined in (A.2) yields

$$T_{11}(i) \leq (\mu_{\varepsilon^\delta})^{q-1} \int_{t-\varepsilon^\delta}^t dr \int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} dz' S(t-r, x_i - z) f(|z - z'|)$$
$$\times S(t-r, x_i - z') E\left(\left|\sigma(u(r, z))\sigma(u(r, z'))\right|^q\right).$$

Schwarz's inequality and the property (1.8) imply

$$T_{11}(i) \leq C\mu^q_{\varepsilon^\delta}. \tag{3.4}$$

Let $(e_k, k \geq 0)$ be a CONS of $\mathscr{H}(\varepsilon, \delta)$. We apply the Hilbert-valued version of Burkholder's inequality to the stochastic integral $\int_{t-\varepsilon^\delta}^t \int_{\mathbf{R}^2} S(t-s, x_j - y)\, Du(s, y)\sigma'(u(s, y)) F(ds, dy)$ and then Hölder's inequality. We obtain

$$T_{12}(j) \leq CE\left(\left|\sum_{k\geq 0} \int_{t-\varepsilon^\delta}^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy' S(t-s, x_j - y) D_{e_k} u(s, y)\sigma'(u(s, y))\right.\right.$$
$$\left.\left. \times f\left(|y - y'|\right)\sigma'(u(s, y'))\, D_{e_k} u(s, y') S(t-s, x_j - y')\right|^q\right)$$
$$= CE\left(\left|\int_{t-\varepsilon^\delta}^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy' S(t-s, x_j - y)\sigma'(u(s, y)) f\left(|y - y'|\right)\right.\right.$$
$$\left.\left. \times \sigma'(u(s, y')) S(t-s, x_j - y')\left\langle Du(s, y), Du(s, y')\right\rangle_{\mathscr{H}(\varepsilon,\delta)}\right|^q\right)$$
$$\leq C\mu^{q-1}_{\varepsilon^\delta} \int_{t-\varepsilon^\delta}^t ds \int_{\mathbf{R}^2} dy \int_{\mathbf{R}^2} dy'\, S(t-s, x_j - y)$$
$$\times f\left(|y - y'|\right) S(t-s, x_j - y')$$
$$\times E\left(\left|\left\langle Du(s, y),\ Du(s, y')\right\rangle_{\mathscr{H}(\varepsilon,\delta)}\right|^q\right).$$

The inequality (2.1) in Theorem 2.2 implies

$$\sup_{t-\varepsilon^\delta\le s\le t}\ \sup_{y\in\mathbf{R}^2} E\big(\|Du(s,y)\|^{2q}_{\mathscr{H}(\varepsilon,\delta)}\big)\le C\mu^q_{\varepsilon^\delta}. \tag{3.5}$$

Hence, Schwarz's inequality yields

$$T_{12}(j)\le C\mu^{2q}_{\varepsilon^\delta}, \tag{3.6}$$

and therefore,

$$T_1(i,j)\le C\mu^{3q/2}_{\varepsilon^\delta}. \tag{3.7}$$

The structure of the term $T_2(i,j)$ is similar to that of $T_1(i,j)$, the stochastic integral being replaced by a Lebesgue integral. Thus,

$$T_2(i,j)\le T_{11}(i)^{1/2}T_{22}(j)^{1/2},$$

with

$$T^{(j)}_{22}=E\bigg(\bigg\|\int_{t-\varepsilon^\delta}^t\int_{\mathbf{R}^2}S(t-s,x_j-y)\,Du(s,y)b'(u(s,y))\,ds\,dy\bigg\|^{2q}_{\mathscr{H}(\varepsilon,\delta)}\bigg).$$

Again Hölder's inequality and (3.5) imply

$$T_{22}(j)\le C\nu^{2q}_{\varepsilon^\delta}\sup_{t-\varepsilon^\delta\le s\le t}\ \sup_{y\in\mathbf{R}^2}E\big(\|Du(s,y)\|^{2q}_{\mathscr{H}(\varepsilon,\delta)}\big)\le C\nu^{2q}_{\varepsilon^\delta}\mu^q_{\varepsilon^\delta}, \tag{3.8}$$

where ν_t is defined by (A.3). Therefore, (3.4) and (3.8) imply

$$T_2(i,j)\le C\nu^q_{\varepsilon^\delta}\mu^q_{\varepsilon^\delta}. \tag{3.9}$$

Schwarz's inequality and (3.6) yield

$$T_3(i,j)\le\big(T_{12}(i)T_{12}(j)\big)^{1/2}\le C\mu^{2q}_{\varepsilon^\delta}. \tag{3.10}$$

Furthermore, Schwarz's inequality implies $T_4(i,j)\le(T_{12}(i)T_{22}(j))^{1/2}$, so that (3.6) and (3.8) yield

$$T_4(i,j)\le C\mu^{(3/2)q}_{\varepsilon^\delta}\nu^q_{\varepsilon^\delta}. \tag{3.11}$$

Finally,

$$T_5(i,j)\le\big(T_{22}(i)T_{22}(j)\big)^{1/2}\le C\nu^{2q}_{\varepsilon^\delta}\mu^q_{\varepsilon^\delta}. \tag{3.12}$$

The inequalities (3.3), (3.7), (3.9)–(3.12), (A.3), (A.5) and (3.1) yield

$$\begin{aligned}P^2_\varepsilon(v)&\le C\varepsilon^{-q}\Big[\mu^{3q/2}_{\varepsilon^\delta}+\nu^q_{\varepsilon^\delta}\mu^q_{\varepsilon^\delta}\Big]\\&\le C\varepsilon^{-q}\Big[\varepsilon^{(3q/2)\delta(a_2+1)}+\varepsilon^{q\delta(3+a_2)}\Big]\\&\le C\varepsilon^{q[-1+(3\delta/2)(a_2+1)]}.\end{aligned} \tag{3.13}$$

We now study $P^1_\varepsilon(v)$. In order to use assumption (iv), set

$$P^1_\varepsilon(v)\le P^{11}_\varepsilon(v)+P^{12}_\varepsilon(v)+P^{13}_\varepsilon(v),$$

with

$$P_\varepsilon^{11}(v) = P\biggl(\sum_{j=1}^d v_j^2 \bigl\|\sigma(u(t,x_j))S(t-\cdot, x_j-*)\bigr\|^2_{\mathscr{H}(\varepsilon,\delta)} < 4\,\varepsilon\biggr),$$

$$P_\varepsilon^{12}(v) = P\biggl(\sum_{i,\,j=1,\,i\neq j}^d \biggl|v_i v_j \int_{t-\varepsilon^\delta}^t dr \int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} dz'\, \sigma(u(t,x_i))S(t-r,x_i-z)$$
$$\times f(|z-z'|)\sigma(u(t,x_j))S(t-r,x_j-z')\biggr| \geq \varepsilon\biggr),$$

$$P_\varepsilon^{13}(v) = P\biggl(\sum_{i,\,j=1}^d \biggl|v_i v_j \int_{t-\varepsilon^\delta}^t dr \int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} dz' S(t-r,x_i-z)S(t-r,x_j-z')$$
$$\times f\bigl(|z-z'|\bigr)\bigl[\sigma(u(r,z))\,\sigma(u(r,z'))$$
$$- \sigma(u(t,x_i))\sigma(u(t,x_j))\bigr]\biggr| \geq \varepsilon\biggr).$$

Assumptions (i) and (iv) and (A.5) imply that, for every $j = 1,\dots,d$,

$$\bigl\|\sigma(u(t,x_j))S(t-\cdot,x_j-*)\bigr\|^2_{\mathscr{H}_{\varepsilon,\delta}} \geq a^2\mu_{\varepsilon^\delta} \geq C_1 a^2 \varepsilon^{\delta(a_1+1)}.$$

