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GROWTH PROFILE AND INVARIANT MEASURES FOR THE WEAKLY SUPERCRITICAL CONTACT PROCESS ON A HOMOGENEOUS TREE[1]

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It is known that the contact process on a homogeneous tree of degree $d + 1 \geq 3$ has a *weak survival* phase, in which the infection survives with positive probability but nevertheless eventually vacates every finite subset of the tree. It is shown in this paper that in the weak survival phase there exists a spherically symmetric invariant measure whose density decays exponentially at infinity, thus confirming a conjecture of Liggett. The proof is based on a study of the relationships between various thermodynamic parameters and functions associated with the contact process initiated by a single infected site. These include (1) the *growth profile*, which determines the exponential rate of growth in space–time on the event of survival, (2) the exponential rate β of decay of the hitting probability function at infinity (also studied by the author) and (3) the exponential rate η of decay in time t of the probability that the initial infected site is infected at time t. It is shown that β is a strictly increasing function of the infection rate λ in the weak survival phase, and that $\beta = 1/\sqrt{d}$ at the upper critical point λ_2 demarcating the boundary between the weak and strong survival phases. It is also shown that $\eta < 1$ except at λ_2, where $\eta = 1$.

1. Introduction. This paper is a sequel to [7], in which the *weakly supercritical* phase of an isotropic contact process on an infinite homogeneous tree was studied. Briefly, an *isotropic contact process* on the homogeneous tree $\mathscr{T} = \mathscr{T}_d$ of degree $d + 1$ is a continuous time Markov process A_t on the set of finite subsets of $\mathscr{T}$ (henceforth, $\mathscr{T}$ will be identified with the vertex set of the tree) that evolves as follows. Infected sites (members of A_t) recover at rate 1 and upon recovery are removed from A_t; healthy sites (members of A_t^c) become infected at rate λ times the number of infected neighbors and upon infection are added to A_t. Under the default probability measure P, the initial state A_0 is the singleton set $\{e\}$ (where e is a distinguished element of $\mathscr{T}$ called the "root"). See [8] and [5] for general information on the contact process, and [7], especially Sections 1 and 2, and also [9] and [11] for background information concerning the contact process on a homogeneous tree. Some of the arguments of this paper (see the proofs of Proposition 8 and Theorem 4 below) are borrowed from [7].

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It was discovered by Pemantle [14] for trees of degree greater than 3 and by Liggett [11] (see also [16] for a relatively simple argument) for the tree of degree 3 that the contact process described above has (at least) three essentially different "phases": (1) *extinction*, (2) *weak survival*, and (3) *strong survival*. In phase (1), $A_t = \varnothing$ eventually, with probability 1; in phase (2), $|A_t| \to \infty$ with positive probability but for every *finite* subset B of $\mathscr{T}$, $A_t \cap B = \varnothing$ eventually, with probability 1 and in phase (3), with positive probability $e \in A_t$ for arbitrarily large values of t. There exist critical constants $0 < \lambda_1 < \lambda_2 < \infty$ such that $\lambda \le \lambda_1$ implies ultimate extinction, $\lambda_1 < \lambda \le \lambda_2$ implies weak survival, and $\lambda > \lambda_2$ implies strong survival. The phase of interest here is *weak survival* (we also refer to the contact process in this phase as *weakly supercritical*). This phase is of interest, among other reasons, because it does not occur for the contact process on Euclidean lattices (e.g., on $\mathbb{Z}^d$). (It *does* occur for certain related stochastic processes, e.g., branching random walks, on state spaces with tree-like geometry: see [6].) Henceforth, we shall discuss only this case. Thus, we make the standing assumption that

$$\lambda \in (\lambda_1, \lambda_2].$$

1.1. *Growth profile and other characteristics.* The main results of this paper concern a function $V(s)$ which we shall call the growth profile of the contact process. Two other parameters also enter into the statements of these results: $\beta = \beta(\lambda)$, which determines the size of the "limit set" of the contact process on the event of its survival (see [7]) and $\eta = \eta(\lambda)$, the exponential rate of decay of $P\{e \in A_t\}$. The definitions are as follows:

$$\beta = \lim_{n = d(x,e) \to \infty} \left(P\{x \in A_t \text{ for some } t\}\right)^{1/n}; \tag{1}$$

$$\eta = \lim_{t \to \infty} \left(P\{e \in A_t\}\right)^{1/t}; \tag{2}$$

$$V(s) = \log \lim_{n = d(x,e) \to \infty} \left(P\{x \in A_{ns}\}\right)^{1/n}. \tag{3}$$

The existence of these limits follows from simple subadditivity arguments [see below for η and $V(s)$, and see [7] for β]. The main result of [7] is that for all values of $\lambda \in (\lambda_1, \lambda_2]$,

$$\beta \le \frac{1}{\sqrt{d}}. \tag{4}$$

THEOREM 1. *The growth profile $V(s)$ is a concave, continuous function of $s > 0$ that is bounded above by* $\log \beta$ *and satisfies*

$$\lim_{s \to 0+} V(s) = -\infty, \tag{5}$$

$$\lim_{s \to \infty} V(s)/s = \log \eta. \tag{6}$$

Theorem 1 will be proved in Section 2—see Propositions 1–4. We have been unable to prove that $V(s)$ is *strictly* concave, but we conjecture that it is.

THEOREM 2. *If* $\beta < 1/\sqrt{d}$, *then* $\eta < 1$. *If* $\beta = 1/\sqrt{d}$, *then* $\eta = 1$.

In view of (6), Theorem 2 implies that if $\beta < 1/\sqrt{d}$, then $V(s) \to -\infty$ as $s \to \infty$, but that if $\beta = 1/\sqrt{d}$ then $V(s)$ is nondecreasing in s (since it is concave). This will be of crucial importance in the proof of Theorem 5 below. Theorem 2 will be proved in Section 4.

THEOREM 3. *In the weak survival phase* $\lambda \in (\lambda_1, \lambda_2]$, *the parameter* β *is a strictly increasing function of* λ.

Theorem 3 will be proved in Section 6 below. Since $\beta \le 1/\sqrt{d}$ for all $\lambda \le \lambda_2$, by the main result of [7], it follows from Theorems 2 and 3 that $\eta < 1$ for all $\lambda \in (\lambda_1, \lambda_2)$.

The function $V(s)$ proscribes the shape and size of the set A_t of infected sites at large times t. Define r_t and R_t to be the *smallest* and *largest* distances $d(x, e)$ among the infected sites $x \in A_t$. Also, for any integer n and any $s > 0$, define $N_n(ns)$ to be the number of vertices $x \in A_{ns}$ at distance n from e that are infected at time ns.

THEOREM 4. *Suppose that* $\beta < 1/\sqrt{d}$. *Then there exist smallest and largest solutions* $0 < s_1 \le s_2 < \infty$ *of* $V(s) = -\log d$. *Almost surely on the event of survival*,

$$\lim_{t \to \infty} r_t/t = 1/s_2 \tag{7}$$

and

$$\lim_{t \to \infty} R_t/t = 1/s_1. \tag{8}$$

Moreover, *for any* $s > 0$ *such that* $V(s) + \log d > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log N_n(ns) = V(s) + \log d. \tag{9}$$

See Section 5 for the proof. This theorem explains the use of the term "growth profile." The function $V(s)$ effectively determines how the contact process "spreads out" in space–time and determines roughly how many vertices at each distance from e are infected at any large time t. Together with Theorem 2, it also implies that A_t recedes linearly from the root vertex e if and only if $\lambda \in (\lambda_1, \lambda_2)$.

