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CHARACTERIZATION OF G-REGULARITY FOR SUPER-BROWNIAN MOTION AND CONSEQUENCES FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

BY JEAN-FRANÇOIS DELMAS[1] AND JEAN-STÉPHANE DHERSIN

MSRI and Université René Descartes

We give a characterization of G-regularity for super-Brownian motion and the Brownian snake. More precisely, we define a capacity on $E = (0, \infty) \times \mathbb{R}^d$, which is not invariant by translation. We then prove that the measure of hitting a Borel set $A \subset E$ for the graph of the Brownian snake excursion starting at $(0, 0)$ is comparable, up to multiplicative constants, to its capacity. This implies that super-Brownian motion started at time 0 at the Dirac mass δ_0 hits immediately A [i.e., $(0, 0)$ is G-regular for A^c] if and only if its capacity is infinite. As a direct consequence, if $Q \subset E$ is a domain such that $(0, 0) \in \partial Q$, we give a necessary and sufficient condition for the existence on Q of a positive solution of $\partial_t u + \frac{1}{2}\Delta u = 2u^2$, which blows up at $(0, 0)$. We also give an estimate of the hitting probabilities for the support of super-Brownian motion at fixed time. We prove that if $d \geq 2$, the support of super-Brownian motion is intersection-equivalent to the range of Brownian motion.

1. Introduction. The purpose of this paper is to give a characterization of the so-called G-regularity for super-Brownian motion introduced by Dynkin [8]. Thus we say that a point $(r, x) \in \mathbb{R} \times \mathbb{R}^d$ is G-regular for a Borel set $A \subset \mathbb{R} \times \mathbb{R}^d$, if a.s. the graph of a super-Brownian motion started at time r with the Dirac mass at x immediately intersects A^c, the complementary of A. In case $A = Q$ is an open set, this is equivalent to the existence of nonnegative solutions of the equation $(\partial u/\partial t) + \frac{1}{2}\Delta u = 2u^2$ on the open set Q, which blow up at $(r, x) \in \partial Q$ (cf. [8]).

Let $E = (0, \infty) \times \mathbb{R}^d$. We prove that $(0, 0)$ is G-regular for a Borel set $A \subset \mathbb{R} \times \mathbb{R}^d$ if and only if the capacity of $A^c \cap E$ is infinite, for the following capacity: for any Borel set $A' \subset E$,

$$\mathrm{cap}(A') = [\inf I(\nu)]^{-1}$$

where

$$I(\nu) = \iint_E ds\, dy\, p(s, y)\left(\iint_E \nu(dt, dx)\frac{p(t-s, x-y)}{p(t, x)}\right)^2,$$

and p denotes the heat kernel,

$$p(t, x) = \begin{cases} (2\pi t)^{-d/2}\exp(-|x|^2/2t), & \text{if } (t, x) \in E, \\ 0, & \text{if } (t, x) \in (-\infty, 0] \times \mathbb{R}^d. \end{cases}$$

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[$|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.] The infimum is taken over all probability measures ν on E such that $\nu(A')=1$. Notice that this capacity is not invariant by translation in time or space. This capacity arises naturally when one considers the Brownian snake, a useful tool to study super-Brownian motion. Indeed, using potential theory of symmetric Markov process, $I(\nu)$ can be viewed as the energy, with respect to the Brownian snake, of a certain probability measure (see Section 4 for more details).

We extend a result due to Dhersin and Le Gall [6] where the authors study G-regularity of $(0,0)$ for sets $Q=\{(s,y)\in E; |y|<\sqrt{s}h(s)\}$, where h is a positive decreasing function defined on $(0,\infty)$. Our result can also be viewed as a parabolic extension of the Wiener's test in [5] in an elliptic setting.

The proof of our results relies on the Brownian snake introduced by Le Gall. We only give the definition and some properties for completeness in this paper and refer to [10], [12] for detailed presentations. We will use time inhomogeneous notations.

Let $(r,x)\in\mathbb{R}\times\mathbb{R}^d$ be a fixed point. We denote by $\mathscr{W}_{r,x}$ the set of all stopped paths in $\mathbb{R}^d$ started at x at time r. An element w of $\mathscr{W}_{r,x}$ is a continuous mapping $w\colon [r,\zeta]\to\mathbb{R}^d$ such that $w(r)=x$, and $\zeta=\zeta_{(w)}\in[r,\infty)$ is called its lifetime. We denote by $\hat{w}$ the end point $w(\zeta)$. With the metric $d(w,w')=\left|\zeta_{(w)}-\zeta_{(w')}\right|+\sup_{s\geq r}\left|w(s\wedge\zeta_{(w)})-w'(s\wedge\zeta_{(w')})\right|$, the space $\mathscr{W}_{r,x}$ is a Polish space. The Brownian snake started at x at time r is a continuous strong Markov process $W=(W_s, s\geq 0)$ with values in $\mathscr{W}_{r,x}$, whose law is characterized by the following two properties.

1. The lifetime process $\zeta=(\zeta_s=\zeta_{(W_s)}, s\geq 0)$ is a reflecting Brownian motion in $[r,\infty)$.
2. Conditionally given $(\zeta_s, s\geq 0)$, the process $(W_s, s\geq 0)$ is a time-inhomogeneous continuous Markov process, such that for $s'\geq s$,
   $W_{s'}(t)=W_s(t)$ for $r\leq t\leq m(s,s')=\inf_{v\in[s,s']}\zeta_v$.
   $(W_{s'}(m(s,s')+t)-W_{s'}(m(s,s')), 0\leq t\leq \zeta_{s'}-m(s,s'))$ is a Brownian motion in $\mathbb{R}^d$ independent of W_s.

From now on we shall consider the canonical realization of the process W defined on the space $\Omega=C(\mathbb{R}^+,\mathscr{W}_{r,x})$, and denote by $\mathscr{E}_w$ the law of W started at $w\in\mathscr{W}_{r,x}$. The trivial path $\mathbf{x}_r$ such that $\zeta_{(\mathbf{x}_r)}=r$, $\mathbf{x}_r(r)=x$ is clearly a regular point for the process $(W,\mathscr{E}_w)$. We denote by $\mathbb{N}_{r,x}$ the excursion measure outside $\{\mathbf{x}_r\}$, normalized by the following: for every $\varepsilon>0$,

$$\mathbb{N}_{r,x}\Big[\sup_{s\geq 0}\zeta_s>\varepsilon+r\Big]=\frac{1}{2\varepsilon}.$$

Notice that $\mathbb{N}_{r,x}$ is an infinite measure. The distribution of W under $\mathbb{N}_{r,x}$ can be characterized as above, except that now the lifetime process ζ is distributed according to the Itô measure of excursions of linear reflecting Brownian motion in $[r,\infty)$. Let $\sigma=\inf\{s>0;\zeta_s=r\}$ denote the duration of the excursion of ζ

under $\mathbb{N}_{r,x}$. The graph $\mathscr{G}^*$ of W is defined under $\mathbb{N}_{r,x}$ by

$$\mathscr{G}^* = \big\{(t, W_s(t)); r < t \leq \zeta_s, 0 < s < \sigma\big\} = \big\{(\zeta_s, \hat{W}_s); 0 < s < \sigma\big\}.$$

We write $\mathscr{G}^*(W)$ for $\mathscr{G}^*$ when there is a risk of confusion.

Let us now explain the connection between the Brownian snake and super-Brownian motion. First, we introduce some notations. We denote by $(M_f, \mathscr{M}_f)$ the space of all finite measures on $\mathbb{R}^d$, endowed with the topology of weak convergence. We denote by $\mathscr{B}(S)$ (resp., $\mathscr{B}_{b+}(S)$) the set of all real measurable [resp., bounded nonnegative measurable] functions defined on a Polish space S. We also denote by $\mathscr{B}(S)$ the Borel σ-field on S. For every measure $\nu \in M_f$, and $f \in \mathscr{B}_{b+}(\mathbb{R}^d)$, we shall write $(\nu, f) = \int f(y)\nu(dy)$. We also denote by $\operatorname{supp}\nu$ the closed support of the measure ν.

We consider under $\mathbb{N}_{r,x}$ the continuous version $(l_s^t, t > r, s \geq 0)$ of the local time of ζ at level t and time s and define the measure-valued process Y on $\mathbb{R}^d$ by setting for every $t > r$, for every $\varphi \in \mathscr{B}_{b+}(\mathbb{R}^d)$,

$$(Y_t, \varphi) = \int_0^\sigma dl_s^t\, \varphi(\hat{W}_s).$$

Let $\mathscr{W}_r = \bigcup_{x\in\mathbb{R}^d} \mathscr{W}_{r,x}$. Let μ be a finite measure on $\mathbb{R}^d$ and $\sum_{i\in I} \delta_{W^i}$ be a Poisson measure on $C(\mathbb{R}^+, \mathscr{W}_r)$ with intensity $\int \mu(dx)\mathbb{N}_{r,x}[\cdot]$. Then the process X defined by $X_r = \mu$ and $X_t = \sum_{i\in I} Y_t(W^i)$ if $t > r$, is a super-Brownian motion started at time r at μ (see [10], [12]). We shall denote by $\mathbf{P}_{r,\mu}$ (resp., $\mathbf{P}_{r,x}$) the law of the super-Brownian motion started at time r at μ (resp., at the Dirac mass δ_x). We deduce from the normalization of $\mathbb{N}_{r,x}$ that, for every $t > r$, $\mathbb{N}_{r,x}[Y_t \neq 0] = 1/2(t-r) < \infty$. This implies that, for $t > r$, there is only a finite number of indices $i \in I$ such that $\mathscr{G}^*(W^i) \cap [t, \infty) \times \mathbb{R}^d$ is nonempty.