Thus, since $\sum_{j=1}^d v_j^2 = 1$, for ε small enough

$$P_\varepsilon^{11}(v) = 0 \quad \text{if } \delta(1+a_1) < 1. \tag{3.14}$$

Set $m_1 = \inf\{|x_i - x_j|; i \neq j\}$ and $m_2 = \sup\{|x_i - x_j|; i \neq j\}$; then $4\varepsilon^\delta < m_1$, $|z - x_i| < \varepsilon^\delta$, $|z' - x_j| < \varepsilon^\delta$ for $i \neq j$ imply $m_1/2 \leq |z'-z| \leq 3m_2/2$, so that $f(|z-z'|)$ is bounded by some constant C depending on m and M. Hence for $q \in [1,\infty[$, (1.8) and Chebyshev's inequality imply

$$\begin{aligned} P_\varepsilon^{12}(v) &\leq C\varepsilon^{-q} \sup_{1\leq i\leq d}\bigl(1 + E(|u(t,x_i)|^{2q})\bigr) \\ &\quad\times \biggl(\int_0^{\varepsilon^\delta}\biggl(\int_{|z|<r} \frac{dz}{\sqrt{r^2-|z|^2}}\biggr)^2 dr\biggr)^q \\ &\leq C\varepsilon^{-q+3\delta q}. \end{aligned} \tag{3.15}$$

Finally, Chebyshev's inequality implies that, for $q \in [1,\infty[$,

$$P_\varepsilon^{13}(v) \leq C\varepsilon^{-q} \sup_{1\leq i,\,j\leq d} E\bigl(|I_{ij}|^q\bigr),$$

with

$$I_{ij} = \int_{t-\varepsilon^\delta}^t dr \int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} dz'\bigl[\sigma(u(r,z))\,\sigma(u(r,z')) - \sigma(u(t,x_i))\sigma(u(t,x_j))\bigr]$$
$$\times S(t-r,x_i-z)f\bigl(|z-z'|\bigr)S(t-r,x_j-z').$$

By Remark 1.5(ii), the conclusion of Proposition 1.4 is satisfied with $\beta = a_2$; thus Hölder's inequality and the moment estimates (1.8), (1.17) and (1.18) yield, for $\gamma \in]0, a_2/2(1+a_2)[$,

$$P_\varepsilon^{13}(v) \le C\varepsilon^{-q}\mu_{\varepsilon^\delta}^q \varepsilon^{\delta q \gamma} \le C\varepsilon^{q[\delta(1+a_2+\gamma)-1]}. \tag{3.16}$$

Choose δ, $\gamma > 0$ satisfying the following conditions:

$$0 < \gamma < \frac{a_2}{2(1+a_2)}, \frac{3\delta}{2}(1+a_2) > 1, \qquad 3\delta > 1,$$
$$\delta(1+a_1) < 1, \qquad \delta(1+a_2+\gamma) > 1.$$

It is easy to check that such a choice is possible, due to the constraints on a_1, a_2 given in hypothesis (i). Then (3.13)–(3.16) show the estimate (3.2) and conclude the proof. □

APPENDIX

In this section we show several technical results which are needed in the proofs of this paper. First we introduce some notation. For $s > 0$, set

$$J(s) = \int_{|y|<|x|<s} f(|x-y|)\frac{1}{\sqrt{s^2-|x|^2}}\,\frac{1}{\sqrt{s^2-|y|^2}}\,dx\,dy \tag{A.1}$$

and for $t > 0$, let

$$\mu_t = \int_0^t ds \int_{\mathbf{R}^2} dx \int_{\mathbf{R}^2} dy\, f(|x-y|)S(s,x)S(s,y) = \frac{1}{2\pi^2}\int_0^t J(s)\,ds. \tag{A.2}$$

Finally, set

$$\nu_t = \frac{1}{2\pi}\int_0^t ds \int_{|x|<s} \frac{dx}{\sqrt{s^2-|x|^2}} = \frac{t^2}{2}, \qquad t \ge 0. \tag{A.3}$$

Our first purpose is to study sufficient conditions on f ensuring $\sup_{0\le s\le T} J(s) \le C_T$, for some positive and finite constant depending on T. This property plays an essential role in the proof of existence and uniqueness of solution for (1.3). We are also interested in establishing lower and upper bounds for μ_t in order to prove the existence and smoothness of density for the law of the solution to (1.3).

LEMMA A1. *Fix $T > 0$. There exist positive constants $C_1(T)$, $C_2(T)$ depending only on T, such that*:

(a) *For every* $0 \le s \le T$,

$$\begin{aligned} C_1(T)\int_0^s rf(r)\ln\left(1+\frac{s}{r}\right)dr &\le J(s) \\ &\le C_2(T)\int_0^{2s} rf(r)\ln\left(1+\frac{s}{r}\right)dr. \end{aligned} \tag{A.4}$$

(b) *For all* $0 \le t \le T$,

$$\begin{aligned} C_1(T)t\int_0^{t/3} rf(r)\ln\left(1+\frac{t}{r}\right)dr &\le \mu_t \\ &\le C_2(T)t\int_0^{2t} rf(r)\ln\left(1+\frac{t}{r}\right)dr. \end{aligned} \tag{A.5}$$

REMARK A2. *Condition* (1.7) *is equivalent to* $\int_0^{2t} rf(r)\ln(1+t/r)\,dr < \infty$, *for any* $t \ge 0$. *Then* (A.4) *implies*

$$J(t) \le C_T \quad \text{and} \quad \mu_t \le C_T t,\ 0 \le t \le T. \tag{A.6}$$

PROOF OF LEMMA A1. Let K be a compact subset of $\mathbf{R}^n$, $n \ge 1$, and f, g real functions defined on K. The notation $f(s) \asymp g(s)$, $\forall s \in K$, means that there exist positive constants c and C depending on K such that

$$cg(s) \le f(s) \le Cg(s) \qquad \forall s \in K.$$

Let $z = x - y$ and consider the change of variables

$$\begin{aligned} x &= (u\cos\theta_0, u\sin\theta_0), \\ z &= (v\cos(\theta+\theta_0), v\sin(\theta+\theta_0)). \end{aligned}$$

Then

$$J(s) = C\pi\varphi(s), \tag{A.7}$$

with

$$\varphi(s) = \int_0^s \frac{u\,du}{\sqrt{s^2-u^2}}\int_0^{2u} vf(v)\,dv \int_0^{\arccos(v/(2u))}\frac{d\theta}{\sqrt{s^2-u^2-v^2+2uv\cos\theta}}.$$

Let $r = \cos\theta$; then since $1 \le \sqrt{1+r} \le \sqrt{2}$ for $r \in [0,1]$, we obtain

$$\int_0^{\arccos(v/(2u))}\frac{d\theta}{\sqrt{s^2-u^2-v^2+2uv\cos\theta}} \asymp A(u,v)$$

with

$$A(u,v) = \int_{\frac{v}{2u}}^1 \frac{dr}{\sqrt{(1-r)(s^2-u^2-v^2+2uvr)}}.$$

Let

$$a^2 = \frac{[s^2-(u-v)^2]^2}{8uv}, \qquad b^2 = 2uv, \qquad c = \frac{s^2-(u+v)^2}{4uv};$$

then $(b/a)(1+c)=1$ and

$$A(u,v)=\int_{v/(2u)}^{1}\frac{dr}{\sqrt{a^2-b^2(r+c)^2}}$$
$$=\frac{1}{b}\int_{(b/a)(v/(2u)+c)}^{1}\frac{dr}{\sqrt{1-r^2}}=A_1(u,v)+A_2(u,v),$$

where

$$A_1(u,v)=\frac{1}{b}\biggl[1_{\{v/(2u)+c\geq 0\}}\int_{(b/a)(v/(2u)+c)}^{1}\frac{dr}{\sqrt{1-r^2}}+1_{\{v/(2u)+c<0\}}\int_0^1\frac{dr}{\sqrt{1-r^2}},\biggr],$$
$$A_2(u,v)=\frac{1}{b}1_{\{v/(2u)+c<0\}}\int_{(b/a)(v/(2u)+c)}^{0}\frac{dr}{\sqrt{1-r^2}}.$$

Clearly, $(1-r^2)^{-1/2}\asymp(1-r)^{-1/2}$, $\forall\, r\in[0,1)$ and $(1-r^2)^{-1/2}\asymp(1+r)^{-1/2}$, $\forall\, r\in(-1,0]$. Consequently,

$$A_1(u,v)\asymp\frac{1}{b}\biggl[1_{\{v/(2u)+c\geq 0\}}\sqrt{1-\frac{b}{a}\biggl(\frac{v}{2u}+c\biggr)}+1_{v/(2u)+c<0}\biggr],$$
$$A_2(u,v)\asymp\frac{1}{b}1_{\{v/(2u)+c<0\}}\biggl[1-\sqrt{1+\frac{b}{a}\biggl(\frac{v}{2u}+c\biggr)}\biggr].$$

