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AN INVARIANCE PRINCIPLE FOR DIFFUSION IN TURBULENCE

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We prove an almost sure invariance principle for diffusion driven by velocities with unbounded stationary vector potentials. The result generalizes to multiple particles motion, driven by a common velocity field and independent molecular Brownian motions.

CONTENTS

1. Introduction. Turbulent dispersion is one of the fundamental problems in statistical fluid dynamics. Its simplest model takes the form of a diffusive particle convected by a random velocity field with prescribed statistics. The problem then is to determine the long-time behavior of the particle motion from the statistics of velocity.

More specifically, let the sample path of the particle $\{\boldsymbol{x}(t)\}_{t\geq 0}$ be a solution of the following Itô stochastic differential equation,

$$d\boldsymbol{x}(t) = \boldsymbol{b}(t, \boldsymbol{x}(t))\, dt + \sqrt{2}\, d\boldsymbol{w}(t), \tag{1}$$

where $\{\boldsymbol{w}(t)\}_{t\geq 0}$ is the standard d-dimensional Brownian motion and the vector field $\boldsymbol{b}(t, \boldsymbol{x})$, $(t, \boldsymbol{x}) \in R^1 \times R^d$ is space-homogeneous, time-stationary, ergodic, incompressible and of zero mean. One would like to find sufficiently general conditions on the velocity $\boldsymbol{b}$ under which the rescaled processes

$$\boldsymbol{x}_\varepsilon(t) = \varepsilon \boldsymbol{x}(t/\varepsilon^2) \tag{2}$$

converge to a Brownian motion. The scaling in (2) is known as diffusive scaling. Sometimes a space-homogeneous, time-stationary function is simply referred to as a stationary function.

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Long-time diffusive limit for steady (time-independent) flows have been previously obtained [Papanicolaou and Varadhan (1982), Oelschläger (1988), Osada (1982), Fannjiang and Komorowski (1996), Fannjiang and Papanicolaou (1996) and the references therein]. The conditions on $\boldsymbol{b}$ are naturally phrased in terms of a stream matrix $\mathbf{H}$. A stream matrix $\mathbf{H}$ for the velocity $\boldsymbol{b}$ is a skew-symmetric matrix and satisfies

$$\nabla \cdot \mathbf{H} = \boldsymbol{b}.$$

This determines $\mathbf{H}$ up to a Coulomb gauge. Without loss of generality, we assume that $\mathbf{H}$ is expressed in terms of a solenoidal vector potential with zero mean, so it is unique. The stream matrix is not homogeneous in space in general even if $\boldsymbol{b}$ is homogeneous. However, in dimension three or higher, the stream matrix is homogeneous and square integrable provided that the velocity correlations decay faster than the power of two at large distance [Fannjiang and Papanicolaou (1996)]. For nonhomogeneous stream matrices, anomalous diffusions, rather than the normal diffusion, are expected [Bouchaud and Georges (1990), Fannjiang (1998), Koch and Brady (1989)].

For steady flows, long-time behavior is diffusive *in probability* with respect to the velocity ensemble if $\mathbf{H}(\boldsymbol{x})$ is square integrable [Fannjiang and Papanicolaou (1996)]. For almost sure convergence, the best condition to date is that $\mathbf{H}$ has finite pth-moment with $p > d$, the dimension [Fannjiang and Komorowski (1996)].

The time dependence in velocity, hence in stream matrix, introduces additional difficulties, which are not addressed by previous techniques. Recently, Landim, Olla and Yau (1996) showed the convergence to Brownian motion in the case of time-dependent, uniformly bounded fields with bounded stream matrices.

In the present paper, we show the almost sure convergence to Brownian motion (an invariance principle) for velocity fields with unbounded stream matrices. The stream matrices are space homogeneous, time stationary and ergodic and satisfy the integrability condition

$$\limsup_{\varepsilon \downarrow 0} \int_0^1 \left(\int_{|\boldsymbol{x}|<1} \left| \mathbf{H}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon) \right|^p d\boldsymbol{x} \right)^{q/p} dt < \infty, \tag{3}$$

with

$$\frac{d}{p} + \frac{2}{q} < 1 \tag{4}$$

in addition to the standard regularity conditions on the velocity (Section 2). From this it follows that the invariance principle holds for stream matrices with finite pth-moment,

$$\mathbf{E}|\mathbf{H}|^p < \infty, \qquad p = q > d + 2. \tag{5}$$

When $q = \infty$ in (3), then $p > d$ and the condition for steady flows described in Fannjiang and Komorowski (1996) is essentially recovered.

Unbounded velocity fields such as Gaussian fields or those given by Poisson point vortices are commonly used in turbulence modeling and computation [Chorin (1994), Frisch (1996), McComb (1990)]. The resulting stream matrices are unbounded in far fields with probability 1. In this regard, it is essential to deal with unbounded stream matrices. A stationary Gaussian velocity field, with correlations decaying faster than the power of two in three or higher dimensions, has a stationary Gaussian stream matrix satisfying (3) and (4). Outside the realm of Gaussian fields, we show that, for dimension $d > 2$, any stationary velocity field $\boldsymbol{b}$, with finite pth moment, $p > 2d + 4$, and with the spatial α-mixing coefficient decaying sufficiently fast gives rise to a stationary stream matrix satisfying (5) [see the remark after condition (H2)].

Condition (3)–(4) does not take into account temporal randomness, which may compensate lack of space decorrelation in velocity and is probably far from being optimal.

The invariance principle can be easily extended, in two directions:

1. The case with an unbounded symmetric part (Section 4). If the molecular diffusivity is a positive-definite, square integrable (i.e., $\mathbf{E}|\mathbf{S}|^2 < \infty$) random matrix $\mathbf{S}(t, \boldsymbol{x})$ bounded away from zero such that

$$\limsup_{\varepsilon \downarrow 0} \int_0^1 dt \left(\int_{|\boldsymbol{x}|<1} \left| \mathbf{S}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon) \right|^{p/2} d\boldsymbol{x} \right)^{q/p} < \infty, \tag{6}$$

and, if the unbounded stream matrix $\mathbf{H}$ is square integrable (i.e., $\mathbf{E}|\mathbf{H}|^2 < \infty$) and

$$\limsup_{\varepsilon \downarrow 0} \int_0^1 dt \left(\int_{|\boldsymbol{x}|<1} \left| \sqrt{\mathbf{S}^{-1}}\, \mathbf{H}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon) \right|^{p} d\boldsymbol{x} \right)^{q/p} < \infty, \tag{7}$$

with p, q as in (4) then the invariance principle holds (See Section 4).
2. Multiparticle motions in flows, driven by independent molecular Brownian motions. Our estimates show that, in the diffusive limit (2), the joint process of N particles essentially comprises N d-dimensional martingales w.r.t. independent d-dimensional Brownian motions and, thus becomes N independent diffusion processes in the limit. Thereby we obtain the invariance principles for multiparticle motions in flows.

1.1. *Outline of the method.* The standard approach to diffusive limit theorems consists of two parts: the weak compactness of the rescaled process $\boldsymbol{x}_\varepsilon(t)$ and the identification of limit. For a flow with a uniformly bounded stream matrix, be it steady or not, the compactness is readily available by either the Aronson–Nash estimate for Green's function [Aronson (1967)] and/or other techniques for estimating the modulus of continuity [Olla (1994)]. When it comes to identification of limit, time dependence introduces additional difficulties and the previous techniques fall short of establishing the so-called sublinear growth estimate for the difference between $\boldsymbol{x}_\varepsilon(t)$ and a particular martingale, which is supposed to capture the long-time asymptotics of $\boldsymbol{x}_\varepsilon(t)$.

When the stream matrix is time dependent and unbounded, both compactness and identification of limit are in question. As in Osada (1982), Kozlov and Molchanov (1984), Kozlov (1985), Fannjiang and Komorowski (1996) and Landim, Olla and Yau (1996), we decompose the process $\boldsymbol{x}_\varepsilon(t)$ into a martingale part, called the harmonic coordinates, and a fluctuation, called the correctors. This decomposition can be done if velocity is locally Lipschitz, and the stream matrix is locally bounded. The hard part of the approach is to obtain uniform estimates on the far field behaviors of correctors, which is the difference between the harmonic coordinates and $\boldsymbol{x}$ [i.e., (Y3) of the Main Lemma in Section 2]. Under the integrability condition (3)–(4), this can be done by generalizing to the current setting the classical estimates for the maximum and continuity moduli of Moser (1964) and Kružkov (1963), respectively. This enables us to show that the fluctuation is uniformly small with probability 1 in the velocity ensemble. The standard martingale invariance principle then gives the desired result.

Moser and Kružkov proved maximum and uniform Hölder continuity estimates, respectively, for positive subsolutions of parabolic, divergence form equations with bounded coefficients. For the bounded stream matrix, their estimates easily lead to the desired results. When Moser's and Kružkov's arguments are used in the context of unbounded coefficients with (6) and (7), the estimate for maximum norm remains intact but the Hölderian estimate is weakened to that for the continuity modulus (Lemmas 2 and 3 in Section 4).

The major difference in techniques between the present paper and Fannjiang and Komorowski (1996) lies in the asymptotic uniform linearity estimate for the harmonic coordinates [(Y3) of the Main Lemma in Section 2]. In the case of steady flows, by Sobolev's theorem of compact embedding, the estimate for the maximum modulus is sufficient to show that the correctors die out uniformly on any compact sets. In the case of unsteady flows, due to the lack of ellipticity in the time variable, the compact embedding fails in space–time and the estimate for the maximum modulus (Lemma 2) gives only a bound on the correctors. To resolve this problem we prove estimates for the modulus of continuity (Lemma 3) which, together with the estimate for the maximum modulus (Lemmas 1 and 2), yields the compactness in the space of continuous functions. Then by a weak convergence result (Proposition 3), we show that, in the diffusive limit (2), the harmonic coordinates become uniformly close to the original ones almost surely. Thus, in the limit, the displacement is essentially a martingale. The desired invariance principle for single-particle motion follows from the standard martingale invariance principle.

The invariance principle for multiple particles, each driven by the common velocity field and independent molecular Brownian motions, also follows easily from our approach since, in the limit, the joint displacements become independent martingales. In general, multiparticle motion in turbulence is much more difficult to study, since the system of equations for the joint displacements is not space homogeneous.

2. Main results. Let the triplet $(\Omega, \mathscr{V}, P)$ be a probability space and $\mathbf{E}$ the associated expectation. Let $(\Omega, \mathscr{V}, P)$ be endowed with a group of P-measure preserving transformations $\tau_{t,\boldsymbol{x}}$ with parameters $(t, \boldsymbol{x}) \in R^1 \times R^d$. The group $\tau_{t,\boldsymbol{x}}$ is assumed to be *ergodic*, that is, if $P[\tau_{t,\boldsymbol{x}}(A) \triangle A] = 0$ ($\triangle$ here means the set operation of symmetric difference) for all $(t, \boldsymbol{x}) \in R^1 \times R^d$, then either $P[A] = 0$ or $P[A] = 1$. Moreover, $\tau_{t,\boldsymbol{x}}$ is assumed to be stochastically continuous. This implies that the group of isometries $U_{t,\boldsymbol{x}}$ on $L^2(\Omega)$ space given by

$$U_{t,\boldsymbol{x}} f(\omega) = f(\tau_{t,\boldsymbol{x}}(\omega)), \qquad (t, \boldsymbol{x}) \in R^1 \times R^d$$

is strongly continuous. We have the spectral resolution

$$U_{t,\boldsymbol{x}} = \int\int \exp\{i(t\tau + \boldsymbol{x}\boldsymbol{\xi})\} \mathscr{U}(d\tau, d\boldsymbol{\xi}), \qquad (t, \boldsymbol{x}) \in R^1 \times R^d,$$

where $\mathscr{U}(d\tau, d\xi)$ is the spectral measure of $U_{t,\boldsymbol{x}}$.

For a fixed $k \in \{0, \ldots, d\}$ we define the skew-adjoint operator,

$$D_k = i\int\int \xi_k \mathscr{U}(d\tau, d\boldsymbol{\xi}),$$

with $\xi_0 = \tau$. For $k = 1, \ldots, d$, D_k generates the subgroup $\{U_{0, x\mathbf{e}_k}\}_{x \in R}$ and D_0 the subgroup $\{U_{t,0}\}_{t \in R}$. Here $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_d = (0, \ldots, 0, 1) \in R^d$.

Let $L^p(\Omega)$, $p > 0$, be the space of functions on $(\Omega, \mathscr{V}, P)$ with finite pth moment. Let $H^1(\Omega)$ be the space of square integrable functions with square integrable spatial gradient

$$H^1(\Omega) = \left\{ g \in L^2(\Omega) \,\middle|\, \int |D_k g|^2(\omega)\, dP(\omega) < \infty, k = 1, \ldots, d. \right\}.$$

A function $g(t, \boldsymbol{x}; \omega)$ is space homogeneous and time stationary if it is the translate of a function $g \in L^2(\Omega)$,

$$g(t, \boldsymbol{x}; \omega) := g(\tau_{t,\boldsymbol{x}}(\omega)), \qquad (t, \boldsymbol{x}) \in R^1 \times R^d.$$

Such functions are also referred to as stationary functions. With a slight abuse of notation we denote the function and its translate by the same notation, distinguished only by the display of independent variables.

Let $\mathbf{H}(\omega) = [H_{k,l}(\omega)] \in (L^2(\Omega))^{d \times d}$ be a $d \times d$ antisymmetric matrix satisfying the following two conditions almost surely w.r.t. P.

(H1) $\mathbf{EH} = \mathbf{0}$ and

$$\limsup_{\varepsilon \downarrow 0} \int_0^1 \left(\int_{|\boldsymbol{x}|<1} \left| \mathbf{H}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon) \right|^p d\boldsymbol{x} \right)^{q/p} dt < \infty \tag{8}$$

for

$$\frac{d}{p} + \frac{2}{q} < 1, \qquad p, q > 0.$$

(H2) The matrix $\mathbf{H}(t, \boldsymbol{x}; \omega)$ is continuous in t and $\boldsymbol{x}$. Its realizations are continuously differentiable in $\boldsymbol{x}$.

