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HARNACK INEQUALITIES FOR LOG-SOBOLEV FUNCTIONS AND ESTIMATES OF LOG-SOBOLEV CONSTANTS[1]

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By using the maximum principle and analysis of heat semigroups, Harnack inequalities are studied for log-Sobolev functions. From this, some lower bound estimates of the log-Sobolev constant are presented by using the spectral gap inequality and the coupling method. The resulting inequalities either recover or improve the corresponding ones proved by Chung and Yau. Especially, Harnack inequalities and estimates of log-Sobolev constants can be dimension-free.

1. Introduction. Let M be a d-dimensional connected, compact Riemannian manifold with boundary ∂M either convex or empty. Consider $L = \Delta + \nabla V$ for some $V \in C^\infty(M)$. Let $d\mu = Z^{-1}\exp(V(x))\,dx$ with $Z = \int \exp(V(x))\,dx$, and let $K(V) \in \mathbb{R}$ be such that $\text{Ric–Hess}_V \geq -K(V)$. Simply denote $K(0) = K$ The log-Sobolev constant for L is defined as

$$\alpha = \inf_{f\in\mathscr{F}} \frac{2\mu(|\nabla f|^2)}{\mu(f^2 \log f^2)}, \tag{1.1}$$

where $\mathscr{F} = \{f \in C^1(M)\colon \mu(f^2) = 1,\ f \neq \text{constant}\}$. A function in $\mathscr{F}$ is called a log-Sobolev function (LSF) if it achieves the log-Sobolev constant.

Following the arguments of [5] and [10], a LSF is a solution to

$$Lf = -\frac{\alpha}{2} f \log f^2. \tag{1.2}$$

Conversely, a nonconstant normalized solution to (1.2) is a LSF. By an observation of [5], $f \neq 0$ if $f \in \mathscr{F}$ solves (1.2). Without loss of generality, we may only consider positive LSF's. Starting from this point of view, [5] proved that if $V = 0$, $\partial M = \varnothing$ and $K \leq 0$, then for any $f > 0$ solving (1.2),

$$\sup f \leq \exp(d/2) \tag{1.3}$$

and

$$|\nabla \log f|^2 + \alpha \log f \leq \alpha d/2. \tag{1.4}$$

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These then were used there to derive the following lower bound estimate of α ($V=0,\ \partial M=\varnothing,\ K\le 0$):

$$\alpha \ge \min\left\{\frac{\lambda_1}{4\mathrm{e}}, \frac{2}{dD^2}\right\}, \tag{1.5}$$

where λ_1 is the first eigenvalue (spectral gap) of Δ and D is the diameter of M.

Reference [5] suggested an alternative method to estimate α, although the estimate (1.5) is not as sharp as some known ones. For instance, Deuschel–Stroock's estimate [7] (noting that $\lambda_1 \ge \pi^2/D^2$ for $K\le 0$): if $V=0$, then

$$\alpha \ge \max\left\{\frac{\lambda_1}{d} - K, \frac{3\lambda_1 - dK}{d+2}\right\}. \tag{1.6}$$

See [1], [4], [11] and [12] for more estimates, especially in the negative curvature case.

In this paper, we adopt both the maximum principle and a dimension-free Harnack inequality for heat semigroups due to [12]. Our first result is an extension of (1.4): if $V=0$ and $\partial M=\varnothing$, then

$$|\nabla \log f|^2 + \left(\alpha + \frac{2}{3}K\right)\log f \le \frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2 \tag{1.7}$$

for any $f>0$ solving (1.2). This implies

$$\sup f \le \exp\left[\frac{d}{2} + \frac{dK}{3\alpha}\right] \tag{1.8}$$

and the Harnack inequality

$$f(x) \le f^{1-\varepsilon}(y)\exp\left[\left(\alpha + \frac{2}{3}K\right)\left(\frac{\rho(x,y)^2}{4\varepsilon} + \frac{d\varepsilon}{2\alpha}\right)\right] \tag{1.9}$$

for any $\varepsilon\in(0,1)$, where ρ is the Riemannian distance.