Substituting a,b,c by their respective values and using the equality $(b/a)(1+c)=1$, one easily obtains

$$A_1(u,v)\asymp A_{11}(u,v)+A_{12}(u,v),$$

with

$$A_{11}(u,v)=\biggl(\frac{2u-v}{u(s^2-(u-v)^2)}\biggr)^{1/2}1_{\{s^2-(u+v)^2+2v^2\geq 0\}},$$
$$A_{12}(u,v)=\frac{1}{\sqrt{uv}}1_{\{s^2-(u+v)^2+2v^2<0\}}$$

and

$$A_2(u,v)\asymp\frac{1}{\sqrt{uv}}\frac{|s^2-(u+v)^2+2v^2|}{s^2-(u-v)^2}1_{\{s^2-(u+v)^2+2v^2<0\}}. \tag{A.8}$$

We study the contribution of $A_{11}(u,v)$, $A_{12}(u,v)$ and $A_2(u,v)$ to the integral $\varphi(s)$ defined by (A.7).

CONTRIBUTION OF $A_{12}(u,v)$. Set

$$\varphi_{1,2}(s)=\int_0^s\frac{u\,du}{\sqrt{s^2-u^2}}\int_0^{2u}vf(v)A_{12}(u,v)\,dv.$$

Fubini's theorem yields

$$\varphi_{1,2}(s)=\int_0^{2s}dv\,v^{1/2}f(v)\int_{\sqrt{2v^2+s^2}-v}^{s}\biggl(\frac{u}{s^2-u^2}\biggr)^{1/2}du.$$

The function $u \in]-s, +\infty[\to u/(s+u)$ increases. Thus

$$\begin{aligned}
&\int_0^{2s} dv\, v^{1/2} f(v) \int_{\sqrt{2v^2+s^2}-v}^{s} \frac{1}{\sqrt{s-u}} \left(\frac{\sqrt{2v^2+s^2}-v}{s+\sqrt{2v^2+s^2}-v} \right)^{1/2} du \\
&\qquad \le \varphi_{1,2}(s) \le \frac{1}{\sqrt{2}} \int_0^{2s} dv\, v^{1/2} f(v) \int_{\sqrt{2v^2+s^2}-v}^{s} \frac{1}{\sqrt{s-u}}\, du \qquad \text{(A.9)} \\
&\qquad = \sqrt{2} \int_0^{2s} dv\, v^{1/2} f(v) \big(s+v-\sqrt{2v^2+s^2}\big)^{1/2}\, dy.
\end{aligned}$$

For $v \in [0, 2s]$, $s^2 + 2v^2 \le (s+v)^2$, so that $s+v+\sqrt{2v^2+s^2} \asymp s+v$, and for all $v \in [0, 2s)$,

$$s+v-\sqrt{2v^2+s^2} = \frac{v(2s-v)}{s+v+\sqrt{2v^2+s^2}} \asymp \frac{v(2s-v)}{s+v}. \qquad \text{(A.10)}$$

Since $\sup_{s>0}(2s-v)/(s+v) = 2$, (A.9) and (A.10) imply

$$\varphi_{12}(s) \le 2\int_0^{2s} v f(v)\, dv. \qquad \text{(A.11)}$$

The first inequality in (A.9) provides a lower bound of $\varphi_{12}(s)$. Indeed, fix $v \ge 0$ and set

$$G_v(s) = \frac{\sqrt{2v^2+s^2}-v}{s+\sqrt{2v^2+s^2}-v}, \qquad s \ge 0.$$

A simple computation shows that $(\partial/\partial s)G_v(s) > 0$ if and only if $s > v\sqrt{2}$. Hence for $s \in [v/2, T]$, $G_v(s) \ge G_v(v\sqrt{2}) = \sqrt{2}-1$. These inequalities, (A.9) and (A.10), imply

$$\begin{aligned}
\varphi_{12}(s) &\ge \int_0^{2s} dv\, v^{1/2} f(v) \int_{\sqrt{2v^2+s^2}-v}^{s} \left(\frac{G_v(s)}{s-u} \right)^{1/2} du \\
&\ge C_T \int_0^{2s} v f(v) \left(\frac{2s-v}{s+v} \right)^{1/2} dv \qquad \text{(A.12)} \\
&\ge C_T \int_0^{3s/2} v f(v)\, dv.
\end{aligned}$$

CONTRIBUTION OF $A_2(u, v)$. Set

$$\varphi_2(s) = \int_0^s \frac{u\, du}{\sqrt{s^2-u^2}} \int_0^{2u} v\, f(v)\, A_2(u, v)\, dv.$$

(A.8) implies

$$\varphi_2(s) \asymp \int_0^s \frac{u^{1/2}\, du}{\sqrt{s^2-u^2}} \int_0^{2u} v^{1/2} f(v) \frac{|s^2-(u+v)^2+2v^2|}{s^2-(u-v)^2} 1_{\{s^2-(u+v)^2+2v^2<0\}}\, dv.$$

Fubini's theorem implies

$$\varphi_2(s) \asymp \int_0^{2s} dv\, v^{1/2} f(v) \int_{\sqrt{2v^2+s^2}-v}^{s} \left(\frac{u}{s^2-u^2} \right)^{1/2} \frac{(u+v)^2-s^2-2v^2}{s^2-(u-v)^2}\, du.$$

The functions

$$\psi_1(u) = \frac{u}{s+u} \quad \text{and} \quad \psi_2(u) = \frac{(u+v)^2 - s^2 - 2v^2}{s^2 - (u-v)^2}$$

are increasing on $[0, s]$ for $0 \le v \le 2u$; hence on $[0, s]$, $\psi_1(u)\psi_2(u) \le \psi_1(s)\psi_2(s) = \frac{1}{2}$, which implies

$$\varphi_2(s) \le C \int_0^{2s} v^{1/2} f(v)\big(s + v - \sqrt{2v^2 + s^2}\big)^{1/2}\, dv.$$

Since $\sup_{0\le v\le 2s}(2s - v)/(s + v) = 2$, (A.10) implies

$$0 \le \varphi_2(s) \le C \int_0^{2s} v f(v)\, dv. \tag{A.13}$$

Contribution of $A_{11}(u, v)$. Set

$$\varphi_{11}(s) = \int_0^s \frac{u\, du}{\sqrt{s^2 - u^2}} \int_0^{2u} v f(v) A_{11}(u, v)\, dv$$

Fubini's theorem implies

$$\begin{aligned} \varphi_{11}(s) = &\int_0^{2s} dv\, v f(v) \\ &\times \int_{v/2}^{\sqrt{2v^2+s^2}-v} \left(\frac{u(2u - v)}{(s^2 - u^2)(s^2 - (u - v)^2)} \right)^{1/2} du. \end{aligned} \tag{A.14}$$

If $s \in [0, T]$ and $0 \le v < 2s$, then $\sqrt{2v^2 + s^2} - v \le s$ and the function $u \mapsto u/(s + u)$ and $u \mapsto (2u - v)/(s + u - v)$ increase on $[v/2, s[$. Hence, in this interval, $0 \le u(2u - v)/((s + u)(s + u - v)) \le 1/2$, while $(s - u)(s - u + v) = (s + v/2 - u)^2 - v^2/4 > 0$. Therefore,

$$\varphi_{11}(s) \le C \int_0^{2s} v f(v) B_1(v)\, dv,$$

where

$$B_1(v) = \int_{v/2}^{\sqrt{2v^2+s^2}-v} \frac{du}{\sqrt{(s - u)(s - u + v)}}.$$

Assume first that $s \le v \le 2s$, and set $z = s + v/2 - u$; then

$$B_1(v) = \int_{s+3v/2-\sqrt{2v^2+s^2}}^{s} \frac{dz}{\sqrt{z^2 - v^2/4}} = \ln \phi(s),$$

with

$$\phi(s) = \frac{s + \sqrt{s^2 - v^2/4}}{s + 3v/2 - \sqrt{2v^2 + s^2} + \big(\big(s + 3v/2 - \sqrt{2v^2 + s^2}\big)^2 - v^2/4\big)^{1/2}}.$$