1.2. *Invariant measures for the contact process*. The set of invariant measures in the phase of strong survival has a relatively simple structure: every such measure is a mixture of the point mass at $\emptyset$ and the "upper invariant measure" (see [17]). For weakly supercritical contact processes, the set of invariant measures seems in general to be much more complex. Durrett and Schinazi [3] proved that it has infinitely many extreme points. Liggett [9] showed that, at least for $\lambda > \lambda_1$ sufficiently close to λ_1, there are invariant measures ν that are spherically symmetric (about the root vertex e) and have exponentially receding densities, that is, are such that for all $x \in \mathscr{T}$,

$$C_1 z^{d(x,e)} \le \nu\{A\colon x \in A\} \le C_2 z^{d(x,e)}, \tag{10}$$

where $0 < C_1 < C_2 < \infty$ are constants independent of x and $d(x,e)$ denotes the distance from x to e in $\mathscr{T}$. Liggett also conjectured that such invariant measures exist for *all* $\lambda \in (\lambda_1, \lambda_2)$, and gave a sufficient condition for their existence. This condition involves the function

$$\varphi(z) = \lim_{t\to\infty} \Bigl(E \sum_{x \in A_t} z^{L(x)}\Bigr)^{1/t}, \tag{11}$$

where $L(x)$ is the *depth* of the vertex x in the tree (see [9] or Section 3 for the definition). Liggett proved that if there exists a solution $z \in (1/\sqrt{d}, 1)$ of the equation

$$\varphi(z) = 1, \tag{12}$$

then there exists a spherically symmetric invariant measure for the contact process that satisfies the exponential decay law (10). The next result shows that the growth profile determines Liggett's function φ.

THEOREM 5. *If $\beta < 1/\sqrt{d}$, then for every $z > 1/\sqrt{d}$,*

$$\varphi(z) = \max_{0<s<\infty} \exp\left(\frac{V(s) + \log(dz)}{s}\right). \tag{13}$$

The proof will be given in Section 3, along with the proof of the following corollary.

COROLLARY 1. *If $\beta < 1/\sqrt{d}$, then $\varphi(1/d\beta) = 1$.*

In view of (4) and Theorem 3, this implies that Liggett's conjecture is true.

COROLLARY 2. *For every $\lambda \in (\lambda_1, \lambda_2)$ there is a spherically symmetric invariant measure for the contact process whose density decays exponentially at infinity.*

COROLLARY 3. *If $\lambda = \lambda_2$, then $\beta = 1/\sqrt{d}$.*

PROOF. Suppose, to the contrary, that $\beta < 1/\sqrt{d}$; then the interval $J = (1/\sqrt{d}, 1/d\beta)$ is nonempty. By Corollary 1 and Proposition 1.19(e) of [9], $\varphi(z) < 1$ for all $z \in J$. For each $z \ne 1/\sqrt{d}$, the function $\varphi(z) = \varphi(z,\lambda)$ is continuous in λ, by Proposition 1.19(d) of [9], so it follows that for some $\lambda > \lambda_2$ there exists $z \in J$ such that $\varphi(z,\lambda) < 1$. However, if $\varphi(z,\lambda) < 1$ for some $z > 0$, then the contact process cannot survive strongly with positive probability, by (11). Since for $\lambda > \lambda_2$ the contact process does indeed survive with positive probability, this is a contradiction. (T. Liggett communicated this elegant argument to the author after reading the first draft of this paper.) □

Schonmann [15] has recently proved that $\beta(\lambda)$ is continuous for $\lambda \in (0, \lambda_2]$, and that $\lim_{\lambda \to 0+} \beta(\lambda) = 0$.

2. Growth profile: basic properties. By the isotropy of the contact process, the probability that the set A_t of sites infected at time t contains a particular vertex $x \in \mathscr{T}$ depends only on t and $|x|$, and similarly the probability that x is *ever* infected depends only on $|x|$. Thus, we may define

$$u_n(t) = P\{x \in A_t\} \quad \text{and} \quad u_n = P\Big\{x \in \bigcup_{t \geq 0} A_t\Big\},$$

where x is any vertex at distance n from the root e. Recall that $\beta = \lim_{n \to \infty} u_n^{1/n}$. The Markov and monotonicity properties of the contact process imply that for all nonnegative integers m, n and all times $s, t \geq 0$,

$$u_{m+n}(s + t) \geq u_m(s) u_n(t). \tag{14}$$

Consequently, by the basic convergence theorem for subadditive sequences, for every $s > 0$,

$$\lim_{n \to \infty} u_n(ns)^{1/n} \triangleq U(s) \triangleq \exp\{V(s)\} \tag{15}$$

exists, and for every $s > 0$ and every integer $n \geq 0$,

$$u_n(ns) \leq U(s)^n. \tag{16}$$

Proposition 1. $V(s) \leq \log \beta$.

Proof. It is clear that $u_n(ns) \leq u_n$ for any $n \in \mathbb{N}$ and any $s > 0$. Taking nth roots on both sides of this inequality and letting $n \to \infty$, one obtains the desired result. □

Proposition 2. *$V(s)$ is a concave and, therefore, continuous function of $s > 0$.*

Proof. According to the fundamental inequality (14), for all $0 < s < t < \infty$ and all nonnegative integers m, n,

$$u_m(ms) u_n(nt) \leq u_{m+n}(ms + nt).$$

Taking the $(m + n)$th root of both sides and letting $m, n \to \infty$ in such a way that the ratio $m/(m + n)$ converges to $\alpha \in (0, 1)$, one gets

$$U(s)^{\alpha} U(t)^{1-\alpha} \leq U(s + t),$$

which implies that V is concave. □

Proposition 3. $\lim_{s \to 0+} V(s) = -\infty$.