In the sequel ∂_m, $m = 1, \dots, d$, denote the partial derivatives in space and ∂_t the time derivative. As usual, $\nabla = (\partial_1, \dots, \partial_d)$ and $\Delta = \nabla \cdot \nabla = \sum_{m=1}^{d} \partial_m^2$.

Thanks to (H1) and (H2), the velocity field $\boldsymbol{b} = \nabla \cdot \mathbf{H}$ is well defined and is divergence free.

REMARK. Let us give a condition on the velocity field $\boldsymbol{b}$ which guarantees (H1). For arbitrary $\mathbf{x} \in R^d$ and $h > 0$ we denote by $B_-(\mathbf{x}) = \{\mathbf{y} = (y_1, \dots, y_d) : y_1 \le x_1, \dots, y_d \le x_d\}$ and $B_+(\boldsymbol{x}, h) = \{\boldsymbol{y}\colon \operatorname{dist}(\boldsymbol{y}, B_-(\boldsymbol{x})) > h\}$. The spatial α-mixing coefficient of the field is defined as $\alpha(h) := \inf_{\boldsymbol{x}} \alpha(h, \boldsymbol{x})$, where $\alpha(h, \boldsymbol{x}) = \inf |P(A \cap B) - P(A)P(B)|$. Here the infimum is taken over all events A and B in the σ-algebras generated by the field restricted to $B_-(\boldsymbol{x})$ and $B_+(\boldsymbol{x}, h)$, respectively.

Let $\mathscr{L}$ be the L^p-generator of the Markov process $\eta(t) = \tau_{0, \boldsymbol{w}(t)}(\omega)$ where $\boldsymbol{w}(t)$, $t \ge 0$ is a standard Brownian motion independent of ω. It is easy to see that $\boldsymbol{b}$ gives rise to a stream matrix satisfying (H1) if $\boldsymbol{b} = (-\mathscr{L})^{1/2} \boldsymbol{g}$ for some $\boldsymbol{g} \in (L^p(\Omega))^d$. This condition in turn is implied by

$$\int_0^\infty \frac{\mathbf{E}\{|P^t \boldsymbol{b}|^p\}^{1/p}}{\sqrt{t}}\, dt < \infty. \tag{9}$$

Here P^t is the semigroup of transition probability associated with η. One can show that $\mathbf{E}\{|P^t \boldsymbol{b}|^p\}^{1/p} \sim t^{-d/4}$ for large t if $\boldsymbol{b} \in L^q$, for any $q > 2p$, and $\sup_{h>0} h^n \alpha(h) < \infty$ for n sufficiently large [see Ibragimov and Linnik (1971), Theorem 17.2.2]. For such $\mathbf{b}$ and $d > 2$, (9) holds. □

The velocity field $\boldsymbol{b}$ is assumed to satisfy the following condition.

(B) The field $\boldsymbol{b}(t, \boldsymbol{x}; \omega)$ is P a.s. Hölder continuous in t and Lipschitz continuous in $\boldsymbol{x}$ on any compact subset of $R^1 \times R^d$.

REMARK. One can show by using a truncation argument and a result of Port and Stone (1976) that the solutions $\boldsymbol{x}(t)$ of (1) do not explode with probability 1 under stationarity and (B) without the usual linear growth condition.

Let the triplet $(W, \mathscr{M}, Q)$ be the underlying probability space of a standard Brownian motion $\boldsymbol{w}(t)$, $t \ge 0$ and let M denote the expectation w.r.t. the measure Q. We assume that the measure Q is independent of the measure P.

Let $\boldsymbol{x}^{(j)}(t; \omega, w^{(j)})$, $w^{(j)} \in W^{(j)}$, $j = 1, \dots, N$ be the solutions of (1) with the Brownian motion $\boldsymbol{w}(t)$ replaced by N *independent* Brownian motions $\boldsymbol{w}^{(j)}(t)$ with their corresponding probability space denoted by $(W^{(j)}, \mathscr{M}^{(j)}, Q^{(j)})$, $j = 1, \dots, N$. The equation of motion for the joint displacement of N particles,

$$\boldsymbol{X}(t; \omega, \tilde{w}) = \left(\boldsymbol{x}^{(1)}(t; \omega, w^{(1)}), \dots, \boldsymbol{x}^{(N)}(t; \omega, w^{(N)})\right),$$

takes the form

$$d\boldsymbol{X}(t; \omega, \tilde{w}) = \mathbf{B}(t, \boldsymbol{X}(t); \omega, \tilde{w})\, dt + \sqrt{2}\, d\boldsymbol{W}(t; \tilde{w}), \tag{10}$$

where $\boldsymbol{B}(t, \boldsymbol{X}; \omega, \tilde{w}) = (\boldsymbol{b}(t, \boldsymbol{x}^{(1)}; \omega, w^{(1)}), \boldsymbol{b}(t, \boldsymbol{x}^{(2)}; \omega, w^{(2)}), \ldots, \boldsymbol{b}(t, \boldsymbol{x}^{(N)}; \omega, w^{(N)})$ and $\boldsymbol{W}(t; \tilde{w}) = (\boldsymbol{w}^{(1)}(t; w^{(1)}), \ldots, \boldsymbol{w}^{(N)}(t; w^{(N)}))$. As usual, we drop $\omega, w^{(j)}$ from the notations where there is no risk of confusion. Let $\boldsymbol{x}_\varepsilon^{(j)}(t) = \varepsilon \boldsymbol{x}^{(j)}(t/\varepsilon^2)$, $j = 1, \ldots, N$, be the rescaled displacement and let $\boldsymbol{X}^\varepsilon(t) = (\boldsymbol{x}_\varepsilon^{(1)}(t), \ldots, \boldsymbol{x}_\varepsilon^{(N)}(t))$ be the rescaled, joint displacement.

It appears that (10) is of the form similar to that of the single-particle motion (1), except that the joint velocity field $\boldsymbol{B}(t, \boldsymbol{X})$ is no longer homogeneous in $\boldsymbol{X}$. This is one of the difficulties in studying multiparticle motions. However, this difficulty makes no difference in our approach because of the sample-wise nature of the results in the Main Lemma.

MAIN THEOREM. *Under conditions* (H1), (H2) *and* (B), *the following statements hold almost surely w.r.t. the measure* P:

(i) *The limit*

$$\lim_{t \uparrow \infty} \mathrm{M} \frac{x_i(t) x_j(t)}{t} = d_{ij}, \qquad i, j = 1, \ldots, d \tag{11}$$

exists and is deterministic.

(ii) *The family of processes* $\{\boldsymbol{x}_\varepsilon(t)\}_{t \geq 0}$, *given by* (10), *satisfies the invariance principle, as* $\varepsilon \to 0$, *with the limiting Wiener process having the covariance matrix* $\mathbf{D} = [d_{ij}]$.

(iii) *The family of processes* $\{\boldsymbol{X}_\varepsilon(t)\}_{t \geq 0}$ *satisfies the invariance principle, as* $\varepsilon \to 0$, *with the limiting Wiener process whose covariance matrix is block diagonal with each block given by* $\mathbf{D} = [d_{ij}]$.

The key ingredient of the proof is the Main Lemma about the existence of a special random coordinate system $\boldsymbol{y}(t, \boldsymbol{x})$, known as harmonic coordinates [Kozlov and Molchanov (1984)].

MAIN LEMMA. *Under conditions* (H1), (H2) *and* (B), *the harmonic coordinates* $\boldsymbol{y}(t, \boldsymbol{x}, \omega) = (y_1(t, \boldsymbol{x}, \omega), \ldots, y_d(t, \boldsymbol{x}, \omega))$ *exist, almost surely w.r.t.* P, *with the properties* (Y1), (Y2) *and* (Y3).

(Y1) *The harmonic coordinates* $\boldsymbol{y}$ *are continuously differentiable in t and twice continuously differentiable in* $\boldsymbol{x}$, *and they solve the equation*

$$\partial_t \boldsymbol{y}(t, \boldsymbol{x}) + \Delta \boldsymbol{y}(t, \boldsymbol{x}) + (\boldsymbol{b}(t, \boldsymbol{x}), \nabla) \boldsymbol{y}(t, \boldsymbol{x}) = 0, \qquad \boldsymbol{y}(0, \mathbf{0}) = \mathbf{0}. \tag{12}$$

(Y2) *The Jacobian matrix* $\nabla \boldsymbol{y}(t, \boldsymbol{x})$ *is stationary, square integrable and has the mean* $\mathbf{E}(\nabla \boldsymbol{y}) = \mathbf{I}$.

(Y3) *The rescaled harmonic coordinates* $\boldsymbol{y}_\varepsilon(t, \boldsymbol{x}) = \varepsilon \boldsymbol{y}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon)$ *have the asymptotics*

$$\lim_{\varepsilon \downarrow 0} \sup_{\Omega_{T,R}} |\boldsymbol{y}_\varepsilon(t, \boldsymbol{x}) - \boldsymbol{x}| = 0,$$

where $\Omega_{T,R}(t, \boldsymbol{x})$ *is the cylinder* $[t, T + t] \times \{\boldsymbol{y} \in R^d \mid |\boldsymbol{y} - \boldsymbol{x}| < R\}$.

By (B), the harmonic coordinates $\boldsymbol{y}(t, \boldsymbol{x})$ are essentially the same as the Euclidean ones in the limit. Now the sample path in the harmonic coordinate system is a martingale almost surely and the desired invariance principle follows from the standard martingale invariance principle. The rest of the proof goes as follows.

2.1. *Proof of the main theorem.* In this section, we make ω-dependence explicit in the notation and write, for example, $\mathbf{x}^{\omega}(t)$, $\mathbf{y}^{\omega}(t)$, instead of $\mathbf{x}(t)$, $\mathbf{y}(t)$.

PROOF OF PART (i).
Using Itô's formula and (Y1), we get that

$$\text{(13)} \quad \begin{aligned} &\mathrm{M}\frac{y_k(t,\mathbf{x}^{\omega}(t))\,y_l(t,\mathbf{x}^{\omega}(t))}{t} \\ &\quad = \frac{2}{t}\int_0^t \mathrm{M}\big(\nabla y_k(s,\mathbf{x}^{\omega}(s)),\nabla y_l(s,\mathbf{x}^{\omega}(s))\big)\,ds. \end{aligned}$$

Let us denote by $\mathscr{M}_s$, $s \geq 0$, the filtration of σ-subalgebras of $\mathscr{M}$ generated by $\boldsymbol{w}(t)$, $t \leq s$. Note that

$$\eta^{\omega}(t;\sigma) = \tau_{t,\mathbf{x}^{\omega}(t;\sigma)}(\omega), \qquad t \geq 0$$

is an Ω-valued, stationary Markov process with respect to the filtration $\mathscr{M}_s$, $s \geq 0$.

Since the group $\tau_{t,\mathbf{x}}$ is stochastically continuous,

$$\mathbf{P}^t f(\omega) = \mathrm{M} f(\eta^{\omega}(t)), \qquad t \geq 0$$

is a strongly continuous semigroup of Markov operators in $L^2(\Omega)$.

Proposition 1 is well known in the time-independent case [Osada (1982)] and the proof can be easily adapted to our setting.

PROPOSITION 1. *Measure P is invariant and ergodic for the family $\{\eta^{\omega}(t)\}_{t\geq 0}$, $\omega \in \Omega$, that is,*

$$\mathbf{E}\mathbf{P}^t f = \mathbf{E} f \quad \textit{for } f \in L^2(\Omega) \textit{ and } t \geq 0$$

and if for some $A \in \mathscr{V}$,

$$\mathbf{E}\left|\mathbf{P}^t \chi_A - \chi_A\right| = 0 \quad \textit{for } t \geq 0,$$

then $P(A) = 0$ or $P(A) = 1$.

By (Y2) the right side of (13) equals

$$\frac{2}{t}\int_0^t \mathbf{P}^s f_{kl}(\omega)\,ds$$

for any $t \geq 0$ where

$$f_{kl}(\omega) = \sum_{m=1}^{d} \partial_m y_k(0,\mathbf{0},\omega)\,\partial_m y_l(0,\mathbf{0},\omega).$$

From Proposition 1 and the individual ergodic theorem [see, e.g., Krengel (1985), Corollary 3.8] we have that for P-a.s. ω,

$$\lim_{t\uparrow+\infty} \mathrm{M}\frac{y_k(t,\mathbf{x}^\omega(t))\,y_l(t,\mathbf{x}^\omega(t))}{t} = 2\mathbf{E}f_{kl}. \tag{14}$$

Define $\mathbf{y}^\omega(t) = \boldsymbol{y}(t, \boldsymbol{x}^\omega(t))$. Then $\mathbf{y}^\omega(t)$ is a continuous local martingale for almost all ω. Moreover, since

$$\mathrm{M}\left[\int_0^T |\nabla \mathbf{y}(t,\mathbf{x}^\omega(t))|^2\,dt\right] < \infty$$

for any $T > 0$, $\mathbf{y}_\varepsilon^\omega(t)$ is a continuous martingale.

Next, we show

$$\limsup_{t\uparrow+\infty} \mathrm{M}\frac{|\mathbf{x}^\omega(t)|^2}{t} < \infty \tag{15}$$

almost surely in P.

