*The Annals of Probability*
2002, Vol. 30, No. 1, 437–442

# ON THE NONUNIQUENESS OF THE INVARIANT PROBABILITY FOR I.I.D. RANDOM LOGISTIC MAPS

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Let $\{X_n\}_0^\infty$ be a Markov chain with values in $[0, 1]$ generated by the iteration of random logistic maps defined by $X_{n+1} = f_{C_{n+1}}(X_n) \equiv C_{n+1}X_n(1 - X_n)$, $n = 0, 1, 2, \dots,$ with $\{C_n\}_1^\infty$ being independent and identically distributed random variables with values in $[0, 4]$ and independent of $X_0$. This paper provides a class of examples where $C_i$ take only two values $\lambda$ and $\mu$ such that there exist two distinct invariant probability distributions $\pi_0$ and $\pi_1$ supported by the open interval $(0, 1)$. This settles a question raised by R. N. Bhattacharya.

**1. Introduction.** The logistic map $f_c(x) \equiv cx(1-x)$ maps the interval $[0, 1]$ into itself provided $0 \le c \le 4$. The dynamical system generated by $f_c(\cdot)$, that is, the iteration sequence $\{f_c^{(n)}(x) : n = 0, 1, 2, \dots\}$ defined by $f_c^{(0)}(x) \equiv x$, $f_c^{(n+1)}(x) = f_c(f_c^{(n)}(x))$ for $n = 0, 1, 2, \dots,$ is well studied in the literature. The bifurcation phenomenon as $c$ increases from 0 to 3.5699 is well known; see, for example, [7]. The study of the case when the parameter $c$ changes randomly at each step was initiated by Bhattacharya and Rao [4] and has had contributions from Bhattacharya and Majumdar [3], Bhattacharya and Waymire [5], Athreya and Dai [1] and Dai [6]. Since $f_c(0) = 0$ for all $c$, the delta measure $\delta_0$ at 0 is clearly an invariant probability for the $[0, 1]$-valued Markov chain defined by

$$X_{n+1} = C_{n+1}X_n(1 - X_n), \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $\{C_n\}_1^\infty$ is a sequence of independent and identically distributed random variables with values in $[0, 4]$ and independent of $X_0$ with values in $[0, 1]$. Athreya and Dai [1] showed that (a) if there exists an invariant probability measure $\pi$ for $\{X_n\}$ such that $\pi(0, 1) = 1$, then it is necessary that $E \ln C_1 > 0$ and $\int -\ln(1 - x)\pi(dx) = E \ln C_1$, and (b) if $E \ln C_1 > 0$ and $E|\ln(4 - C_1)| < \infty$, then there exists an invariant probability measure $\pi$ such that $\pi(0, 1) = 1$.

This paper addresses the problem of finding an example of nonuniqueness of the invariant measure $\pi$ such that $\pi(0, 1) = 1$.

Sufficient conditions for the uniqueness of such a nontrivial invariant $\pi$ have been given by Bhattacharya and Rao [4], Dai [6] and Bhattacharya and Waymire [5] (see also Bhattacharya and Majumdar [3]). Bhattacharya and Rao [4]

Received May 2000; revised May 2001.
[1]Supported in part by NSF Grant CCR-9610461.
*AMS 2000 subject classifications.* Primary 60J05, 92D25; secondary 60F05.
*Key words and phrases.* Logistic maps, invariant probability, uniqueness.

considered the case when $C_1$ takes only two values, say, $\mu$ and $\lambda$. Using the results of Dubins and Freedman [8] on the iteration of i.i.d. monotone maps on finite intervals satisfying the so-called splitting condition, they established the following:

(a) If $1 < \mu < \lambda \leq 2$, then there exists a unique $\pi$ such that $\pi(0, 1) = 1$ and then $\pi$ is nonatomic. If, in addition, $\lambda^{-2} - \lambda^{-3} < \mu^{-2} - \mu^{-3}$, then the support $J$ of $\pi$ is a Cantor set (i.e., a closed nowhere dense set with no isolated points) contained in $[p_\mu, p_\lambda]$, where $p_\theta = 1 - \theta^{-1}$. If $\lambda(2 - \mu) < \mu(\mu - 1)$, then $J$ has Lebesgue measure 0. If $(\lambda - 1)\mu^2 < 2\lambda^2(\mu - 1)$ and $\lambda(2 - \mu)^2 < \mu$, then $J = [p_\mu, p_\lambda]$.