The proof of (1.7) is based on the maximum principle as in [5]. Next, we use a different method to derive a dimension-free Harnack inequality. Let P_t be the heat semigroup of L with reflecting boundary when $\partial M\ne\varnothing$ and recall that [12] presented the following inequality ([12], Lemma 2.1): for any positive $f\in C(M)$ and $\delta>1,\ t\ge 0$, and for any positive $g\in C[0,t]$,

$$(P_t f(x))^\delta \le P_t f^\delta(y)\exp\left[\frac{\delta\rho(x,y)^2\int_0^t g(s)^2\,ds}{4(\delta-1)(\int_0^t g(s)\exp(-K(V)s)\,ds)^2}\right]. \tag{1.10}$$

From this we prove that if $f>0$ solves (1.2), then

$$(\sup f)^{1-2\alpha t} \le (\inf f)^{1-\alpha t}\exp\left[\frac{K(V)D^2}{1-\exp(-2K(V)t)}\right], \qquad t\in(0,1/(2\alpha)). \tag{1.11}$$

Then, by using the above inequalities, we obtain some lower bound estimates of α which can be dimension-free. For instance, if $K(V)\le 0$ we have (see Corollary 4.2)

$$\alpha > \frac{2.73}{D^2}. \tag{1.12}$$

It has been claimed in [5] that (1.4) and hence (1.3) also hold for the convex boundary case, since the authors believed that the maximum points for the test functions, that is, ϕ and ψ in Section 2, are interior points of the manifold. But this observation seems suspicious in general; at least the following example shows that a LSF may have no maximum points in the interior.

EXAMPLE 1.1. Let $M = [a, b]$ with $b > a$ and take $V = 0$. Then, for any LSF f, $f' \neq 0$ in (a, b). Actually, by the Neumann boundary condition, we have $f'(a) = f'(b) = 0$. Assume that there exists $c \in (a, b)$ such that $f'(c) = 0$. For any $a', b' \in [a, b]$ with $a' < b'$, let $I(a', b') = (1/(b' - a')) \int_{a'}^{b'} f^2(s)\, ds$. We have $I(a, b) = 1$. From this we see that $\min\{I(a, c), I(c, b)\} \leq 1$. Without loss of generality, we assume that $I(c, b) \leq 1$. Denoting by $\alpha(c, b)$ the log-Sobolev constant on $[c, b]$, we have

$$\frac{\pi^2}{(b-c)^2} = \alpha(c, b) \leq \frac{2\int_c^b f'(s)^2\, ds}{\int_c^b (f^2 \log f^2)(s)\, ds - (b-c)I(c, b)\log I(c, b)}$$
$$\leq \frac{2\int_c^b f'(s)^2\, ds}{\int_c^b (f^2 \log f^2)(s)\, ds} = \alpha = \frac{\pi^2}{(b-a)^2}.$$

This is a contradiction.

It should be pointed out that the author is not sure yet whether there always exists a LSF. It seems that the LSF may not exist when $V = 0$ and $M = \mathbb{S}^{d-1}$. Actually, for this case we have $\alpha = \lambda_1 = d$ and α can be attained by

$$f_\varepsilon(x) = \frac{1 + \varepsilon f(x)}{\|1 + \varepsilon f\|_2} \quad \text{as } \varepsilon \downarrow 0,$$

where f is the normalized first eigenfunction. Obviously, f_ε goes to constant as $\varepsilon \downarrow 0$. From this we guess that α could not be attained by any nonconstant function.

The nonexistence of the LSF will, of course, cause difficulty when we try to use a LSF to estimate α. But we will see in Section 3 that this can be overcome by an approximation argument.

Finally, the relationship between a LSF and (1.2) may help us find more examples with an exact evaluation of α.

EXAMPLE 1.2. Take $M = [a, b]$ and $L = (d^2/dx^2) - (\pi\varepsilon/(b-a))(\sin(\pi x/(b-a)))(d/dx)$, $\varepsilon \in \mathbb{R}$. Then $V = \varepsilon \cos(\pi x/(b-a))$ and $\alpha = \pi^2/(b-a)^2$. We need only consider the case that $\varepsilon \neq 0$. Let $f(x) = \exp[-\varepsilon \cos(\pi x/(b-a))]$, then f solves (1.2) with $\alpha = \pi^2/(b-a)^2$ and $\mu(f^2) = 1$. Hence f is a LSF of L and $\alpha = \pi^2/(b-a)^2$.

2. Harnack inequalities.

THEOREM 2.1. *Suppose that $V = 0$ and $\partial M = \varnothing$. If $f > 0$ solves* (1.2), *then* (1.7) *and* (1.8) *hold*.