We consider the graph of X,

$$\mathscr{G}(X) = \bigcup_{\varepsilon > r} \left(\overline{\bigcup_{t\geq\varepsilon} \{t\} \times \operatorname{supp} X_t} \right) = \bigcup_{i\in I} \mathscr{G}^*(W^i),$$

where $\bar{A}$ denotes the closure of A. A set $A \subset \mathscr{B}(\mathbb{R} \times \mathbb{R}^d)$ is called G-polar if $\mathbf{P}_{r,x}[\mathscr{G}(X) \cap A \neq \varnothing] = 0$ for every $(r, x) \in \mathbb{R} \times \mathbb{R}^d$. From Poisson measure theory, we have

$$\mathbf{P}_{r,x}\big[\mathscr{G}(X) \cap A \neq \varnothing\big] = 1 - \exp\left(-\mathbb{N}_{r,x}[\mathscr{G}^* \cap A \neq \varnothing]\right).$$

Hence A is G-polar if and only if $\mathbb{N}_{r,x}[\mathscr{G}^* \cap A \neq \varnothing] = 0$ for all $(r, x) \in \mathbb{R} \times \mathbb{R}^d$. We consider the capacity defined by the following: for $A \in \mathscr{B}(\mathbb{R} \times \mathbb{R}^d)$,

$$\operatorname{cap}'(A) = \Bigg[\inf \iint_{\mathbb{R}\times\mathbb{R}^d} ds\, dy \left(\iint_{(s,\infty)\times\mathbb{R}^d} \nu(dt, dx) p(t-s, x-y) \exp\left(-(t-s)/2\right) \right)^2 \Bigg]^{-1},$$

where the infimum is taken over all probability measures ν on $\mathbb{R}\times\mathbb{R}^d$ such that $\nu(A)=1$. Dynkin proved (see Theorem 3.2 in [7]) that $A\in\mathscr{B}(\mathbb{R}\times\mathbb{R}^d)$ is G-polar if and only if $\mathrm{cap}'(A)=0$. (We have $\mathrm{cap}'(A)=0 \Leftrightarrow \mathbb{N}_{r,x}[\mathscr{G}^*\cap A\neq\varnothing]=0$ for all $(r,x)\in\mathbb{R}\times\mathbb{R}^d$.) It is easy to check that if $A\subset E$ is a compact set, then

$$\mathrm{cap}'(A)=0 \quad\Leftrightarrow\quad \mathrm{cap}(A)=0.$$

This can be extended to all Borel subsets of E since the two capacities are inner capacities (see [13]). In fact, it seems more relevant to consider the capacity cap to characterize G-regularity, as we shall see. We have the following quantitative theorem.

THEOREM 1. *There exists a constant C_0 such that for any $A\in\mathscr{B}(E)$,*

$$4^{-1}\,\mathrm{cap}(A)\leq \mathbb{N}_{0,0}[\mathscr{G}^*\cap A\neq\varnothing]\leq C_0\,\mathrm{cap}(A).$$

If Q is a domain of $\mathbb{R}\times\mathbb{R}^d$, it is well known that the function $(r,x)\mapsto \mathbb{N}_{r,x}[\mathscr{G}^*\cap Q^c\neq\varnothing]$ is the maximal nonnegative solution u_M of $\partial u/\partial t+\frac{1}{2}\Delta u=2u^2$ in Q. Hence, we deduce from Theorem 1 that if $(0,0)\in Q$, then $u_M(0,0)$ and $\mathrm{cap}(Q^c\cap E)$ are comparable up to multiplicative constants which are independent of Q.

The proof of Theorem 1 is split in two parts. In Section 2, we introduce a capacity associated with a weighted Sobolev space, which is equivalent to the capacity cap. In Section 3, using the connections between super-Brownian motion and partial differential equations, we prove the upper bound with this new capacity and hence for the capacity cap. The lower bound is obtained in Section 4 by using additive functionals of the Brownian snake introduced in [5].

Now, for $A\in\mathscr{B}(\mathbb{R}\times\mathbb{R}^d)$, we consider under $\mathbf{P}_{r,x}$ the random time

$$\tau_A=\inf\bigl\{t>r,(\{t\}\times\mathrm{supp}\,X_t)\cap A\neq\varnothing\bigr\}.$$

Arguments similar to those of [5] yield that τ_A is a stopping time for the natural filtration of X completed the usual way. Thus we have $\mathbf{P}_{r,x}(\tau_A=r)=1$ or 0. Following Dynkin [8], Section II-6, we say a point $(r,x)\in\mathbb{R}\times\mathbb{R}^d$ is G-regular for A^c if $\mathbf{P}_{r,x}$-a.s. $\tau_A=r$. Let A^{Gr} denote the set of all points that are G-regular for A^c. From the known path properties of super-Brownian motion it is obvious that $\mathrm{int}(A)\subset A^{Gr}\subset\bar{A}$, where $\mathrm{int}(A)$ denotes the interior of A. We set $T_A=\inf\{s>0,(\zeta_s,\hat{W}_s)\in A\}$. Following [5], it is easy to deduce from Theorem 1 the next result.

PROPOSITION 2. *Let $A\in\mathscr{B}(\mathbb{R}\times\mathbb{R}^d)$. The following properties are equivalent*:

(i) (r,x) *is G-regular for* A^c;
(ii) $\mathbb{N}_{r,x}[\mathscr{G}^*\cap A\neq\varnothing]=\infty$;
(iii) $\mathscr{E}_{\mathbf{x}_r}$*-a.s.* $T_A=0$;
(iv) $\mathrm{cap}(A_{r,x}\cap E)=\infty$, *where* $A_{r,x}=\{(s,y);(s+r,y+x)\in A\}$.

We can give a straightforward analytic consequence of Proposition 2 and the link between super-Brownian motion and nonlinear differential equations.

COROLLARY 3. *Let Q be a domain in E such that $(0,0) \in \partial Q$. The following three conditions are equivalent*:

(i) $(0,0)$ *is G-regular for* Q;
(ii) $\operatorname{cap}(Q^c \cap E) = \infty$;
(iii) *there exists a nonnegative solution of* $\partial u/\partial t + \frac{1}{2}\Delta u = 2u^2$ *in* Q *such that*

$$\lim_{(s,y)\to(0,0),\,(s,y)\in Q} u(s,y) = \infty.$$

The equivalence of assertions (i) and (iii) is due to Dynkin [8], Theorem II.6.1. The equivalence of (i) and (ii) is given by Proposition 2.

Finally, using Theorem 1, we give in Section 5 an estimate of the hitting probability of the support of X_1, and we prove that in dimension $d \geq 2$, the support of super-Brownian motion and the range of d-dimensional Brownian motion are intersection-equivalent.

2. Equivalence of capacities for a weighted Sobolev space. In this section, we introduce a new capacity, associated with a weighted Sobolev space, which is equivalent to the capacity cap. This capacity will be very useful in the next section to prove the upper bound for Theorem 1.

If S is an open subset of $\mathbb{R}^r$, we denote by $C_0^\infty(S)$ the set of all functions of class C^∞ defined on S with compact support. If f is a measurable function defined on S then $\|f\|_\infty = \sup_{x\in S}|f(s)|$. We consider the Hilbert space $L^2(p) = \{f \in \mathscr{B}(E); \|f\|_{(p)} < \infty\}$, where $\|f\|_{(p)}^2 = \iint_E dt\,dx\; p(t,x)f(t,x)^2$.

Notice the kernel defined on $E \times E$ by $k(t,x;s,y) = p(t-s,x-y)p(t,x)^{-1}$ is nonnegative and lower semicontinuous. Thus we can introduce the operator Λ defined on the set of nonnegative functions $f \in \mathscr{B}(E)$ by

$$\Lambda(f) = p^{-1}[p * (pf)] = \iint_E ds\,dy\,k(\cdot,\cdot;s,y)p(s,y)f(s,y),$$

where $*$ denotes the usual convolution product on E. Furthermore, the function $\Lambda(f)$ is even lower semicontinuous (see [9], Lemma 2.2.1).

We define the capacity Cap on E in the following way: if $A \subset E$, then

$$\operatorname{Cap}(A) = \inf\{\|f\|_{(p)}^2; f \geq 0, f \in L^2(p), \Lambda(f) \geq 1 \text{ on } A\},$$

with the convention $\inf\varnothing = \infty$. Notice this capacity is not invariant by translation in time or space. This capacity is an outer capacity (see [13], Theorem 1). Moreover, it coincides with the capacity cap on the analytic sets (see [13], Theorem 14). Now, we want to connect this capacity to an analytic capacity (see [3] for similar results but with different norms). Therefore we consider the weighted Sobolev space W_D which is the completion of $C_0^\infty(E)$ with respect to the norm $\|\cdot\|_D$, defined by

$$\|\varphi\|_D^2 = \|\partial_t\varphi\|_{(p)}^2 + \sum_{i=1}^d \|\partial_i(\log p)\partial_i\varphi\|_{(p)}^2 + \sum_{i=1}^d \|\partial_{ii}^2\varphi\|_{(p)}^2, \qquad \varphi \in C_0^\infty(E),$$

with the usual notations $\partial_t g(t,x) = (\partial g/\partial t)(t,x)$, $\partial_i g(t,x) = (\partial g/\partial x_i)(t,x)$ for $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ and $\partial_{ii}^2 = \partial_i\partial_i$. Notice the nonzero constants do not belong to W_D. We can introduce the outer capacity cap_D associated to W_D defined as follows. For any compact set $K \subset E$, we set

$$\begin{aligned}\mathrm{cap}_D(K) &= \inf\{\|\varphi\|_D^2; \varphi \in C_0^\infty(E), \varphi \ge 0, \varphi \ge 1 \text{ on } K\}\\ &= \inf\{\|\varphi\|_D^2; \varphi \in C_0^\infty(E), \varphi \ge 0, \varphi \ge 1 \text{ on a neighborhood of } K\}.\end{aligned}$$

Then we set for any open set $G \subset E$,

$$\mathrm{cap}_D(G) = \sup\{\mathrm{cap}_D(K); K \subset G, K \text{ compact}\}, \tag{1}$$

and, for any analytic set $A \subset E$,

$$\mathrm{cap}_D(A) = \inf\{\mathrm{cap}_D(G); A \subset G, G \text{ open}\}.$$

Notice the definition is consistent (see [2], for example).