(b) If $2 < \mu < \lambda < 1 + \sqrt{5}$ and $8 < \lambda(4 - \lambda)\mu$, then there is a unique invariant $\pi$ such that $\pi(0, 1) = 1$ and this $\pi$ is nonatomic with support concentrated in $[\frac{1}{2}, \frac{\lambda}{4}]$. Dai [6] showed that if there exist constants $1 < a < b < 4$, an interval $I \subset (1, 3)$ and a $\delta > 0$ such that $P(a \leq C_1 \leq b) = 1$ and all Borel sets $B$ in $R$, $P(C_1 \in B) \geq \delta m(B \cap I)$, where $m(\cdot)$ is the Lebesgue measure, then there is a unique nontrivial invariant probability. Bhattacharya and Waymire [5] extended Dai's result by modifying $I$ to require only that there is a $\gamma$ in the interior of $I$ such that $f_\gamma$ has an attractive periodic orbit of some period $2^n$, $n \geq 0$, and dropping $I \subset (1, 3)$ but retaining all other conditions. Bhattacharya [2] raised the question of nonuniqueness of the invariant probability. Bhattacharya and Waymire stated in [5] that "although we have no example for which there exists more than one invariant probability on $S = (0, 1)$ we believe that there are lots of $Q$'s for which there are more than one invariant probability." This belief is substantiated in this paper.

The construction is based on the following ideas. Let $\mu_0 = 3.67\ldots$ be the solution of the equation $x^3(4 - x) - 16 = 0$ that is in $(3, 4)$. [This $\mu_0$ and this equation will be used later, in the proof of (ii) of Proposition 2.] For $3 < \mu < \mu_0$, choosing $\lambda$ such that $\lambda^{-1} + \mu^{-1} = 1$ makes $f_\lambda$ map $\{a, b\}$ to $b$ and $f_\mu$ map $\{a, b\}$ to $a$, where $a = \lambda^{-1}$ and $b = \mu^{-1}$. This renders the measure $\pi$ defined by $\pi\{a\} = 1 - \eta$, $\pi\{b\} = \eta$ stationary if $P(C_1 = \lambda) = \eta$ and $P(C_1 = \mu) = 1 - \eta$. It turns out that the $f_\lambda$ and $f_\mu$ leave the interval $I = [1 - \mu/4, \mu/4]$ invariant. Since $f_\lambda$ is attractive and $f_\mu$ is repelling near $\{a, b\}$, by making $\eta$ small, the chain $\{X_n\}$ can be made to get away from $\{a, b\}$ whenever it gets close to it. This will entail that the occupation measures $\mu_{n,x}(A) \equiv \frac{1}{n}\sum_0^{n-1} P_x(X_j \in A)$ have a vague limit $\pi$ such that $\pi\{a, b\} = 0$ and $\pi(0, 1) = \pi(I) = 1$. A Foster–Liapounov-type argument is used to establish this.

## 2. The main result.

Let

$$b = \frac{1}{\mu}, \qquad a = 1 - b, \qquad \lambda = \frac{1}{a}, \tag{1}$$

with $3 < \mu < \mu_0$, where $\mu_0 = 3.67\ldots$ is the solution of the equation $x^3(4 - x) - 16 = 0$ that is in $(3, 4)$. Let $0 < \eta < 1$ and let $\{C_i\}_1^\infty$ be i.i.d. random variables with distribution

$$P(C_1 = \lambda) = \eta \quad \text{and} \quad P(C_1 = \mu) = 1 - \eta. \tag{2}$$

Let

$$f_c(x) \equiv cx(1-x), \qquad 0 \le x \le 1, \tag{3}$$

with $0 \le c \le 4$.