PROOF. This is equivalent to showing that $\lim_{s\to 0+} U(s) = 0$. Recall that

$$U(s) = \lim_{n\to\infty} u_n(ns)^{1/n}$$

and that $u_n(ns)$ is the probability that a particular vertex x at distance n from e will be infected at time ns. Let $x_0, x_1, x_2, \dots, x_n$ be the successive vertices on the geodesic segment from $e = x_0$ to $x = x_n$, and for each $1 \le m \le n$, define τ_m to be the elapsed time between the first infection of x_{m-1} and the first infection of x_m. In order that x_n be infected at time ns, it must be the case that at least half of the times τ_m are no greater than $2s$. But the distribution of τ_m, conditional on the history of the contact process up to the time of first infection of x_{m-1}, is stochastically larger than an exponential random variable with mean $1/\lambda$, so the conditional probability that $\tau_m \le 2s$ is no larger than $1 - \exp(-2\lambda s)$. Thus, the probability that at least half of the random variables τ_m, $1 \le m \le n$, are $\le 2s$ is no larger than

$$\sum_{k=[n/2]}^{n} \binom{n}{k} (1 - \exp(-2\lambda s))^k (\exp(-2\lambda s))^{n-k}$$
$$\le (1 - \exp(-2\lambda s))^{[n/2]} \sum_{k=[n/2]}^{n} \binom{n}{k} \le 2^n (1 - \exp(-2\lambda s))^{[n/2]}.$$

Taking nth roots, one finds that

$$U(s) = \limsup_{n\to\infty} (u_n(ns))^{1/n} \le 2\sqrt{1 - \exp(-2\lambda s)},$$

which converges to 0 as $s \to 0+$. □

By definition, $u_0(t) = P\{e \in A_t\}$ is the probability that the root vertex e is infected at time t. If e is infected at time t and if there is an infection trail from (e, t) to $(e, t + s)$, then clearly e is infected at time $t + s$. Hence, by the Markov and monotonicity properties of the contact process, $u_0(t + s) \ge u_0(t)u_0(s)$ for all $s, t > 0$. (Recall from [7] that an infection trail is a connected path in the percolation structure that does not pass through any death events. The (Harris) percolation structure is the system of death marks and infection arrows used to define the contact process; see, e.g., [8].) It follows that

$$\eta = \lim_{t\to\infty} u_0(t)^{1/t} \le 1$$

exists, and that $u_0(t) \le \eta^t$ for all $t > 0$.

PROPOSITION 4. $\lim_{s\to\infty} V(s)/s = \log \eta$.

PROOF. The concavity of V implies that the limit exists and is nonpositive, so it suffices to consider only integer values of s. Fix $\varepsilon = 1/m > 0$ smaller than 1, and let $n > 0$ be an integer such that $n\varepsilon \in \mathbb{N}$. Let x be a vertex at distance $n\varepsilon$ from the root e. If there are infection trails that extend from $(e, 0)$ to $(e, n - n\varepsilon)$ and from $(e, n - n\varepsilon)$ to (x, n), then their concatenation is an infection trail from $(e, 0)$ to (x, n). Hence, by the Markov and monotonicity properties,

$$u_{n\varepsilon}(n) \geq u_0(n - n\varepsilon) u_{n\varepsilon}(n\varepsilon).$$

Taking nth roots of both sides and letting $n \to \infty$ through integer multiples of $1/\varepsilon$ gives

$$U(1/\varepsilon)^{\varepsilon} \geq \eta^{1-\varepsilon} U(1)^{\varepsilon}.$$

Taking logarithms and letting $\varepsilon \to 0$ yields

$$\lim_{s \to \infty} \frac{V(s)}{s} \geq \log \eta.$$

The reverse inequality is obtained in much the same way. Let x be a vertex at distance $n\varepsilon$ from the root ε. If there are infection trails that extend from $(e, 0)$ to (x, n) and from (x, n) to $(e, n + n\varepsilon)$ then their concatenation is an infection trail from $(e, 0)$ to $(e, n + n\varepsilon)$; consequently,

$$u_{n\varepsilon}(n) u_{n\varepsilon}(n\varepsilon) \leq u_0(n + n\varepsilon).$$

Taking nth roots and letting $n \to \infty$ gives

$$U(1/\varepsilon)^{\varepsilon} U(1)^{\varepsilon} \leq \eta^{1+\varepsilon}$$

and letting $\varepsilon \to 0$ yields

$$\lim_{s \to \infty} \frac{V(s)}{s} \leq \log \eta. \qquad \square$$

PROPOSITION 5. *If* $\lim_{s \to \infty} V(s) = -\infty$, *then*

$$\max_{0 < s < \infty} V(s) = \log \beta.$$

PROOF. By Proposition 3, $V(s) \to -\infty$ as $s \to 0+$, and by Proposition 2, $V(s)$ is continuous in s. Consequently, if $V(s) \to -\infty$ as $s \to \infty$ then $V(s)$ attains its maximum value $v_{\max}$. Moreover, since V is concave, the hypothesis that $V \to -\infty$ as $s \to \infty$ implies that there exist constants $a < 0$ and $0 < s_* < \infty$ such that

$$V(s) \leq as - as_* + v_{\max} - 1 \quad \text{if } s_* \leq s < \infty. \tag{17}$$

Let x be any vertex of $\mathscr{T}$. If x is infected for a first time at time τ, then conditional on the history of the contact process up to time τ the probability that x will remain infected until time $\tau + 1$ is at least $\exp\{-1\}$, because the death rate is 1. On this event, x will remain infected at the first *integer* time after τ. Hence,

$$\begin{aligned} u_n &\leq \exp\{1\} \sum_{m=1}^{\infty} u_n(m) \\ &\leq \exp\{1\} \sum_{m=1}^{\infty} \exp\{nV(m/n)\} \\ &= \exp\{1\} \Bigg(\sum_{m \leq ns_*} + \sum_{m > ns_*} \Bigg). \end{aligned} \tag{18}$$

By (17), the second sum (over $m > ns_*$) is dominated by a convergent geometric series whose first term is no larger than $\exp\{n(v_{\max} - 1)\}$. On the other hand, the first sum (over $m \le ns_*$) contains at most $ns_* + 1$ terms, none larger than $\exp\{nv_{\max}\}$. Thus,

$$\sum_{m \le ns_*} \le \exp\{nv_{\max}\}(ns_* + 1),$$

$$\sum_{m > ns_*} \le \exp\{nv_{\max} - n\}/(1 - \exp\{-a\})$$

and so the dominant term is the first of these sums. Taking nth roots in (18) and letting $n \to \infty$ gives $\log \beta \le v_{\max}$. Proposition 1 implies the reverse inequality. □

3. The growth profile and Liggett's φ-function. In this section we prove Theorem 5, thus making the connection between the growth profile V and Liggett's φ-function. Recall [9] that φ is defined by

$$\varphi(z) = \lim_{t \to \infty} \left(Ew_z(A_t)\right)^{1/t} = \lim_{t \to \infty} \left(E \sum_{x \in A_t} z^{L(x)}\right)^{1/t},$$

where $L(x)$ is the *depth* of the vertex x. The definition of the depth function L depends on an arrangement of the vertices of $\mathscr{T}$ in *levels* $(\mathscr{L}_n)_{n \in \mathbb{Z}}$ so that (1) distinct levels $\mathscr{L}_n$ and $\mathscr{L}_m$ do not intersect; (2) every vertex $y \in \mathscr{L}_n$ has exactly d neighbors in $\mathscr{L}_{n+1}$ and exactly one neighbor in $\mathscr{L}_{n-1}$ and (3) the root vertex e is in level $\mathscr{L}_0$. The *depth* $L(x)$ of a vertex x is then defined to be the index n of the level $\mathscr{L}_n$ in which x lies. Observe that there are infinitely many distinct arrangements of the vertices in levels, but that the expectations in the definition of $\varphi(z)$ do not depend on which arrangement is used, by the isotropy of the contact process.