PROPOSITION 4. *There exists a constant C such that for any set $A \subset E$,*

$$\mathrm{Cap}(A) \le \mathrm{cap}_D(A) \le C\ \mathrm{Cap}(A).$$

PROOF. Since the two capacities are outer capacities, it is enough to consider open sets. Now, using (1) and [13], Theorem 8, we see it is enough to consider compact sets.

Let us introduce the operator $H = \partial_t - \frac{1}{2}\Delta$. We consider a nonempty compact set $K \subset E$. Let $\varphi \in C_0^\infty(E)$ be such that $\varphi \ge 0$ and $\varphi \ge 1$ on K. Notice that (in the distribution sense) $Hp = \delta_{(0,0)}$, where $\delta_{(0,0)}$ is the Dirac mass at $(0,0) \in \mathbb{R}\times\mathbb{R}^d$. Then we have $p * [H(p\varphi)] = (Hp) * (p\varphi) = p\varphi$. The function $f = p^{-1}|H(p\varphi)| = |H(\varphi) - (\nabla \log p, \nabla\varphi)|$ is nonnegative and

$$\Lambda(f) = p^{-1}(p * |H(p\varphi)|) \ge p^{-1}(p * H(p\varphi)) = \varphi.$$

Thus we have $\Lambda(f) \ge 1$ on K. We also have

$$\|f\|_{(p)} \le \|\partial_t\varphi - \tfrac{1}{2}\Delta\varphi - (\nabla \log p, \nabla\varphi)\|_{(p)} \le \|\varphi\|_D.$$

Hence we have $\mathrm{Cap}(K) \le \|\varphi\|_D^2$. The first inequality follows by taking a sequence (φ_n) such that $\|\varphi_n\|_D^2$ converges to $\mathrm{cap}_D(K)$.

To prove the other inequality, let us consider a nonnegative function $f_1 \in L^2(p)$, such that $\Lambda(f_1) \ge 1$ on K. Notice this implies $\|f_1\|_{(p)} > 0$. Let $\delta > 0$. It is easy to construct a function $\varepsilon \in L^2(p)$ such that $\varepsilon > 0$ on E and $\|\varepsilon\|_{(p)} \le \delta\|f_1\|_{(p)}$. We set $f_2 = f_1 + \varepsilon$. Since the function $\Lambda(f_2)$ is lower semicontinuous, the set $\{(t,x) \in E; \Lambda(f_2) > 1\}$ is open and it also contains K. It is then obvious that for $\delta' > 0$ small enough, if we set $f_3(t, x) = f_2(t, x)\mathbf{1}_{\{\delta' < t < \delta'^{-1}, \left|x\right| < \delta'^{-1}\}}$ for $(t, x) \in E$, we get $\Lambda(f_3) > 1$ on an open set containing K. Let us introduce a nonnegative function $h \in C_0^\infty(E)$ such that $\int\int_E h(t, x)\,dt\,dx = 1$. For $\theta > 0$, we write $h_\theta(t, x) = \theta^{-d-1}h(t/\theta, x/\theta)$. Now using the uniform continuity of p on $[\delta'/2, \infty) \times \mathbb{R}^d$, it is easy to see that if $f = h_\theta * f_3$, then $\Lambda(f) > 1$ on an open set containing K for θ small enough. The function f is nonnegative, belongs to $C_0^\infty(E)$, and the function $\Lambda(f)$ is of class C^∞. We can choose δ and θ small enough so that $\|f\|_{(p)} \le 2\|f_1\|_{(p)}$.

Let $\alpha \in C_0^\infty([0, \infty))$ such that $0 \le \alpha \le 1$, $\alpha = 1$ on $[0, 1/2]$ and $\alpha = 0$ on $[1, \infty)$. Let $\xi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \xi \le 1$ and $\xi = 1$ in a neighborhood of 0. We define $\alpha_n(t) = \alpha(t/n)$ and $\xi_n(x) = \xi(x/n)$. The function $\varphi_n = \alpha_n\xi_n\Lambda(f)$ belongs to $C_0^\infty(E)$, is nonnegative, and $\varphi_n \ge 1$ on a neighborhood of K for n large enough.

Let us now give two key lemmas. If M is a bounded operator from $L^2(p)$ into itself, we denote by $\| M \|_{(p)} = \sup\{\|M(f)\|_{(p)}; f \in L^2(p), \|f\|_{(p)} = 1\}$ its norm. We define the operator Λ_0: for $f \in \mathscr{B}(E)$ nonnegative, $\Lambda_0(f)(t, x) = t^{-1}\Lambda(f)(t, x)$, $(t, x) \in E$. For $T > 0$, let us introduce $E_T = (0, T) \times \mathbb{R}^d$.

Lemma 5. *The operators $\mathbf{1}_{E_T}\Lambda$ and Λ_0 are bounded operators from $L^2(p)$ into itself. Furthermore, we have $\| \mathbf{1}_{E_T}\Lambda \|_{(p)} \le T/\sqrt{2}$ and $\| \Lambda_0 \|_{(p)} \le 2$.*

Proof. Let $f \in L^2(p)$. We have

$$
\begin{aligned}
\left\|\Lambda_0(f)\right\|_{(p)}^2 &= \iint_E dt\,dx\,t^{-2}p(t, x)^{-1}\left[\iint_E ds\,dy\; p(t-s, x-y)p(s, y)f(s, y)\right]^2 \\
&\le \iint_E dt\,dx\,t^{-2}p(t, x)^{-1}\left[\iint_E ds\,dy\; p(t-s, x-y)p(s, y)\right. \\
&\qquad\qquad \left.\times \left(s^{-1/2}\mathbf{1}_{s\le t}\right)^{1/2}\left(s^{1/2}f(s, y)^2\mathbf{1}_{s\le t}\right)^{1/2}\right]^2 \\
&\le \iint_E dt\,dx\,t^{-2}p(t, x)^{-1} \\
&\qquad \times \iint_E ds'\,dy'\; p(t-s', x-y')p(s', y')s'^{-1/2}\mathbf{1}_{s'\le t} \\
&\qquad\quad \times \iint_E ds\,dy\; p(t-s, x-y)p(s, y)s^{1/2}f(s, y)^2\mathbf{1}_{s\le t} \\
&\le \iint_E dt\,dx\,t^{-2}\int_0^t ds'\,s'^{-1/2} \\
&\qquad\qquad \times \iint_E ds\,dy\; p(t-s, x-y)p(s, y)s^{1/2}f(s, y)^2\mathbf{1}_{s\le t} \\
&= 2\iint_E dt\,dx\,t^{-2}t^{1/2}\iint_E ds\,dy\; p(t-s, x-y)p(s, y)s^{1/2}f(s, y)^2\mathbf{1}_{s\le t} \\
&\le 2\iint_E ds\,dy\; p(s, y)f(s, y)^2 s^{1/2}\int_s^\infty t^{-3/2}\,dt \le 4\left\|f\right\|_{(p)}^2,
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality for the second inequality. Hence the operator Λ_0 is a bounded operator from $L^2(p)$ into itself. And we have $\| \Lambda_0 \|_{(p)} \leq 2$. The operator $\mathbf{1}_{E_T}\Lambda$ can be handled in a very similar way. □

LEMMA 6. *The operators defined on $C_0^\infty(E)$ by $g \in C_0^\infty(E)$,*

$$\begin{aligned} & & \Lambda_1(g) &= \partial_t \Lambda(g), \\ & \textit{for } i \in \{1, \dots, d\}, & \Lambda_{2,i}(g) &= \tfrac{1}{2}\partial_{ii}^2 \Lambda(g), \\ & \textit{and for } i \in \{1, \dots, d\}, & \Lambda_{3,i}(g) &= \partial_i(\log p)\partial_i \Lambda(g), \end{aligned}$$

can be uniquely extended into bounded operators from $L^2(p)$ into itself. Moreover we have

$$\| \Lambda_1 \|_{(p)} \leq 1 + 3d, \tag{2}$$

$$\textit{for } i \in \{1, \dots, d\}, \quad \| \Lambda_{2,i} \|_{(p)} \leq 1, \tag{3}$$

$$\textit{for } i \in \{1, \dots, d\}, \quad \| \Lambda_{3,i} \|_{(p)} \leq 4. \tag{4}$$

The proof of this lemma is given in the Appendix.

