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BOOK REVIEW

PAUL MALLIAVIN, *Stochastic Analysis*. Springer, New York, 1997, 370 pages, $125.00.

REVIEW BY DENIS BELL

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This book is an exposition of some important topics in stochastic analysis and stochastic geometry. In reviewing the book, it is as well to start with Malliavin's main contribution to the field.

The *Malliavin calculus*, introduced in the mid-seventies in the papers [9] and [10], was motivated by the following link between stochastic analysis and partial differential equations: consider the Stratonovich stochastic differential equation

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i(t) + X_0(x_t)\, dt \tag{1}$$

where $X_0, \dots, X_n$ are smooth vector fields defined on $\mathbb{R}^d$ and $w = (w_1, \dots, w_n)$ is a standard Wiener process. Let L denote the second-order differential operator

$$\frac{1}{2}\sum_{i=1}^{n} X_i^2 + X_0.$$

It has been known since the early work of Itô that the solution process x_t in (1) is Markov and that its transition probabilities yield the fundamental solution (in the sense of distributions) to the heat equation

$$\frac{\partial u}{\partial t} = Lu.$$

Suppose now that the vector fields $X_1, \dots, X_n$, together with all the Lie brackets generated by $X_0, \dots, X_n$, span $\mathbb{R}^d$ at every point [we'll refer to this assumption as (HC)]. Then an application of Hörmander's hypoellipticity theorem implies that the transition probabilities $p(t, x, dy)$ of x_t admit smooth densities $p(t, x, y)$, for all positive t. Malliavin reversed this flow of information from PDE theory to probability by proving *directly* that condition (HC) implies the existence of smooth densities for the process x_t. From here standard techniques can be used to deduce that the operator $\partial/\partial t - L$ is hypoelliptic, thus yielding a probabilistic proof of Hörmander's theorem.

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A major obstacle is encountered in trying to implement this program. Let ν denote the measure $p(t, x, .)$. The natural way to establish the required regularity of ν is to obtain estimates of the form

$$\left| \int_{\mathbb{R}^d} D^{(\alpha)} \phi \, d\nu \right| \leq C_\alpha \|\phi\|_\infty \tag{2}$$

for all test functions ϕ and multi-indices α, where C_α are constants depending only on α. Now the measure ν is the image of Wiener measure on the space of continuous paths P under the *Itô map* $I : w \in P \mapsto x$, composed with evaluation at time t (denote the composition by I_t). The derivation of estimates such as (2) necessarily requires an *integration by parts* calculation involving the map I_t. However I, and hence I_t, are *nonsmooth* in the classical sense. Prior to Malliavin's work, the pathological nature of the Itô map had prevented probabilists from directly obtaining regularity results for random variables arising from stochastic differential equations. Malliavin solved this problem by constructing an extended calculus for Wiener functionals based on the *number operator*, an object from quantum mechanics.

Integration by parts in this setting produces a stochastic $p \times p$ matrix σ_t, now known as the *Malliavin covariance matrix*. The absolute continuity of the measure ν follows from the condition

$$\sigma_t \in GL(d) \quad \text{a.s.} \tag{3}$$

It is relatively easy to prove that (3) holds under (HC). In order to show that ν has a *smooth* density and thus prove Hörmander's theorem probabilistically, one needs the stronger quantitative result that

$$\text{(HC) implies } (\det \sigma_t)^{-1} \in \bigcap_{p \geq 1} L^p. \tag{4}$$

The proof of (4) is difficult and requires intricate stochastic analysis. Kusuoka and Stroock [8] first proved (4) in 1985.

After the appearance of Malliavin's original papers, easier methods were discovered to implement the program outlined above. Stroock [15] produced a functional analytic treatment in which the operator used to effect the key integration by parts step was explicitly identified as the number operator (it had been introduced in Malliavin's papers as the generator of a path-valued Ornstein–Uhlenbeck process). Bismut [3] formulated an alternative variational approach, based upon perturbation of the system (1) by a suitably chosen family of drifts. The reviewer [1] found an elementary way to perform the integration by parts, using *classical* finite-dimensional calculus. This approach exploits the fact that the Itô map admits a smooth restriction to the *Cameron–Martin space*, the subspace of P consisting of absolutely continuous paths with square integrable derivatives

(Shigekawa [14] had independently used a similar idea to study the existence of densities for multiple Wiener integrals).

A common theme emerged from the foregoing works: namely, the extended calculus on Wiener space *need only operate in the Cameron–Martin directions*. This idea had, in fact, appeared before Malliavin's papers in the work of Krée and Gross, but it had not previously been used to study the existence of densities.

Malliavin's work inspired many new results in stochastic analysis. Examples include filtering theorems (Michel [11]), a deeper understanding of the Skorohod integral and the development of an anticipating stochastic calculus (Nualart and Pardoux [12]), an extension of Clark's formula (Ocone [13]), Bismut's probabilistic analysis of the small-time asymptotics of the heat kernel of the Dirac operator on a Riemannian manifold [4] and his subsequent proof of the associated index theorem [5], and a sharp hypoellipticity theorem for Hörmander operators with hypersurfaces of infinite type (Bell and Mohammed [2]).