PROOF. Let $\psi = \log f$ and

$$\phi = |\nabla\psi|^2 + (\alpha + rK)\psi,$$

where $r \in (0, 1)$ is to be determined. By (1.2) we have

$$\Delta\psi = -|\nabla\psi|^2 - \alpha\psi = -\phi + Kr\psi.$$

Letting x_0 be the maximum point of ϕ, we have, at x_0,

$$\begin{aligned}
0 \geq \Delta\phi &= 2\sum_{i,j}\psi_{ij}^2 + 2\sum_j \psi_j(\Delta\psi)_j + 2\operatorname{Ric}(\nabla\psi, \nabla\psi) + (\alpha + rK)\Delta\psi \\
&\geq \frac{2}{d}(\Delta\psi)^2 - 2K|\nabla\psi|^2 + 2Kr|\nabla\psi|^2 - (\alpha + Kr)\phi + Kr(\alpha + Kr)\psi \\
&= \frac{2}{d}\phi^2 - \left\{\frac{4Kr}{d}\psi + K(2-r) + \alpha\right\}\phi + \frac{2K^2r^2}{d}\psi^2 + K(\alpha + Kr)(2-r)\psi.
\end{aligned}$$

Letting $s = \sqrt{K\,d\alpha(r-1)\psi + (d^2/16)[K(2-r)+\alpha]^2}$, we obtain

$$\begin{aligned}
\phi(x_0) &\leq Kr\psi + \frac{d}{4}[K(2-r)+\alpha] + s \\
&= -\frac{r}{(1-r)d\alpha}s^2 + s + \frac{d}{4}[K(2-r)+\alpha] + \frac{rd}{16(1-r)\alpha}[K(2-r)+\alpha]^2 \\
&\leq \frac{d(1-r)}{4r\alpha}\left[\alpha + \frac{r(K(2-r)+\alpha)}{2(1-r)}\right]^2.
\end{aligned}$$

This proves (1.7) by taking $r = 2/3$.

Finally, letting y_0 be the maximum point of f, by (1.7) we have

$$\left(\alpha + \frac{2}{3}K\right)\log f(y_0) \leq \phi(x_0) \leq \frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2.$$

This proves (1.8) by the fact that $\alpha + \frac{2}{3}K > 0$ following from (1.6). □

COROLLARY 2.2. *Under the assumption of Theorem* 2.1, *if $f > 0$ solves* (1.2), *then* (1.9) *holds*.

PROOF. By (1.7) we have

$$|\nabla \log f| \leq \sqrt{\frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2 - \left(\alpha + \frac{2}{3}K\right)\log f}\,. \tag{2.1}$$

For any $x, y \in M$ with $\rho(x, y) > 0$, let γ: $[0, \rho(x, y)] \to M$ be any minimal geodesic from x to y. Choose s_1 and s_2 such that

$$f(\gamma_{s_1}) = \max_{[0, \rho(x, y)]} f(\gamma_s), \qquad f(\gamma_{s_2}) = \min_{[0, \rho(x, y)]} f(\gamma_s). \tag{2.2}$$

Noting that $\alpha + \frac{2}{3}K > 0$, by (2.1) we obtain

$$\begin{aligned}\log \frac{f(\gamma_{s_1})}{f(\gamma_{s_2})} &= \int_{s_2}^{s_1} \frac{d}{ds} \log f(\gamma_s)\, ds \\ &\le \rho(x, y)\sqrt{\frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2 - \left(\alpha + \frac{2}{3}K\right)\log f(\gamma_{s_2})}.\end{aligned}$$

Therefore,

$$\begin{aligned}f(x) &\le f(\gamma_{s_1}) \le f(\gamma_{s_2}) \exp\left[\rho(x, y)\sqrt{\frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2 - \left(\alpha + \frac{2}{3}K\right)\log f(\gamma_{s_2})}\right] \\ &\le f^{1-\varepsilon}(\gamma_{s_2}) \exp\left[\varepsilon \log f(\gamma_{s_2}) \right.\\ &\qquad\left. + \rho(x, y)\sqrt{\frac{d}{2\alpha}\left(\alpha + \frac{2}{3}K\right)^2 - \left(\alpha + \frac{2}{3}K\right)\log f(\gamma_{s_2})}\right].\end{aligned}$$