We now bound $\|\varphi_n\|_D$. Lemma 5 provides an upper bound for $\|\partial_t \varphi_n\|_{(p)}$,

$$\begin{aligned} \|\partial_t \varphi_n\|_{(p)} &\leq \|\partial_t \alpha_n\|_\infty \|\mathbf{1}_{E_n}\Lambda(f)\|_{(p)} + \|\Lambda_1 f\|_{(p)} \\ &\leq \left(\|\partial_t \alpha\|_\infty 2^{-1/2} + \| \Lambda_1 \|_{(p)}\right)\|f\|_{(p)}. \end{aligned} \tag{5}$$

Using Lemma 5 we derive an upper bound for $\sum_{i=1}^d \|\partial_i(\log p)\partial_i \varphi_n\|_{(p)}$,

$$\begin{aligned} \sum_{i=1}^d \|\partial_i(\log p)\partial_i\varphi_n\|_{(p)} &\leq \sum_{i=1}^d \Big(\|\Lambda_{3,i}(f)\|_{(p)} + \sup_{x\in\mathbb{R}^d}|x_i\partial_i\xi(x)| \|\Lambda_0(f)\|_{(p)}\Big) \\ &\leq \sum_{i=1}^d \Big(\| \Lambda_{3,i} \|_{(p)} + \sup_{x\in\mathbb{R}^d}|x_i\partial_i\xi(x)| \, \| \Lambda_0 \|_{(p)}\Big)\|f\|_{(p)}. \end{aligned} \tag{6}$$

In order to give an upper bound for $\sum_{i=1}^d \|\partial_{ii}^2 \varphi_n\|_{(p)}$, we need an intermediary lemma.

LEMMA 7. *There exists a constant c_1 (depending on ξ) such that for all $n \geq 1$, $g \in C_0^\infty(E)$, $i \in \{1, \dots, d\}$,*

$$\|\mathbf{1}_{E_n}\partial_i \xi_n \, \partial_i \Lambda(g)\|_{(p)} \leq c_1 n^{-1/2} \|g\|_{(p)}.$$

PROOF. Recall that ξ_n has compact support. Then, an integration by parts, the Cauchy–Schwarz inequality and Lemma 5 give for $1 \leq i \leq d$,

$$\begin{aligned} \|\mathbf{1}_{E_n}\partial_i\xi_n\partial_i\Lambda(g)\|_{(p)}^2 = &- \iint_E p\mathbf{1}_{E_n}\Lambda(g)(\partial_i\xi_n)^2\partial_{ii}^2\Lambda(g) \\ &- \iint_E p\mathbf{1}_{E_n}\Lambda(g)(\partial_i\xi_n)^2\partial_i\Lambda(g)\partial_i \log p \\ &- 2\iint_E p\mathbf{1}_{E_n}\Lambda(g)\partial_i\xi_n\partial_i\Lambda(g)\partial_{ii}^2\xi_n \end{aligned}$$

$$
\begin{aligned}
&\le \|\partial_i \xi_n\|_\infty^2 \|\mathbf{1}_{E_n}\Lambda(g)\|_{(p)} \|\partial_{ii}^2\Lambda(g)\|_{(p)}\\
&\quad + \|\partial_i \xi_n\|_\infty^2 \|\mathbf{1}_{E_n}\Lambda(g)\|_{(p)} \|\partial_i\Lambda(g)\partial_i \log p\|_{(p)}\\
&\quad + 2\|\partial_{ii}^2 \xi_n\|_\infty \|\mathbf{1}_{E_n}\Lambda(g)\|_{(p)} \|\mathbf{1}_{E_n}\partial_i\xi_n\partial_i\Lambda(g)\|_{(p)}\\
&\le 2^{-1/2}n^{-1}\big[2\parallel \Lambda_{2,\,i}\parallel_{(p)} + \parallel \Lambda_{3,\,i}\parallel_{(p)}\big]\|\partial_i\xi\|_\infty^2 \|g\|_{(p)}^2\\
&\quad + 2^{1/2}n^{-1}\|\partial_{ii}^2\xi\|_\infty \|g\|_{(p)} \|\mathbf{1}_{E_n}\partial_i\xi_n\partial_i\Lambda(g)\|_{(p)}.
\end{aligned}
$$

Notice that if a, b, c are positive then $a^2 \le c^2 + ba$ implies $a \le c + b$. Thus we get

$$
\begin{aligned}
&\|\mathbf{1}_{E_n}\partial_i\xi_n\partial_i\Lambda(g)\|_{(p)}\\
&\qquad \le 2^{-1/4}n^{-1/2}\big[2\parallel \Lambda_{2,\,i}\parallel_{(p)} + \parallel \Lambda_{3,\,i}\parallel_{(p)}\big]^{1/2}\|\partial_i\xi\|_\infty \|g\|_{(p)}\\
&\qquad\quad + 2^{1/2}n^{-1}\|\partial_{i,\,i}^2\xi\|_\infty \|g\|_{(p)},
\end{aligned}
$$

which, thanks to Lemma 6, ends the proof. □

Using this lemma and Lemma 5, we get that

$$
\begin{aligned}
\sum_{i=1}^d \|\partial_{ii}^2\varphi_n\|_{(p)} &\le \sum_{i=1}^d \Big[2\|\Lambda_{2,\,i}(f)\|_{(p)} + \|\partial_{ii}^2\xi_n\|_\infty \|\mathbf{1}_{E_n}\Lambda(f)\|_{(p)}\\
&\qquad\qquad + 2\|\mathbf{1}_{E_n}\partial_i\xi_n\partial_i\Lambda(f))\|_{(p)}\Big]\\
&\le \sum_{i=1}^d \Big[2\parallel \Lambda_{2,\,i}\parallel_{(p)} + 2^{-1/2}n^{-1}\|\partial_{ii}^2\xi\|_\infty + 2c_1 n^{-1/2}\Big]\|f\|_{(p)}.
\end{aligned}
\tag{7}
$$

Then we deduce from (5), (6), (7) and Lemma 6 that there exists a constant c_2 independent of f and $n \ge 1$ such that

$$\|\varphi_n\|_D \le c_2\|f\|_{(p)}.$$

Thus we have $\|\varphi_n\|_D \le 2c_2\|f_1\|_{(p)}$. The second inequality of the proposition is then obvious with $C = 4\,c_2^2$. □

We shall need the following lemma.

LEMMA 8. *For any compact set* $K \subset E$ *with* $\mathrm{cap}_D(K) > 0$, *there exists* $\varphi \in C_0^\infty(E)$ *such that*:

(i) $0 \le \varphi \le 1$;
(ii) $\varphi = 1$ *on a neighborhood of* K;
(iii) $\|\varphi\|_D^2 \le \gamma\,\mathrm{cap}_D(K)$,

where γ *is a constant independent of* K *and* φ.

The proof is classic, but we give it for completeness.

PROOF. Let $h \in C^\infty([0,\infty))$ such that $0 \le h \le 1$, $h = 0$ on $[0, 1/4]$ and $h = 1$ on $[3/4, \infty)$. Since $\operatorname{cap}_D(K) > 0$, there exists $g \in C_0^\infty(E)$ such that $g \ge 0$, $g \ge 1$ in a neighborhood of K, and $2\operatorname{cap}_D(K) \ge \|g\|_D^2$. Let $\varphi = h \circ g$. The function $\varphi \in C_0^\infty(E)$ satisfies (i) and (ii). Let us check (iii). We have

$$\begin{aligned} \left\|\partial_t \varphi\right\|_{(p)} &\le \left\|h'\right\|_\infty \left\|\partial_t g\right\|_{(p)}, \\ \left\|\partial_i(\log p)\partial_i \varphi\right\|_{(p)} &\le \left\|h'\right\|_\infty \left\|\partial_i(\log p)\partial_i g\right\|_{(p)} \\ \left\|\partial_{ii}^2 \varphi\right\|_{(p)} &\le \left\|h'\right\|_\infty \left\|\partial_{ii}^2 g\right\|_{(p)} + \left\|(h'' \circ g)(\partial_i g)^2\right\|_{(p)}. \end{aligned}$$

Only the upper bound for the second right-hand side term of the last inequality is not obvious. We first search an upper bound for $\|(\partial_i \varphi_1)^2/(1+\varphi_1)\|_{(p)}$, where $\varphi_1 \in C_0^\infty(E)$ is a nonnegative function. An integration by parts and the Cauchy–Schwarz inequality give

$$\begin{aligned} \iint_E p\frac{(\partial_i\varphi_1)^4}{(1+\varphi_1)^2} &= 3\iint_E p\partial_{ii}^2\varphi_1 \frac{(\partial_i\varphi_1)^2}{1+\varphi_1} + \iint_E p\partial_i(\log p)\partial_i\varphi_1\frac{(\partial_i\varphi_1)^2}{1+\varphi_1} \\ &\le \Big(3\left\|\partial_{ii}^2\varphi_1\right\|_{(p)} + \left\|\partial_i(\log p)\,\partial_i\varphi_1\right\|_{(p)}\Big)\left\|(\partial_i\varphi_1)^2/(1+\varphi_1)\right\|_{(p)}. \end{aligned}$$

Thus we get

$$\left\|(\partial_i\varphi_1)^2/(1+\varphi_1)\right\|_{(p)} \le 3\Big(\left\|\partial_{ii}^2\varphi_1\right\|_{(p)} + \left\|\partial_i(\log p)\partial_i\varphi_1\right\|_{(p)}\Big). \tag{8}$$