The book under review is divided into five parts. The objective of the first part is to present what might be called the *fundamental theorem* of Malliavin calculus. This is done in the setting of an abstract Gaussian space X with a designated subspace of *admissible* directions H (ultimately X is chosen to be the Wiener space and H the Cameron–Martin space). A functional f on X is defined to be *smooth* if its derivatives exist a.s. in the H directions. This is Gross' H*-derivative*, introduced in the late sixties in the context of his theory of abstract Wiener spaces [7]. Let $F\colon\ X \mapsto \mathbb{R}^d$ denote a functional on X, smooth in the aforementioned sense with H-derivative DF. The Malliavin covariance matrix $[\sigma_{ij}]$ corresponding to F is defined, where $\sigma_{ij} \equiv (DF_i, DF_j)_H$. Setting aside some nontrivial technical considerations, the main result is as follows: if σ is a.s. nondegenerate, then the law of F is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. This part of the book also includes a discussion of several related topics, for example, Hermite polynomials, the Ornstein–Uhlenbeck process, abstract Wiener spaces, and the Krée–Meyer inequalities.

Part II deals with the subject of quasi-sure analysis, a theory initiated (in the context of stochastic analysis) in the mid-eighties by Fukushima, Kaneko and Takeda. The basic idea is to use the theory of capacity to define a class of negligible sets in the Wiener space, called *slim* sets. The class of slim sets is smaller than the usual class of sets of Wiener measure zero. A property that holds outside of a slim set is said to hold *quasi-surely*. A number of technical results about capacities are proved. It is then shown that if g is a smooth nondegenerate function from the Wiener space into $\mathbb{R}^d$ and π is a property that holds quasi-surely, then, conditioned on any nonempty level set $\{g = \xi\}$, property π holds *almost surely*. Thus a nondegenerate Wiener functional g gives rise to a disintegration of the Wiener measure into measures of finite energy supported on the fibers generated by g. The coarea formula of Bouleau and Hirsch and a version of Stokes' formula are proved within this context.

The theory of stochastic integration is discussed in Part III. The usual introductory material to this topic is covered, including the definition of the Wiener, Itô and Stratonovich integrals and Itô's formula. Much additional material is also presented, for example, a description of Wiener's theory of homogeneous chaos and Fock space, a brief discussion of anticipating stochastic integration, and the relationship of this theory to a result of Gaveau and Trauber identifying the Skorohod integral with the divergence operator on Wiener space. Also included are a formula of Stroock that computes the action of the number operator on the Itô integral, Ramer's transformation theorem for abstract Wiener measure (as extended by Kusuoka), and the Clark–Bismut–Ocone formula.

Part IV treats stochastic differential equations. A result of Gaveau is presented that gives an explicit formula for the solution of a Stratonovich equation when the diffusion coefficients satisfy a second-level nilpotency condition (a more general result of this nature has been proved by Kunita). The main result of Part IV is a limit theorem which asserts that the flow on $\mathbb{R}^d$ defined by the sequence of ordinary differential equations

$$d\xi_t^{(n)} = A_0(\xi_t^{(n)})\,dt + \sum_{i=1}^{n} A_i(\xi_t^{(n)})\,dw_i^{(n)}(t), \qquad t \in [0, T],$$

converges uniformly on compact subsets of $[0, T] \times \mathbb{R}^d$ to the stochastic flow of the Stratonovich equation

$$d\xi_t = A_0(\xi_t)\,dt + \sum_{i=1}^{n} A_i(\xi_t) \circ dw_i(t).$$

Here $w_i^{(n)}$ are a sequence of piecewise linearizations of w_i formed by linearly interpolating on a sequence of partitions of $[0, T]$ with mesh tending to zero. Malliavin first proves this for the torus, then extends the result to Euclidean space. A brief discussion of stochastic differential equations on manifolds is given, and their lifting to the diffeomorphism group in the spirit of Eells–Elworthy.

Part V returns to the problem of establishing probabilistically the regularity of the heat kernel corresponding to a degenerate differential operator of Hörmander type. It is shown that the map $w \mapsto x_t$ defined by equation (1) is smooth in the sense of Part I. The limit theorem in Part IV is used to compute the associated matrix σ_t. A proof is then given that (HC) implies condition (3). Curiously, Kusuoka and Stroock's proof of the central result (4) is *not* included in the book. The variational formulation of Bismut is briefly discussed and a summary of Malliavin's original approach is given.

In the last chapter of the book the integration theory associated with Wiener measure γ on the path space $P(M)$ of a Riemannian manifold is discussed. The goal of this active line of research is to carry over the analysis that has been so fully developed for the classical Wiener space into the nonlinear manifold setting. The stochastic development of *Eells–Elworthy* allows the Wiener measure on Euclidean space to be intrinsically transferred to $P(M)$. However, a major problem exists in that the underlying geometric structure (i.e., the Cameron–Martin space) is not preserved by this process. A breakthrough in this area was made by Driver [6] who, in 1992, proved an integration by parts formula for Wiener measure on $P(M)$. The book closes with a discussion of such integration by parts formulae and related results. The case where M is a Lie group, which had been addressed by Shigekawa prior to Driver's work, and the more general Riemannian manifold setting, are treated separately.

In conclusion, Malliavin's book is a good exposition of an interesting body of work. It is written in a lively and engaging style and includes enough background material to make it accessible to a (sophisticated) nonspecialist mathematical audience. It will be a valuable addition to your bookshelves!

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