If $v \le 2s$, then $s + 3v/2 - \sqrt{2v^2 + s^2} \ge v/2$. Thus, for $v/2 \le s \le v$, $\phi(s) \le v(1 + \sqrt{3}/2)/(v/2) = 2 + \sqrt{3}$. This implies that

$$\int_s^{2s} vf(v)B_1(v)\,dv \le C\int_s^{2s} vf(v)\,dv. \tag{A.15}$$

Suppose now that $0 < v < s$. Then $B_1(v) = B_{11}(v) + B_{12}(v)$, with

$$B_{11}(v) = \int_{v/2}^{s/2} \frac{du}{\sqrt{(s-u)(s-u+v)}},$$

$$B_{12}(v) = \int_{s/2}^{\sqrt{2v^2+s^2}-v} \frac{du}{\sqrt{(s-u)(s-u+v)}}.$$

Clearly, $B_{11}(v) \le (s-v)/(s(s+2v))^{1/2} \le 1$. The change of variable $z = s + v/2 - u$ and a computation similar to that of $B_1(v)$ when $s \le v < 2v$ yield

$$B_{12}(v) = \int_{s+3v/2-\sqrt{2v^2+s^2}}^{(s+v)/2} \frac{dz}{\sqrt{z^2 - v^2/4}} = \ln\left(\frac{\psi(s)}{2}\right),$$

with

$$\psi(s) = \left(s + v + \sqrt{s^2 + 2sv}\right)\psi_1(s)^{-1} \tag{A.16}$$

and

$$\psi_1(s) = s + \frac{3v}{2} - \sqrt{2v^2 + s^2} + \left(\left(s + \frac{3v}{2} - \sqrt{2v^2 + s^2}\right)^2 - \frac{v^2}{4}\right)^{1/2}. \tag{A.17}$$

For $v < s$, it is easy to see that $s + 3v/2 - \sqrt{2v^2 + s^2} \ge v/2$, which implies $\psi(s) \le C(s+v)/v$. Therefore,

$$\int_0^s vf(v)B_1(v)\,dv \le C\left[\int_0^s vf(v)\,dv + \int_0^s vf(v)\ln\left(1 + \frac{s}{v}\right)dv\right]. \tag{A.18}$$

The inequalities (A.15) and (A.18) yield

$$\varphi_{11}(s) \le C\int_0^{2s} vf(v)\ln\left(1 + \frac{s}{v}\right)dv. \tag{A.19}$$

Thus, the inequalities (A.11), (A.13) and (A.19) imply the upper bound in (A.4). We now prove the lower bound of $J(s)$ in (A.4). Let $0 < v < s$; the function $u \mapsto u(2u - v)/(s+u)(s+u-v)$ is increasing on $[s/2, \sqrt{2v^2+s^2} - v]$. Hence (A.14) yields

$$\begin{aligned}\varphi_{11}(s) &\ge \int_0^s dv\, vf(v)\int_{s/2}^{\sqrt{2v^2+s^2}-v} du\left(\frac{u(2u-v)}{(s+u)(s+u-v)}\right)^{1/2}\\ &\qquad \times\left(\frac{1}{(s-u)(s-u+v)}\right)^{1/2}\\ &\ge \int_0^s v\,f(v)\sqrt{\frac{2}{3}}\left(\frac{s-v}{3s-2v}\right)^{1/2} B_{12}(v)\,dv.\end{aligned} \tag{A.20}$$

Let $0 < a < 1/5$; since the function $v \to (s-v)/(3s-2v)$ is decreasing on $[0,s]$, from (A.20) it follows that

$$\varphi_{11}(s) \geq C\int_0^{as} vf(v)B_{12}(v)\,dv,$$

where $C = (2(1-a)/3(3-2a))^{1/2}$. For $s > v$, $\psi_1(s)$ defined in (A.17) satisfies $\psi_1(s) \leq 2(s+3v/2-\sqrt{2v^2+s^2}) \leq 3v$, which implies $\psi(s) \geq \frac{1}{3}(1+s/v)$ and $B_{12}(v) \geq \ln[\frac{1}{6}(1+s/v)]$. Therefore

$$\varphi_{11}(s) \geq C\int_0^{as} vf(v)\ln\left(1+\frac{s}{v}\right)dv. \tag{A.21}$$

The inequalities (A.12) and (A.21) yield

$$\begin{aligned} J(s) &\geq C\big(\varphi_{11}(s)+\varphi_{11}(s)\big) \\ &\geq C\left[\int_0^{as} vf(v)\ln\left(1+\frac{s}{v}\right)dv + \int_0^{3s/2} vf(v)\,dv\right] \\ &\geq C\int_0^{s} vf(v)\ln\left(1+\frac{s}{v}\right)dv. \end{aligned}$$

The proof of the upper bound in (A.5) is a consequence of (A.4) and Fubini's theorem. For the lower bound, we first apply Fubini's theorem and do an explicit computation of the integral in the s variable. We obtain

$$\int_0^t ds\int_0^s rf(r)\ln\left(1+\frac{s}{r}\right)dr = \int_0^t rf(r)I(r)\,dr,$$

with $I(r) = (r+t)\ln(r+t) - 2r\ln(2r) - (t-r) - (t-r)\ln r$. It is easy to check that, for any $r \in [0, t/3]$, $I(r) \geq Ct\ln(1+t/r)$, for some positive constant C; this proves the lower bound in (A.5). □

LEMMA A3. *For any* $0 \leq t' < t \leq T$, $1 \leq p < 4/3$,

$$\int_{|z|<t'}\left|\frac{1}{\sqrt{t'^2-|z|^2}} - \frac{1}{\sqrt{t^2-|z|^2}}\right|^p dz \leq C|t-t'|^{1-p/2}, \tag{A.22}$$

for some positive constant depending on p *and* T.

PROOF. If $1 \leq p < 4/3$, $1-p/2 > 0$, and $1-3p/2 > -1$; hence

$$\begin{aligned} &\int_{|z|<t'}\left|\frac{1}{\sqrt{t'^2-|z|^2}} - \frac{1}{\sqrt{t^2-|z|^2}}\right|^p dz \\ &\quad = C\int_0^{t'}\left|\int_{t'}^{t}\frac{x\,dx}{(x^2-\rho^2)^{3/2}}\right|^p \rho\,d\rho \\ &\quad \leq C|t-t'|^{p-1}\int_{t'}^{t} x^p\left(\int_0^{t'}\frac{\rho\,d\rho}{(x^2-\rho^2)^{3p/2}}\right)dx \end{aligned}$$

$$\leq C|t-t'|^{p-1}\int_{t'}^{t} x^p (x^2-t'^2)^{-3p/2+1}\,dx$$

$$\leq C|t-t'|^{p-1}\int_0^{t-t'}\left(\frac{t'+x}{t'+2x}\right)^p (t'+2x)^{-p/2+1}x^{-3p/2+1}\,dx$$

$$\leq C|t-t'|^{p-1}T^{1-p/2}|t-t'|^{-3p/2+2}$$

$$\leq C|t-t'|^{1-p/2}. \qquad \square$$

LEMMA A4. *For $T>0$, $x\in\mathbf{R}^2$, $1<p<2$, $\zeta<2-p$, $0<s\leq T$,*

$$\int_{|z-x|<|z|<s}\left|\frac{1}{\sqrt{s^2-|z|^2}}-\frac{1}{\sqrt{s^2-|z-x|^2}}\right|^p dz \leq C|x|^{\zeta/2}, \tag{A.23}$$

$$\int_{|z-x|<|z|<s} dz\int_{|x|/2}^{s} dr\left(\frac{1}{\sqrt{r^2-|z|^2}}-\frac{1}{\sqrt{r^2-|z-x|^2}}\right) \leq C|x|^{1/2}, \tag{A.24}$$

where C is a positive constant depending on p and T.