Indeed, we have

$$\begin{aligned}\mathrm{M}\frac{|\mathbf{x}^\omega(t)|^2}{t} &= \mathrm{M}\left[\frac{|\mathbf{x}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]}\right] + \mathrm{M}\left[\frac{|\mathbf{x}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|\le\sqrt{t}]}\right]\\ &\le \mathrm{M}\left[\frac{|\mathbf{x}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]}\right] + 1.\end{aligned}$$

Notice that

$$\begin{aligned}&\frac{|\mathbf{x}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]}\\ &\quad\le 2\left[\frac{|\mathbf{x}^\omega(t)-\mathbf{y}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]} + \frac{|\mathbf{y}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]}\right].\end{aligned} \tag{16}$$

As remarked before, the sample path $\mathbf{x}^\omega(t)$ does not explode for almost all ω. Thus $\mathbf{x}^\omega(t)$ is finite with probability 1 and we set $1/|\mathbf{x}^\omega(t)| = \varepsilon(\omega,\sigma)$. With this the first term on the right side of (16) can be written as

$$\begin{aligned}&\frac{|\mathbf{x}^\omega(t)-\mathbf{y}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]}\\ &\quad=\left|\mathbf{z}_\varepsilon\left(\frac{t}{|\mathbf{x}^\omega(t)|^2},\frac{\mathbf{x}^\omega(t)}{|\mathbf{x}^\omega(t))|}\right)\right|^2\frac{|\mathbf{x}^\omega(t)|^2}{t}\chi_{[|\mathbf{x}^\omega(t)|>\sqrt{t}]},\end{aligned} \tag{17}$$

where

$$\mathbf{z}_\varepsilon(t,\boldsymbol{x}) \equiv \mathbf{y}_\varepsilon(t,\boldsymbol{x}) - \boldsymbol{x}.$$

Since $|\mathbf{x}^{\omega}(t)| > \sqrt{t}$, we get from (17) that

$$\begin{aligned}
&\frac{|\mathbf{x}^{\omega}(t) - \mathbf{y}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| > \sqrt{t}]} \\
&\qquad \le \sup_{0<\varepsilon<1/\sqrt{t}} \sup_{\Omega_{1,1}} |\mathbf{z}_{\varepsilon}(s, \boldsymbol{x})|^2 \frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| > \sqrt{t}]}. \qquad (18)
\end{aligned}$$

By (Y3) there exists $t_0(\omega)$ such that for $t > t_0(\omega)$ the right side of (18) is less than or equal to

$$\frac{1}{4}\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| > \sqrt{t}]}.$$

Then (16) implies that

$$\limsup_{t\uparrow+\infty} \mathrm{M}\left[\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| > \sqrt{t}]}\right] \le 4 \lim_{t\uparrow+\infty} \mathrm{M}\frac{|\mathbf{y}^{\omega}(t)|^2}{t},$$

and, hence (15).

To end the proof, we note that

$$\begin{aligned}
&\mathrm{M}\left[\frac{x_k^{\omega}(t)x_l^{\omega}(t)}{t}\right] \\
&\quad = \mathrm{M}\left\{\frac{[x_k^{\omega}(t) - y_k^{\omega}(t)][x_l^{\omega}(t) - y_l^{\omega}(t)]}{t}\right\} \\
&\qquad + \mathrm{M}\left\{\frac{y_k^{\omega}(t)[x_l^{\omega}(t) - y_l^{\omega}(t)]}{t}\right\} + \mathrm{M}\left\{\frac{y_l^{\omega}(t)[x_k^{\omega}(t) - y_k^{\omega}(t)]}{t}\right\} \qquad (19)\\
&\qquad + \mathrm{M}\left[\frac{y_k^{\omega}(t)y_l^{\omega}(t)}{t}\right].
\end{aligned}$$

The first term on the right side of (19) can be estimated as above by

$$\begin{aligned}
&\mathrm{M}\left[\frac{|\mathbf{x}^{\omega}(t) - \mathbf{y}^{\omega}(t)|^2}{t}\right] \\
&\quad = \mathrm{M}\left[\frac{|\mathbf{x}^{\omega}(t) - \mathbf{y}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| > \sqrt{t}]}\right] + \mathrm{M}\left[\frac{|\mathbf{x}^{\omega}(t) - \mathbf{y}^{\omega}(t)|^2}{t}\chi_{[|\mathbf{x}^{\omega}(t)| \le \sqrt{t}]}\right] \\
&\quad \le \sup_{0<\varepsilon<1/\sqrt{t}} \sup_{\Omega_{1,1}} \left|\mathbf{z}_{\varepsilon}(s, \boldsymbol{x})\right|^2 \mathrm{M}\left[\frac{|\mathbf{x}^{\omega}(t)|^2}{t}\right] + \sup_{B_1}\left|\mathbf{z}_{1/\sqrt{t}}(1, \boldsymbol{x})\right|,
\end{aligned}$$

which vanishes as $t\uparrow+\infty$. The same conclusion can be reached for the second and third terms. By (14) we have the proof of part (i).

PROOF OF PARTS (ii) AND (iii).

The preceding analysis shows that

$$\mathbf{D} = \mathbf{E}\Big[\nabla \boldsymbol{y}(0,\mathbf{0})(\nabla \boldsymbol{y}(0,\mathbf{0}))^T\Big] \geq \mathbf{I},$$

where $\mathbf{D} = [d_{kl}]_{k,l=1,\dots,d}$, with d_{kl} given by (11). Thus $\mathbf{D}$ is invertible.

Denote $\mathbf{y}_\varepsilon^\omega(t) = \varepsilon \mathbf{y}^\omega(t/\varepsilon^2)$, $\varepsilon > 0$. By Itô's formula and (Y1),

$$\mathbf{y}_\varepsilon^\omega(t) = \sqrt{2}\,\varepsilon \int_0^{t/\varepsilon^2} \big(\nabla \mathbf{y}(s, \mathbf{x}^\omega(s)), \sqrt{2}\, d\mathbf{w}(s)\big).$$

Set $\mathbf{y}_\varepsilon^{\omega,\mathbf{v}}(t) = (\mathbf{D}^{-1/2}\mathbf{y}_\varepsilon^\omega(t), \mathbf{v})$, $t \geq 0$. The processes $\mathbf{y}_\varepsilon^{\omega,\mathbf{v}_1}(t)$, $\mathbf{y}_\varepsilon^{\omega,\mathbf{v}_2}(t)$ are stationary, mean-zero, square integrable and continuous martingales with joint quadratic variation

$$\langle \mathbf{y}_\varepsilon^{\omega,\mathbf{v}_1}, \mathbf{y}_\varepsilon^{\omega,\mathbf{v}_2}\rangle_t = \varepsilon^2 \int_0^{t/\varepsilon^2} \Big(\mathbf{D}^{-1}\nabla \boldsymbol{y}(s,\mathbf{x}^\omega(s))[\nabla \boldsymbol{y}(s,\mathbf{x}^\omega(s))]^T, \mathbf{v}_1 \otimes \mathbf{v}_2\Big)\, ds.$$

By Proposition 1 and the individual ergodic theorem again,

$$\lim_{\varepsilon \downarrow 0} \langle \mathbf{y}_\varepsilon^{\omega,\mathbf{v}_1}, \mathbf{y}_\varepsilon^{\omega,\mathbf{v}_2}\rangle_t = t(\mathbf{v}_1, \mathbf{v}_2).$$

By a modification of Theorem 5.4 of Helland (1982) [Fannjiang and Komorowski (1996)] the family $\{\mathbf{D}^{-1/2}\mathbf{y}_\varepsilon^\omega(t)\}_{t\geq 0}$, $\varepsilon > 0$ converges to the standard Brownian motion and the invariance principle holds.

Fix $T, N > 0$ and denote by $\tau_{N,\varepsilon}^\omega$ the exit time of $\mathbf{x}_\varepsilon^\omega(t)$, $t \geq 0$ from the ball $\overline{B}_N$. By (Y3), $\lim_{\varepsilon\downarrow 0} \sup_{0\leq t\leq \tau_{N,\varepsilon}^\omega \wedge T} |\mathbf{z}_\varepsilon^\omega(t)| = 0$, where $\mathbf{z}_\varepsilon^\omega(t) = \mathbf{z}_\varepsilon(t, \mathbf{x}_\varepsilon^\omega(t))$. Suppose that $\sup_{0\leq t\leq \tau_{N,\varepsilon}^\omega\wedge T} |\mathbf{z}_\varepsilon^\omega(t)| < 1$ for $0 < \varepsilon < \varepsilon_0(\omega)$. Then, for $0 < \varepsilon < \varepsilon_0(\omega)$, we have

$$\begin{aligned} Q\Big[\sup_{0\leq t\leq T} |\mathbf{x}_\varepsilon^\omega(t)| \geq N\Big] &= Q[\tau_{N,\varepsilon}^\omega \leq T] \\ &= Q\Big[\tau_{N,\varepsilon}^\omega \leq T, \sup_{0\leq t\leq T} |\mathbf{y}_\varepsilon^\omega(t)| \geq N-1\Big] \\ &\leq Q\Big[\sup_{0\leq t\leq T} |\mathbf{y}_\varepsilon^\omega(t)| \geq N-1\Big]. \end{aligned}$$

Since $\{\mathbf{y}_\varepsilon^\omega(t)\}_{t\geq 0}$ converges to a nondegenerate d-dimensional Brownian motion, there exist constants γ, $C > 0$ independent of ε, ω such that

$$\limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\leq t\leq T} |\mathbf{x}_\varepsilon^\omega(t)| \geq N\Big] \leq Ce^{-\gamma N}$$

[Chung and Zhao (1995), Proposition 1.16, page 20]. Thus, for any $\rho > 0$,

$$\begin{aligned} &\limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\leq t\leq T} |\mathbf{z}_\varepsilon^\omega(t)| \geq \varrho\Big] \\ &\quad\leq \limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\leq t\leq \tau_{N,\varepsilon}^\omega\wedge T} |\mathbf{z}_\varepsilon^\omega(t)| \geq \varrho\Big] + \limsup_{\varepsilon\downarrow 0} Q\Big[\sup_{0\leq t\leq T} |\mathbf{x}_\varepsilon^\omega(t)| \geq N\Big] \\ &\quad\leq Ce^{-\gamma N}. \end{aligned}$$

Since $N > 0$ can be chosen arbitrarily, we conclude that

$$\lim_{\varepsilon \downarrow 0} Q\left[\sup_{0 \le t \le T} |\mathbf{z}_\varepsilon^\omega(t)| \ge \varrho \right] = 0$$

for any $\rho > 0$.

Thus $\{\mathbf{x}_\varepsilon^\omega(t)\}_{t \ge 0}$ satisfies the invariance principle as $\varepsilon \downarrow 0$ for almost all ω. The covariance matrix of the limiting Wiener measure is $\mathbf{D}$.

3. Proof of the main lemma. Let $B_R(\boldsymbol{x})$ be the ball $\{\boldsymbol{y} \in R^d \mid |\boldsymbol{y} - \boldsymbol{x}| < R\}$ and $\Omega_{T,R}(t, \boldsymbol{x})$ the cylinder $[t, T + t] \times \overline{B}_R(\boldsymbol{x})$. We write B_R and $\Omega_{T,R}$ in case $\boldsymbol{x} = \mathbf{0}$, $t = 0$.

We denote the $L^{p,q}(\Omega_{T,R}(t, \boldsymbol{x}))$ norm by

$$\|f\|_{p,q,\Omega_{T,R}(t,\boldsymbol{x})} = \left\{ \frac{1}{T} \int_t^{t+T} ds \left[\frac{1}{|B_R|} \int_{B_R(\boldsymbol{x})} |f(s, \boldsymbol{y})|^p \, d\boldsymbol{y} \right]^{q/p} \right\}^{1/q}$$

and just write $\|f\|_{p,\Omega_{T,R}(t,\boldsymbol{x})}$, if $p = q$. Likewise $\|f\|_{p,B_R(\boldsymbol{x})}$ has an analogous meaning. For p and/or $q = \infty$, we mean the essential supremum.

A weak solution (subsolution) of the equation

$$\partial_t u(t, \boldsymbol{x}) + \sum_{k,l=1}^d \partial_k [a_{kl}(t, \boldsymbol{x}) \partial_l u(t, \boldsymbol{x})] = 0 \tag{20}$$

with $\mathbf{a} = [a_{ij}] = \mathbf{I} + \mathbf{H}$ is a measurable function u having locally square integrable space-gradient,

$$\|u\|_{2,\infty,\Omega_{T,R}} + \|\nabla u\|_{2,2,\Omega_{T,R}} < \infty \tag{21}$$

and satisfies

$$\iint_{\Omega_{T,R}} \Big[-\partial_t u(t, \boldsymbol{x}) \eta(t, \boldsymbol{x}) + \sum_{k,l=1}^d a_{kl}(t, \boldsymbol{x}) \partial_l u(t, \boldsymbol{x}) \partial_k \eta(t, \boldsymbol{x}) \Big] \, dt \, d\boldsymbol{x} = 0 (\le 0) \tag{22}$$

for all $T, R > 0$ and nonnegative smooth functions η with compact supports in $\boldsymbol{x}$ for each t. Any (weak) solution u can be written as $u = v - w$ where both $v = \sqrt{u^2 + \delta}$, $w = \sqrt{u^2 + \delta} - u$ are (weak) positive subsolutions and δ is any positive number. This decomposition is used several times in the sequel.

3.1. *Existence of the classical solution*: *Proof of* (Y1) *and* (Y2). We first use a cut-off argument to construct solutions to (12). We need to show that the limit solution, as the cut-off is removed, is a weak solution. Then by a classical theorem a weak solution is a classical solution also (Proposition 2).

Let $\varphi_n \colon R \to R$ be a smooth, odd function such that $\varphi_n(x) = x$, for $|x| \le n$, $\varphi_n(x) = n + 1$, for $x \ge n + 1$ and $\varphi_n(x) = -n - 1$, for $x \le -n - 1$ and $|\varphi_n'(x)| \le 2$, $x \in R$. We define the cut-off matrix $\mathbf{H}^{(n)} = [H_{kl}^{(n)}]_{k,l=1,\dots,d}$ with $H_{kl}^{(n)} = \varphi_n(H_{kl})$.