Letting $s = \sqrt{(d/2\alpha)(\alpha + \frac{2}{3}K)^2 - (\alpha + \frac{2}{3}K)\log f(\gamma_{s_2})}$, we have

$$\begin{aligned}f(x) &\le f^{1-\varepsilon}(y) \exp\left[-\frac{\varepsilon s^2}{\alpha + \frac{2}{3}K} + \rho(x, y)s + \frac{d\varepsilon}{2\alpha}\left(\alpha + \frac{2}{3}K\right)\right] \\ &\le f^{1-\varepsilon}(y) \exp\left[\left(\alpha + \frac{2}{3}K\right)\left(\frac{\rho(x, y)^2}{4\varepsilon} + \frac{d\varepsilon}{2\alpha}\right)\right].\end{aligned}$$

□

THEOREM 2.3. *Suppose that $V \in C^2(M)$ and ∂M is either convex or empty. If $f > 0$ solves* (1.2), *then* (1.11) *holds.*

PROOF. Let x_0 and y_0 be, respectively, the maximum point and the minimum point of f. By (1.2) we have

$$\begin{aligned}P_t f(x_0) - P_s f(x_0) &= \int_s^t E^{x_0} Lf(x_u)\, du = -\alpha \int_s^t E^{x_0} f(x_u) \log f(x_u)\, du \\ &\ge -\alpha \log f(x_0) \int_s^t P_u f(x_0)\, du, \qquad t \ge s \ge 0.\end{aligned}$$

This implies

$$P_t f(x_0) \ge f(x_0) \exp[-\alpha t \log f(x_0)] = f^{1-\alpha t}(x_0). \tag{2.3}$$

Similarly, we have

$$P_t f(y_0) \le f^{1-\alpha t}(y_0). \tag{2.4}$$

On the other hand, by taking $\delta = 2$ and $g(s) = \exp(-K(V)s)$ in (1.10), we obtain

$$\begin{aligned}(P_t f(x_0))^2 &\le P_t f^2(y_0) \exp\left[\frac{K(V)D^2}{1-\exp(-2K(V)t)}\right]\\ &\le f(x_0) P_t f(y_0) \exp\left[\frac{K(V)D^2}{1-\exp(-2K(V)t)}\right].\end{aligned}$$

The proof is now complete by combining this with (2.3) and (2.4). □

COROLLARY 2.4. *Under the assumption of Theorem* 2.3, *if* $f > 0$ *is a LSF, then*

$$\sup \log f \le D^2\alpha + D^2 K(V)^+, \tag{2.5}$$

$$\inf \log f \ge -\frac{27}{16} D^2 \alpha - \frac{9}{8} D^2 K(V)^+. \tag{2.6}$$

PROOF. Let x_0 and y_0 be, respectively, the maximum and minimum points of f. By taking $\delta = 2$ and $g(s) = \exp(-K(V)s)$ in (1.10), we have

$$(P_t f(x_0))^2 \le P_t f^2(x) \exp\left[\frac{K(V)D^2}{1-\exp(-2K(V)t)}\right].$$

By combining this with (2.3) and the fact that $\mu(P_t f^2) = \mu(f^2) = 1$, we obtain

$$\log f(x_0) \le \frac{K(V)D^2}{2(1-\alpha t)(1-\exp(-2K(V)t))}, \qquad \alpha t \in (0,1).$$

Next, it is easy to check that

$$\frac{K}{1-\exp(-2Kt)} \le \frac{1}{2t} + K^+, \qquad t > 0. \tag{2.7}$$

Then

$$\log f(x_0) \le \frac{D^2((1/2t) + K(V)^+)}{2(1-\alpha t)} \le D^2\alpha + D^2 K(V)^+$$

by taking $t = (1/2\alpha)$.