LEMMA 1. *Define M_ν^k to be the number of vertices of $\mathscr{T}$ at distance k from the root e and at depth ν. Then*

$$M_\nu^k = 1, \quad \text{if } \nu = -k; \tag{19}$$

$$M_\nu^k = d^k \quad \text{if } \nu = k; \tag{20}$$

$$M_\nu^k = (d-1)d^{j-1} \le d^j \quad \text{if } \nu = -k + 2j \text{ and } -k < \nu < k;$$

$$M_\nu^k = 0 \quad \text{otherwise}.$$

PROOF. Let x be a vertex at distance n from the root e. The geodesic segment from e to x makes j steps "up" and $n - j$ steps "down" for some integer $0 \le j \le n$, with the j up steps preceding the $n - j$ down steps (otherwise the path would retrace some of its steps, contradicting the supposition that it is a geodesic segment). The depth at termination of any such geodesic segment is $(n - j) - j$, and the number of such distinct geodesic segments (with j up steps followed by $n - j$ down steps) is d^n if $j = 0$ and

$(d-1)d^{n-j-1}$ if $j \geq 1$. Since geodesic segments starting at e are in one-to-one correspondence with their endpoints, the formulas for M_ν^k follow. □

The proof of Theorem 5 will use the following intermediate characterization of φ.

PROPOSITION 6. *For every* $z > 1/\sqrt{d}$,

$$\varphi(z) = \lim_{t\to\infty}\left(\sum_{n=0}^{\infty} u_n(t)(dz)^n\right)^{1/t}.$$

PROOF. For any finite set A of vertices and any $z \in \mathbb{R}$, define $w_z(A) = \sum_{x\in A} z^{L(x)}$. Then $\varphi(z) = \lim_{t\to\infty}(Ew_z(A_t))^{1/t}$, where A_t denotes the set of infected vertices at time t. Arranging vertices by distance from the root gives

$$Ew_z(A_t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} z^{-k+2j} M^k_{-k+2j} u_k(t).$$

Since $M_k^k = d^k$, it follows directly that

$$Ew_z(A_t) \geq \sum_{k=0}^{\infty} z^k d^k u_k(t).$$

On the other hand, Lemma 1 and the hypothesis $z^2 d > 1$, together with the change of variable $j \to k-j$, imply that

$$\begin{aligned} Ew_z(A_t) &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{k} z^k d^k u_k(t) z^{-2j} d^{-j} \\ &\leq \sum_{k=0}^{\infty} z^k d^k u_k(t)\left(1-(z^2 d)^{-1}\right)^{-1}. \end{aligned}$$

Since $\lim_{t\to\infty}(Ew_z(A_t))^{1/t}$ exists and equals $\varphi(z)$ (see [9]), the desired result follows from the last two displayed inequalities. □

PROOF OF THEOREM 5. Fix $z > 1/\sqrt{d}$, set $V_z(s) = V(s) + \log(dz)$ and define

$$\sigma = \sup_{s>0} V_z(s)/s = \sup_{s>0}(V(s) + \log(dz))/s.$$

By Propositions 3 and 4 and Theorems 1 and 2, $V_z(s)/s$ converges to $-\infty$ as $s \to 0+$ and $V_z(s)/s$ converges to $\log\eta < 0$ as $s\to\infty$. Consequently, $\sigma < \infty$.

By definition of σ and the intermediate value theorem, for all $\sigma_* < \sigma$ sufficiently close to σ, the line of slope σ_* through the origin must intersect the graph of V_z. Thus, there exists $s_* \in (0,\infty)$ such that $V_z(s_*) = \sigma_*$. Now

$$\begin{aligned} \left(\sum_{n=0}^{\infty} u_n(t) d^n z^n\right)^{1/t} &\geq \left(u_{[t/s_*]}(t)(dz)^{[t/s_*]}\right)^{1/t} \\ &\to \left(U(s_*)dz\right)^{1/s_*} \\ &= \exp\{\sigma_*\} \end{aligned}$$

as $t \to \infty$. This implies, by Proposition 6, that $\varphi(z) \geq e^{\sigma_*}$. Since $\sigma_* < \sigma$ was arbitrary, it follows that

$$\varphi(z) \geq e^{\sigma}.$$

The proof of the reverse inequality is similar to the proof of Proposition 5. First note that by Proposition 3, $V(s) \to -\infty$ as $s \to 0+$, so for any $z > 0$ there exists $\delta \in (0,1)$ sufficiently small that for all $s \in (0, \delta)$,

$$V(s) + \log(dz) < \min(-1, \sigma - 1).$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(t) d^n z^n &\leq \sum_{n=0}^{\infty} U(t/n)^n d^n z^n \\ &= \sum_{n=0}^{\infty} \exp\{n(V(t/n) + \log(dz))\} \\ &= \sum_{n \leq \delta t} + \sum_{n > \delta t} .\end{aligned}$$

There are at most δt terms in the first sum, none larger than $e^{\sigma t}$, since $\sigma t \geq V(t) + \log(dz)$ by definition of σ. By our choice of δ, the terms of the second are dominated by the terms of a geometric series with ratio $\leq e^{-1}$ and with initial term no larger than $e^{t(\sigma-1)/\delta} < e^{t(\sigma-1)}$. Hence, by Proposition 6,

$$\varphi(z) = \limsup_{t \to \infty} \left(\sum_{n=0}^{\infty} u_n(t) d^n z^n \right)^{1/t} \leq e^{\sigma}. \qquad \square$$

Proof of Corollary 1. By Theorems 1 and 2, if $\beta < 1/\sqrt{d}$, then $\lim_{s \to \infty} V(s) = -\infty$. Thus, Theorem 5 and the continuity of $\varphi(z)$ imply that for every $z \geq 1/\sqrt{d}$ the value of $\varphi(z)$ is given by (13). By Proposition 5, the function $V(s)$ attains its maximum value of $\log \beta$ at some $s = s_* \in (0, \infty)$. When $z = 1/(d\beta)$,

$$\max_{s > 0} (V(s) + \log(dz)) = (V(s_*) + \log(dz)) = 0,$$

so by (13), $\varphi(z) = 1$. $\square$

4. The growth profile at ∞.

Proposition 7. *If* $\beta < 1/\sqrt{d}$, *then* $\eta < 1$.