Let $\mathscr{E}_{\sigma,n,\beta}(f,g)$, $\sigma > 0$, $\beta > 0$ be the closed bilinear form given

$$\begin{aligned}\mathscr{E}_{\sigma,n,\beta}(f,g) \\ = \sigma\mathbf{E}(D_0 f D_0 g) + \sum_{k,l=1}^{d}\mathbf{E}\left[\left(\delta_{kl} + H_{kl}^{(n)}\right)D_l f D_k g\right] \\ - \mathbf{E}(D_0 fg) + \beta\mathbf{E}(fg),\end{aligned} \tag{23}$$

for $f, g \in H^1(\Omega)$. The first term on the right side of (23) regularizes the time variable. Now that the coefficients are bounded, by one Lax–Millgram lemma, a unique $f_{k,\sigma,n,\beta} \in H^1(\Omega)$ exists with

$$\mathscr{E}_{\sigma,n,\beta}\left(f_{k,\sigma,n,\beta}, g\right) = -\sum_{l=1}^{d}\mathbf{E}\left(H_{kl}^{(n)}D_l g\right), \tag{24}$$

for all $g \in H^1(\Omega)$. Thus the equation is satisfied in the sense of distribution,

$$\begin{aligned}\sigma\partial_t^2 f_{k,\beta,\sigma,n}(t,\boldsymbol{x};\omega) + \sum_{l,m=1}^{d}\partial_l\left[\left(\delta_{lm} + H_{lm}^{(n)}(t,\boldsymbol{x};\omega)\right)\partial_m f_{k,\beta,\sigma,n}(t,\boldsymbol{x};\omega)\right] \\ + \partial_t f_{k,\beta,\sigma,n}(t,\boldsymbol{x};\omega) - \beta f_{k,\beta,\sigma,n}(t,\boldsymbol{x};\omega) = b_k^{(n)}(t,\boldsymbol{x};\omega)\end{aligned}$$

almost surely.

By the classical regularity theory for divergence form equations with bounded and measurable coefficients [Ladyzhenskaya, Solonnikov and Uralćeva (1968)], we may assume that $f_{k,\sigma,n,\beta}(t,\boldsymbol{x};\omega)$, $k = 1,\cdot, d$ are locally Hölder continuous and its first derivatives are locally square integrable.

Substituting $g = f_{k,\sigma,n,\beta}$ into (24) we have that

$$\sigma\mathbf{E}\left\{\left(D_0 f_{k,\sigma,n,\beta}\right)^2\right\} + \sum_{l=1}^{d}\mathbf{E}\left\{\left(D_l f_{k,\sigma,n,\beta}\right)^2\right\} + \beta\mathbf{E}\left\{\left(f_{k,\sigma,n,\beta}\right)^2\right\} \le C$$

uniformly in all parameters. Extract an L^2-weakly convergent subsequence $(D_1 f_{k,\sigma',n',\beta'},\dots,D_d f_{k,\sigma',n',\beta'})$ with limit $\mathbf{F}_k = (F_{k,1},\dots,F_{k,d}) \in (L^2(\Omega))^d$ and set

$$f_k(t,\boldsymbol{x}) = \sum_{l=1}^{d} i\int_R\int_{R^d} e^{it\tau}\frac{e^{i\boldsymbol{x}\boldsymbol{\xi}} - 1}{|\boldsymbol{\xi}|^2}\xi_l\mathscr{U}(d\tau d\boldsymbol{\xi})F_{k,l}. \tag{25}$$

Here $f_k(t,\boldsymbol{x};\omega)$ is not stationary in general [note $f_k(0,\mathbf{0}) = 0$]. For a fixed $(t,\boldsymbol{x})$, integral (25) defines an elements of $L^2(\Omega)$. Clearly, we have that $f_k(t,\mathbf{0}) = 0$ and $\nabla f_k(t,\boldsymbol{x};\omega) = \mathbf{F}_k(t,\boldsymbol{x};\omega)$. Thus, the function

$$y_k^0(t,\boldsymbol{x}) = x_k + f_k(t,\boldsymbol{x}) \tag{26}$$

and its spatial gradient are locally square integrable and $y_k^0(t,\mathbf{0}) = 0$.

Let $y_k(t,\cdot)$, $k = 1,\dots,d$, be the distribution given by

$$\begin{aligned}y_k(t,\eta) = \int_{R^d} y_k^0(0,\boldsymbol{x})\eta(\boldsymbol{x})\,d\boldsymbol{x} \\ + \sum_{l,m=1}^{d}\int_0^t\int_{R^d}\left[\delta_{lm} + H_{lm}(s,\boldsymbol{x})\right]\partial_m y_k^0(t,\boldsymbol{x})\partial_l\eta(\boldsymbol{x})\,ds\,d\boldsymbol{x},\end{aligned} \tag{27}$$

for any $\eta \in C_0^\infty(R^d)$. Since y_k^0 are distributional solutions, we have

$$-y_k(t, \partial_l \eta) = \int_{R^d} \partial_l y_k^0(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \, d\boldsymbol{x}, \qquad l = 1, \dots, d.$$

Thus $y_k(t, \eta)$ can be written as $\int y_k(\boldsymbol{x}) \eta(\boldsymbol{x}) \, d\boldsymbol{x}$ with a measurable function

$$y_k(t, \boldsymbol{x}) = y_k(t, \mathbf{0}) + y_k^0(t, \boldsymbol{x}). \tag{28}$$

Note that $y_k(0, 0) = 0$. Equation (27) now becomes

$$\begin{aligned} &\int_{R^d} y_k(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \, d\boldsymbol{x} \\ &\quad = \int_{R^d} y_k(0, \boldsymbol{x}) \eta(\boldsymbol{x}) \, d\boldsymbol{x} \\ &\qquad + \sum_{l, m = 1}^{d} \int_0^t \int_{R^d} [\delta_{lm} + H_{lm}(s, \boldsymbol{x})] \, \partial_m y_k(t, \boldsymbol{x}) \, \partial_l \eta(\boldsymbol{x}) \, ds \, d\boldsymbol{x}, \end{aligned} \tag{29}$$

for all $\eta \in C_0^\infty(R^d)$. Because $\mathbf{H}(t, \boldsymbol{x})$ is locally bounded almost surely we have from classical regularity theory that $\boldsymbol{y}(t, \boldsymbol{x}) = (y_k(t, \boldsymbol{x}))$ is twice Hölder continuously differentiable in space and once in time locally provided that y_k are weak solutions.

PROPOSITION 2. *The harmonic coordinates* $y_k(t, \boldsymbol{x})$, $k = 1, \dots, d$, *given by* (28), *are local*, *classical solutions of* (12) *almost surely*.

PROOF. It suffices to show (21). It is easy to see from (29) that $y_k(t, \boldsymbol{x})$ is locally integrable almost surely. Let $\chi(t, \boldsymbol{x})$ be a nonnegative, smooth, compactly supported function satisfying $\chi(-t, -\boldsymbol{x}) = \chi(t, \boldsymbol{x})$ and $\iint \chi(t, \boldsymbol{x}) \, dt \, d\boldsymbol{x} = 1$. We set $\chi_\delta(t, \boldsymbol{x}) = (1/\delta^{d+1}) \chi(t/\delta, \boldsymbol{x}/\delta)$ and $y_{k,\delta} = \chi_\delta * y_k$. Here $*$ denotes the convolution. We have the equation for $y_{k,\delta}$.

$$\partial_t y_{k,\delta} + \Delta y_{k,\delta} = f_\delta,$$

with $f_\delta = \sum_{l=1}^d [(b_l \partial_l y_k) * \chi_\delta]$, in the classical sense.

Because $\sup_{\delta > 0} \| f_\delta \|_{2,2,\Omega_{T,R}} < \infty$, we have, by standard estimates for the heat equation, that

$$\sup_{\delta > 0} \| \nabla y_{k,\delta} \|_{2,2,\Omega_{T,R}} < \infty,$$

$$\sup_{\delta > 0} \sup_{s \in [0,T]} \| y_{k,\delta}(s, \cdot) \|_{2, B_R(\boldsymbol{x}_0)} < \infty$$

almost surely. Thus the limit $y_k = \lim_{\delta \to 0} y_{k,\delta}$ must satisfy the same estimates, namely, (21). □

To see that (Y2) is satisfied: $\nabla y_k(t, \boldsymbol{x}, \omega) = \nabla y_k^0(t, \boldsymbol{x}, \omega) = \mathbf{F}_k(t, \boldsymbol{x}, \omega) = \mathbf{F}_k(\tau_l, \boldsymbol{x}^\omega) \in L^2(\Omega)$.

3.2. *Asymptotic behavior*: *Proof of* (Y3). In the sequel we write $\mathbf{a} = [a_{lm}]$, with

$$a_{lm}(t, \boldsymbol{x}) = \delta_{lm} + H_{lm}(t, \boldsymbol{x}),$$

and $\mathbf{a}_\varepsilon = [a_{lm,\varepsilon}]$, with

$$a_{lm,\varepsilon}(t,\boldsymbol{x}) = \delta_{lm} + H_{lm}(t/\varepsilon^2, \boldsymbol{x}/\varepsilon).$$

The rescaled harmonic coordinates then satisfy the equation

$$\partial_t y_{k,\varepsilon}(t,\boldsymbol{x}) + \sum_{l,m=1}^{d} \partial_l\big[a_{lm,\varepsilon}(t,\boldsymbol{x})\partial_m y_{k,\varepsilon}(t,\boldsymbol{x})\big] = 0. \tag{30}$$

We first prove a weaker version of (Y3), by showing weak convergence of the rescaled coordinates. Then, in the next step, we strengthen the sense of convergence.

PROPOSITION 3. *For any $\eta \in C(\overline{B}_R)$ and $\phi \in C(\Omega_{T,R})$,*

$$\lim_{\varepsilon\downarrow 0}\int_{B_R} y_{k,\varepsilon}(0,\boldsymbol{x})\eta(\boldsymbol{x})\,d\boldsymbol{x} = \int_{B_R} x_k\eta(\boldsymbol{x})\,d\boldsymbol{x}$$

and

$$\lim_{\varepsilon\downarrow 0}\int_{-T}^{T}\int_{B_R} y_{k,\varepsilon}(t,\boldsymbol{x})\phi(t,\boldsymbol{x})\,dt\,d\boldsymbol{x} = \int_{-T}^{T}\int_{B_R} x_k\phi(t,\boldsymbol{x})\,dt\,d\boldsymbol{x}. \tag{31}$$

To show Proposition 3 we need a standard averaging lemma which is closely related to the individual ergodic theorem.

PROPOSITION 4. *Suppose $\tilde{f}(\omega) \in L^2(\Omega)$ and $\phi(t,\boldsymbol{x}) \in L^2(\Omega_{T,R})$. Then*

$$\lim_{\varepsilon\downarrow 0}\int\!\!\int_{\Omega_{T,R}} U_{t/\varepsilon^2,\boldsymbol{x}/\varepsilon}\tilde{f}(\omega)\phi(t,\boldsymbol{x})\,dt\,d\boldsymbol{x} = \mathbf{E}\{\tilde{f}\}\int\!\!\int_{\Omega_{T,R}}\phi(t,\boldsymbol{x})\,dt\,d\boldsymbol{x} \text{ and} \tag{32}$$

$$\lim_{\varepsilon\downarrow 0}\int_{B_R} U_{0,\boldsymbol{x}/\varepsilon}\tilde{f}(\omega)\phi(0,\boldsymbol{x})\,d\boldsymbol{x} = \mathbf{E}\{\tilde{f}\mid\mathscr{V}_s\}\int_{B_R}\phi(0,\boldsymbol{x})\,d\boldsymbol{x}, \tag{33}$$

for almost all ω. Here $\mathscr{V}_s$ is the sub-σ-algebra of sets invariant under space translations.

PROOF OF PROPOSITION 3. Set $y^0_{k,\varepsilon}(t,\boldsymbol{x}) := \varepsilon y^0_k(t/\varepsilon^2,\boldsymbol{x}/\varepsilon)$. It is useful to write

$$\int_{B_R} y^0_{k,\varepsilon}(0,\boldsymbol{x})\eta(\boldsymbol{x})\,d\boldsymbol{x} = \sum_{l=1}^{d}\int_0^1 ds\int_{B_R}\partial_l y^0_{k,\varepsilon}(0,s\boldsymbol{x})x_l\eta(\boldsymbol{x})\,d\boldsymbol{x}. \tag{34}$$

Using $\nabla y_{k,\varepsilon}(t,\boldsymbol{x}) = \mathbf{e}_k + \mathbf{F}_k(t/\varepsilon^2,\boldsymbol{x}/\varepsilon)$ and passing the limit by (33) we have that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{B_R} y^0_{k,\varepsilon}(0,\boldsymbol{x})\eta(\boldsymbol{x})\,d\boldsymbol{x}\\ &\qquad= \int_{B_R} x_k\eta(\boldsymbol{x})\,d\boldsymbol{x} + \sum_{l=1}^{d}\mathbf{E}\big[F^k_l\mid\mathscr{V}_s\big]\int_{B_R} x_l\eta(\boldsymbol{x})\,d\boldsymbol{x}\end{aligned} \tag{35}$$

for any $\eta \in C(\overline{B}_R)$. The remainder of the proof is to show that the second term on the right side of (35) vanishes.

By (26) and (32) we have, after a standard computation [cf. Fannjiang and Komorowski (1996)], that

$$\lim_{\varepsilon \downarrow 0} \int_{-T}^{T} \int_{B_R} y_{k,\varepsilon}^{0}(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \varphi(t)\, dt\, d\boldsymbol{x} = \int_{-T}^{T} \varphi(t)\, dt \int_{B_R} x_k \eta(\boldsymbol{x})\, d\boldsymbol{x}, \tag{36}$$

for any $\eta \in C(\overline{B}_R)$ and $\varphi \in C[-T, T]$.