Next, By (2.3), (2.4) and taking $g(s) = \exp(-K(V)s)$ in (1.10), we obtain

$$1 \le f^{1-\delta\alpha t}(x_0) \le f^{1-\alpha t}(y_0) \exp\left[\frac{\delta K(V) D^2}{4(\delta-1)(1-\mathrm{e}^{-2K(V)t})}\right], \qquad \delta\alpha t \le 1.$$

By taking $\delta = 1/\alpha t$, we obtain

$$\log f(y_0) \ge \frac{-K(V)D^2}{2(1-\alpha t)^2(1-\exp(-2K(V)t))}, \qquad \alpha t < 1.$$

By (2.7) and taking $t = 1/3\alpha$, we prove (2.6). □

3. Estimates of the log-Sobolev constant. In case the LSF may not exist, we turn to consider the weak log-Sobolev constant α_ε for small $\varepsilon > 0$, which is the largest possible constant such that

$$\mu(f^2 \log f^2) \le \frac{2}{\alpha_\varepsilon}\mu(|\nabla f|^2) + \varepsilon, \qquad f \in \mathscr{F}.$$

Then $\alpha_\varepsilon \downarrow \alpha$ as $\varepsilon \downarrow 0$. By [10], the constant α_ε can be achieved and the corresponding LSF could not be constant since $\varepsilon \ne 0$. Let f_ε be such a LSF, then by [10],

$$Lf_\varepsilon = -\frac{\alpha_\varepsilon}{2} f_\varepsilon \log f_\varepsilon^2 + \frac{1}{2}\alpha_\varepsilon \varepsilon f_\varepsilon. \tag{3.1}$$

By the remark in [5] following Theorem 1, we may assume that $f_\varepsilon > 0$. Therefore, by the proofs of Theorem 2.1 and Corollary 2.4, we have

$$\sup \log f_\varepsilon \le D^2\alpha + D^2 K(V)^+ + h_\varepsilon, \tag{3.2}$$

$$\sup \log f_\varepsilon \le \frac{d}{2} + \frac{dK}{3\alpha} + h_\varepsilon \quad \text{if } V = 0,\ \partial M = \varnothing, \tag{3.3}$$

where $h_\varepsilon \to 0$ as $\varepsilon \to 0$.

THEOREM 3.1. *Under the assumption of Theorem* 2.3, *let* λ_1 *be the first eigenvalue of* L. *We have*

$$\alpha \ge \frac{\sqrt{(1 + D^2K(V)^+)^2 + 4D^2\lambda_1} - 1 - D^2K(V)^+}{2D^2}. \tag{3.4}$$

Especially, *if* $K(V) \le 0$, *then*

$$\alpha \ge \frac{\sqrt{1 + 4D^2\lambda_1} - 1}{2D^2} \ge \frac{\sqrt{1 + 4\pi^2} - 1}{2D^2} > \frac{2.68}{D^2}. \tag{3.5}$$

Finally, *for the case that* $V = 0$ *and* $\partial M = \varnothing$, *we have*

$$\alpha \ge \frac{6\lambda_1 - 2dK}{3(d+2)}. \tag{3.6}$$

PROOF. By the spectral theory we have

$$\mu\{(Lf_\varepsilon)^2\} \ge \lambda_1 \mu(|\nabla f_\varepsilon|^2). \tag{3.7}$$

Next, by (3.1) we have

$$\begin{aligned} \mu\{(Lf_\varepsilon)^2\} &= \alpha_\varepsilon^2 \mu\{f_\varepsilon^2(\log f_\varepsilon)^2\} + \frac{\varepsilon^2\alpha_\varepsilon^2}{4} - \varepsilon\alpha_\varepsilon^2 \mu\{f_\varepsilon^2 \log f_\varepsilon\} \\ &= \alpha_\varepsilon^2 \mu\{f_\varepsilon^2(\log f_\varepsilon)^2\} - \varepsilon\alpha_\varepsilon \mu(|\nabla f_\varepsilon|^2) - \frac{\varepsilon^2\alpha_\varepsilon^2}{4}. \end{aligned} \tag{3.8}$$

Noting that

$$f_\varepsilon L(f_\varepsilon \log f_\varepsilon) = -\alpha_\varepsilon f_\varepsilon^2(\log f_\varepsilon)^2 + \frac{\varepsilon\alpha_\varepsilon}{2} f_\varepsilon^2 \log f_\varepsilon + f_\varepsilon L f_\varepsilon + |\nabla f_\varepsilon|^2,$$

we have

$$\begin{aligned}\alpha_\varepsilon\mu\{f_\varepsilon^2(\log f_\varepsilon)^2\} &= \frac{\varepsilon\alpha_\varepsilon}{2}\mu\{f_\varepsilon^2\log f_\varepsilon\} + \mu\{(1+\log f_\varepsilon)|\nabla f_\varepsilon|^2\}\\ &= \frac{\varepsilon}{2}\mu(|\nabla f_\varepsilon|^2) + \frac{\varepsilon^2\alpha_\varepsilon}{4} + \mu\{(1+\log f_\varepsilon)|\nabla f_\varepsilon|^2\}\\ &\le \left(1+\frac{\varepsilon}{2}+\sup\log f_\varepsilon\right)\mu(|\nabla f_\varepsilon|^2) + \frac{\varepsilon^2\alpha_\varepsilon}{4}.\end{aligned}$$