Proof. It suffices to prove that if $\beta < 1/\sqrt{d}$, then there exists a value of $z > 0$ such that $\varphi(z) < 1$, because for any $z > 0$,

$$\eta = \lim_{t \to \infty} (P\{e \in A_t\})^{1/t} \leq \lim_{t \to \infty} (E w_z(A_t))^{1/t} = \varphi(z).$$

According to Proposition 1.19(e) of [9], if $\lambda < \lambda_2$ and if $\varphi(z_*) \leq 1$ for some $z_* > 1/\sqrt{d}$, then $\varphi(z) < 1$ for every $z \in [1/\sqrt{d}, z_*)$. In fact, the hypothesis that $\lambda < \lambda_2$ is extraneous; the proof requires only that the probability of strong survival at λ is 0, and this is true for $\lambda = \lambda_2$ by Corollary 1 of [7]. (Thanks to T. Liggett for pointing this out.) Hence, it suffices to prove that if $\beta < 1/\sqrt{d}$, then for some $z > 1/\sqrt{d}$,

$$\varphi(z) = \lim_{t \to \infty} \left(Ew_z(A_t) \right)^{1/t} \leq 1. \tag{21}$$

Define A_* to be the (random) set consisting of all vertices that are *ever* infected. Obviously, $A_t \subset A_*$ for every $t < \infty$, and so it is also the case that $Ew_z(A_t) \leq Ew_z(A_*)$. We will show that if $\beta < 1/\sqrt{d}$ and $z\beta < 1/d$, then $Ew_z(A_*) < \infty$. It will then follow that (21) holds for all $z \in [1/\sqrt{d}, 1/d\beta)$. Summing over vertices at fixed distances from the root e as in the proof of Proposition 6 gives

$$\begin{aligned} Ew_z(A_*) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} z^{-k+2j} M^k_{-k+2j} u_k \\ &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{k} z^k d^k \beta^k z^{-2j} d^{-j} \\ &\leq \sum_{k=0}^{\infty} z^k d^k \beta^k \left(1 - (z^2 d)^{-1}\right)^{-1} \\ &= (1 - z\beta d)^{-1} \left(1 - (z^2 d)^{-1}\right)^{-1}. \qquad \square \end{aligned}$$

The remainder of this section is devoted to the proof of the following proposition, which will complete the proof of Theorem 2. The argument is largely borrowed from [7], Section 4.

PROPOSITION 8. *If* $\beta = 1/\sqrt{d}$, *then* $\eta = 1$.

For any integer $n \geq 1$, define F_n to be the event that there is an infection trail that begins at the root e at time $t = 0$, reaches a vertex x at distance n from e and then returns from x to e. Since the contact process is, by hypothesis, weakly supercritical, $P(F_n) \to 0$ as $n \to \infty$ (see [7], Section 4). Define

$$\zeta = \limsup_{n \to \infty} P(F_n)^{1/n}.$$

LEMMA 2. *If* $\zeta = 1$ *then* $\eta = 1$.

PROOF. Let H_n be the event that $e \in A_t$ for some $t \geq n$, and let H_n' be the event that $e \in A_t$ for some *integer* $t \geq n$. Since the recovery rate in the contact process is 1, for any n the conditional probability is at least $\exp\{-1\}$ that e remains infected for $n \leq t \leq n+1$, given that e is infected at time n. Hence, $P(H_n) \leq \exp\{1\} P(H_n')$, and it follows that

$$P(H_n) \leq \exp\{1\} \sum_{m=0}^{\infty} P(e \in A_{n+m}) \leq \exp\{1\} \eta^n / (1 - \eta).$$

Thus, to prove that $\eta = 1$ it suffices to prove that

$$\limsup_{n\to\infty} P(H_n)^{1/n} = 1.$$

Fix $\varepsilon > 0$ and let G_n be the event that there is an infection trail starting at $(e, 0)$ that reaches a vertex at distance n from e and then returns to e by time εn. On the event G_n, some vertex at distance n from e is infected at some time $t < \varepsilon n$; hence, summing over all such vertices and all integer times $< \varepsilon n$, using again the fact that the recovery rate is 1, we obtain that

$$P(G_n) < (d+1)d^{n-1}\left(\exp\{1\}\sum_{m=1}^{\varepsilon n}\exp\{nV(m/n)\} + \lambda^n\exp\{-\lambda\}/n!\right).$$

(Here $\lambda^n e^{-\lambda}/n!$ is an upper bound for the probability that a particular vertex x at distance n from e is infected before time 1). Consequently, by Proposition 3, if $\varepsilon > 0$ is sufficiently small then

$$\limsup_{n\to\infty} P(G_n)^{1/n} < 1.$$

Now by hypothesis, $\limsup_{n\to\infty} P(F_n)^{1/n} = 1$. Consequently, if $\varepsilon > 0$ is sufficiently small, then, by the result of the preceding paragraph, $\limsup_{n\to\infty} P(F_n \cap G_n^c)^{1/n} = 1$. But the event $F_n \cap G_n^c$ is contained in $H_{[\varepsilon n]}$, so it follows that

$$\limsup_{n\to\infty} P\big(H_{[\varepsilon n]}\big)^{1/n} = 1. \qquad \square$$

Proof of Proposition 8. By Lemma 2, it suffices to show that $\zeta = 1$. For this we adapt from [7], Section 3, the notion of a *downward* infection trail. Let $x, y \in \mathscr{T}$ be vertices such that $L(x) = m < L(y) = m + k$, where as earlier $L(z)$ denotes the *depth* of the vertex z (the index of its *level* $\mathscr{L}_{L(z)}$). A *downward infection trail* from x to y is an infection trail that begins at x, never enters level $\mathscr{L}_{m-1}$ and first reaches $\mathscr{L}_{m+k}$ at y, where it terminates. By the isotropy of the contact process, the probability that there is a downward infection trail from x to y depends only on $k = d(x, y)$, so we may define

$$w_k = P\{\exists \text{ downward infection trail } x \to y\}.$$

By Proposition 1 of [7], $w_k^{1/k} \to \beta$ as $k \to \infty$. Thus, the probability that there is a downward infection trail from e to x is, in exponential size, just as large as the probability that there is *any* infection trail from e to x.

If there is a downward infection trail from an infected site x at level $L(x) \geq 0$ (beginning at x during the *first* infection epoch of x) to a site $x^* \in \mathscr{T}(x)$, say that x^* is a *descendant* of x. Fix an integer $L \geq 1$ and define Y_L to be the number of descendants of e at depth L. Then by Proposition 2 of [7], there is a Galton–Watson process Z_n^L with mean offspring number $d^L w_L$ such that $Y_{nL} \geq Z_n^L$ for all n. Since $w_L^{1/L} \to \beta = 1/\sqrt{d}$ as $L \to \infty$, if L is sufficiently large, then the Galton–Watson Z_n^L is supercritical. Hence, for any

$\beta_- < \beta$, if k is sufficiently large, then

$$p_k \triangleq P\{Y_k > (d\beta_-)^k\} > \varepsilon > 0, \tag{22}$$

where $\varepsilon > 0$ is the probability that the Galton–Watson process Z_n^L does not reach extinction.