From (30) it follows that

$$\begin{aligned}
&\int_{-T}^{T} \int_{B_R} y_{k,\varepsilon}(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \varphi(t)\, dt\, d\boldsymbol{x} \\
&\quad = \int_{B_R} y_{k,\varepsilon}^{0}(0, \boldsymbol{x}) \eta(\boldsymbol{x})\, d\boldsymbol{x} \int_{-T}^{T} \varphi(t)\, dt \\
&\qquad + \sum_{l,m=1}^{d} \int_{-T}^{T} \varphi(t)\, dt \left\{ \int_0^t \int_{B_R} a_{lm,\varepsilon}(s, \boldsymbol{x}) \partial_m y_{k,\varepsilon}(s, \boldsymbol{x}) \partial_l \eta(\boldsymbol{x})\, ds\, d\boldsymbol{x} \right\}
\end{aligned}$$

for any $\eta(\boldsymbol{x}) \in C_0^\infty(B_R)$, $\varphi \in C[-T, T]$. By (32) the second term on the right side tends to

$$\sum_{k=1}^{d} \int_0^t \int_{B_R} \mathbf{E}[\mathbf{a}\mathbf{F}_k] \nabla \eta(\boldsymbol{x})\, ds\, d\boldsymbol{x} = 0$$

by integrating by parts.

Thus (35) implies

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{-T}^{T} \int_{B_R} y_{k,\varepsilon}(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \varphi(t)\, dt\, d\boldsymbol{x} \\
&\quad = \left\{ \int_{B_R} x_k \eta(\boldsymbol{x})\, d\boldsymbol{x} + \sum_{l=1}^{d} \mathbf{E}\left[F_l^k \mid \mathscr{V}_s\right] \int_{B_R} x_l \eta(\boldsymbol{x})\, d\boldsymbol{x} \right\} \int_{-T}^{T} \varphi(t)\, dt,
\end{aligned} \tag{37}$$

for all $T > 0$.

From (28) we have

$$\lim_{\varepsilon \downarrow 0} \int_{-T}^{T} \int_{B_R} y_{k,\varepsilon}(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \varphi(t)\, dt\, d\boldsymbol{x} = \lim_{\varepsilon \downarrow 0} \int_{-T}^{T} \int_{B_R} y_{k,\varepsilon}^{0}(t, \boldsymbol{x}) \eta(\boldsymbol{x}) \varphi(t)\, dt\, d\boldsymbol{x}$$

for any $\eta \in C_0^\infty(B_R)$, $\int_{B_R} \eta(\boldsymbol{x})\, d\boldsymbol{x} = 0$, and $\varphi \in C[-T, T]$.

Equations (36) and (37) then imply

$$\sum_{l=1}^{d} \mathbf{E}\left[F_l^k \mid \mathscr{V}_s\right] \int_{B_R} x_l \eta(\boldsymbol{x})\, d\boldsymbol{x} = 0 \tag{38}$$

for all $\eta \in L^2(B_R)$ with $\int_{B_R} \eta(\boldsymbol{x})\, d\boldsymbol{x} = 0$. Thus $\mathbf{E}[F_l^k \mid \mathscr{V}_s] = 0$, for all $k, 1, \dots, d$. We have (31) by (37) and (38). □

We shall show that the weak convergence of the rescaled harmonic coordinates can be strengthened to the convergence in L^∞ norm. We first prove an estimate needed for compactness in L^∞ norm of the harmonic coordinates.

LEMMA 1. *Under conditions* (H1), (H2) *and* (B), *we have*

$$\limsup_{\varepsilon\downarrow 0} \| y_{k,\varepsilon}\|_{\infty,\infty,\Omega_{T,R}} < \infty,$$

for any $T, R > 0$, *almost surely.*

PROOF. By the Gagliardo–Nirenberg inequality [Ladyzhenskaya, Solonnikov and Ural'ceva (1968), Theorem 2.2, page 62 and Remark 2.11], we have that

$$\| \tilde{y}_{k,\varepsilon}(t,\cdot)\|_{s,B_R} \le c\|\nabla y_{k,\varepsilon}(t,\cdot)\|^{\alpha}_{2,B_R}\| \tilde{y}_{k,\varepsilon}(t,\cdot)\|^{1-\alpha}_{1,B_R}, \tag{39}$$

with

$$\alpha = \left(1 - \frac{1}{s}\right)\frac{2d}{d+2}$$

for

$$\tilde{y}_{k,\varepsilon}(t,\boldsymbol{x}) = y_{k,\varepsilon}(t,\boldsymbol{x}) - \int_{B_R} y_{k,\varepsilon}(t,\boldsymbol{x})\,d\boldsymbol{x}$$

and

$$1 \le s < \frac{2d}{d-2}.$$

Estimating $L^{s,r}(\Omega_{T,R})$-norm by using (39) and the Hölder inequality, we get

$$\| \tilde{y}_{k,\varepsilon}\|_{s,r,\Omega_{T,R}} \le c \sup_{0\le t\le T} \| \tilde{y}_{k,\varepsilon}(t,\cdot)\|^{1-\alpha}_{1,B_R}\|\nabla y_{k,\varepsilon}\|^{\alpha}_{2,2,\Omega_{T,R}}. \tag{40}$$

We take

$$s = 2p^*, \qquad r = 2q^*$$

with

$$p^* := \frac{p}{p-2}, \qquad q^* := \frac{q}{q-2}$$

in (40). For $p > d$, $1 \le s < 2d/(d-2)$.

Note that $\nabla y_{k,\varepsilon}(t,\boldsymbol{x}) = \boldsymbol{e}_k + (\nabla y_k)(t/\varepsilon^2, \boldsymbol{x}/\varepsilon)$ so the term $\|\nabla y_{k,\varepsilon}\|_{2,2,\Omega_{T,R}}$ is uniformly bounded by Proposition 1 and Proposition 3.

We claim that

$$\limsup_{\varepsilon\downarrow 0} \sup_{0\le t\le T} \|y_{k,\varepsilon}(t,\cdot)\|_{1,B_R} < \infty, \tag{41}$$

for any $T, R > 0$.

From (40) and (41) it follows that

$$\limsup_{\varepsilon\downarrow 0} \| \tilde{y}_{k,\varepsilon}\|_{2p^*,2q^*,\Omega_{T,R}} < \infty.$$

The proof of Lemma 1 will be complete once (41) and Lemma 2 (and the Remark) in Section 4 are proved.

PROOF OF (41). We write the solution $y_{k,\varepsilon}$ as $y_{k,\varepsilon} = u_{k,\varepsilon} - v_{k,\varepsilon}$, where $u_{k,\varepsilon} = \sqrt{y_{k,\varepsilon}^2 + 1}$ and $v_{k,\varepsilon} = \sqrt{y_{k,\varepsilon}^2 + 1} - y_{k,\varepsilon}$ are positive subsolutions.

Let us consider in (22) for $u_{k,\varepsilon}$ the test function $\phi = \phi_1(t)\phi_2(\boldsymbol{x})$ where ϕ_1 is one for $t \le T$, zero for $t \ge 2T$ and linearly interpolated in between, and ϕ_2 is one in B_R, zero outside B_{2R} and linearly interpolated in between. We get

$$\begin{aligned}\sup_{[0,T]} \int_{B_R} u_{k,\varepsilon}\, d\boldsymbol{x} &\le \int_0^{2T}\!\!\int |u_{k,\varepsilon}\phi_t|\, d\boldsymbol{x}\, dt + \int_0^{2T}\!\!\int \left|\mathbf{a}_\varepsilon(t,\boldsymbol{x})\nabla u_{k,\varepsilon}\nabla\phi\right| d\boldsymbol{x}\, dt \\ &\le \int_0^{2T}\!\!\int |u_{k,\varepsilon}\phi_t|\, d\boldsymbol{x}\, dt + \frac{2}{R}\left(\int_0^{2T}\!\!\int_{B_{2R}} |\mathbf{a}_\varepsilon|^2\, d\boldsymbol{x}\, dt\right)^{1/2} \\ &\quad \times \left(\int_0^{2T}\!\!\int_{B_{2R}} |\nabla y_{k,\varepsilon}|^2\, d\boldsymbol{x}\, dt\right)^{1/2}. \end{aligned} \tag{42}$$

Here the identity

$$\nabla u_{k,\varepsilon} = \frac{y_{k,\varepsilon}\nabla y_{k,\varepsilon}}{\sqrt{y_{k,\varepsilon}^2 + 1}}$$

is used.

The individual ergodic theorem (Proposition 4), applied to $|\nabla y_{k,\varepsilon}|^2$ and $|\mathbf{a}_\varepsilon|^2$, gives the supremum limit of the second term of the right side of (42),

$$\frac{2T}{R}|B_R|\, \|\mathbf{a}\|_{2,2,\Omega_{2T,2R}} \|\nabla y_k\|_{2,2,\Omega_{2T,2R}}$$

as ε tends to zero.

To bound the first integral on the right side of (42), we apply the Cauchy–Schwartz inequality

$$\left(\iint_{\Omega_{2T,2R}} |u_{k,\varepsilon}\phi_t|\, d\boldsymbol{x}\, dt\right)^2 \le \iint_{\Omega_{2T,2R}} u_{k,\varepsilon}^2\, d\boldsymbol{x}\, dt \iint_{\Omega_{2T,2R}} \phi_t^2\, d\boldsymbol{x}\, dt \tag{43}$$

and then bound the right side of (43) by the Poincaré inequality,

$$\begin{aligned}\iint_{\Omega_{2T,2R}} u_{k,\varepsilon}^2\, d\boldsymbol{x}\, dt &= \iint_{\Omega_{2T,2R}} \left(y_{k,\varepsilon}^2 + 1\right) d\boldsymbol{x}\, dt \\ &\le c\Bigg[\left(\iint_{\Omega_{2T,2R}} y_{k,\varepsilon}\, d\boldsymbol{x}\, dt\right)^2 \\ &\qquad + \iint_{\Omega_{2T,2R}} \left|(\nabla y_k)(t/\varepsilon^2, \boldsymbol{x}/\varepsilon)\right|^2 d\boldsymbol{x}\, dt + 1\Bigg]\end{aligned} \tag{44}$$

where c is independent of ε. By Propositions 3 and 4 the integral $\iint_{\Omega_{2T,2R}} u_{k,\varepsilon}^2\, d\boldsymbol{x}\, dt$ is bounded uniformly as $\varepsilon \to 0$.

The proof of (41) is complete in view of the pointwise estimate $|y_{k,\varepsilon}| \le |u_{k,\varepsilon}|$. □

PROPOSITION 5. *Under conditions* (H1), (H2) *and* (B) *the family of functions* $\mathbf{y}_\varepsilon(\cdot,\cdot;\omega)$, $\varepsilon > 0$ *is equicontinuous on* $\Omega_{T,R}$, *for any* $T, R > 0$.

PROOF. In this proof the notations are the same as in Lemma 3.

Without loss of generality, assume $T = R^2$. Let us define, following (67),

$$K(\mathbf{a}) = \limsup_{\varepsilon \downarrow 0} \{K(\mathscr{C}_\varepsilon, \mathbf{a}_\varepsilon)\}$$

with $\mathscr{C}_\varepsilon$ given by Lemma 3 for $\mathbf{a}_\varepsilon$. By (H1) and (H2), $K(\mathbf{a}) < \infty$ almost surely.

Let $\sigma > 0$ be arbitrary. Let n be so large that

$$C\{1 - C_1 \exp[-CK^\kappa(\mathbf{a})]\}^{[n/2]} \limsup_{\varepsilon \downarrow 0} \|y_{k,\varepsilon}\|_{\infty,\infty,\Omega_{2R^2,2R}} < \sigma,$$

where C_1, C, κ are as in Lemma 3.

Choose θ and a finite covering $\mathscr{C}$ according to Lemma 3 (with $\lambda = 1$). For some sufficiently small $\varepsilon < \varepsilon_0(\omega)$, $K(\mathscr{C}, \mathbf{a}_\varepsilon) < 2K(\mathscr{C}, \mathbf{a})$. By Lemma 3, $w_{y_{k,\varepsilon}, \Omega_{R^2,R}}(\theta^{2n}R^2, \theta^n R) < \sigma$, for all $\varepsilon < \varepsilon_0(\omega)$ [see (68) for a definition].

For $\varepsilon \ge \varepsilon_0(\omega)$, since the matrix $\mathbf{H}_\varepsilon$ is locally bounded for ε away from zero, by the classical Hölder continuity estimate [Kružkov (1963), Moser (1964)], the modulus of continuity satisfies $w_{y_{k,\varepsilon}, \Omega_{T,R}}(\tau, \varrho) < \sigma$, for sufficiently small $\tau, \varrho > 0$. As a result, the family $y_{k,\varepsilon}(\cdot,\cdot;\omega)$ is equicontinuous for almost all ω. □

Lemma 1 and Proposition 5 yield the compactness of the sequence $y_{k,\varepsilon}$ in the space $C(\Omega_{T,R})$ by the Ascoli–Arzela lemma. The limit points of the sequence are identified as x_k by Proposition 3 which then implies (Y3).

4. Estimates for maximum and continuity moduli. In this section we state and prove the main estimates for a general $d \times d$ matrix $\mathbf{a}(t, \boldsymbol{x}) = [a_{kl}(t, \boldsymbol{x})]$. Let $\mathbf{S}$ and $\mathbf{H}$ be the symmetric and antisymmetric parts of $\mathbf{a}$, respectively, with the following properties.

(A1) Uniform ellipticity:

$$(\mathbf{a}(t, \boldsymbol{x})\boldsymbol{\xi}, \boldsymbol{\xi}) \ge \lambda |\boldsymbol{\xi}|^2,$$

for some constant $\lambda > 0$ independent of $\boldsymbol{\xi} \in R^d$ and $(t, \boldsymbol{x})$.

(A2) Integrability:

$$\|\mathbf{a}\|_{2,2,\Omega_{T,R}}, \|\mathbf{S}\|_{p/2,q/2,\Omega_{T,R}}, \left\|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right\|_{p,q,\Omega_{T,R}} < \infty$$

for

$$\frac{d}{p} + \frac{2}{q} < 1$$

for all $T, R > 0$.