PROOF. Let $J(s,x)$ be the left-hand side of (A.23) and set $x=(|x|\cos\theta_0, |x|\sin\theta_0)$ and $z=(\rho\cos(\theta+\theta_0), \rho\sin(\theta+\theta_0))$. Then $|z-x|<|z|<s$ if and only if $|x|<2\rho$ and $\cos\theta>|x|/(2\rho)$. Hence, for $|x|>2s$, $J(s,x)=0$, while for $|x|<2s$,

$$J(s,x)=2\int_{|x|/2}^{s}\rho\,d\rho\int_0^{\arccos(|x|/(2\rho))}d\theta \times\left|\frac{1}{\sqrt{s^2-\rho^2}}-\frac{1}{\sqrt{s^2-\rho^2+2\rho|x|\cos\theta-|x|^2}}\right|^p.$$

Let $\lambda\in(0,1)$ be such that $(p/2)(1+\lambda)<1$ and $\lambda p>\zeta$. Then

$$J(s,x)\leq C\int_0^{\arccos(|x|/(2s))}d\theta\int_{|x|/(2\cos\theta)}^{s}\frac{\rho|x|^{\lambda p/2}\big(2\rho\cos\theta-|x|\big)^{\lambda p/2}\,d\rho}{\big(s^2-\rho^2\big)^{p/2}\big(s^2-\rho^2+2\rho|x|\cos\theta-|x|^2\big)^{\lambda p/2}}$$

$$\leq C|x|^{\lambda p/2}s^{\lambda p/2}\int_0^{\arccos(|x|/(2s))}d\theta\int_{|x|/(2\cos\theta)}^{s}\frac{\rho d\rho}{(s^2-\rho^2)^{p/2(1+\lambda)}}$$

$$\leq C|x|^{\lambda p/2}\leq C|x|^{\zeta/2}.$$

This completes the proof of (A.23). We now check (A.24). Let $K(s,x)$ denote the left-hand side of (A.24). Then a similar change of variables as that of the first part of the proof yields

$$K(s,x)=2\int_{|x|/2}^{s}\rho\,d\rho\int_0^{\arccos(|x|/(2\rho))}d\theta \times\int_\rho^s\left[\frac{1}{\sqrt{r^2-\rho^2}}-\frac{1}{\sqrt{r^2-\rho^2+2\rho|x|\cos\theta-|x|^2}}\right]dr.$$

Since $\int (r^2+ar+b)^{-1/2}\,dr = \ln(a+2r+2\sqrt{r^2+ar+b})$,

$$K(s,x) = 2\int_{|x|/2}^{s} \rho\,d\rho \times \int_0^{\arccos(|x|/(2\rho))} \ln\biggl(\frac{\bigl(s+\sqrt{s^2-\rho^2}\bigr)\bigl(\rho+\sqrt{2\rho|x|\cos\theta-|x|^2}\bigr)}{\bigl(s+\sqrt{s^2-\rho^2+2\rho|x|\cos\theta-|x|^2}\bigr)\rho}\biggr)\,d\theta.$$

Since $2\rho\cos\theta > |x|$, we have

$$\frac{s+\sqrt{s^2-\rho^2}}{s+\sqrt{s^2-\rho^2+2\rho|x|\cos\theta-|x|^2}} < 1.$$

Hence,

$$K(s,x) \le C|x|^{1/2}\int_{|x|/2}^{s}\rho\,d\rho\int_0^{\arccos(|x|/(2\rho))}\frac{(2\rho\cos\theta-|x|)^{1/2}}{\rho}\,d\theta \le C|x|^{1/2}. \qquad \square$$

The following lemma is used in the proof of Proposition 1.4. Let us first recall some notation used in the proof of this proposition. For $t>0$, $h>0$ and $\xi\in\mathbf{R}^2$, we have set

$$\mu_{t,h} = \int_0^t ds\int_{|y|<s} dy\int_{|z|<s} dz\bigl[S(s,y)-S(s+h,y)\bigr] \times f(|y-z|)\bigl[S(s,z)-S(s+h,z)\bigr],$$

$$\mu_{t,h} = \int_0^t ds\int_{s<|y|<s+h} dy\int_{s<|z|<s+h} dz\,S(s+h,y)f(|y-z|)S(s+h,z),$$

$$M_{t,\xi} = \int_0^t ds\int_{|y|<s,\,|y-\xi|>s} dy\int_{|z|<s,\,|z-\xi|>s} dz\,S(s,y)f(|y-z|)S(s,z),$$

$$N_{t,\xi} = \int_{|\xi|/2}^t ds\int_{|y|<s,\,|y-\xi|<s} dy\int_{|z|<s,\,|z-\xi|<s} dz\bigl|S(s,y)-S(s,y-\xi)\bigr| \times f(|y-z|)\bigl|S(s,z)-S(s,z-\xi)\bigr|.$$

LEMMA A5. *Suppose that there exist $b\in]0,1[$, $C>0$ and $t_0>0$ such that for $0<t<t_0$,*

$$\int_0^t rf(r)\ln\Bigl(1+\frac{t}{r}\Bigr)\,dr \le Ct^b. \tag{A.25}$$

Then, for any fixed $T>0$, $a<b/(1+b)$, there exists $C>0$ such that for every $t\le T$, $h\vee|\xi|\le\frac{1}{2}$,

$$\mu_{t,h}+\tilde{\mu}_{t,h} \le Ch^a, \tag{A.26}$$

$$N_{t,\xi}+M_{t,\xi} \le C|\xi|^a. \tag{A.27}$$

PROOF. Using Lemma 4 in [4], we have

$$\mu_{t,h} \le 2\int_{|z|<|y|<t} dy\,dz \frac{f(|y-z|)}{|y|} \ln\left(1+\frac{2ht+h^2}{|y|^2-|z|^2}\right).$$

Let $y=(\rho\cos\theta_0, \rho\sin\theta_0)$, $y-z=(r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))$. Then

$$\mu_{t,h} \le 2\int_0^t \rho\,d\rho \int_0^{2\rho} rf(r)\,dr \int_0^{2\pi} d\theta_0 \int_0^{\arccos(r/2\rho)} d\theta \frac{1}{\rho}\ln\left(1+\frac{2ht+h^2}{r(2\rho\cos\theta-r)}\right).$$

Set $v=\cos\theta-r/2\rho$ in the last integral; given $0<\gamma<1$, we deduce that $\mu_{t,h}\le C(\mu_{t,h}^1+\mu_{t,h}^2)$, where (using the convention $\int_A^B \varphi(x)\,dx=0$ if $A>B$),

$$\mu_{t,h}^1 = \int_0^{h^\gamma} rf(r)\,dr \int_{r/2}^t d\rho \int_0^{1-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}},$$

$$\mu_{t,h}^2 = \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r/2}^t d\rho \int_0^{1-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}}.$$

Since for $v\in\,]0,\frac12(1-r/(2\rho))[$, $\sqrt{1-(r/(2\rho)+v)^2}\ge C\sqrt{1-r/(2\rho)}$ and for $v\in\,]\frac12(1-r/(2\rho)), 1-(r/(2\rho))[$, $\ln(1/v)\le\ln(4\rho/(2\rho-r))$, we have, using (A.25),

$$\begin{aligned}
\mu_{t,h}^1 &\le C\int_0^{h^\gamma} rf(r)\,dr\int_{r/2}^t d\rho \\
&\quad\times\left[\ln\left(\frac{Ch^\gamma}{\rho r}\right)\int_{r/(2\rho)}^1\frac{dv}{\sqrt{1-v^2}}+\frac{C}{\sqrt{1-r/(2\rho)}}\int_0^{(1-r/(2\rho))/2}\ln\left(\frac1v\right)dv\right.\\
&\qquad\left.+\ln\left(\frac{4\rho}{2\rho-r}\right)\int_{(1-r/(2\rho))/2}^{1-r/(2\rho)}\frac{dv}{\sqrt{1-(r/(2\rho)+v)}}\right]
\end{aligned}$$

$$\begin{aligned}
&\le C\int_0^{h^\gamma} rf(r)\,dr\int_{r/2}^t\left[\ln\left(\frac{Ch^\gamma}{r}\right)+\ln\left(\frac1\rho\right)+\ln\left(\frac{4\rho}{2\rho-r}\right)\right] \\
&\qquad\times\left(1-\frac{r}{2\rho}\right)^{1/2}d\rho \qquad\text{(A.28)}\\
&\le C\int_0^{h^\gamma} rf(r)\left[\ln\left(\frac{h^\gamma}{r}\right)+1\right]dr\\
&\le C\int_0^{h^\gamma} rf(r)\ln\left(1+\frac{h^\gamma}{r}\right)dr \le Ch^{\gamma b}.
\end{aligned}$$