Consider now the event F_n^k that there is an infection trail that begins at the root e at time $t = 0$, reaches a vertex $x \in Y_{nk}$ and then returns from x to e. Clearly, $P(F_n^k) \le P(F_{nk})$, where F_{nk} is as in Lemma 2 above. Then event F_n^k will occur if there is *just one* descendant x of e at depth k such that (1) there is a descendant x^* of x at depth kn and an infection trail from x^* to x, beginning at the instant of initial infection of x^* and (2) there is a subsequent infection trail from x to e. Now conditional on the set $\mathscr{Y}_k$ of descendants of e at depth k, the events (1) for the different $x^* \in \mathscr{Y}_k$ are mutually independent (since they involve nonoverlapping parts of the percolation structure), and each has probability $P(F_{n-1}^k)$ (by the isotropy of the contact process). Moreover, given that event (1) occurs for some $x \in Y_k$, the conditional probability of event (2) is at least u_k. Consequently, if $r_n = P(F_n^k)$, then

$$\begin{aligned} r_n &\ge P\{Y_k \ge (d\beta_-)^k\} P\Bigl(\bigcup_{x \in \mathscr{Y}_k} (1)\ \&\ (2) | Y_k \ge (d\beta_-)^k\Bigr) \\ &\ge p_k\Bigl(1 - (1 - r_{n-1})^{(d\beta_-)^k}\Bigr)u_k \\ &\ge p_k u_k\Bigl(1 - \exp\{-(d\beta_-)^k r_{n-1}\}\Bigr). \end{aligned}$$

For each fixed k, $r_n = P(F_n^k) \to 0$ as $n \to \infty$, as the contact process is weakly supercritical. This implies that the last exponential displayed above is well approximated by the first term of its Taylor series around 0. Furthermore, if k is sufficiently large, then $u_k > \beta_-^k$, and $p_k > \varepsilon$ by the last paragraph. Thus, for all large n,

$$r_n \ge \varepsilon(d\beta_-^2)^k r_{n-1}/2.$$

It follows that $\liminf_{n\to\infty} P(F_n^k)^{1/n} \ge \varepsilon(d\beta_-^2)^k/2$. Since $P(F_n^k) \le P(F_{nk})$, this implies that $\limsup_{n\to\infty} P(F_n)^{1/n} \ge d\beta_-^2$, but $\beta_- < \beta = 1/\sqrt{d}$ was arbitrary, so this proves that

$$\limsup_{n\to\infty} P(F_n)^{1/n} = \eta = 1. \qquad \square$$

5. The growth profile and the spread of the infection.

Proof of Theorem 4. When $\beta < 1/\sqrt{d}$, by Propositions 5 and 7, $\max_s V(s) = \log \beta$. Also, $\beta \ge -\log d$ because otherwise the expected number of sites *ever* infected would be finite, which would preclude the possibility of weak survival. Consequently, there is at least one solution s of $V(s) = -\log d$. Let s_1 be the smallest solution and s_2 the largest solution. Recall that r_t and R_t are the smallest and largest distances $d(x, e)$ among the infected sites

$x \in A_t$, and for any interval (a, b), $N_t(a, b)$ is the number of vertices $x \in A_t$ such that $at < d(x, e) < bt$.

PROOF OF RELATIONS (7) AND (8) (Sketch). We will show that a.s. on the event of survival, for any $\varepsilon > 0$, $N_t(t/s_1 + \varepsilon t, \infty) = 0$ eventually and $N_t(0, t/s_1 - \varepsilon t) = 0$ eventually. This will imply that a.s. on survival, $\limsup R_t/t \le 1/s_1$ and $\liminf r_t/t \ge 1/s_2$. The reverse inequalities $\liminf R_t/t \ge 1/s_1$ and $\limsup r_t/t \le 1/s_2$ will follow from relation (9) proved below.

A routine argument using estimates like those developed in the proof of Proposition 3 shows that it is enough to consider integer times t (the probability that an infection trail moves a distance $\ge \varepsilon t$ in time 1 decreases more rapidly than any exponential e^{-ct}). For a fixed large integer t, the probability that $N_t(t/s_1 + \varepsilon t, \infty) > 0$ is smaller than

$$\sum_{n \ge t/s_1 + \varepsilon t} u_n(t) d^n (d+1).$$

But $u_n(t) \le \exp\{nV(t/n)\}$ so, by an argument like that used in the proof of Theorem 5, the terms in the above sum are bounded by the terms of a geometric series with sum smaller than $O(e^{-t\delta})$ for some $\delta > 0$, depending only on ε. Since $\sum_{t \in \mathbb{N}} e^{-t\delta} < \infty$, the Borel–Cantelli lemma implies that a.s. $N_t(t/s_1 + \varepsilon t, \infty) = 0$ eventually. A similar argument proves that a.s. $N_t(t/s_1 + \varepsilon t, \infty) = 0$ eventually.

PROOF OF RELATION (9) (Sketch). Fix $s > 0$ such that $V(s) + \log d > 0$. The probability $u_n(ns)$ that a particular vertex at distance n from the root will be infected at time ns is no larger than $\exp\{nV(s)\}$. The number of vertices in $\mathscr{T}$ at distance n from e is $(d+1)d^{n-1}$. It follows that the expected number of such vertices that are infected at time ns is no larger than

$$(d+1) d^n \exp(nV(s)).$$

Consequently, by a routine application of the Borel–Cantelli lemma and the Markov inequality,

$$\limsup_{n \to \infty} \frac{1}{n} \log N_n(ns) \le V(s) + \log d.$$

The proof of the reverse direction uses again the notion of a downward infection trail, which was introduced in the proof of Proposition 8. Let x be a vertex at depth $n > 0$ and let $u_n^*(t)$ be the probability that there is a downward infection trail from $(e, 0)$ to (x, t). Then for any $s > 0$,

$$\lim_{n \to \infty} u_n^*(ns)^{1/n} = \exp(V(s)). \tag{23}$$

This may be proved in much the same manner as Proposition 1 of [7].

The virtue of considering only vertices infected via downward infection trails is that the existence (or nonexistence) of these infection trails depends only on the part of the percolation structure above the vertices of the tree in the sector between the initial and terminal vertices. Thus, for any distinct vertices $x, x', \dots$ at depth m and any times t, s, the numbers $\xi_x(s,t), \xi_{x'}(s,t), \dots$ of vertices $y, y', \dots$ that are infected at time t by downward infection trails starting at $(x,s), (x',s), \dots,$ respectively, are mutually independent. Consequently, if $N_{nm}(nms)$ denotes the number of vertices at distance nm that are infected at time nms [by infection trails starting at $(e,0)$], then $N_{nm}(nms)$ dominates a Galton–Watson chain Z_n whose mean offspring number is

$$d^m u_m^*(ms).$$

(See [7], Proposition 2, for a similar result about time-independent infection trails.) It follows that on the event of nonextinction of this Galton–Watson process,

$$\liminf_{n\to\infty} \frac{1}{n} \log N_{nm}(nms) \geq \log d^m u_m^*(ms).$$

By choosing m large, we can (1) make the event of nonextinction of the Galton–Watson process arbitrarily close to the event of nonextinction of the contact process and (2) push $u_m^*(ms)^{1/m}$ close to $\exp(V(s))$, by (23). Therefore, almost surely on the event of nonextinction,

$$\liminf_{t\to\infty} \frac{1}{t} \log N_n(ns) \geq V(s) + \log d. \qquad \square$$

6. Strict monotonicity of β.

Fix $\lambda \in (\lambda_1, \lambda_2]$, so that the contact process with infection rate λ and recovery rate 1 is weakly supercritical. In this section we will prove the following proposition, which implies Theorem 3.

PROPOSITION 9. *If* $\lambda_* < \lambda$, *then* $\beta(\lambda_*) < \beta(\lambda)$.