LEMMA 2. *Let u be a nonnegative subsolution of* (20) *under* (A1) *and* (A2). *Then*

$$(45)\qquad \|u\|_{\infty,\infty,\Omega_{R^2/2,R/2}(t,\boldsymbol{x})} \le C(\lambda,d)^{1/2(1+d/2\nu)} K \|u\|_{2p^*,2q^*,\Omega_{R^2,R}(t,\boldsymbol{x})}$$

for all $(t,\boldsymbol{x})$ where

$$\nu = 1 - \frac{d}{p} - \frac{2}{q} > 0$$

and the constant K, given by (66), *depends only on d, $\gamma = 1 + 2\nu/d$ and*

$$S(p,q,\Omega) \equiv \|\mathbf{S}\|_{p/2,q/2,\Omega} + \left\|\sqrt{\mathbf{S}^{-1}}\mathbf{H}\right\|^2_{p,q,\Omega}, \qquad \Omega_{R^2/2,R/2} \subset \Omega \subset \Omega_{R^2,R},$$

and the constant $C(\lambda,d)$ only on λ, d.

REMARK. The same estimate applies to solutions such as the harmonic coordinates y_k. To see this we write, as before, $y_k = u_k - v_k$ where $u_k = \sqrt{y_k^2 + \delta}$, $\delta > 0$, and $v_k = u_k - y_k$ are positive subsolutions. Clearly, $0 < v_k < \delta$. Thus (45) holds for y_k up to an error δ. Since δ is arbitrary, (45) holds for y_k exactly.

PROOF. In the proof, $c, c', c_1, c_2, c_3, \dots$ stand for constants depending only on the dimension unless otherwise specified in their arguments, such as in $C(\lambda, d)$.

Without loss of generality we set $(t,\boldsymbol{x}) = (0,\mathbf{0})$ and $R = 1$. The result for the general case can be obtained by translation and rescaling.

Let

$$R_n = \frac{1}{2} + \frac{1}{2^{n+1}}, \qquad T_n = \frac{1}{2} + \frac{1}{4^{n+1}}, n = 0,1,2,3\dots$$

and $B_n = B_{R_n}$, $\Omega_n = \Omega_{T_n,R_n}$.

Since u is a subsolution of (20), the following inequality holds:

$$(46)\qquad \iint_{\Omega_n} \left(-\varphi\partial_t u + (\nabla\varphi, \mathbf{a}\nabla u)\right) dt\, d\boldsymbol{x} \le 0$$

for any nonnegative, differentiable function $\varphi(t,\boldsymbol{x})$ vanishing on the boundary of $\Omega_{1,1}$ except possibly at $t = 0$. Consider $\varphi(t,\boldsymbol{x}) = u(t,\boldsymbol{x})\eta^2(t,\boldsymbol{x})$ with piecewise differentiable function $\eta(t,\boldsymbol{x})$ vanishing on the boundary of $\Omega_{1,1}$, except at $t = 0$.

From (46) we get

$$(47)\qquad \begin{aligned} &-\frac{1}{2}\iint_{\Omega_n} \eta^2 \partial_t u^2\, dt\, d\boldsymbol{x} + \iint_{\Omega_n} (\eta\nabla u, \mathbf{S}\eta\nabla u)\, dt\, d\boldsymbol{x} \\ &\quad \le -2\iint_{\Omega_n} (u\nabla\eta, \mathbf{a}\eta\nabla u)\, dt\, d\boldsymbol{x}. \end{aligned}$$

By the Hölder inequality we have

$$\left|\iint_{\Omega_n} (u\nabla\eta, \mathbf{a}\eta\nabla u)\, dt\, d\boldsymbol{x}\right|$$

$$\tag{48} \le \left[\iint_{\Omega_n} (\eta\nabla u, \mathbf{S}\eta\nabla u)\, dt\, d\boldsymbol{x}\right]^{1/2}\left[\iint_{\Omega_n} (u\nabla\eta, \mathbf{S}u\nabla\eta)\, dt\, d\boldsymbol{x}\right]^{1/2} + \left[\iint_{\Omega_n} (\eta\nabla u, \mathbf{S}\eta\nabla u)\, dt\, d\boldsymbol{x}\right]^{1/2}\left[\iint_{\Omega_n} |u\nabla\eta|^2\left|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right|^2 dt\, d\boldsymbol{x}\right]^{1/2}.$$

By the inequality $AB \le \frac{1}{8}A^2 + 2B^2$, the right side of (48) is less than or equal to

$$\frac{1}{4}\iint_{\Omega_n} (\eta\nabla u, \mathbf{S}\eta\nabla u)\, dt\, d\boldsymbol{x} + 2\iint_{\Omega_n} (u\nabla\eta, \mathbf{S}u\nabla\eta)\, dt\, d\boldsymbol{x} + 2\iint_{\Omega_n} |u\nabla\eta|^2\left|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right|^2 dt\, d\boldsymbol{x}.$$

The last two integrals can be bounded as

$$\tag{49} \frac{1}{|\Omega_n|}\iint_{\Omega_n} |u\nabla\eta|^2\left|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right|^2 dt\, d\boldsymbol{x} \le \left\|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right\|^2_{p,q,\Omega_n}\|u\nabla\eta\|^2_{2p^*,2q^*,\Omega_n}$$

and

$$\tag{50} \frac{1}{|\Omega_n|}\iint_{\Omega_n} (u\nabla\eta, \mathbf{S}u\nabla\eta) \le \|\mathbf{S}\|_{p/2,q/2,\Omega_n}\|u\nabla\eta\|^2_{2p^*,2q^*,\Omega_n}$$

by the Hölder inequality.

Integrating by parts in the first integral on the left side of (47) and estimating the right side by (49), (50) we get that

$$\tag{51} \begin{aligned} &\frac{1}{2|\Omega_n|}\int_{B_n} u^2(0,\boldsymbol{x})\eta^2(0,\boldsymbol{x})\, d\boldsymbol{x} + \frac{1}{2|\Omega_n|}\iint_{\Omega_n} (\eta\nabla u, \mathbf{S}\eta\nabla u)\, dt\, d\boldsymbol{x} \\ &\quad\le -\frac{1}{2|\Omega_n|}\iint_{\Omega_n} u^2\frac{\partial\eta^2}{\partial t}\, dt\, d\boldsymbol{x} + 4\left\|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\right\|^2_{p,q,\Omega_n}\|u\nabla\eta\|^2_{2p^*,2q^*,\Omega_n} \\ &\qquad + 4\|\mathbf{S}\|_{p/2,q/2,\Omega_n}\|u\nabla\eta\|^2_{2p^*,2q^*,\Omega_n}. \end{aligned}$$

Inequality (51) can be easily generalized to any cylinder $[s, T_n] \times B_n$, $s \le T_n$. The first integral on the left side of (51) is then replaced by $\int_{B_n} u^2(s,\boldsymbol{x})\eta^2(s,\boldsymbol{x})\, d\boldsymbol{x}$ and all other terms by corresponding intervals over $[s, T_n] \times B_n$.

Define the function $\eta(t,\boldsymbol{x}) = \psi_1(t)\psi_2(|\boldsymbol{x}|)$ with $\psi_1(t) = 1$ in $[0, T_{n+1})$, $\psi_1(t) = 0$ for $t \ge T_n$, linearly interpolated in $[T_{n+1}, T_n]$, and the function $\psi_2(r) = 1$ for $r < R_{n+1}$, $\psi_2(r) = 0$ for $r \ge R_n$, linearly interpolated in $[R_{n+1}, R_n]$.

From (A1) and (51) we have

$$\lambda\|\nabla u\|^2_{2,2,\Omega_{n+1}} \le c_1 4^n\Big[S(p,q,\Omega_n)\|u\|^2_{2p^*,2q^*,\Omega_n} + \|u\|^2_{2,2,\Omega_n}\Big], \tag{52}$$

where

$$S(p,q,\Omega_n) \equiv \|\mathbf{S}\|_{p/2,q/2,\Omega_n} + \Big\|\sqrt{\mathbf{S}^{-1}}\,\mathbf{H}\Big\|^2_{p,q,\Omega_n}.$$

Applying the remark after (51) we obtain

$$\begin{aligned}\sup_{t\in[0,T_{n+1}]} \frac{1}{|\Omega_n|}\int_{B_{n+1}} u^2(t,\boldsymbol{x})\,d\boldsymbol{x}\\ \le c_2 4^n\Big[S(p,q,\Omega_n)\|u\|^2_{2p^*,2q^*,\Omega_n} + \|u\|^2_{2,2,\Omega_n}\Big].\end{aligned} \tag{53}$$

Since u^α, $\alpha > 1$ are still positive subsolutions, it follows from (52) that

$$\lambda\|\nabla u^\alpha\|^2_{2,2,\Omega_{n+1}} \le c_1 4^n\Big[S(p,q,\Omega_n)\|u\|^{2\alpha}_{2\alpha p^*,2\alpha q^*,\Omega_n} + \|u\|^{2\alpha}_{2\alpha,2\alpha,\Omega_n}\Big] \tag{54}$$

and from (53),

$$\begin{aligned}\sup_{t\in[0,T_{n+1}]} \int_{B_{n+1}} u^{2\alpha}(t,\boldsymbol{x})\,d\boldsymbol{x}\\ \le c_2 4^n\Big[S(p,q,\Omega_n)\|u\|^{2\alpha}_{2\alpha p^*,2\alpha q^*,\Omega_n} + \|u\|^{2\alpha}_{2\alpha,2\alpha,\Omega_n}\Big].\end{aligned} \tag{55}$$

Now we combine (54) and (55) into one inequality to be iterated indefinitely to yield the result. To this end we choose the numbers $a, b, m > 0$ such that

$$\frac{a}{b} = \frac{p^*}{q^*}, \qquad \frac{bd^*}{am'} = 1, \qquad \frac{1}{am} + \frac{1}{b} = 1, \tag{56}$$

with $m' = m/(m-1)$, $p^* = p/(p-2)$, $q^* = q/(q-2)$. Equation (56) uniquely determines a, b, m. For $d > 2$,

$$\begin{aligned} m &= \frac{1}{1 - p^*/d^*q^*} = 1 + \frac{1}{(d^*q^*/p^*) - 1},\\ a &= \frac{d^*}{m'} + \frac{1}{m}, \qquad b = \frac{am'}{d^*}\end{aligned} \tag{57}$$

with $d^* = d/(d-2)$. For $d = 2$, set

$$m = 1, \qquad a = 1 + p^*/q^*, \qquad b = 1 + q^*/p^*. \tag{58}$$

Since $d/p + 2/q < 1$,

$$m \in \left(1 + \frac{1}{d^*(p/d - 1)}, 1 + \frac{1}{d^*(q^* - 1)}\right) \quad\text{and}\quad \frac{d}{a} + \frac{2}{b} = d.$$

Consequently, $a/p^* = b/q^* > 1$.

First, consider the case $d > 2$. By the Hölder inequality and (57),

$$\int_0^{T_{n+1}} \left(\int_{B_{n+1}} w^{2a}\, d\boldsymbol{x} \right)^{b/a} dt$$

$$\leq \int_0^{T_{n+1}} \left(\int_{B_{n+1}} w^2\, d\boldsymbol{x} \right)^{b/am} \left(\int_{B_{n+1}} \left(w^{2(a-1/m)}\right)^{m'} d\boldsymbol{x} \right)^{b/am'} dt$$

$$\leq \left(\sup_{t\in[0,T_{n+1}]} \int_{B_{n+1}} w^2(t,\boldsymbol{x})\, d\boldsymbol{x} \right)^{b/am} \int_0^{T_{n+1}} \left(\int_{B_{n+1}} w^{2d^*}\, d\boldsymbol{x} \right)^{1/d^*} dt \tag{59}$$

$$\leq c_3 \left(\sup_{t\in[0,T_{n+1}]} \int_{B_{n+1}} w^2(t,\boldsymbol{x})\, d\boldsymbol{x} \right)^{b/am}$$

$$\times \int_0^{T_{n+1}} \left(\int_{B_{n+1}} w^2\, d\boldsymbol{x} + \int_{B_{n+1}} |\nabla w|^2\, d\boldsymbol{x} \right)^{bd^*/am'} dt$$

$$\leq c_3 \left(\sup_{t\in[0,T_{n+1}]} \int_{B_{n+1}} w^2(t,\boldsymbol{x})\, d\boldsymbol{x} \right)^{b/am} \tag{60}$$

$$\times \int_0^{T_{n+1}} \left(\int_{B_{n+1}} w^2\, d\boldsymbol{x} + \int_{B_{n+1}} |\nabla w|^2\, d\boldsymbol{x} \right) dt.$$

Here we use the classical Sobolev inequality

$$\left(\frac{1}{R_n^d} \int_{B_n} w^{2d^*}\, d\boldsymbol{x} \right)^{1/d^*} \leq \frac{c_3}{R_n^d} \left(\int_{B_n} w^2\, d\boldsymbol{x} + R_n^2 \int_{B_n} |\nabla w|^2\, d\boldsymbol{x} \right), \tag{61}$$

for $n = 0, 1, 2, \ldots$ and any square-integrable functions w with square-integrable gradient ∇w. Note that $R_n^2 < 1$ since $1/2 < R_n < 1$.

Thus for $w = u^\alpha$,

$$\|u\|_{2a\alpha,2b\alpha,\Omega_{n+1}}^{2\alpha} \leq c_3 \|u^\alpha\|_{2,\infty,\Omega_{n+1}}^{2/am} \left(\|\nabla u^\alpha\|_{2,2,\Omega_{n+1}}^2 + \|u^\alpha\|_{2,2,\Omega_{n+1}}^2 \right)^{1/b}. \tag{62}$$

Using (54), (55) and the inequality $\|w\|_{2,2,\Omega_{n+1}} \leq c_4 \|w\|_{2p^*,2q^*,\Omega_{n+1}}$ we have from (62) that

$$\begin{aligned} \|u\|_{2a\alpha,2b\alpha,\Omega_{n+1}}^{2\alpha} &\leq c_5 \left[4^n \left(S(p,q,\Omega_n) + c_4 \right) \right]^{1/am} \|u\|_{2\alpha p^*,2\alpha q^*,\Omega_n}^{2\alpha/am} \\ &\quad \times \left[\frac{4^n}{\lambda} \left(S(p,q,\Omega_n) + c_4 \right) + c_4 \right]^{1/b} \|u\|_{2\alpha p^*,2\alpha q^*,\Omega_n}^{2\alpha/b} \\ &\leq c_5 \frac{4^n}{\lambda^{1/b}} \left(S(p,q,\Omega_n) + c_4 \left(1 + \frac{\lambda}{4^n} \right) \right) \|u\|_{2\alpha p^*,2\alpha q^*,\Omega_n}^{2\alpha}. \end{aligned} \tag{63}$$

In a more concise form,

$$\begin{aligned} \|u\|_{2a\alpha,2b\alpha,\Omega_{n+1}} &\leq C(\lambda,d)^{1/2\alpha} 2^{n/\alpha} \\ &\quad \times \left[S(p,q,\Omega_n) + c_4 \left(1 + \frac{\lambda}{4^n} \right) \right]^{1/2\alpha} \|u\|_{2\alpha p^*,2\alpha q^*,\Omega_n}. \end{aligned} \tag{64}$$

For $d = 2$, (64) remains true after a minor change in the use of the Sobolev inequality. Since $a > p^*$ and $b > q^*$, (64) bounds higher norms in smaller domains by lower ones in bigger domains so it can be iterated indefinitely to bound $L^{\infty,\infty}$ norm.