Let $\{X_n\}_0^\infty$ be a Markov chain with state space [0, 1] defined by the random iteration scheme

$$X_{n+1} \equiv f_{C_{n+1}}(X_n) \equiv C_{n+1}X_n(1-X_n). \tag{4}$$

Since $a + b = 1$, $f_\lambda\{a, b\} = \{b\}$ and $f_\mu\{a, b\} = \{a\}$. Thus, if

$$\pi_1\{a\} = 1 - \eta, \qquad \pi_1\{b\} = \eta, \tag{5}$$

then $P(X_0 \in \{a, b\}) = 1$ implies $X_1 \sim \pi_1$ and hence $\pi_1$ is a stationary distribution for $\{X_n\}$. The main result of this paper is that for $\eta > 0$ small there is another invariant measure $\pi_0$ such that $\pi_0(0, 1) = 1$.

THEOREM 1. *Let $b$, $a$, $\lambda$ be as in* (1), *$\eta$ as in* (2) *and $\{X_n\}_1^\infty$ as in* (4). *Then for $\eta > 0$ sufficiently small there exist two distinct probability measures $\pi_0$ and $\pi_1$ such that $\pi_0(0, 1) = 1 = \pi_1(0, 1)$ and $\pi_0 \ne \pi_1$.*

Since $\pi_1$ defined in (5) is invariant, it suffices to show that there is an invariant probability measure $\pi_0$ such that $\pi_0(0, 1) = 1$ and $\pi_0 \ne \pi_1$.

The proof of this is based on the following two propositions.

PROPOSITION 1. *Both $f_\lambda$ and $f_\mu$ leave $I = [1 - \mu/4, \mu/4]$ invariant.*

PROOF. For $x \in I$, $f_\lambda(x) < f_\mu(x) \le \mu/4$. Since $1/\mu + 1/\lambda = 1$, $\frac{1}{\mu}\frac{1}{\lambda} \le \frac{1}{4}$. So $\mu\lambda \ge 4$. Therefore $f_\mu(x) > f_\lambda(x) \ge f_\lambda(\mu/4) = \lambda(\mu/4)(1 - \mu/4) \ge 1 - \mu/4$. □

COROLLARY. *Let $\{C_i\}_1^\infty$ be i.i.d. random variables that satisfy* (2). *Then $I$ is a closed set for the Markov chain $\{X_n\}_0^\infty$ defined in* (4). *That is, $P_x(X_n \in I$ for $n \ge 1) = 1$ for all $x$ in $I$.*

PROOF. Since $P(C_i = \lambda \text{ or } \mu) = 1$ for all $i \ge 1$, the corollary follows from Proposition 1. □

We assume in the following that $3 < \mu < \mu_0$ is fixed.

PROPOSITION 2. *Let*

$$h(x) = \begin{cases} \ln \dfrac{1}{|x-a|}, & \text{if } 0 < |x-a| < \varepsilon_1, \\ \ln \dfrac{1}{|x-b|}, & \text{if } 0 < |x-b| < \varepsilon_1, \\ 0, & \text{otherwise}, \end{cases}$$

*where* $0 < \varepsilon_1 < (a-b)/2$ *is such that* $(a - 2\varepsilon_1, a + 2\varepsilon_1) \subset I = [1 - \mu/4, \mu/4]$ *and* $(b - 2\varepsilon_1, b + 2\varepsilon_1) \subset I$. *Then there exist* $0 < \varepsilon_3 < \varepsilon_1$, $\eta > 0$, $0 < \gamma$, $\delta < \infty$ *such that* $\varphi(x) \equiv E_x h(X_1) - h(x)$ *satisfies*:

(i) $\varphi(x) \le -\delta$ *for all* $x$ *such that either* $0 < |x-a| < \varepsilon_3$ *or* $0 < |x-b| < \varepsilon_3$;
(ii) $\varphi(x) \le \gamma$ *for all* $x \in I$.