We begin the proof by introducing an augmentation of the usual percolation structure used in the construction of the contact process. Recall that the percolation structure is a system of independent Poisson processes attached to vertices and ordered pairs of neighboring vertices. For each vertex x of $\mathscr{T}$, the Poisson process attached to x has rate 1 and determines the recovery times at x; in particular, at every occurrence time, site x recovers if it is infected. For each ordered pair (x,y) of neighboring vertices, the Poisson process attached to (x,y) has rate λ; the occurrence times are precisely those times when an infection at x may jump to y. Occurrences in these Poisson processes are marked on a system of directed rays $\{x\} \times [0,\infty)$ connected to the vertices x of $\mathscr{T}$, in such a way that (1) at each occurrence time t of the Poisson process attached to (x,y) an "infection arrow" is drawn from (x,t) (the "tail") to (y,t) (the "head") in $\mathscr{T} \times (0,\infty)$ and (2) at each occurrence time t of the Poisson process attached to x a "recovery mark" $*$ is attached to (x,t). Note that there are no simultaneous occurrences of infection arrows and/or recovery marks in the percolation structure. The state A_t of the contact process at time t may now be defined to be the set of all vertices y for which there is a (directed) path through the percolation structure (the system of rays and arrows described above) that begins at e, ends at (y, t), and does not pass through any recovery marks $*$.

In its augmented form, the percolation structure is provided with a collection of Bernoulli-p random variables ξ_α, one for each infection arrow α. These are conditionally independent, given the realization of the percolation structure. The value of p is chosen so that

$$\lambda_* = \frac{\lambda p}{1 + \lambda q},$$

where $q = 1 - p$. A version A_t^* of the contact process with infection rate λp and recovery rate $1 + \lambda q$ may be constructed using the augmented percolation structure by (1) first creating a modified percolation structure by changing every infection arrow α such that $\xi_\alpha = 0$ to a recovery mark $*$, then (2) defining A_t^* to be the set of all vertices y for which there exists a directed path from $(e, 0)$ to (y, t) in the modified percolation structure that does not pass through any recovery marks $*$. This new contact process A_t^*, with infection rate λp and recovery rate $1 + \lambda q$, is a time-changed version of a contact process with infection rate λ_* and recovery rate 1. Consequently, the value of the parameter $\beta(\lambda_*)$ satisfies

$$\beta(\lambda_*) = \lim_{n \to \infty} (u_n^*)^{1/n},$$

where u_n^* is the probability that the contact process A_t^* ever infects vertex e_n at distance n from e.

Let G_n be the event that the contact process A_t infects vertex e_n at some finite time t, and let G_n^* be the corresponding event for the contact process A_y^* [thus, $P(G_n) = u_n$ and $P(G_n^*) = u_n^*$]. By construction, $G_n^* \subset G_n$, because every infection arrow in the modified percolation structure occurs also in the unmodified percolation structure, and every recovery mark $*$ in the unmodified percolation structure is retained in the modified percolation structure. On the event G_n, there is at least one directed path through the unmodified percolation structure that leads from $(e, 0)$ to $e_n \times (0, \infty)$ (and in fact there may be many overlapping such paths). Call an infection arrow α in the unmodified percolation structure *essential* for the event G_n if event G_n occurs, and if changing α from an infection arrow to a recovery mark would destroy all directed paths from $(e, 0)$ to $e_n \times (0, \infty)$. Define N_n to be the number of essential arrows for the event G_n when G_n occurs and $N_n = 0$ when G_n does not occur. Since modification of any one of the essential arrows would disconnect $(e, 0)$ from $e_n \times (0, \infty)$, in order that event G_n^* occur it is necessary that $N_n \geq 1$ and that $\xi_\alpha = 1$ for *every* essential arrow α. The conditional probability of this, given the realization of the unmodified percolation structure, is p^{N_n}; hence,

$$u_n^* = P(G_n^*) = Ep^{N_n} 1_{G_n}.$$

If it were the case that $N_n \geq Cn$ on G_n for some $C > 0$, then it would follow that $u_n^* \leq u_n p^{Cn}$ and consequently that $\beta(\lambda_*) \leq \beta(\lambda) p^C$. Alternatively, if for some $\rho < 1$ it were the case that $P(N_n \le Cn|G_n) \le \rho^n$, then it would follow that $u_n^* \le u_n(p^{Cn} + \rho^n)$, and again $\beta(\lambda_*) < \beta(\lambda)$. Thus, to prove Proposition 9, it suffices to prove the following lemma.

LEMMA 3. *There exist constants* $C < \infty$ *and* $0 < \rho < 1$ *such that for all* n *sufficiently large,*

$$P(N_n \le Cn|G_n) \le \rho^n.$$

Lemma 3 is a statement that refers only to the unmodified percolation structure. The remaining arguments make no further reference to the augmented or modified percolation structures. Henceforth, the term *directed path* will mean a directed path in the (unmodified) percolation structure that does not pass through any recovery marks $*$.

The proof of Lemma 3 is based on a continuous-time adaptation of (a part of) Menshikov's proof of the uniqueness of the critical point for bond percolation on the integer lattices $\mathbb{Z}^d$, $d \ge 2$ (see [13]; see also [4], Section 3.1, for a detailed exposition). This machinery has also been used in the context of the contact process on $\mathbb{Z}^d$ by Bezuidenhout and Grimmett [1]; the argument here is simpler, however. The key tool is a continuous-time version of the BKF inequality, as stated in Section 2.2 of [1], which provides a bound for the probability of disjoint occurrence of two events. Recall [1] that for any two events F, G, the event $F \circ G$, the event that F and G occur disjointly, is defined to be the set of all realizations ω of the percolation structure such that there exist disjoint measurable subsets Γ, Δ of $\mathscr{T} \times [0, \infty)$ such that the cylinders ω_Γ and ω_Δ are contained in F and G, respectively. (The cylinder ω_Γ is defined to be the set of all realizations of the percolation structure that agree with ω in Γ.) For any sets A, B of vertices, define $H(A, B)$ to be the event that there is a directed path from $A \times \{0\}$ to $B \times (0, \infty)$. Then the event

$$H(A_1, B_1) \circ \big(H(A_2, B_2) \cap H(A_3, B_3)^c\big)$$

is the event that there exist nonoverlapping directed paths from $A_1 \times \{0\}$ to $B_1 \times (0, \infty)$ and from $A_2 \times \{0\}$ to $B_2 \times (0, \infty)$ and *no* directed path from $A_3 \times \{0\}$ to $B_3 \times (0, \infty)$.

LEMMA 4 (BKF inequality).

$$\begin{aligned} P\big(H(A_1, B_1) \circ \big(H(A_2, B_2) \cap H(A_3, B_3)^c\big)\big) \\ \le P(H(A_1, B_1))P\big(H(A_2, B_2) \cap H(A_3, B_3)^c\big). \end{aligned}$$

In this form, the lemma follows almost immediately from the statement of the BKF inequality in Section 2.2 of [1], as the events $H(A, B)$ are all increasing, and therefore their complements $H(A, B)^c$ are decreasing. Although the event $H(A, B)$ may depend on the entire percolation structure and not just on a compact part of it (as required by the hypotheses of the BKF inequality in [1]), $H(A, B)$ is clearly a monotone limit of increasing events

$H_n(A, B)$, each of which depends only on a compact part of the percolation structure. Thus, Lemma 4 follows from the BKF inequality in [1] by the monotone convergence theorem.