Since we have $\left|h''(t)\right| \le 2(1+t)^{-1}\|h''\|_\infty$, taking $\varphi_1 = g$ in the above inequality we deduce that

$$\begin{aligned} \left\|(h''\circ g)(\partial_i g)^2\right\|_{(p)} &\le 2\left\|(\partial_i g)^2/(1+g)\right\|_{(p)}\left\|h''\right\|_\infty \\ &\le 6\left(\left\|\partial_{ii}^2 g\right\|_{(p)} + \left\|\partial_i(\log p)\,\partial_i g\right\|_{(p)}\right)\left\|h''\right\|_\infty. \end{aligned}$$

The previous inequalities imply there exists a constant c depending only on h and d such that $\|\varphi\|_D \le c\|g\|_D$. Thus (iii) holds with $\gamma = 2c^2$. □

3. Upper bound for hitting probabilities. In this section we prove the second inequality of Theorem 1 for compact sets. Let us introduce $K \subset E_T$, a compact set such that $\operatorname{cap}_D(K) > 0$. Let φ be as in Lemma 8. We set $\varphi = 0$ outside E. We introduce the function $\psi = 1 - \varphi$, which takes values in $[0, 1]$. We consider the function u defined on $\mathbb{R} \times \mathbb{R}^d$ by $u(t,x) = \mathbb{N}_{t,x}[\mathscr{G}^* \cap K \neq \varnothing]$ $(\in [0,\infty])$. With the convention $0.\infty = 0$, the function $u\psi^4$ is bounded nonnegative and of class C^∞ on $\mathbb{R}\times\mathbb{R}^d$. Let $(B_t, t \ge 0)$ denote under P_0 a d-dimensional Brownian motion started from 0. Itô's formula implies that for all $t \ge 0$, P_0-a.s.,

$$\begin{aligned} u\psi^4(t, B_t) = u\psi^4(0,0) &+ \int_0^t \partial_t(u\psi^4)(s, B_s)\,ds \\ &+ \int_0^t \frac{\Delta}{2}(u\psi^4)(s,B_s)\,ds + \int_0^t \nabla(u\psi^4)(s,B_s)\,dB_s. \end{aligned}$$

Consider the stopping time $T_a = T \wedge \inf\{t > 0; |B_t| \geq a\}$. We can then apply the optional stopping theorem at time T_a and get

$$\begin{aligned}
&\mathrm{E}_0\, u\psi^4(T_a, B_{T_a}) \\
&\quad = u(0,0) + \mathrm{E}_0 \int_0^{T_a} \partial_t(u\psi^4)(s, B_s)\, ds + \mathrm{E}_0 \int_0^{T_a} \frac{\Delta}{2}(u\psi^4)(s, B_s)\, ds \\
&\quad = u(0,0) + \mathrm{E}_0 \int_0^{T_a} \Big[2u^2\psi^4 + 4u\psi^3\partial_t\psi + 4(\nabla u, \nabla\psi)\psi^3 \\
&\qquad\qquad\qquad + 6u\psi^2(\nabla\psi, \nabla\psi) + 2u\psi^3\Delta\psi \Big](s, B_s)\, ds.
\end{aligned}$$

We have used that $\partial_t u + \frac{1}{2}\Delta u = 2u^2$ to get the last equality. Notice that each integrand is either nonnegative or bounded. By dominated convergence and monotone convergence, we get, as a goes to infinity,

$$\begin{aligned}
&u(0,0) + 2\|u\psi^2 \mathbf{1}_{E_T}\|_{(p)}^2 \\
&\quad = \mathrm{E}_0\, u\psi^4(T, B_T) \\
&\qquad - \iint_{E_T} p\Big[4u\psi^3\partial_t\psi + 4(\nabla u, \nabla\psi)\psi^3 + 6u\psi^2(\nabla\psi, \nabla\psi) + 2u\psi^3\Delta\psi\Big].
\end{aligned}$$

Since $K \subset E_T$, we deduce that $u(t, x) = 0$ for $t \geq T$. Thus we have

$$\begin{aligned}
&u(0,0) + 2\|u\psi^2\|_{(p)}^2 \\
&\quad = -\iint_E p\Big[4u\psi^3\partial_t\psi + 4(\nabla u, \nabla\psi)\psi^3 + 6u\psi^2(\nabla\psi, \nabla\psi) + 2u\psi^3\Delta\psi\Big].
\end{aligned} \tag{9}$$

We now bound the right-hand side. Using the Cauchy–Schwarz inequality, that $0 \leq \psi \leq 1$ and that $-\varphi$ and ψ have the same derivatives, we get

$$-\iint_E pu\psi^3\partial_t\psi \leq \|u\psi^2\|_{(p)} \|\partial_t\varphi\|_{(p)},$$
$$-\iint_E pu\psi^3\partial_{ii}^2\psi \leq \|u\psi^2\|_{(p)} \|\partial_{ii}^2\varphi\|_{(p)}$$

and

$$\begin{aligned}
-\iint_E pu\psi^2(\nabla\psi, \nabla\psi) &\leq \|u\psi^2\|_{(p)} \sum_{i=1}^d \|(\partial_i\varphi)^2\|_{(p)} \\
&\leq 2\|u\psi^2\|_{(p)} \sum_{i=1}^d \|(\partial_i\varphi)^2/(1+\varphi)\|_{(p)} \\
&\leq 6\|u\psi^2\|_{(p)} \sum_{i=1}^d \left(\|\partial_{ii}^2\varphi\|_{(p)} + \|\partial_i(\log p)\partial_i\varphi\|_{(p)} \right),
\end{aligned}$$

where we have used (8) with $\varphi_1 = \varphi$ for the last inequality. Now an integration by parts and the Cauchy–Schwarz inequality give

$$\begin{aligned}
&-\iint_E p\psi^3(\nabla u, \nabla\psi)\\
&\quad= \iint_E pu\psi^2\big[\psi(\nabla \log p, \nabla\psi) + 3(\nabla\psi, \nabla\psi) + \psi\Delta\psi\big]\\
&\quad\le \big\|u\psi^2\big\|_{(p)} \sum_{i=1}^{d}\Big[\big\|\partial_i(\log p)\partial_i\varphi\big\|_{(p)} + 3\big\|(\partial_i\varphi)^2\big\|_{(p)} + \big\|\partial^2_{ii}\varphi\big\|_{(p)}\Big]\\
&\quad\le 19\big\|u\psi^2\big\|_{(p)} \sum_{i=1}^{d}\Big[\big\|\partial_i(\log p)\partial_i\varphi\big\|_{(p)} + \big\|\partial^2_{ii}\varphi\big\|_{(p)}\Big],
\end{aligned}$$

where we have used again (8) for the last inequality. Taking those results together, we deduce from (9) that

$$u(0,0) + 2\big\|u\psi^2\big\|^2_{(p)} \le c_3\big\|u\psi^2\big\|_{(p)}\big\|\varphi\big\|_D,$$

where the constant c_3 depends only on d. Since $\|u\psi^2\|_{(p)}$ is finite (recall $u\psi$ is bounded, and zero on $[T,\infty)\times\mathbb{R}^d$), this implies that $\|u\psi^2\|_{(p)} \le c_3\|\varphi\|_D$ and hence $u(0,0) \le c_3^2\|\varphi\|_D^2$. This last inequality and the definition of φ imply that

$$\mathbb{N}_{0,0}\big[\mathscr{G}^* \cap K \ne \varnothing\big] = u(0,0) \le c_3^2\gamma\,\mathrm{cap}_D(K) \le c_3^2\gamma C\,\mathrm{Cap}(K) = c_3^2\gamma C\,\mathrm{cap}(K).$$

4. Lower bound for hitting probabilities and proof of Theorem 1. In this section, we prove the first inequality of Theorem 1 for compact sets. Let us introduce a compact set $K \subset E$, ν a probability measure on K and $T > 0$ such that $K \subset E_T$. We consider the probability measure μ defined on $\mathscr{W}_{0,0}$ by

$$\mu(dw) = \iint_E \nu(dt, dx)\mathrm{P}_0^{t,x}(dw),$$

where $\mathrm{P}_0^{t,x}$ is the law on $\mathscr{W}_{0,0}$ of the Brownian bridge starting at time 0 at point 0 and ending at time t at point x. Notice that the measure μ is in fact a measure on $\mathscr{W}^*_{0,0}$, the set of nontrivial path in $\mathscr{W}_{0,0}$ (a trivial path is a path of lifetime zero). The measure $\mathrm{P}_0^{t,x}$ can also be viewed as a probability measure on the canonical space $C(\mathbb{R}^+, \mathbb{R}^d)$ endowed with the filtration $(\mathscr{C}_t)$ generated by the coordinate mappings. Let P_0 be the law on the canonical space of the standard Brownian motion. For $s \in [0, t)$, we have

$$\mathrm{P}_0^{t,x}(dw)_{|\mathscr{C}_s} = \frac{p(t-s, x-w(s))}{p(t,x)}\mathrm{P}_0(dw)_{|\mathscr{C}_s}.$$

We consider the energy of μ with respect to the process (W_s) (see [11] for a precise description and definition). Thanks to [11], Proposition 1.1, we have

$$\mathscr{E}(\mu) = 2\int_0^\infty ds\ \mathrm{P}_0\bigg[\bigg(\iint_E \nu(dt,dx)p(t-s, x-w(s))/p(t,x)\bigg)^2\bigg] = 2I(\nu).$$

Now, using [5], Proposition 5, we know there exists an additive functional A of the Brownian snake killed when its lifetime reaches 0 such that:

1. For every Borel function $F \geq 0$ on $\mathscr{W}^*_{0,0}$,

$$\mathbb{N}_{0,0}\left[\int_0^\infty F(W_s)\,dA_s\right] = \int \mu(dw)F(w).$$

2. $\mathbb{N}_{0,0}[A^2_\infty] = 2\mathscr{E}(\mu)$.