PROOF. Since $f_\lambda(x) \equiv \lambda x(1-x)$ and $f_\mu(x) \equiv \mu x(1-x)$ are both continuous on [0, 1] and satisfy $f_\lambda(a) = f_\lambda(b) = b$ and $f_\mu(a) = f_\mu(b) = a$, there exists an $\varepsilon_2$, $0 < \varepsilon_2 < \varepsilon_1$, such that

$$|x-a| < \varepsilon_2 \quad \text{or} \quad |x-b| < \varepsilon_2 \Rightarrow |f_\lambda(x) - b| < \varepsilon_1 \quad \text{and} \quad |f_\mu(x) - a| < \varepsilon_1.$$

So, for $0 < |x-a| < \varepsilon_2$,

$$\begin{aligned} \varphi(x) = E_x h(X_1) - h(x) &= \eta \ln \frac{1}{|f_\lambda(x) - b|} + (1-\eta) \ln \frac{1}{|f_\mu(x) - a|} - \ln \frac{1}{|x-a|} \\ &= \eta \ln \frac{|x-a|}{|f_\lambda(x) - f_\lambda(a)|} + (1-\eta) \ln \frac{|x-a|}{|f_\mu(x) - f_\mu(a)|}. \end{aligned}$$

Thus

$$\lim_{x\to a} \varphi(x) = \eta \ln \frac{1}{|f'_\lambda(a)|} + (1-\eta) \ln \frac{1}{|f'_\mu(a)|}.$$

Now

$$|f'_\lambda(a)| = |f'_\lambda(b)| = \left| \frac{2a-1}{a} \right| > 0$$

and

$$|f'_\mu(a)| = |f'_\mu(b)| = \left| \frac{2a-1}{b} \right| = |\mu[2(1-b) - 1]| = \mu - 2 > 1.$$

Hence

$$\eta \ln \frac{1}{|f'_\lambda(a)|} + (1-\eta) \ln \frac{1}{|f'_\mu(a)|} = \eta \ln \frac{1}{|f'_\lambda(a)|} - (1-\eta) \ln(\mu - 2) < 0$$

for $\eta > 0$ and small. Since $\varphi(x) = \varphi(1-x)$ and $b = 1-a$, $\lim_{x\to b}\varphi(x) = \lim_{x\to a}\varphi(x)$. Thus there exist an $\eta > 0$, an $\varepsilon_3 > 0$ and a $\delta > 0$ such that $0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ and

$$\varphi(x) = E_x h(X_1) - h(x) \le -\gamma$$

for all $x$ such that $0 < |x - a| < \varepsilon_3$ or $0 < |x - b| < \varepsilon_3$, thus proving (i).

To show (ii), we claim first that

$$x \in I - \{a, b\} \Rightarrow f_\mu(x) \quad \text{and} \quad f_\lambda(x) \notin \{a, b\}.$$

To see this, note that $f_\lambda(x)$ is a quadratic and $f_\lambda(a) = f_\lambda(b) = b$ implies $f_\lambda(x) = b$ only for $x = a$ or $b$. Next, the equation $f_\lambda(x) = a$ has no solutions in $I$ since $f_\lambda(x) \le \lambda/4 < a$. Similarly, $f_\mu(x)$ is a quadratic and $f_\mu(a) = f_\mu(b) = a$ implies $f_\mu(x) = a$ only for $x = a$ or $b$. Let $g(x) = x^3(4 - x) - 16$. Then $g(3) > 0 > g(4)$, $g$ has a root $\mu_0 = 3.67\ldots$ in $(3, 4)$. Since $3 < \mu < \mu_0$, $g(\mu) > 0$. So $\mu^3(4 - \mu) > 16$, that is, $(\mu^2/4)(1 - \mu/4) > 1/\mu = b$. So, for $x \in I$, $f_\mu(x) \ge f_\mu(\mu/4) = \mu(\mu/4)(1 - \mu/4) > b$. Therefore the equation $f_\mu(x) = b$ has no solutions in $I$. Next, let $J \equiv \{x : x \in I,\ |x - a| \ge \varepsilon_3,\ |x - b| \ge \varepsilon_3\}$. Since $J$ is compact, $\inf_{x\in J} |f_\lambda(x) - a|$, $\inf_{x\in J} |f_\lambda(x) - b|$, $\inf_{x\in J} |f_\mu(x) - a|$, $\inf_{x\in J} |f_\mu(x) - b|$ are all strictly positive and hence $E_x h(X_1)$ and $h(x)$ are all bounded above on $J$. Thus $\varphi(x) = E_x h(X_1) - h(x)$ is bounded above on $J$ and hence by some $\gamma$ on $I$. This completes the proof of (ii) and hence Proposition 2. □