Set $\gamma = a/p^* = b/q^*$. We write it as $\gamma = 1 + 2\nu/d > 1$ with $\nu = 1 - d/p - 2/q > 0$. Let $\alpha_n = \gamma^n$, $n = 0, 1, 2, 3 \dots$.

Inequality (64) now takes the form

$$\begin{aligned} &\|u\|_{2p^*\alpha_{n+1}, 2q^*\alpha_{n+1}, \Omega_{n+1}} \\ &\quad \le C(\lambda, d)^{1/2\alpha_n} 2^{n/\alpha_n} \left[S(p, q, \Omega_n) + c_4\left(1 + \frac{\lambda}{4^n}\right)\right]^{1/2\alpha_n} \\ &\qquad \times \|u\|_{2p^*\alpha_n, 2q^*\alpha_n, \Omega_n}. \end{aligned} \tag{65}$$

Iterating (65) we have

$$\|u\|_{\infty,\infty,\Omega_{1/2,1/2}} = \lim_{n\to\infty} \|u\|_{2p^*\alpha_n, 2q^*\alpha_n, \Omega_n} \le C(\lambda, d)^{p_1} 2^{p_2} K \|u\|_{2p^*, 2q^*, \Omega_{1,1}},$$

where

$$K = \prod_{n=0}^{\infty} \left[S(p, q, \Omega_n) + c_4\left(1 + \frac{\lambda}{4^n}\right)\right]^{1/2\gamma^n} < \infty \tag{66}$$

and

$$p_1 = \frac{1}{2} \sum_0^\infty \frac{1}{\gamma^i} = \frac{\gamma}{2(\gamma - 1)} = \frac{1}{2}\left(1 + \frac{d}{2\nu}\right), \qquad p_2 = \sum_0^\infty \frac{i}{\gamma^i} < \infty.$$

The proof is now complete. □

Let $\mathscr{F}(T, R)$ be the family of all finite coverings of the cylinder $\Omega_{T,R}$ by closed cylinders. For any covering $\mathscr{C} \in \mathscr{F}(T, R)$ define

$$K(\mathscr{C}, \mathbf{a}) = \max_{\Omega \in \mathscr{C}} \{ K(\Omega), \|\mathbf{a}\|_{2,2,\Omega} \}, \tag{67}$$

where the constant $K(\Omega)$ is given by (66) with $\Omega_{R^2,R}$ replaced by Ω.

We denote the modulus of continuity of a function f in a domain $D \subset R^{d+1}$ by

$$\begin{aligned} w_{f,D}(\tau, \varrho) = \sup\{ |f(t, \boldsymbol{x}) - f(s, \boldsymbol{y})| \colon (t, \boldsymbol{x}), (s, \boldsymbol{y}) \in D, \\ |t - s| \le \tau, |\boldsymbol{x} - \boldsymbol{y}| \le \varrho \}. \end{aligned} \tag{68}$$

LEMMA 3. *Let u be a weak solution of* (20) *with* (A1) *and* (A2). *Then there exists a covering $\mathscr{C} \in \mathscr{F}(R^2, R)$ such that*

$$\begin{aligned} &w_{u,\Omega_{R^2/2,R/2}}(\theta^{2n} R^2, \theta^n R) \\ &\qquad \le \{ 1 - C_1 \exp\{ -C K^\kappa(\mathscr{C}, \mathbf{a}) \} \}^{[n/2]} \|u\|_{\infty,\infty,\Omega_{R^2,R}}, \end{aligned} \tag{69}$$

for some $\kappa > 0$, all $R > 0$ and all positive integers n, where constants $\theta, C_1 \in (0,1)$ and $C > 0$ depend only on λ, d, ν.

PROOF. We adopt the approach of Kružkov (1963).

Let α, β be numbers such that

$$0 < \alpha < \tfrac{1}{2}, \qquad 0 < \beta < 1, \qquad 2^{-1}(1-\alpha)^{-1}\beta^{-d} < \tfrac{2}{3}. \tag{70}$$

Set $\theta = \min(\sqrt{\alpha/2}, \beta/2)$.

Let $\mathscr{C}$ be a covering whose members are the cylinders

$$\Omega_{i,k,n} := \Omega_{\theta^{2n-2k}R^2,\, \theta^{n-k}R}(t_i, \boldsymbol{x}_i), \qquad i = 1, \dots, N, k = 0, \dots, n$$

for some positive integers n and N.

Set

$$M_{i,k}^{(n)} = \sup_{(t,\boldsymbol{x})\in\Omega_{i,k,n}} u(t,\boldsymbol{x}), \qquad m_{i,k}^{(n)} = \inf_{(t,\boldsymbol{x})\in\Omega_{i,k,n}} u(t,\boldsymbol{x}).$$

Let $\omega_{i,k}^{(n)} = M_{i,k}^{(n)} - m_{ik}^{(n)}$. Without any loss of generality, we can assume that $M_{i,k}^{(n)} = \frac{1}{2}\omega_{i,k}^{(n)} = -m_{i,k}^{(n)}$. Then

$$u_{i,k}^{(n)} = 1 + \frac{u}{M_{i,k}^{(n)}}, \qquad v_{i,k}^{(n)} = 1 - \frac{u}{M_{i,k}^{(n)}} \tag{71}$$

are two nonnegative solutions of (20) such that $0 \le u_{i,k}^{(n)}, v_{i,k}^{(n)} \le 2$ and $u_{i,k}^{(n)} + v_{i,k}^{(n)} = 2$ in $\Omega_{i,k,n}$. At least one of them, say $u_{i,k}^{(n)}$, is greater than or equal to 1 on a set $N \subseteq \Omega_{i,k,n}$ such that $\mu_{d+1}(N) \ge \frac{1}{2}\mu_{d+1}(\Omega_{i,k,n})$. Both here and in the sequel μ_m stands for the Lebesgue measure on R^m.

We want to show that the function $u_{i,k}^{(n)}$ is bounded away from zero in the cylinder $\Omega_{i,k-2,n}$; that is, there exists constants $C > 0$, $\kappa > 0$, $1 > C_1' > 0$ depending only on ν, d, λ such that

$$u_{i,k}^{(n)}(t,\boldsymbol{x}) \ge C_1' \exp\{-CK^{\kappa}(\mathscr{C}, \mathbf{a})\} \quad \text{in } \Omega_{i,k-2,n}, \tag{72}$$

for $1 \le i \le N$, $2 \le k \le n$. This in turn implies that

$$\omega_{i,k-2}^{(n)} \le \omega_{i,k}^{(n)}\left\{1 - \frac{C_1'}{2}\exp\{-CK^{\kappa}(\mathscr{C}, \mathbf{a})\}\right\}, \tag{73}$$

for all $i = 1, \dots, N$, $2 \le k \le n$. Clearly (73) implies (69) with $C_1 = C_1'/2$.

Define

$$w_\delta := f_\delta\left(u_{i,k}^{(n)}\right),$$

where

$$f_\delta(x) = \max\left[\ln\frac{\delta}{x+\delta^2}, 0\right]$$

and δ is a small number to be determined. Note that w_δ is a subsolution of (20) since f_δ is convex. Clearly,

$$w_\delta \le \ln\frac{1}{\delta} \quad \text{in } \Omega_{i,k,n}. \tag{74}$$

We will show a stronger upper bound for w_δ in a smaller region,

$$w_\delta \le LK(\Omega_{i,k,n})\ln^r\left(\frac{1}{\delta}\right) \quad \text{in } \Omega_{i,k-2,n}, \tag{75}$$

for $0 < \gamma < 1$ and δ sufficiently small, where constants L, r depend only on λ, p, q, d. This would eventually lead to the desired lower bound (72) for $u_{i,k}^{(n)}$. Without loss of generality we consider $t_i = 0$, $\boldsymbol{x}_i = \mathbf{0}$ only.

Define a test function for (22), with u replaced by $u_{i,k}^{(n)}$,

$$\varphi(t,\boldsymbol{x}) = f_\delta'\big(u_{i,k}^{(n)}\big)\eta^2(\boldsymbol{x})\zeta(t), \tag{76}$$

where $\zeta(t) \equiv 1$, for $0 \le t \le \alpha\theta^{2n-2k}R^2$ and 0 otherwise, and $\eta(\boldsymbol{x}) \equiv 0$, for $|\boldsymbol{x}| \ge 0^{n-k}R$, $\eta(\boldsymbol{x}) \equiv 1$ for $|\boldsymbol{x}| \le \beta\theta^{n-k}R$ and linearly interpolated in between.

Substituting (76) into (22) and using that $f_\delta'' \ge (f_\delta')^2$, we have that

$$\begin{aligned} 0 \ge& \iint_{\Omega(\alpha,1)} \big(\mathbf{a}\nabla u_{i,k}^{(n)}, \nabla u_{i,k}^{(n)}\big)\big[f_\delta'\big(u_{i,k}^{(n)}\big)\big]^2\eta^2\,dt\,d\boldsymbol{x}\\ &+ 2\iint_{\Omega(\alpha,1)} \big(\mathbf{a}\nabla u_{i,k}^{(n)}, \nabla\eta\big) f_\delta'\big(u_{i,k}^{(n)}\big)\eta\,dt\,d\boldsymbol{x}\\ &- \iint_{\Omega(\alpha,1)} \partial_t u_{i,k}^{(n)} f_\delta'\big(u_{i,k}^{(n)}\big)\eta^2\,dt\,d\boldsymbol{x}, \end{aligned} \tag{77}$$

where

$$\Omega(\tau,\sigma) = \Omega_{\tau\theta^{2n-2k}R^2,\,\sigma\theta^{n-k}R},$$

for any $\tau, \sigma > 0$. Since $|\nabla\eta| \le 1/((1-\beta)\theta^{n-k}R)$ we then have

$$\begin{aligned} &\iint_{\Omega(\alpha,1)} (\mathbf{a}\eta\nabla w_\delta, \eta\nabla w_\delta)\,dt\,d\boldsymbol{x} + \int_{B_{\theta^{n-k}R}} w_\delta(0,\boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x}\\ &\quad\le \frac{2}{(1-\beta)\theta^{n-k}R}\iint_{\Omega(\alpha,1)} |\mathbf{a}\eta\nabla w_\delta|\,dt\,d\boldsymbol{x}\\ &\qquad+ \int_{B_{\theta^{n-k}R}} w_\delta(\alpha\theta^{2n-2k}R^2,\boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x}. \end{aligned} \tag{78}$$

By (A1) and the Cauchy–Schwartz inequality we get

$$\begin{aligned} &\lambda\iint_{\Omega(\alpha,1)} |\eta\nabla w_\delta|^2\,dt\,d\boldsymbol{x}\\ &\quad\le \frac{2\sqrt{\alpha}}{1-\beta}\|\mathbf{a}\|_{2,2,\Omega(\alpha,1)}|B_{\theta^{n-k}R}|^{1/2}\left(\iint_{\Omega(\alpha,1)} |\eta\nabla w_\delta|^2\,dt\,d\boldsymbol{x}\right)^{1/2}\\ &\qquad+ \int_{B_{\theta^{n-k}R}} w_\delta(\alpha\theta^{2n-2k}R^2,\boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x}. \end{aligned} \tag{79}$$

By the inequality $AB \le (3\lambda/4)A^2 + (1/3\lambda)B^2$ and (74), the right side of (79) is less than or equal to

$$\frac{3\lambda}{4}\iint_{\Omega(\alpha,1)} |\eta\nabla w_\delta|^2\,dt\,d\boldsymbol{x} + \frac{4\alpha}{3\lambda(1-\beta)^2}\|\mathbf{a}\|^2_{2,2,\Omega(\alpha,1)}|B_{\theta^{n-k}R}| + \ln\frac{1}{\delta}|B_{\theta^{n-k}R}|.$$

Finally,

$$\iint_{\Omega(\alpha,\beta)} |\nabla w_\delta|^2 \, dt\, d\boldsymbol{x} \le \iint_{\Omega(\alpha,1)} |\eta \nabla w_\delta|^2 \, dt\, d\boldsymbol{x} \le \left[\frac{16\alpha}{3\lambda^2(1-\beta)^2} \|\mathbf{a}\|^2_{2,2,\Omega(\alpha,1)} + \frac{4}{\lambda} \ln \frac{1}{\delta} \right] |B_{\theta^{n-k}R}| \tag{80}$$

or, equivalently,

$$\|\nabla w_\delta\|_{2,2,\Omega(\alpha,\beta)} \le \frac{1}{\beta^{d/2}\sqrt{\alpha}\,\theta^{n-k}R} \times \left[\frac{16\alpha}{3\lambda^2(1-\beta)^2} \|\mathbf{a}\|^2_{2,2,\Omega(\alpha,1)} + \frac{4}{\lambda} \ln \frac{1}{\delta} \right]^{1/2}. \tag{81}$$

Since w_δ is a nonnegative subsolution. Lemma 2 implies that

$$\|w_\delta\|_{\infty,\infty,\Omega(\alpha/2,\beta/2)} \le CK(\Omega(\alpha,\beta)) \|w_\delta\|_{2p^*,2q^*,\Omega(\alpha,\beta)} \le CK(\Omega(\alpha,\beta)) \|w_\delta\|_{2\tilde{p}^*,2\tilde{q}^*,\Omega(\alpha,\beta)} \tag{82}$$

with

$$\frac{d}{\tilde{p}} + \frac{2}{\tilde{q}} = 1, \qquad \frac{d}{\tilde{p}} - \frac{d}{p} = \frac{2}{\tilde{q}} - \frac{2}{q},$$

with $\tilde{p}^* := \tilde{p}/(\tilde{p}-2)$, $\tilde{q}^* := \tilde{q}/(\tilde{q}-2)$. In fact, $\tilde{p}^* = a$, $\tilde{q}^* = b$ as given by (57) or (58), hence $\tilde{p}^* > p^*$, $\tilde{q}^* > q^*$. The constants C, C' depend only on d, λ and ν.