We deduce from (1) that the additive functional increases only when $\hat{W}_s \in \operatorname{supp}\nu \subset K$. Therefore, using the Cauchy–Schwarz inequality, we get

$$\mathbb{N}_{0,0}[\mathscr{G}^* \cap K \neq \varnothing] \geq \mathbb{N}_{0,0}[A_\infty > 0] \geq \mathbb{N}_0[A_\infty]^2/\mathbb{N}_0[A^2_\infty].$$

We get $\mathbb{N}_{0,0}[\mathscr{G}^* \cap K \neq \varnothing] \geq [4I(\nu)]^{-1}$. Since the above inequality is true for any probability ν on K, we get that

$$\mathbb{N}_{0,0}\big[\mathscr{G}^* \cap K \neq \varnothing\big] \geq 4^{-1}\ \mathrm{Cap}(K) = 4^{-1}\ \mathrm{cap}(K).$$

Proof of Theorem 1. Notice the application defined on $\mathscr{B}(E)$ by $T(A) = \mathbb{N}_{0,0}[\mathscr{G}^* \cap A \neq \varnothing]$ for $A \in \mathscr{B}(E)$ is a Choquet capacity (see [4], Théorème 1). Since the capacity cap is an inner capacity (see [13], Theorem 12), it is enough to prove the theorem for compact subsets of E. The result is then given by the previous section (with $C_0 = c_3^2\gamma C$) and the above result. □

5. Brownian range and support of X_1. In this section, we first give an estimate for the hitting probabilities of the support of X_1. Then we prove that the range of Brownian motion and the support of super-Brownian motion at fixed time are intersection-equivalent.

Let us fix $d \geq 2$. We denote by cap_{d-2} the usual Newtonian (logarithmic if $d = 2$) capacity in $\mathbb{R}^d$,

$$\mathrm{cap}_{d-2}(A) = \left[\inf \iint_{\mathbb{R}^d\times\mathbb{R}^d} \rho(dx)\rho(dy)h_{d-2}(|x-y|)\right]^{-1},$$

with $h_\gamma(r) = r^{-\gamma}$ if $\gamma > 0$ and $h_0(r) = \log_+(1/r)$. The infimum is taken over all probability measures ρ on $\mathbb{R}^d$ such that $\rho(A) = 1$. Let $B(0,h)$ be the open ball of $\mathbb{R}^d$ centered at 0 with radius h.

Proposition 9. *Let $M > 0$. There exist two positive constants a and b such that for any Borel set $A \subset B(0,1)$, for any finite measure μ on $B(0,1)$, with $(\mu, \mathbf{1}) \leq M$, we have*

$$a(\mu,\mathbf{1})\,\mathrm{cap}_{d-2}(A) \leq \mathbf{P}_{0,\mu}\big[\operatorname{supp} X_1 \cap A \neq \varnothing\big] \leq b(\mu,\mathbf{1})\,\mathrm{cap}_{d-2}(A).$$

Proof. Let $A \subset B(0,2)$ be a Borel set. Let ν be a probability measure on E such that $\nu(\{1\}\times A) = 1$. Then we have $\nu = \delta_{\{1\}}\times\rho$, where ρ is a probability measure on $\mathbb{R}^d$ such that $\rho(A) = 1$. We get

$$\begin{aligned} I(\nu) = &\iint_{(0,1)\times\mathbb{R}^d} ds\,dy\,p(s,y)\\ &\times \iint_{A\times A} \rho(dx)\rho(dx')p(1-s,x-y)p(1-s,x'-y)p(1,x)^{-1}p(1,x')^{-1}. \end{aligned}$$

Since x, x' are in $B(0,2)$ and since $s \in (0,1)$, it is easy to see there exist two positive constants a_1 and b_1 (independent of A and ρ) such that

$$a_1 I(\nu) \le \iint_{A\times A} \rho(dx)\rho(dx')h_{d-2}(|x-x'|) \le b_1 I(\nu).$$

This implies that for any Borel set $A \subset B(0,2)$,

$$a_1 \operatorname{cap}_{d-2}(A) \le \operatorname{cap}(\{1\}\times A) \le b_1 \operatorname{cap}_{d-2}(A).$$

Since the capacity cap_{d-2} is invariant by translation, we get that for any Borel set $A \subset B(0,1)$, for any $x \in B(0,1)$,

$$a_1 \operatorname{cap}_{d-2}(A) \le \operatorname{cap}(\{1\}\times A_x) \le b_1 \operatorname{cap}_{d-2}(A),$$

where $A_x = \{y; y - x \in A\}$. We deduce from Theorem 1 that

$$4^{-1} a_1 \operatorname{cap}_{d-2}(A) \le \mathbb{N}_{0,x}\big[\mathscr{G}^* \cap (\{1\}\times A) \neq \varnothing\big] \le C_0 b_1 \operatorname{cap}_{d-2}(A).$$

Since $X_1 = \sum_{i\in I} Y_1(W^i)$, where $\sum_{i\in I}\delta_{W^i}$ is a Poisson measure on $C(\mathbb{R}^+, \mathscr{W}_0)$ with intensity $\int \mu(dx)\mathbb{N}_{0,x}[\cdot]$, we have

$$\mathbf{P}_{0,\mu}\big[\operatorname{supp} X_1 \cap A \neq \varnothing\big] = 1 - \exp\bigg(-\int \mu(dx)\mathbb{N}_{0,x}\big[\operatorname{supp} Y_1 \cap A \neq \varnothing\big]\bigg).$$

Notice that $\mathbb{N}_{0,x}$-a.e., $\{1\}\times(\operatorname{supp} Y_1 \cap A) = \mathscr{G}^* \cap (\{1\}\times A)$. Since $(\mu, \mathbf{1}) < M$, we then easily get the result. $\square$

Intersection-equivalence between random sets has been defined by Peres [15]. Two random Borel sets F_1 and F_2 in $\mathbb{R}^d$ are intersection-equivalent in an open set U, if there exist positive constants a and b such that, for any Borel set $A \subset U$,

$$a\,\mathrm{P}[A\cap F_1] \le \mathrm{P}[A \cap F_2] \le b\,\mathrm{P}[A\cap F_1].$$

If π is a probability measure on $B(0,1)$, then we denote by P_π the law of a d-dimensional Brownian motion $(B_t, t\ge 0)$ started with the law π. For $d \ge 3$ the range of Brownian motion is defined by $\mathscr{R}_B = \{B_t, t \ge 0\}$ in $\mathbb{R}^d$. For $d = 2$, we also denote by $\mathscr{R}_B$ the set $\mathscr{R}_B = \{B_t, t\in[0,\xi]\}$, where ξ is an exponential random variable of parameter 1 independent of $(B_t, t \ge 0)$.

COROLLARY 10. *Let $M > 0$. There exist two positive constants a and b such that for any Borel set $A \subset B(0,1)$, for any absolutely continuous probability measure π on $B(0,1)$ with density bounded by M, for any finite measure μ on $B(0,1)$, with $(\mu,\mathbf{1}) \le M$, we have*

$$a(\mu,\mathbf{1})\,\mathrm{P}_\pi[\mathscr{R}_B \cap A \neq \varnothing] \le \mathbf{P}_{0,\mu}\big[\operatorname{supp} X_1 \cap A \neq \varnothing\big] \le b(\mu,\mathbf{1})\,\mathrm{P}_\pi[\mathscr{R}_B\cap A\neq\varnothing].$$

PROOF. This is a consequence of Proposition 9 and the fact that there exist two positive constants a_2 and b_2 such that for any Borel set $A \subset B(0,1)$, for any absolutely continuous probability measure π on $B(0,1)$ with density bounded by M,

$$a_2 \operatorname{cap}_{d-2}(A) \le \mathrm{P}_\pi\big[\mathscr{R}_B \cap A \neq \varnothing\big] \le b_2 \operatorname{cap}_{d-2}(A)$$

(see, e.g., [15], Proposition 3.2, for $d \ge 3$ and [14] for $d = 2$). $\square$

APPENDIX

In this section, we give the proof of Lemma 6, which relies on the properties of the Hermite polynomials. We first recall the definition and some properties of those polynomials.

A.1. Hermite polynomials. For $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, we set $|n| = \sum_{i=1}^d n_i$, $n! = \prod_{i=1}^d n_i!$ and $\sum_{n \ge 0} = \sum_{i=1}^d \sum_{n_i=0}^\infty$. For $j \in \{1, \dots, d\}$, let $\delta(j)$ be the element of $\mathbb{N}^d$ such that $\delta(j)_i = \delta_{i,j}$, the standard Kronecker symbol. If $z = (z_1, \dots, z_d)$ is an element of $\mathbb{R}^d$, then we set $z^n = \prod_{i=1}^d z_i^{n_i}$. Let $(\cdot,\cdot)$ be the Euclidean product on $\mathbb{R}^d$.