By combining this with (3.6) and (3.7), we obtain

$$\lambda_1\mu(|\nabla f_\varepsilon|^2) \le \alpha_\varepsilon(1+\sup\log f_\varepsilon)\mu(|\nabla f_\varepsilon|^2).$$

This implies

$$\alpha_\varepsilon \ge \frac{\lambda_1}{1+\sup\log f_\varepsilon}. \tag{3.9}$$

By (3.2) and letting $\varepsilon \downarrow 0$, we obtain

$$\alpha \ge \frac{\lambda_1}{1+D^2\alpha + D^2K(V)^+}.$$

This proves (3.4), and (3.5) then follows from the fact that $\lambda_1 \ge \pi^2/D^2$ for $K(V) \le 0$; see, for example, [3]. Finally, (3.6) follows from (3.3) and (3.9). □

We remark that, for the free boundary case, one may also extend (3.6) to the case $V \ne 0$. But the resulting estimate will depend on both $K(V)$ and $\|\nabla V\|_\infty$, due to the maximum principle. We omit this extension here since the resulting estimate of α is usually less sharp.

4. Estimates of α by using coupling. The coupling method has been used successfully to estimate the first eigenvalue; see, for example, [2] and [3]. The Harnack inequalities proved in Section 2 enable us to use this method to estimate α. By the approximation procedure as in Section 3, we may assume that the LSF exists. The coupling method then works as follows.

THEOREM 4.1. *Let $f > 0$ be a LSF. Define $\beta_1 = \sup f$ and $\beta_2 = \sup|1+\log f|$. Next, let (x_t, y_t) be a coupling for the L-diffusion process with coupling time $T = \inf\{t \ge 0\colon x_t = y_t\}$. We have*:

(i) $\alpha \ge \{\sup_{x,y\in M} E^{x,y}T\}^{-1}(\beta_1 - 1)/(\beta_1\log\beta_1)$.

(ii) *If there exists $\bar\rho \in C(M\times M)$ with $\bar\rho \ge c\rho$ for some $c > 0$ such that*

$$E^{x,y}\bar\rho(x_t, y_t) \le \bar\rho(x,y)\exp(-\delta t) \tag{4.1}$$

for some $\delta > 0$ and all $t \ge 0$. Then $\alpha \ge \delta/\beta_2$.

PROOF. (i) Let x_0 and y_0 be, respectively, the maximum point and the minimum point of f. We have

$$1 = \frac{1}{f(x_0) - f(y_0)} \left\{ E^{x_0, y_0}[f(x_t) - f(y_t)] + \int_0^t E^{x_0, y_0}[Lf(x_s) - Lf(y_s)]\, ds \right\}$$
$$\leq P^{x_0, y_0}(T > t) + \frac{\alpha \delta(f \log f)}{f(x_0) - f(y_0)} \int_0^t P^{x_0, y_0}(T > s)\, ds,$$

where

$$\begin{aligned}
\delta(f \log f) &:= \sup f \log f - \inf f \log f \\
&= f(x_0) \log f(x_0) - [f(y_0) \vee \exp(-1)] \log[f(y_0) \vee \exp(-1)] \\
&= \int_{f(y_0) \vee \exp(-1)}^{f(x_0)} (1 + \log s)\, ds \\
&\leq (f(x_0) - f(y_0) \vee \exp(-1)) \frac{1}{\beta_1 - 1} \int_1^{\beta_1} (1 + \log s)\, ds \\
&\leq (f(x_0) - f(y_0)) \frac{\beta_1 \log \beta_1}{\beta_1 - 1}.
\end{aligned}$$

Here, we have used the facts that $f(y_0) \leq 1$ and $[1/(\beta_1 - r)] \int_r^{\beta_1} (1 + \log s)\, ds$ is increasing in r. Therefore

$$\alpha \geq \frac{(\beta_1 - 1) P^{x_0, y_0}(T \leq t)}{\beta_1 \log \beta_1 E^{x_0, y_0} T}.$$

This proves (i) by letting $t \uparrow \infty$.