We state now an inequality applicable to w_δ which is closely related to the Sobolev inequality [Moser (1960), Lemma 2, page 461],

$$\left(\int_{B_{\beta\theta^{n-k}R}} w_\delta^{2d^*}(t,\boldsymbol{x})\, d\boldsymbol{x} \right)^{1/d^*} \le c_d \int_{B_{\beta\theta^{n-k}R}} |\nabla w_\delta(t,\boldsymbol{x})|^2 \, d\boldsymbol{x}, \tag{83}$$

or

$$\|w_\delta(t,\cdot)\|_{2d^*, B_{\beta\theta^{n-k}R}} \le \sqrt{c_d}\, \beta\theta^{n-k} R \|\nabla w_\delta(t,\cdot)\|_{2, B_{\beta\theta^{n-k}R}}, \tag{84}$$

for $0 \le t \le \alpha\theta^{2(n-k)}R^2$, where the constant c_d depends only on the dimension.

Inequality (84) holds because of the following property of w_δ.

CLAIM.

$$\mu_d(N_\delta(t)) \ge \tfrac{1}{4} |B_{\beta\theta^{n-k}R}| \tag{85}$$

for the set

$$N_\delta(t) \equiv \left[\boldsymbol{x} \in B_{\beta\theta^{n-k}R} \;\middle|\; u^{(n)}_{i,k}(t,\boldsymbol{x}) \ge \delta \right]$$

and for all $0 \le t \le \alpha\theta^{2n-2k}R^2$,

$$0 < \delta < E \equiv \exp\left\{ -12 \left[\ln 2 + 4\|\mathbf{a}\|_{2,2,\Omega_{T,R}} / \left(3\lambda(1-\beta)^2 \beta^d \right) \right] \right\}.$$

Note that w_δ vanishes on $N_\delta(t)$.

The proof of the claim is postponed until the end of the section. Let us continue with the proof of the bound (75).

The argument used to obtain (59) can be repeated here to yield

$$\|w\|_{2\tilde{p}^*, 2\tilde{q}^*, \Omega(\alpha,\beta)} \le \|w\|_{2,\infty,\Omega(\alpha,\beta)}^{1-1/\tilde{q}^*} \|w\|_{2d^*,2,\Omega(\alpha,\beta)}^{1/\tilde{q}^*}. \tag{86}$$

From (82), (86), (74) and (84) it follows that

$$\begin{aligned} &\|w_\delta\|_{\infty,\infty,\Omega(\alpha/2,\beta/2)} \\ &\quad \le CK(\Omega(\alpha,\beta))\left(\ln\frac{1}{\delta}\right)^{1-1/\tilde{q}^*} \|w_\delta\|_{2d^*,2,\Omega(\alpha,\beta)}^{1/\tilde{q}^*} \\ &\quad \le C_3 K(\Omega(\alpha,\beta))\left(\ln\frac{1}{\delta}\right)^{1-1/\tilde{q}^*} \left(\beta\theta^{n-k}R\right)^{1/\tilde{q}^*} \|\nabla w_\delta\|_{2,2,\Omega(\alpha,\beta)}^{1/\tilde{q}^*}. \end{aligned} \tag{87}$$

By (81),

$$\begin{aligned} \|w_\delta\|_{\infty,\infty,\Omega(\alpha/2,\beta/2)} &\le C_3 K(\Omega(\alpha,\beta))\left(\ln\frac{1}{\delta}\right)^{1-1/\tilde{q}^*} \left(\frac{1}{\beta^{d/2-1}\sqrt{\alpha}}\right)^{1/\tilde{q}^*} \\ &\qquad \times \left[\frac{16\alpha}{3\lambda^2(1-\beta)^2}\|\mathbf{a}\|_{2,2,\Omega(\alpha,1)}^2 + \frac{4}{\lambda}\ln\frac{1}{\delta}\right]^{1/(2\tilde{q}^*)} \\ &\le C_3 K(\Omega(\alpha,\beta))\left(\frac{1}{\beta^{d/2-1}\sqrt{\alpha}}\right)^{1/\tilde{q}^*} \left(\frac{8}{\lambda}\right)^{1/2q^*} \left(\ln\frac{1}{\delta}\right)^{1-1/2\tilde{q}^*} \end{aligned}$$

provided that

$$\delta \le \min\left\{E, \exp\left\{-\frac{4\alpha}{3\lambda(1-\beta)^2}\|\mathbf{a}\|_{2,2,\Omega_{i,k,n}}^2\right\}\right\}. \tag{88}$$

Therefore, (75) holds with $r = 1 - 1/2\tilde{q}^*$, $L = C_3(8/\beta^{d-2}\alpha\lambda)^{1/2\tilde{q}^*}$ for δ satisfying (88).

The rest of the argument is split into two cases.

Case 1. When $u_{i,k}^{(n)}(t,\boldsymbol{x}) + \delta^2 < \delta$, that is, $w_\delta > 0$, then by (75) we have

$$\frac{\delta}{u_{i,k}^{(n)}(t,\boldsymbol{x}) + \delta^2} \le \exp\left\{LK(\Omega_{i,k,n})\left(\ln\frac{1}{\delta}\right)^r\right\}$$

so

$$\begin{aligned} u_{i,k}^{(n)}(t,\boldsymbol{x}) &\ge \delta\exp\left\{-LK(\Omega_{i,k,n})\left(\ln\frac{1}{\delta}\right)^r\right\} - \delta^2 \\ &\ge \frac{1}{2}\delta\exp\left\{-LK(\Omega_{i,k,n})\left(\ln\frac{1}{\delta}\right)^r\right\}, \end{aligned}$$

provided that

$$\begin{aligned} \delta \le \delta_0 = \min\Big\{&E, \exp\left\{-L^{2\tilde{q}^*}K^{1/1-r}(\Omega_{i,k,n})\right\}, \\ &\exp\left(-4\alpha\|\mathbf{a}\|_{2,2,\Omega_{i,k,n}}^2/3\lambda(1-\beta)^2\right)\Big\}. \end{aligned}$$

Case 2. When $u_{i,k}^{(n)}(t, \boldsymbol{x}) + \delta^2 \geq \delta$, we have that $u_{i,k}^{(n)}(t, \boldsymbol{x}) \geq \frac{1}{2}\delta$ provided that $\delta \leq 1/2$.

Therefore, for $\delta = \min\{\delta_0, 1/2\}$, we have that

$$u_{i,k}^{(n)}(t, \boldsymbol{x}) \geq C_1' \exp\{-CK^{\kappa}(\mathscr{C}, \mathbf{a})\},$$

where C_1', C, κ depend only on λ, ν, d.

Proof of Claim (85). For any $0 \leq t \leq \theta^{2n-2k}R^2$ let us denote

$$N_1(t) = \left[\boldsymbol{x} \in B_{\theta^{n-k}R}\colon u_{i,k}^{(n)}(t, \boldsymbol{x}) \geq 1\right].$$

Let $\mu_d(N_1(t_0))$ be the maximum of $\mu_d(N_1(t))$ in $\alpha\theta^{2n-2k}R^2 \leq t \leq \theta^{2n-2k}R^2$. We show now that

$$\mu_d(N_1(t_0)) \geq \left(\tfrac{1}{2} - \alpha\right)(1-\alpha)^{-1}|B_{\theta^{n-k}R}|. \tag{89}$$

Indeed,

$$\begin{aligned}
&\mu_d(N_1(t_0))\theta^{2n-2k}R^2(1-\alpha)\\
&\quad \geq \int_{\alpha\theta^{2n-2k}R^2}^{\theta^{2n-2k}R^2} \mu_d(N_1(t))\,dt\\
&\quad \geq \mu_{d+1}(N) - \alpha\theta^{2n-2k}R^2|B_{\theta^{n-k}R}| \geq \left(\tfrac{1}{2} - \alpha\right)|B_{\theta^{n-k}R}|R^2.
\end{aligned}$$

Here the set N is defined in the remark on the selection of $u_{i,k}^{(n)}$ or $v_{i,k}^{(n)}$ in (71). Consider

$$v_\delta = g_\delta\left(u_{i,k}^{(n)}\right),$$

where

$$g_\delta(x) = \max\left[\ln\frac{1}{x+\delta}, 0\right].$$

Let us set $\varphi(t, \boldsymbol{x}) = g_\delta'(u_{i,k}^{(n)}(t, \boldsymbol{x}))\eta^2(\boldsymbol{x})\zeta(t)$, where $\zeta(t) \equiv 1$, for $s \leq t \leq t_0$, 0 otherwise and η is the same as in (76). A computation similar to the one leading to (77) yields

$$\begin{aligned}
&\int_s^t \int_{B_{\theta^{n-k}R}} (\mathbf{a}\eta\nabla v_\delta, \eta\nabla v_\delta)\,dt\,d\boldsymbol{x} + \int_{B_{\theta^{n-k}R}} v_\delta(s, \boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x}\\
&\quad \leq \frac{2\sqrt{t_0 - s}}{\theta^{n-k}R(1-\beta)}\|\mathbf{a}\|_{2,2,\Omega_{i,k,n}}|B_{\theta^{n-k}R}|^{1/2}\left(\int_s^{t_0}\int_{B_{\theta^{n-k}R}} |\eta\nabla v_\delta|^2\right)^{1/2}\\
&\qquad + \int_{B_{\theta^{n-k}R}} v_\delta(t_0, \boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x}.
\end{aligned}$$

Using again Young's inequality and (A1), we obtain

$$
\begin{aligned}
&\int_{B_{\beta\theta^{n-k}R}} v_\delta(s,\boldsymbol{x})\eta^2(\boldsymbol{x})\,d\boldsymbol{x} + \frac{\lambda}{4}\int_s^t \int_{B_{\beta\theta^{n-k}R}} |\nabla v_\delta|^2\,dt\,d\boldsymbol{x}\\
&\qquad \le \frac{4(t_0 - s)}{3\lambda(1-\beta)^2\theta^{2(n-k)}R^2\beta^d}\|\mathbf{a}\|_{2,2,\Omega^2_{i,k,n}} |B_{\beta\theta^{n-k}R}| \\
&\qquad\quad + \int_{B_{\theta^{n-k}R}} v_\delta(t_0,\boldsymbol{x})\,d\boldsymbol{x}.
\end{aligned}
\tag{90}
$$

For $\boldsymbol{x}\notin N_\delta(s)$ we have $v_\delta(s,\boldsymbol{x})\ge \ln(1/2\delta)$, for $\boldsymbol{x}\notin N_1(t_0)$ we have $v_\delta(t_0,\boldsymbol{x})\le \ln(1/\delta)$ and for $\boldsymbol{x}\in N_1(t_0)$, $v_\delta(t_0,\boldsymbol{x}) = 0$. Using this and (90) we get

$$
\begin{aligned}
&\mu_d\big(B_{\beta\theta^{n-k}R}\setminus N_\delta(s)\big)\ln\frac{1}{2\delta}\\
&\quad\le \frac{4(t_0-s)}{3\lambda(1-\beta)^2\beta^d R^2\theta^{2(n-k)}}\|\mathbf{a}\|^2_{2,2,\Omega_{i,k,n}}|B_{\beta\theta^{n-k}R}| + \mu_d\big(B_{\theta^{n-k}R}\setminus N_1(t_0)\big)\ln\frac{1}{\delta}\\
&\quad\le \left[\frac{4(t_0-s)}{3\lambda(1-\beta)^2\theta^{2(n-k)}R^2\beta^d}\|\mathbf{a}\|^2_{2,2,\Omega_{i,k,n}} + \frac{1}{2}(1-\alpha)^{-1}\beta^{-d}\ln\frac{1}{\delta}\right]|B_{\beta\theta^{n-k}R}|.
\end{aligned}
$$

The last inequality follows from (89). Finally, we obtain

$$
\begin{aligned}
&\mu_d\big(B_{\beta\theta^{n-k}R}\setminus N_\delta(s)\big)\\
&\quad\le \left[\frac{1}{\ln(1/\delta)}\left(\frac{4(t_0-s)\|\mathbf{a}\|^2_{2,2,\Omega_{i,k,n}}}{3\lambda(1-\beta)^2\theta^{2(n-k)}R^2\beta^d} + \ln 2\right)\right.\\
&\qquad\qquad \left. + \frac{1}{2}(1-\alpha)^{-1}\beta^{-d}\right]|B_{\beta\theta^{n-k}R}|\\
&\quad\le \left[\frac{2}{3} + \frac{1}{\ln(1/\delta)}\left(\frac{4\|\mathbf{a}\|^2_{2,2,\Omega_{i,k,n}}}{3\lambda(1-\beta)^2\beta^d} + \ln 2\right)\right]|B_{\beta\theta^{n-k}R}|.
\end{aligned}
$$

The last inequality follows from (70) and $((t_0 - s)/\theta^{2(n-k)}R^2) < 1$.

For $0 < \delta < E$,

$$
\mu_d\big(N_\delta(s)\big) \ge \tfrac{1}{4}|B_{\beta\theta^{n-k}R}|,
$$

for all $0 \le s \le t_0$. □

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