The function $\varphi(z) = \exp(-[|z|^2 - 2(z,x)]/2)$ is an entire function defined on $\mathbb{R}^d$. We have

$$\exp\Big(-[|z|^2 - 2(x,z)]/2\Big) = \sum_{n \ge 0} \frac{1}{n!} z^n He_n(x), \tag{10}$$

where the nth term $He_n(x)$ is a polynomial of $(x_1, \dots, x_d)$ of degree $|n|$ called the nth Hermite polynomial. Those polynomials can easily be expressed with the usual one-dimensional Hermite polynomials $(He_k^{(1)}, k \in \mathbb{N})$: $He_n(x) = \prod_{i=1}^d He_{n_i}^{(1)}(x_i)$, where $x = (x_1, \dots, x_d)$.

Now let us recall some basic properties of the polynomials He_n. The following recurrence formula can be deduced from (10) by differentiating w.r.t. z_i: for all $n \in \mathbb{N}^d$ such that $n_i > 0$,

$$He_n(x) = x_i He_{n-\delta(i)}(x) - (n_i - 1) He_{n-2\delta(i)}(x) \qquad \forall x \in \mathbb{R}^d, \tag{11}$$

where by convention $He_{n-k\delta(i)} = 0$ if $n_i - k < 0$. The differential formula can be deduced from (10) by differentiating w.r.t. x_i: for all $n \in \mathbb{N}^d$,

$$\partial_i He_n = n_i He_{n-\delta(i)}. \tag{12}$$

We also recall the upper bound for He_n (see [1], 22.14.17); there exists a universal constant $1 < c_0 < 2$ such that

$$|He_n(x)| \le c_0^d \sqrt{n!} \exp\Big(|x|^2/4\Big) \quad \text{for all } x \in \mathbb{R}^d, n \in \mathbb{N}^d. \tag{13}$$

Using the definition of the Hermite polynomials, it is also easy to prove that

$$\int dx\, p(t,x) He_n(x/\sqrt{t}) He_m(x/\sqrt{t}) = n! \prod_{i=1}^d \delta_{n_i, m_i}. \tag{14}$$

It is also well known that the Hermite polynomials are a complete orthogonal system in $L^2(\mathbb{R}^d, \exp\left(-|x|^2/2\right) dx)$. Finally, standard arguments on Hilbert spaces show that if $f \in L^2(p)$ then

$$f(t,x) = \sum_{n\geq 0} f_n(t,x) = \sum_{n\geq 0} He_n(x/\sqrt{t}) g_n(t),$$

where $g_n(t) = (n!)^{-1} \int dx\ p(t,x) He_n(x/\sqrt{t}) f(t,x)$ and $g_n \in L^2((0,\infty))$. Furthermore, we have

$$\|f\|^2_{(p)} = \sum_{n\geq 0} n! \int_0^\infty dt\, g_n(t)^2. \tag{15}$$

Since $C_0^\infty((0,\infty))$ is dense in $L^2((0,\infty))$, it is clear that the set $\mathscr{A}$ of functions $f(t,x) = \sum_{n\geq 0} He_n(x/\sqrt{t}) g_n(t)$ where $g_n \in C_0^\infty((0,\infty))$ is nonzero for a finite number of indices n, is dense in $L^2(p)$.

A.2. Proof of Lemma 6. First step. We prove there exist unique bounded extensions $\tilde{\Lambda}_1$, $\tilde{\Lambda}_{2,i}$ and $\tilde{\Lambda}_{3,i}$ in $L^2(p)$ of the operators Λ_1, $\Lambda_{2,i}$ and $\Lambda_{3,i}$ defined on $\mathscr{A}$. Then in a second step we check that the extensions $\tilde{\Lambda}_1$, $\tilde{\Lambda}_{2,i}$ and $\tilde{\Lambda}_{3,i}$ and the operators Λ_1, $\Lambda_{2,i}$ and $\Lambda_{3,i}$, which are also defined on $C_0^\infty(E)$, agree on $C_0^\infty(E)$.

First step. Let us compute $\Lambda(f)$ for very particular functions $f \in \mathscr{A}$. Let $g \in C_0^\infty((0,\infty))$, α and β be two positive reals such that $\operatorname{supp} g \in [\alpha, \beta]$, and $G(t) = \int_0^t ds\, g(s)$. For $n \in \mathbb{N}^d$, and $(t,x) \in E$, we set

$$h_{n,g}(t,x) = He_n(x/\sqrt{t}) t^{-|n|/2} g(t).$$

Let us prove that

$$\Lambda(h_{n,g}) = h_{n,G}. \tag{16}$$

For $z \in \mathbb{R}$, we introduce the function $\mathscr{H}_{g,z}$ defined on E by

$$\mathscr{H}_{g,z}(t,x) = \sum_{n\geq 0} \frac{1}{n!} z^n h_{n,g}(t,x) = g(t) \exp\left(-[|z|^2 - 2(x,z)]/2t\right).$$

Then we have

$$\begin{aligned}
\Lambda(\mathscr{H}_{g,z})(t,x) &= \frac{1}{p(t,x)} \int_0^t ds \int dy\ p(t-s, x-y) p(s,y) \mathscr{H}_{g,z}(s,y) \\
&= \frac{1}{p(t,x)} \int_0^t ds \int dy\ p\left(\frac{s(t-s)}{t}, y - \frac{sx}{t} - \frac{(t-s)z}{t}\right) \\
&\qquad \times p(t, z-x) g(s) \\
&= \exp\left(-[|z|^2 - 2(x,z)]/2t\right) \int_0^t ds\, g(s) = \mathscr{H}_{G,z}(t,x).
\end{aligned}$$

Using (13), the Chapman–Kolmogorov equation, and that $\operatorname{supp} g \in [\alpha, \beta]$, we get

$$\begin{aligned}\Lambda(|h_{n,g}|)(t,x) &\le \sqrt{n!}c_0^d\, p(t,x)^{-1} \iint_E ds\, dy\, p(t-s, x-y)p(s,y) \\ &\qquad\qquad \times \exp\left(|y|^2/4s\right) s^{-|n|/2}|g(s)| \\ &\le \sqrt{n!}(\sqrt{2}c_0)^d\, p(t,x)^{-1}\int_0^t ds\, p(t+s,x)s^{-|n|/2}|g(s)| \\ &\le \sqrt{n!}(\sqrt{2}c_0)^d \exp\left(|x|^2/4t\right)\int_0^t ds\, s^{-|n|/2}|g(s)| \\ &\le \sqrt{n!}\,(\sqrt{2}c_0)^d \exp\left(|x|^2/4t\right)\|g\|_\infty(\beta-\alpha)\alpha^{-|n|/2}.\end{aligned}$$

The radius of the series $\sum a^k(k!\alpha^k)^{-1/2}$ is infinite. Thus for any $(t,x)\in E$ the series $\sum(n!)^{-1}z^n\Lambda(|h_{n,g}|)(t,x)$ are convergent. Fubini's theorem implies that

$$\sum_{n\ge 0}\frac{1}{n!}z^n\Lambda(h_{n,g})(t,x) = \Lambda\left(\sum_{n\ge 0}\frac{1}{n!}z^n h_{n,g}\right)(t,x) = \mathscr{H}_{G,z}(t,x).$$

Hence the two series $\sum(n!)^{-1}z^n\Lambda(h_{n,g})(t,x)$ and $\sum(n!)^{-1}z^n h_{n,G}(t,x)$ agree. Since their radius of convergence is positive (in fact infinite), we get that (16) is true.

Let us prove that $\Lambda_{2,i}$ has a bounded extension on $L^2(p)$. We deduce from (12) that

$$\begin{aligned}\Lambda_{2,i}(h_{n,g})(t,x) &= \frac{1}{2}\partial_{i\,i}^2\Lambda(h_{n,g})(t,x) \\ &= \frac{1}{2t}n_i(n_i-1)He_{n-2\delta(i)}(x/\sqrt{t})t^{-|n|/2}\int_0^t ds\, g(s).\end{aligned} \tag{17}$$

Let us introduce $f\in\mathscr{A}$, that is, for $(t,x)\in E$, $f(t,x)=\sum_{n\ge 0}He_n(x/\sqrt{t})g_n(t)$, where $g_n\in C_0^\infty((0,\infty))$ and $g_n=0$ except for a finite number of terms. By linearity, we have

$$\Lambda_{2,i}(f)(t,x) = \sum_{n\ge 0}2^{-1}n_i(n_i-1)He_{n-2\delta(i)}(x/\sqrt{t})t^{-1-|n|/2}\int_0^t ds\, s^{|n|/2}g_n(s).$$

Thus, using (14), we have

$$\begin{aligned}\|\Lambda_{2,i}(f)\|_{(p)}^2 &= \sum_{n\ge 0}(n-2\delta(i))!4^{-1}n_i^2(n_i-1)^2 \\ &\qquad\times\int_0^\infty dt\, t^{-2-|n|}\left[\int_0^t ds\, s^{|n|/2}g_n(s)\right]^2 \\ &\le \sum_{n\ge 0}n!\frac{n_i(n_i-1)}{4}\frac{4}{(|n|+1)^2}\int_0^\infty dt\, g_n(t)^2 \\ &\le \|f\|_{(p)}^2,\end{aligned}$$

where we used the Hardy inequality; for $k > -1$,

$$\int_0^\infty dt\, t^{-2-k}\left[\int_0^t s^{k/2}h(s)\,ds\right]^2 \le \frac{4}{(k+1)^2}\int_0^\infty dt\, h(t)^2$$

for the first inequality and (15) for the second one. This means that $\Lambda_{2,i}$, defined on $\mathscr{A}$, can be uniquely extended into a bounded operator $\tilde{\Lambda}_{2,i}$ from $L^2(p)$ into itself. The above inequality implies $\| \tilde{\Lambda}_{2,i} \|_{(p)} \le 1$.