(ii) For any $\varepsilon > 0$, choose $x_\varepsilon \neq y_\varepsilon$ such that

$$\frac{f(x_\varepsilon) - f(y_\varepsilon)}{\bar{\rho}(x_\varepsilon, y_\varepsilon)} \geq \sup \frac{f(x) - f(y)}{\bar{\rho}(x, y)} - \varepsilon := C - \varepsilon.$$

Noting that

$$\frac{|f(x) \log f(x) - f(y) \log f(y)|}{\bar{\rho}(x, y)} \leq \frac{\beta_2 |f(x) - f(y)|}{\bar{\rho}(x, y)} \leq C \beta_2,$$

by (4.1) we obtain

$$\begin{aligned}
(C - \varepsilon) \bar{\rho}(x_\varepsilon, y_\varepsilon) &\leq f(x_\varepsilon) - f(y_\varepsilon) \\
&\leq E^{x_\varepsilon, y_\varepsilon} |f(x_t) - f(y_t)| \\
&\quad + \alpha \int_0^t E^{x_\varepsilon, y_\varepsilon} |(f \log f)(x_s) - (f \log f)(y_s)|\, ds \\
&\leq C E^{x_\varepsilon, y_\varepsilon} \bar{\rho}(x_t, y_t) + \alpha C \beta_2 \int_0^t E^{x_\varepsilon, y_\varepsilon} \bar{\rho}(x_s, y_s)\, ds \\
&\leq C \bar{\rho}(x_\varepsilon, y_\varepsilon) \left[\exp(-\delta t) + \frac{\alpha \beta_2 (1 - \exp(-\delta t))}{\delta} \right].
\end{aligned}$$

The proof of (ii) is then completed by letting $t \uparrow \infty$ and $\varepsilon \downarrow 0$. □

We remark that by [2], we have $\lambda_1 \geq \delta$ if (4.1) holds. Then, (ii) of Theorem 4.1 is implied by the proof of Theorem 3.1. As has been shown in [2], [3] and [13] for λ_1, one may obtain explicit lower bounds of α by Theorem 4.1 and using the coupling by reflection due to [6] and [8]. For instance, we have the following result.

COROLLARY 4.2. (i) *If* $K(V) \leq 0$, *we have*

$$\alpha \geq \frac{c_0}{D^2},$$

where $c_0 > 0$ *solves* $c^2 = 8(1 - e^{-c})$. *It is easy to check that* $c_0 > 2.73$, *then* (1.12) *holds*.

(ii) *Suppose that* $V = 0$ *and* $\partial M = \varnothing$. *If* $K \leq 0$, *then*

$$\alpha \geq \frac{16(1 - e^{-d/2})}{dD^2}.$$

PROOF. For the coupling by reflection, we have (see [2] and [13])

$$(E^{x,y}T)^{-1} \geq \frac{8}{D^2} \quad \text{if } K(V) \leq 0.$$

Then (ii) follows from (1.8) and Theorem 4.1(i). Next, by Corollary 2.4, $\beta_1 \leq \exp(\alpha D^2)$. By Theorem 4.1(i) we obtain

$$\alpha \geq \frac{8(1 - \exp(-\alpha D^2))}{D^4 \alpha}.$$

Letting $c = D^2\alpha$, we have $c > 0$ and

$$c^2 \geq 8(1 - e^{-c}).$$

Therefore, $c \geq c_0$ if $c_0 > 0$ satisfies $c_0^2 = 8(1 - \mathrm{e}^{-c_0})$. $\square$

Finally, we would like to point out that this paper gives a line to estimate the first eigenvalue for the nonlinear problem (1.2). Actually, the coupling method may also apply to a more general version of (1.2),

$$Lf = -\lambda F(f);$$

see, for example, Lu [9]. Especially if $\|F'\|_\infty < \infty$, we have

$$\lambda \geq \delta / \|F'\|_\infty$$

for any positive eigenvalue λ, where δ satisfies (4.1). This then enables one to present a general formula for the lower bound of λ in the spirit of [3], Theorem 1.1.

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