In addition to the BKF inequality, we will need the following elementary consequence of Menger's lemma (see [2], Chapter 2).

LEMMA 5. *Let H be the event that there is a directed path from (x, t) to $\{z\} \times (0, \infty)$, and let $K \subset H$ be the event that the only essential arrow for H is the first infection arrow α leading out of x after time t. Let t'' be the occurrence time of α. Then on the event K, there exist directed paths γ, γ' from (x, t'') to $\{z\} \times (0, \infty)$ that do not intersect except at the endpoints.*

It is possible that this second path γ' might be the trivial path that consists of just α; in this case, it must be the case that $y = z$. In every other case, the path γ' must be nontrivial.

PROOF OF LEMMA 5. Build a directed graph $\mathscr{G}$ from the percolation structure as follows. The vertex set is the set of all $(x, t) \in \mathscr{T} \times [0, \infty)$ such that $t = 0$, or (x, t) is the space–time location of a recovery mark $*$, or (x, t) is the location of either the tail or head of an infection arrow. The directed edges are of two types: (1) the infection arrows and (2) arrows connecting successive vertices (x, t_m) and (x, t_{m+1}) on the same time line, where (x, t_m) is the location of the tail or head of an infection arrow and (x, t_{m+1}) is the location of an occurrence of any type. Thus, there are no directed edges leading out of vertices (x, t) marking the locations of recoveries. It is clear that the paths in the digraph $\mathscr{G}$ coincide with directed paths in the percolation structure that do not pass through recovery marks.

On the event H, the digraph $\mathscr{G}$ is such that there is a path from the last vertex (x, t') before (x, t) [i.e., t' is the maximal $s \le t$ such that (x, s) is a vertex of $\mathscr{G}$] to the set of vertices (z, t_i) with first coordinate z. Clearly, any such path must travel along the vertical edge from (x, t') to (x, t''), where (x, t'') is the first vertex of $\mathscr{G}$ on $\{x\} \times (0, \infty)$ after (x, t') representing an infection arrow *out* of x. This infection arrow must be essential for the event H, because if the arrow were changed to a recovery mark $*$ there would be no way out of (x, t''). On the event K there is no other essential arrow, and so there is no edge of the digraph $\mathscr{G}$ with initial point at time greater than or equal to t'', whose removal would disconnect (x, t'') from $\{z\} \times [t'', \infty)$. Thus, the digraph $\mathscr{G}$ is "two-edge-connected" between (x, t'') and $\{z\} \times [t'', \infty)$ (see [12], page 611). By Menger's theorem ([12], Problem 39(a)), there are two edge-disjoint paths γ, γ' in $\mathscr{G}$ between (x, t'') and $\{z\} \times [t'', \infty)$. However, the paths γ, γ' cannot possibly intersect in a vertex of $\mathscr{G}$ not on the terminal line $\{z\} \times [t'', \infty)$ [except for the initial vertex (x, t'')], because this would require that somewhere in the percolation structure there are simultaneous infection arrows leading in and out of some $y \in \mathscr{T}$. $\square$

PROOF OF LEMMA 3. If the event G_n occurs, then there is at least one essential arrow for G_n, namely, the first infection arrow emanating from the root line $e \times (0, \infty)$. The essential arrows may be ordered in sequence $\alpha_1, \alpha_2, \dots, \alpha_{N_n}$ according to the times $\nu_1, \nu_2, \dots, \nu_{N_n}$ of their occurrence; note that the spatial locations of the essential arrows are not necessarily ordered in any nice way (for instance, by distance from e), nor are the times ν_j stopping times for the contact process.

For each integer $m \geq 1$, define τ_m to be the time of the mth infection arrow α_m^* in the percolation cluster attached to the root $(e, 0)$ [the set of all (x, t) connected to $(e, 0)$ by a directed path] or $\tau_m = \infty$ if there are fewer than m such arrows. Observe that each τ_m *is* a stopping time for the contact process. Moreover, the times ν_j of the essential arrows must occur in the (increasing) sequence $\tau_1, \tau_2, \dots$. Suppose that $\tau_m < \infty$ and let u_m and v_m be the tail and head, respectively, of the infection arrow α_m^*. Note that there is a directed path from $(e, 0)$ to (u_m, τ_m), by definition of τ_m, and that therefore there is also a directed path to (v_m, τ_m). In order that this infection arrow be an essential arrow for G_n (i.e., that α_m^* occurs in the sequence $\alpha_1, \alpha_2, \dots, \alpha_{N_n}$), it is necessary and sufficient that:

(E1) There is a directed path to $\{e_n\} \times (0, \infty)$ from (u_m, τ_m) or from (v_m, τ_m).
(E2) There is *no* directed path to $\{e_n\} \times (0, \infty)$ that does not pass through (u_m, τ_m).

Furthermore, by Lemma 5, given that α_m^* is essential for G_n, in order that the *next* essential arrow has tail w it is necessary that:

(E3) There is a directed path from $\{u_m, v_m\} \times \{\tau_m\}$ to $\{w\} \times (0, \infty)$ that does not intersect the directed path guaranteed by (E1), except possibly at the endpoints.

In order that α_m^* be the *last* essential arrow for G_n (i.e., that $\alpha_m^* = \alpha_{N_n}$), it is necessary that:

(E3′) There is a directed path from $\{u_m, v_m\} \times \{\tau_m\}$ to $\{e_n\} \times (0, \infty)$ that does not intersect the directed path guaranteed by (E1), except possibly at the endpoints.

Assume that the vertices $e_1, e_2, \dots$ lie in succession along a geodesic ray γ_* beginning at $e_0 = e$, so that $d(e_j, e_{j+1}) = 1$ for every $j \geq 0$. Let F_m be the event that the arrow α_m^* is essential for G_n [i.e., the event that (E1) and (E2) are true], and for $k \geq 1$ let $D_{m,k}$ be the event that each of the geodesic segments from u_m and v_m to the tail w of the next essential arrow (or e_n if there is no next essential arrow) intersects γ_* in a segment of length greater than or equal to k. Then by (E3) and (E3′), on the event $F_m \cap D_{m,k}$ there is a directed path from $\{u_m, v_m\} \times \{\tau_m\}$ that travels a distance greater than or equal to k on γ_* and does not intersect the directed path guaranteed by (E1). Consequently, by Lemma 4 and the strong Markov property, if $\mathscr{F}_m$ denotes the σ-algebra generated by the history of the contact process up to time τ_m, then

$$P(D_{m,k} \mid \mathscr{F}_m \wedge F_m) 1_{F_m} \leq 2u_k \leq 2\beta^k.$$

(Compare with Lemma 3.12 of [4].) In words, the conditional distribution of the distance to the next essential arrow is stochastically dominated by a geometric distribution. It follows that the sum of the distances between the first m essential arrows is stochastically dominated by the sum of m i.i.d. random variables with geometrically decaying tail, and Lemma 3 clearly follows. □

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