For $i \in \{1, \dots, d\}$, we set $\Lambda_{4,i} = \Lambda_{3,i} + 2\Lambda_{2,i}$. Using (12) and (11), we deduce from (16) that

$$\begin{aligned}
&\Lambda_{4,i}(h_{n,g})(t,x)\\
&\quad = \left[-(x_i/\sqrt{t})\partial_i He_n(x/\sqrt{t}) + n_i(n_i-1)He_{n-2\delta(i)}(x/\sqrt{t})\right]\\
&\qquad \times t^{-1-|n|/2}\int_0^t ds\, g(s)\\
&\quad = -n_i He_n(x/\sqrt{t})t^{-1-|n|/2}\int_0^t ds\, g(s).
\end{aligned} \tag{18}$$

Arguing as above, we get, for $f \in \mathscr{A}$,

$$\begin{aligned}
\|\Lambda_{4,i}(f)\|_{(p)}^2 &= \sum_{n\ge 0} n! n_i^2 \int_0^\infty dt\, t^{-2-|n|}\left[\int_0^t ds\, s^{|n|/2} g_n(s)\right]^2\\
&\le \sum_{n\ge 0} n! n_i^2 \frac{4}{(|n|+1)^2}\int_0^\infty dt\, g_n(t)^2\\
&\le 4\|f\|_{(p)}^2.
\end{aligned}$$

Thus the operators $\Lambda_{4,i}$ and $\Lambda_{3,i}$, defined on $\mathscr{A}$, can be uniquely extended in bounded operators $\tilde{\Lambda}_{4,i}$ and $\tilde{\Lambda}_{3,i}$ from $L^2(p)$ into itself. Furthermore we have $\| \tilde{\Lambda}_{4,i} \|_{(p)} \le 2$ and $\| \tilde{\Lambda}_{3,i} \|_{(p)} \le \| \tilde{\Lambda}_{4,i} \|_{(p)} + 2 \| \tilde{\Lambda}_{2,i} \|_{(p)} \le 4$.

The proof concerning Λ_1 easily follows from the previous results. From (16), we get

$$\begin{aligned}
&\Lambda_1(h_{n,g})(t,x)\\
&\quad = h_{n,g}(t,x) - \frac{1}{2}\left[|n| He_n(x/\sqrt{t}) + \sum_{i=1}^d \frac{x_i}{\sqrt{t}}\partial_i He_n(x/\sqrt{t})\right] t^{-1-|n|/2}\int_0^t ds\, g(s).
\end{aligned}$$

Then using (17) and (18), we get

$$\Lambda_1(h_{n,g}) = \left[I + \tfrac{1}{2}\sum_{i=1}^d \left[\Lambda_{4,i} + \Lambda_{3,i}\right]\right](h_{n,g}) = \left[I + \sum_{i=1}^d\left[\Lambda_{4,i} - \Lambda_{2,i}\right]\right](h_{n,g}).$$

This means that $\Lambda_1 = I + \sum_{i=1}^d[\Lambda_{4,i} - \Lambda_{2,i}]$ on $\mathscr{A}$. Hence Λ_1 can be uniquely extended in a bounded operator $\tilde{\Lambda}_1$ from $L^2(p)$ into itself and $\tilde{\Lambda}_1 = I + \sum_{i=1}^d[\tilde{\Lambda}_{4,i} - \tilde{\Lambda}_{2,i}]$. We deduce that $\| \tilde{\Lambda}_1 \|_{(p)} \le 1 + 3d$.

Second step. We first consider the operators $\Lambda_{3,i}$ for $i \in \{1, \dots, d\}$. To check that $\Lambda_{3,i}$ and $\tilde{\Lambda}_{3,i}$ agree on $C_0^\infty(E)$, it is enough to check that for $\varphi \in C_0^\infty(E)$, $\Lambda_{3,i}(\varphi)(t,x) = \tilde{\Lambda}_{3,i}(\varphi)(t,x)\ dt\,dx$-a.e. Let $\varphi \in C_0^\infty(E)$. For $k \in \mathbb{N}$, we define,

$$\varphi_k(t,x) = \sum_{|n| \le k} He_n(x/\sqrt{t})(n!)^{-1} \int_{\mathbb{R}^d} dy\, p(t,y) He_n(y/\sqrt{t})\varphi(t,y).$$

The sequence $(\varphi_k, k \ge 0)$ converges in $L^2(p)$ to φ.

If $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $i \in \{1, \dots, d\}$, we denote by $z = \hat{x}_y^i$ the element of $\mathbb{R}^d$ such that $z_i = y$ and $z_j = x_j$ for $j \neq i$. Since $\Lambda_{3,i}(f)(t,x) = -t^{-1}x_i\partial_i\Lambda(f)(t,x)$ for $f \in \mathscr{A} \cup C_0^\infty(E)$, we see that an integration by parts gives

$$\int_0^{x_i} dy\, \Lambda_{3,i}(f)(t, \hat{x}_y^i) = -t^{-1}x_i\Lambda(f)(t,x) + \int_0^{x_i} dy\, t^{-1}\Lambda(f)(t,\hat{x}_y^i). \tag{19}$$

For short we write $P_i(f)$ for the operator $P_i(f)(t,x) = \int_0^{x_i} dy\, f(t,\hat{x}_y^i)$. Let $R > 0$ and $T > \varepsilon > 0$ be fixed. Let $Q = [\varepsilon, T] \times [-R, R]^d$. The heat kernel p is bounded below and above on Q by positive constant, say c_Q and C_Q. Using the Cauchy–Schwarz inequality we have

$$\|\mathbf{1}_Q P_i(f)\|_{(p)}^2 \le C_Q R^2 \iint_Q dt\,dx\, f(t,x)^2 \le C_Q c_Q^{-1} R^2 \|f\|_{(p)}^2.$$

Thus the operator $\mathbf{1}_Q P_i$ is continuous from $L^2(p)$ to $L^2(p)$. Thanks to Lemma 5 and the above first step, we get that the sequences $(\mathbf{1}_Q P_i(\Lambda_0(\varphi_k)), k \ge 0)$ and $(\mathbf{1}_Q P_i(\Lambda_{3,i}(\varphi_k)), k \ge 0)$ converge in $L^2(p)$, respectively, to $\mathbf{1}_Q P_i(\Lambda_0(\varphi))$ and $\mathbf{1}_Q P_i(\tilde{\Lambda}_{3,i}(\varphi))$. Notice also that $(\mathbf{1}_Q \Lambda(\varphi_k), k \ge 0)$ converges in $L^2(p)$ to $\mathbf{1}_Q\Lambda(\varphi)$. Thus, there is a sequence $(\sigma(k), k \ge 0)$ of increasing integers, such that the sequences $(\mathbf{1}_Q P_i(\Lambda_0(\varphi_{\sigma(k)})), k \ge 0)$, $(\mathbf{1}_Q P_i(\Lambda_{3,i}(\varphi_{\sigma(k)})), k \ge 0)$ and $(\Lambda(\varphi_{\sigma(k)}), k \ge 0)$ converge $dt\,dx$-a.e., respectively, to $\mathbf{1}_Q P_i(\Lambda_0(\varphi))$, $\mathbf{1}_Q P_i(\tilde{\Lambda}_{3,i}(\varphi))$ and $\Lambda(\varphi)$. Now (19) holds for $f = \varphi_{\sigma(k)}$; this means that for $(t,x) \in Q$,

$$P_i\big(\Lambda_{3,i}(\varphi_{\sigma(k)})\big)(t,x) = -t^{-1}x_i\Lambda(\varphi_{\sigma(k)})(t,x) + P_i\big(\Lambda_0(\varphi_{\sigma(k)})\big)(t,x).$$

Taking the limit, we get that $dt\,dx$-a.e., in Q,

$$P_i\big(\tilde{\Lambda}_{3,i}(\varphi)\big)(t,x) = -t^{-1}x_i\Lambda(\varphi)(t,x) + P_i\big(\Lambda_0(\varphi)\big)(t,x).$$

Since R, T, ε are arbitrary, the above equality holds $dt\,dx$-a.e. in E. Since (19) holds also for $f = \varphi$, we deduce that $dt\,dx$-a.e.,

$$\int_0^{x_i} dy\, \Lambda_{3,i}(\varphi)(t,\hat{x}_y^i) = \int_0^{x_i} dy\, \tilde{\Lambda}_{3,i}(\varphi)(t,\hat{x}_y^i).$$

Hence we have $dt\,dx$-a.e., $\Lambda_{3,i}(\varphi)(t,x) = \tilde{\Lambda}_{3,i}(\varphi)(t,x)$.

The proofs concerning the operators Λ_1 and $\Lambda_{2,i}$, for $i \in \{1, \dots, d\}$, and their extensions follow the same ideas. □

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ENPC-CERMICS
6 AV. BLAISE PASCAL
CHAMPS-SUR-MARNE
77455 MARNE LA VALLÉE
FRANCE
E-MAIL: delmas@cermics.enpc.fr

UFR DE MATHÉMATIQUES ET D'INFORMATIQUE
UNIVERSITÉ RENÉ DESCARTES
45 RUE DES SAINTS PÈRES
75270 PARIS CEDEX 06
FRANCE
E-MAIL: dhersin@math-info.univ-paris5.fr