Fix $0<\delta<1-\gamma$ and set

$$\begin{aligned}
\mu_{t,h}^{2,1} &= \int_{h^\gamma}^{2t} rf(r)\,dr\int_{r/2}^t d\rho\, 1_{\{h^\delta<(\rho-r)/(2\rho)\}}\\
&\qquad\times\int_0^{h^\delta}\ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}},
\end{aligned}$$

$$\mu_{t,h}^{2,2} = \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r/2}^{t} d\rho\, 1_{\{h^\delta<(\rho-r)/(2\rho)\}} \times \int_{h^\delta}^{1-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-\left(r/(2\rho)+v\right)^2}},$$

$$\mu_{t,h}^{2,3} = \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r}^{t} d\rho\, 1_{\{h^\delta\geq(\rho-r)/(2\rho)\}} \times \int_{0}^{1/2-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}},$$

$$\mu_{t,h}^{2,4} = \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r}^{t} d\rho\, 1_{\{h^\delta\geq(\rho-r)/(2\rho)\}} \times \int_{1/2-r/(2\rho)}^{1-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}},$$

$$\mu_{t,h}^{2,5} = \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r/2}^{r} d\rho\, 1_{\{h^\delta\geq(\rho-r)/(2\rho)\}} \times \int_{0}^{1-r/(2\rho)} \ln\left(1+\frac{Ch}{2\rho r v}\right)\frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}}.$$

Then $\mu_{t,h}^2 = \sum_{i=1}^5 \mu_{t,h}^{2,i}$. Fix $\beta \in]0,1[$; since $\ln(1+x) \leq Cx^\beta$ for $x > 0$, we have

$$\begin{aligned}\mu_{t,h}^{2,1} &\leq C\int_{h^\gamma}^{2t} rf(r)\,dr \int_{r/(1-2h^\delta)\wedge t}^{t} d\rho\, h^\beta \rho^{-\beta} r^{-\beta} \int_0^{h^\delta} v^{-\beta}\,dv \\ &\leq Ch^{\beta+\delta(1-\beta)-\gamma\beta}\int_{h^\gamma}^{2t} rf(r)\,dr \\ &\leq Ch^{\beta(1-\gamma)+\delta(1-\beta)},\end{aligned} \tag{A.29}$$

$$\begin{aligned}\mu_{t,h}^{2,3} &\leq C\int_{h^\gamma}^{2t} rf(r)\,dr \int_{r}^{r/(1-2h^\delta)\wedge t} d\rho\, h^\beta r^{-\beta} \rho^{-\beta} \int_0^{1/2-r/(2\rho)} v^{-\beta}\,dv \\ &\leq C\int_{h^\gamma}^{2t} rf(r)\left(r^{-\beta}h^\beta h^{-\gamma}\int_0^{2h^\delta r/(1-2h^\delta)} u^{1-\beta}\,du\right)dr \\ &\leq Ch^{\beta-\gamma+\delta(2-\beta)}\int_{h^\gamma}^{2t} r^{1+2(1-\beta)}f(r)\,dr = Ch^{\beta-\gamma+\delta(2-\beta)}\end{aligned} \tag{A.30}$$

and

$$\begin{aligned}\mu_{t,h}^{2,4} &\leq C\int_{h^\gamma}^{2t} rf(r)\,dr \int_{r}^{r/(1-2h^\delta)\wedge t} h^\beta r^{-\beta}\rho^{-\beta}\,d\rho \\ &\leq Ch^{\beta-\gamma\beta+\delta(1-\beta)}\int_{h^\gamma}^{2t} r^{2-\beta}f(r)\,dr = Ch^{\beta-\gamma\beta+\delta(1-\beta)}.\end{aligned} \tag{A.31}$$

Furthermore, since $\ln(1+x) \le x$ for $x \ge 0$,

$$
\begin{aligned}
\mu_{t,h}^{2,2} &\le C \int_{h^\gamma}^{2t} rf(r)\,dr \int_{r/(1-2h^\delta)\wedge t}^{t} d\rho\, \frac{h^{1-\delta}}{r\rho} \int_{h^\delta}^{1-r/(2\rho)} \frac{dv}{\sqrt{1-(r/(2\rho)+v)^2}} \\
&\le C h^{1-\delta-\gamma} \int_{h^\gamma}^{2t} rf(r) \ln\left(\frac{t}{r}\right) dr \qquad \text{(A.32)} \\
&\le C h^{1-\delta-\gamma}.
\end{aligned}
$$

Finally, Fubini's theorem implies that, given $0 < \varepsilon' < \varepsilon < 1$, $\frac{1}{2} < \beta < 1$,

$$
\begin{aligned}
\mu_{t,h}^{2,5} &\le C \int_{h^\gamma}^{2t} rf(r)\,dr \int_0^{1/2} dv \int_{r/(2(1-v))}^{r/(1-2h^\delta)\wedge t} \ln\left(1+\frac{Ch}{2\rho r v}\right) \frac{d\rho}{\sqrt{1-(r/(2\rho)+v)^2}} \\
&\le C \int_{h^\gamma}^{2t} r^2 f(r)\,dr \int_0^{1/2} \ln\left(1+\frac{Ch}{2r^2 v}\right) dv \\
&\le C \int_{h^\gamma}^{2t} r^2 f(r)\,dr \left[\int_0^{h^\varepsilon} \ln\left(1+\frac{Ch}{2r^2 v}\right) dv + h^\beta r^{-2\beta} \int_{h^\varepsilon}^{1/2} v^{-\beta}\,dv\right] \qquad \text{(A.33)} \\
&\le C \int_{h^\gamma}^{2t} rf(r) \left[h^\varepsilon \left\{\ln\left(\frac{1}{r}\right) + \ln\left(\frac{1}{h}\right) + 1\right\} + h^{\beta-\gamma(2\beta-1)}\right] dr \\
&\le C\left[h^{\varepsilon'} + h^{\beta-\gamma(2\beta-1)}\right].
\end{aligned}
$$

Inequalities (A.28)—(A.33) imply that given $\alpha < 1$ fixed, choosing $\beta \sim 1$ and $\varepsilon' \sim 1$ and $\delta \sim 0$, one has

$$\mu_{t,h} \le C\left[h^{\gamma b} + h^{\alpha(1-\gamma)}\right].$$

Choose $\gamma = \alpha/(b+\alpha)$; then $\mu_{t,h} \le C h^{\alpha b/(b+\alpha)}$; given $a < b/(b+1)$, it suffices to choose α close enough to 1 to conclude for $0 \le h \le T$,

$$\mu_{t,h} \le C h^a. \qquad \text{(A.34)}$$

We now prove a similar estimation for $\tilde{\mu}_{t,h}$. Clearly, using Fubini's theorem we have

$$
\begin{aligned}
\tilde{\mu}_{t,h} &= \frac{1}{2\pi^2} \int_0^t ds \int\int_{s<|z|<|y|<s+h} dy\,dz\, \frac{1}{\sqrt{(s+h)^2-|y|^2}} \\
&\qquad\qquad \times f(|y-z|) \frac{1}{\sqrt{(s+h)^2-|z|^2}} \\
&= \frac{1}{2\pi^2} \int\int_{|y|-h<|z|<|y|<t+h} dy\,dz\, f(|y-z|) I(h,y,z),
\end{aligned}
$$

where

$$I(h,y,z) = \int_{|y|\vee h}^{(|z|+h)\wedge(t+h)} \frac{du}{\sqrt{u^2-|y|^2}\sqrt{u^2-|z|^2}}.$$

Computations in [4] [proof of Lemma 3, estimation of $E(Y_2^2)$] yield

$$I(h, z, y) \leq \frac{1}{2|y|} \ln\left(1 + \frac{Ch^{1/2}}{|y|^2 - |z|^2}\right),$$

so that

$$\tilde{\mu}_{t,h} \leq C \int\int_{|y|-h<|z|<|y|<t+h} dy\, dz\, f(|y-z|) \frac{1}{|y|} \ln\left(1 + \frac{Ch^{1/2}}{|y|^2 - |z|^2}\right).$$