PROOF OF THEOREM 1. Consider the Markov chain $\{X_n\}_0^\infty$ defined in (4) with state space $I$. By Proposition 1, $x \in I \Rightarrow P_x(X_1 \in I) = 1$ and hence $P_x(X_n \in I \text{ for all } n \ge 1) = 1$. Also, if $g$ is bounded and continuous on $I$, then so is $E_x g(X_1) = Eg(C_1 x(1 - x))$ by the bounded convergence theorem. Thus $\{X_n\}_0^\infty$ is a Feller Markov chain.

Now consider the *occupation measures*

$$\mu_{n,x}(A) \equiv \frac{1}{n} \sum_{0}^{n-1} P_x(X_j \in A)$$

for $x \in I$ and $A$ a Borel subset of $I$. Any vague limit point $\nu$ of the probability measures $\{\mu_{n,x}\}$ is necessarily a probability measure on $I$ since $I$ is compact and invariant for $\{X_n\}$ since it is Feller.

Now we use Proposition 2 to show that there must exist at least one such vague limit $\nu$ such that $\nu(J) > 0$, where $J \equiv \{x : x \in I,\ |x - a| \ge \varepsilon_3,\ |x - b| \ge \varepsilon_3\}$. Indeed,

$$\begin{aligned} E_x h(X_n) - h(x) &= \sum_{1}^{n} E_x\big(h(X_j) - h(X_{j-1})\big) \\ &= \sum_{1}^{n} E_x\big(\varphi(X_{j-1})\big) \quad \text{(by the Markov property)} \\ &= \sum_{1}^{n} E_x\big(\varphi(X_{j-1}) : X_{j-1} \in J\big) + \sum_{1}^{n} E_x\big(\varphi(X_{j-1}) : X_{j-1} \notin J\big). \end{aligned}$$

But by Proposition 2 this yields, for $x$ in $I$,

$$\frac{1}{n}\big(E_x h(X_n) - h(x)\big) \le \gamma \mu_{n,x}(J) - \delta \mu_{n,x}(J^c) \le (\gamma + \delta)\mu_{n,x}(J) - \delta.$$

If $\nu(J) = 0$ for all vague limits $\nu$, then $\lim_n \mu_{n,x}(J) = 0$. Thus

$$\lim_n \frac{1}{n}\big(E_x h(X_n) - h(x)\big) \le -\delta < 0.$$

But the left-hand side equals

$$\lim_n \frac{1}{n} E_x h(X_n),$$

which is greater than or equal to 0, since $h(\cdot) \ge 0$. This contradiction shows that there exists a vague limit point $\nu$ of $\{\nu_{n,x}(\cdot)\}$ such that $\nu(J) > 0$ and hence an invariant probability measure $\pi_0$ for $\{X_n\}$ such that $\pi_0(I) = 1$ and $\pi_0(J) > 0$.

Since $\{a, b\} \subset I$ and $\pi_1\{a, b\} = 1$, it follows that $\pi_1(I) = 1$ and $\pi_1(J) = 0$. But $\pi_0(I) = 1$ and $\pi_0(J) > 0$. So $\pi_0 \ne \pi_1$ and $\pi_1(0, 1) = 1 = \pi_0(0, 1)$. □

**Acknowledgments.** We thank Professor R. N. Bhattacharya of Indiana University for posing the question of nonuniqueness to us and an anonymous referee for many good suggestions. We also thank Ms. Ruth DeBoer of the ISU Mathematics Department for typing this paper.

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