The change of variables $y = (\rho \cos\theta_0, \rho \sin\theta_0)$, $y - z = (r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))$ implies

$$\tilde{\mu}_{t,h} \leq C \int_{h/2}^{t+h} d\rho \int_0^{2\rho} rf(r)\, dr \times \int_{r/(2\rho)\leq\cos\theta<(r/(2\rho)+h(2\rho-h)/(2\rho r))\wedge 1} \ln\left(1 + \frac{Ch^{1/2}}{r(2\rho\cos\theta - r)}\right) d\theta.$$

Fubini's theorem and the change of variable $v = \cos\theta - r/(2\rho)$ show that, given $0 < \gamma < 1$, $\tilde{\mu}_{t,h} \leq C(\tilde{\mu}^1_{t,h} + \tilde{\mu}^2_{t,h})$, where

$$\tilde{\mu}^1_{t,h} = \int_0^{h^\gamma} rf(r)\, dr \int_{r/2}^{t+h} d\rho \int_0^{(h/r)((2\rho-h)/(2\rho))\wedge(1-r/(2\rho))} \ln\left(1 + \frac{Ch^{1/2}}{\rho r v}\right) \times \frac{dv}{\sqrt{1-(v+r/(2\rho))^2}},$$

$$\tilde{\mu}^2_{t,h} = \int_{h^\gamma}^{2(t+h)} rf(r)\, dr \int_{r/2}^{t+h} d\rho \int_0^{(h/r)((2\rho-h)/(2\rho))\wedge(1-r/(2\rho))} \ln\left(1 + \frac{C\, h^{1/2}}{\rho r v}\right) \times \frac{dv}{\sqrt{1-(v+r/(2\rho))^2}},$$

with the convention $\int_A^B \varphi(x)\, dx = 0$ if $A \geq B$. For $\gamma \geq \frac{1}{2}$, computations similar to that of $\mu^1_{t,h}$ yield

$$\begin{aligned}
\tilde{\mu}^1_{t,h} &\leq C \int_0^{h^\gamma} rf(r)\, dr \int_{r/2}^{t} d\rho \ln\left(\frac{Ch^{1/2}}{\rho r}\right) \int_0^{h^{1-\gamma}\wedge(1-r/(2\rho))} \frac{dv}{\sqrt{1-v-r/(2\rho)}} \\
&\quad + C \int_0^{h^\gamma} rf(r)\, dr \int_{r/2}^{t} d\rho \left[-\ln\rho + \ln\left(\frac{4\rho}{2\rho - r}\right)\left(1 - r/(2\rho)\right)^{1/2}\right] \\
&\leq C \int_0^{h^\gamma} rf(r)\, dr \int_{r/2}^{t} d\rho \left[\sqrt{1-r/(2\rho)} - \sqrt{(1-r/(2\rho)-h^{1-\gamma})^+}\right] \\
&\quad \times \ln\left(\frac{Ch^{1/2}}{\rho r}\right) + \int_0^{h^\gamma} rf(r)\, dr
\end{aligned}$$

$$\leq C \int_0^{h^\gamma} rf(r)\, dr \left[1 + \int_{r/2}^{t} d\rho \ln\left(\frac{Ch^{1/2}}{\rho r}\right) \frac{h^{1-\gamma}}{\sqrt{1-r/(2\rho)}}\right] \tag{A.35}$$

$$\le C\int_0^{h^\gamma} rf(r)\,dr\bigg[1+h^{1-\gamma}\ln\bigg(\frac{Ch^{1/2}}{r}\bigg)+h^{1-\gamma}\sup_{0\le\rho\le t}\rho^{1/2}\ln\bigg(\frac{1}{\rho}\bigg)\bigg]$$
$$\le h^{\gamma b}+h^{1-\gamma}\big(h^{\gamma b}+h^{b(\gamma\wedge 1/2)}\big)\le h^{\gamma b}+h^{1-\gamma+b/2}.$$

Furthermore, for $r\ge h^\gamma>h$ (since $\gamma<1$), $(h/r)(2\rho-h/(2\rho))\le 1-(r/(2\rho))$ if and only if $\rho\ge(r+h)/2$. Moreover, if $r\ge h^\gamma$ and $\rho\ge r/(1-2h^{1-\gamma})$, then $(h/r)((2\rho-h)/(2\rho))\le h^{1-\gamma}\le 1/2-r/(2\rho)$, so that for $v\le(h/r)((2\rho-h)/(2\rho))$, $(1-(v+r/(2\rho))^2)^{-1/2}\le C$. Therefore $\tilde\mu^2_{t,h}\le C\sum_{i=1}^3\tilde\mu^{2,i}_{t,h}$, where

$$\tilde\mu^{2,1}_{t,h}=\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_{r/2}^{(r+h)/2}d\rho\int_0^{1-r/(2\rho)}\ln\bigg(1+\frac{C\,h^{1/2}}{\rho rv}\bigg)\frac{dv}{\sqrt{1-(v+r/(2\rho))^2}},$$

$$\tilde\mu^{2,2}_{t,h}=\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_{(r+h)/2}^{r/(1-2h^{1-\gamma})}d\rho\int_0^{(h/r)((2\rho-h)/(2\rho))}\ln\bigg(1+\frac{Ch^{1/2}}{\rho rv}\bigg)\times\frac{dv}{\sqrt{1-(v+r/(2\rho))^2}},$$

$$\tilde\mu^{2,3}_{t,h}=\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_{r/(1-2h^{1-\gamma})}^{t+h}d\rho\int_0^{(h/r)((2\rho-h)/(2\rho))}\ln\bigg(1+\frac{Ch^{1/2}}{\rho rv}\bigg)dv.$$

For $h^\gamma\le r\le 2\rho\le r+h$, $0\le 1-r/(2\rho)\le h^{1-\gamma}$. Hence, since $h^{1-\gamma}\le h^{1/2-2\gamma}$,

$$\tilde\mu^{2,1}_{t,h}\le\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_0^{h^{1-\gamma}}dv\ln\bigg(1+\frac{Ch^{1/2}}{r^2v}\bigg)\int_{r/(2(1-v))}^{(r+h)/2}\frac{\sqrt{2\rho}}{\sqrt{2\rho(1-v)-r}}\,d\rho,$$

$$\tag{A.36}\begin{aligned}&\le C\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_0^{h^{1-\gamma}}\ln\bigg(\frac{C\,h^{1/2-2\gamma}}{v}\bigg)\frac{\sqrt{r+h}}{1-v}\,h^{1/2}\,dv\\&\le Ch^{1/2}\int_{h^\gamma}^{2(t+h)}rf(r)(r+h)^{1/2}h^{1-\gamma}\big(\ln(h^{1/2-2\gamma})+\ln(h^{\gamma-1})+1\big)\,dr\\&\le Ch^{3/2-\gamma}\bigg[1+\ln\bigg(\frac{1}{h}\bigg)\bigg].\end{aligned}$$

If $h^\gamma\le r$ and $r+h\le 2\rho$, then $0\le(h/r)((2\rho-h)/(2\rho))\le h/r\le h^{1-\gamma}$, hence

$$\tilde\mu^{2,2}_{t,h}\le\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_0^{h^{1-\gamma}}dv\ln\bigg(1+\frac{Ch^{1/2}}{r^2v}\bigg)\times\int_0^{r/(1-2h^{1-\gamma})}\frac{\sqrt{2\rho}}{\sqrt{2\rho(1-v)-r}}\,d\rho,$$

$$\tag{A.37}\begin{aligned}&\le C\int_{h^\gamma}^{2(t+h)}rf(r)\,dr\int_0^{h^{1-\gamma}}\ln\bigg(\frac{C\,h^{1/2-2\gamma}}{v}\bigg)\frac{r}{1-v}\,dv\\&\le C\int_{h^\gamma}^{2(t+h)}r^2f(r)\bigg[h^{1-\gamma}\ln\big(h^{1/2-2\gamma}\big)+h^{1-\gamma}\bigg\{\ln\bigg(\frac{1}{h}\bigg)+1\bigg\}\bigg]\,dr\\&\le Ch^{1-\gamma}\bigg[1+\ln\bigg(\frac{1}{h}\bigg)\bigg].\end{aligned}$$

Finally,

$$\begin{aligned}\tilde{\mu}^{2,3}_{t,h} &\le C\int_{h^\gamma}^{2(t+h)} rf(r)\,dr \int_{r/(1-2h^{1-\gamma})}^{t+h} d\rho \int_0^{h^{1-\gamma}} \left[\ln\left(h^{1/2-2\gamma}\right)+\ln\left(\frac{1}{v}\right)\right] dv\\ &\le C\int_{h^\gamma}^{2(t+h)} rf(r)\,dr \int_{r/(1-2h^{1-\gamma})}^{t+h} h^{1-\gamma}\left[\ln\left(\frac{1}{h}\right)+1\right] d\rho \qquad \text{(A.38)}\\ &\le Ch^{1-\gamma}\left[\ln\left(\frac{1}{h}\right)+1\right].\end{aligned}$$

Inequalities (A.35)–(A.38) yield for $\frac{1}{2} \le \gamma < 1$, $0 < h \le \frac{1}{2}$, $\varepsilon > 0$,

$$\tilde{\mu}_{t,h} \le C\left(h^{\gamma b} + h^{1-\gamma+b/2} + h^{1-\gamma-\varepsilon}\right) \le C\left(h^{\gamma b} + h^{1-\gamma-\varepsilon}\right).$$

Fix $a \in [0, b/(1+b)[$ and let $\varepsilon > 0$ be such that $\varepsilon < (1-b)/2 \wedge (1-a(1+b)/b)$. Then $\gamma = (1-\varepsilon)/(1+b) \ge \frac{1}{2}$ and

$$\tilde{\mu}_{t,h} \le Ch^a. \qquad \text{(A.39)}$$

Inequalities (A.34) and (A.39) conclude the proof of (A.26). We now check (A.27). Set $h = |\xi|$; then

$$\begin{aligned}M_{t,\xi} \le 2\int\int_{|y|-h<|z|<|y|<t+h} & f(|y-z|)\\ &\times \left(\int_{|y|}^{(|z|+h)\wedge t} \frac{ds}{\sqrt{s^2-|y|^2}\sqrt{s^2-|z|^2}}\right) dy\,dz.\end{aligned}$$

Estimations similar to those proved for $\tilde{\mu}_{t,h}$ imply that for $0 \le h \le \frac{1}{2}$ and $a \in [0, b/(1+b)[$

$$M_{t,\xi} \le C|\xi|^a. \qquad \text{(A.40)}$$

Similarly, the computations of $E(Z_1^2)$ in the proof of Lemma 3 in [4] imply

$$N_{t,\xi} \le C(N^1_{t,\xi} + N^2_{t,\xi}),$$

with

$$\begin{aligned}N^1_{t,\xi} &= \int\int_{D_1} dy\,dz\,f(|y-z|)\int_{\max(|y|,|z|)}^t \left|S(s,y)-S(s,y-\xi)\right|\\ &\qquad\qquad \times \left|S(s,z)-S(s,z-\xi)\right| ds,\\ N^2_{t,\xi} &= \int\int_{D_3} dy\,dz f(|y-z|)\int_{\max(|y|,|z-\xi|)}^t \left|S(s,y)-S(s,y-\xi)\right|\\ &\qquad\qquad \times \left|S(s,z-\xi)-S(s,z)\right| ds,\end{aligned}$$

where

$$\begin{aligned}D_1 &= \{(y,z)\colon |y-\xi| < |y| < t, |z-\xi| < |z| < t\},\\ D_3 &= \{(y,z)\colon |y-\xi| < |y| < t, |z| < |z-\xi| < t\}.\end{aligned}$$

By Lemma 4 in [4],

$$N^1_{t,\xi} \le C \int\int_{|z-\xi|<|z|<|y|<t} f(|y-z|)\frac{1}{|y|}\ln\left(1+\frac{|z|^2-|z-\xi|^2}{|y|^2-|z|^2}\right)dy\,dz$$
$$\le C\int\int_{|z|<|y|<t} f(|y-z|)\frac{1}{|y|}\ln\left(1+\frac{C|\xi|}{|y|^2-|z|^2}\right)dy\,dz.$$

Therefore as for $\mu_{t,h}$, for $0<a<b/(1+b)$ and $|\xi|<T$,

$$N^1_{t,\xi} \le C|\xi|^a.$$

Set

$$N^2_{t,\xi} = N^{2,1}_{t,\xi} + N^{2,2}_{t,\xi},$$

with

$$N^{2,1}_{t,\xi} = \int\int_{D_3\cap\{|z-\xi|<|y|\}} dy\,dz\, f(|y-z|)\int_{|y|}^t ds S(s,y)\big|S(s,z-\xi)-S(s,z)\big|\,ds,$$
$$N^{2,2}_{t,\xi} = \int\int_{D_3\cap\{|y|<|z-\xi|\}} dy\,dz\, f(|y-z|)\int_{|z-\xi|}^t \big|S(s,y)-S(s,y-\xi)\big| \times S(s,z-\xi)\,ds.$$

As for $N^1_{t,\xi}$, Lemma 4 in [4] yields

$$N^{2,1}_{t,\xi} \le C\int\int_{D_3\cap\{|z-\xi|<|y|\}} f(|y-z|)\,\frac{1}{|y|}\ln\left(1+\frac{|z-\xi|^2-|z|^2}{|y|^2-|z-\xi|^2}\right)dydz$$
$$\le C\int\int_{D_3\cap\{|z-\xi|<|y|\}} f(|y-z|)\frac{1}{|y|}\ln\left(1+\frac{|z-\xi|^2-|z|^2}{|y|^2-|z|^2}\right)dy\,dz$$
$$\le C|\xi|^a.$$

The same arguments imply $N^{2,2}_{t,\xi} \le C|\xi|^a$. Consequently,

$$N_{t,\xi} \le C|\xi|^a. \tag{A.41}$$

Inequalities (A.40) and (A.41) imply (A.27) and complete the proof of the lemma. □

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REFERENCES

[1] ALBEVERIO, S., HABA, A. and RUSSO, F. (1996). Trivial solutions for a non-linear two space dimensional wave equation perturbed by a space-time white noise. *Stochastics Stochastics Rep.* **56** 127–160.

[2] BALLY, V. and PARDOUX, E. (1998). Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.* **9** 27–64.

[3] CARMONA, R. and NUALART, D. (1988). Random nonlinear wave equations: smoothness of the solutions. *Probab. Theory Related Fields* **79** 469–508.
[4] DALANG, R. and FRANGOS, N. (1998). The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26** 187–212.
[5] DUNFORD, N. and SCHWARZ, J. T. (1988). *Linear Operators I: General Theory.* Wiley, New York.
[6] KARCZEWSKA, A. and ZABCZYK, J. (1997). A note on stochastic wave equations. Preprint 574, Polish Academy of Sciences.
[7] MÉTIVIER, M. (1982). *Semimartingales.* de Gruyter, Berlin.
[8] MUELLER, C. (1997). Long time existence for the wave equation with a noise term. *Ann. Probab.* **25** 133–151.
[9] NUALART, D. (1995). *Malliavin Calculus and Related Topics*. Springer, New York.
[10] NUALART, D. (1998). Analysis on the Wiener space and anticipating calculus. *École d'Été de Probabilités de Saint-Flour XXV. Lecture Notes in Math.* **1690** 863–901. Springer, Berlin.
[11] ROVIRA, C. and SANZ-SOLÉ, M. (1996). The law of the solution to a nonlinear hyperbolic SPDE. *J. Theoret. Probab.* **9** 863–901.
[12] STROOCK, D. W. (1993). *Probability Theory: An Analytic View*. Cambridge Univ. Press.
[13] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. *École d'Été de probabilités de Saint-Flour XIV. Lecture Notes in Math.* **1180** 266–437. Springer, Berlin.
[14] WATANABE, S. (1984). Lectures on stochastic differential equations and Malliavin calculus. *Tata Inst. Fund. Res.* Springer, Berlin.
[15] WILCOX, C. H. (1991). The Cauchy problem for the wave equation with distributional data: an elementary approach. *Amer. Math. Monthly* **98** 401–410.

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