The Annals of Probability
2002, Vol. 30, No. 1, 323–363

BOUNDARY AND ENTROPY OF SPACE HOMOGENEOUS MARKOV CHAINS

BY VADIM A. KAIMANOVICH AND WOLFGANG WOESS

CNRS, Université de Rennes-1 and Technische Universität Graz

We study the Poisson boundary ($\equiv$ representation of bounded harmonic functions) of Markov operators on discrete state spaces that are invariant under the action of a transitive group of permutations. This automorphism group is locally compact, but not necessarily discrete or unimodular. The main technical tool is the entropy theory which we develop along the same lines as in the case of random walks on countable groups, while, however, the implementation is different and exploits discreteness of the state space on the one hand and the path space of the induced random walk on the nondiscrete group on the other. Various new examples are given as applications, including a description of the Poisson boundary for random walks on vertex-transitive graphs with infinitely many ends and on the Diestel–Leader graphs.

1. Introduction. The Poisson boundary of a Markov operator P is defined as the space of ergodic components of the time shift in the path space. Via the Poisson formula the space of bounded P-harmonic functions is isometric to the space of bounded measurable functions on the Poisson boundary. This characterizes the Poisson boundary up to a measure-theoretic isomorphism. It also has a topological description in terms of the Martin boundary, where it consists of the set of possible limits of the Markov chain at the boundary together with the family of corresponding harmonic hitting (limit) distributions; see, for example, [14]. However, we emphasize that the Poisson boundary is primarily a measure-theoretical notion and all the objects connected with the Poisson boundary are defined modulo subsets of measure 0 (see [24] for more details).

The principal purpose of this paper is to present methods for determining the Poisson boundary in the case when P is a (*space*) *homogeneous* Markov operator, that is, when there is a locally compact group G of permutations of X that acts transitively on X and leaves P invariant:

$$p(gx, gy) = p(x, y) \qquad \forall x, y \in X,\ g \in G.$$

(In this paper we always deal with Markov chains which are time homogeneous in the traditional sense; that is, their transition probabilities do not depend on time. "Homogeneous" will always mean space homogeneous.) This comprises the case of random walks on countable groups, where the Poisson boundary has been studied extensively by Kaimanovich and Vershik [29], Ledrappier [32] and

Received April 2000; revised January 2001.
AMS 2000 subject classifications. Primary 60J50; secondary 05C25, 22F30, 60B15, 60G50.
Key words and phrases. Markov chain, random walk, group, entropy, Poisson boundary.

Kaimanovich [28]; see also the survey [26]. However, there are also interesting cases that do not arise in this way, where the group G is nondiscrete and may even be a nonunimodular group; that is, its left and right Haar measures do not coincide.

The typical setting where our results apply is when X carries the structure of a connected, locally finite, infinite, vertex-transitive graph and P is adapted to the graph structure in the sense that the transitive group G that leaves P invariant is a subgroup of the graph automorphism group. Recall that the latter is the group of self-isometries of X with respect to the graph metric $d(\cdot, \cdot)$, where $d(x, y)$ is the minimal length (number of edges) of a path connecting x and y. (In this situation the Markov chain is usually called "random walk.") A basic assumption that we need here is finiteness of the first moment of P, that is, the number $\sum_y d(x, y)p(x, y)$.

The reader who is interested in concrete examples might first look at Section 6. Among other things, we show that for homogeneous random walks on graphs with infinitely many ends, the Poisson boundary coincides with the full space of ends with the exception of a "degenerate" (but interesting) treelike case, where the Poisson boundary may be trivial or the full space of ends according to the sign of the *modular drift* (see Section 5 for a definition). Analogous results hold for random walks on hyperbolic graphs, with the hyperbolic boundary in the place of the space of ends, but in this situation we have less knowledge of examples for the "degenerate" case.

Another example is the family of Diestel–Leader graphs, which are in some sense relatives of the amenable Baumslag–Solitar groups, but with nondiscrete automorphism groups. There, the Poisson boundary may have three different forms, again according to the sign of the modular drift.

Homogeneous Markov chains are intermediate between random walks on countable groups and random walks on general locally compact groups: although the state space X is countable, the Poisson boundary of the operator P coincides with the Poisson boundary of a certain induced random walk on the nondiscrete automorphism group G. The main technical tool used in the paper is the entropy theory which we develop along the same lines as in the case of random walks on countable groups (cf. [28] and [29]). However, the implementation is different: instead of the path space of the original Markov chain, we have to work with the path space of the induced random walk on G. On the other hand, the fact that G has a countable homogeneous space X still allows us to avoid technical problems arising in the entropy theory of random walks on general locally compact groups (cf. [24] and [26]).

A natural extension of the class of homogeneous Markov chains is that of *quasi-homogeneous chains* for which the transition probabilities are preserved by a group G acting on X with a finite number of orbits. All the results of our paper carry over to such chains as well (more generally, one may assume that the quotient chain on the space of orbits has a finite stationary measure). This can be done either directly (cf. [33] for the case of covering graphs) or by considering the induced chain on a single orbit (cf. [25]).

The plan of the paper is as follows. In Section 2 we discuss homogeneous Markov operators, their automorphism groups and the associated hypergroups. In Section 3 we present the basic features of the Poisson boundary, and in Section 4 one of the main tools for its study, namely, the entropy. The principal results are given in Section 5, concerning applications of the entropy theory (entropy and growth, rate of escape, modular drift and spectral radius) and, in particular, criteria for identifying the Poisson boundary. As mentioned, Section 6 contains many examples and applications as well as additional remarks and an outline of some further generalizations.

2. Homogeneous Markov operators and their symmetry groups.

2.1. *Permutation groups.* For a countable set X, denote by $\mathfrak{F}(X)$ the semigroup of all self-maps of X endowed with the topology of pointwise convergence, and by $\mathfrak{S}(X) \subset \mathfrak{F}(X)$ the group of all (not necessarily finitely supported!) permutations of X. Note that the group $\mathfrak{S}(X)$ is *not* closed in $\mathfrak{F}(X)$ [if a sequence $g_n \in \mathfrak{S}(X)$ converges to $\varphi \in \mathfrak{F}(X)$, then φ is obviously injective, but does not have to be surjective]. A closed subset $U \subset \mathfrak{F}(X)$ is compact iff all its orbits Ux, $x \in X$, are finite. Given a subgroup $G \subset \mathfrak{S}(X)$, denote by

$$G_x = \{g \in G : gx = x\}, \qquad x \in X,$$

its point stabilizers.

In this paper we shall only consider subgroups $G \subset \mathfrak{S}(X)$ such that

(2.1)
(i) G is transitive on X;
(ii) G is closed in the topology of pointwise convergence in $\mathfrak{F}(X)$;
(iii) all point stabilizers G_x, $x \in X$, act on G with finite orbits.

In particular, if a group G satisfies conditions (2.1), then its point stabilizers are compact, so that G is locally compact. A neighborhood base at the identity of G is given by the family of all pointwise stabilizers of finite sets, so that G is totally disconnected. We emphasize that for us the most interesting situation is when G is uncountable; otherwise, the point stabilizers are finite, and X is just the quotient of a countable group by a finite subgroup.

REMARK. A transitive subgroup of $\mathfrak{S}(X)$ whose point stabilizers have finite orbits need not be closed [for an example take a countable transitive subgroup with infinite point stabilizers in a group satisfying (2.1)]. Also, a closed locally compact transitive subgroup need not have compact point stabilizers. For example, there are groups acting on nonlocally finite trees such that the edge stabilizers are compact, but not the vertex stabilizers. As a matter of fact, any finitely generated group with infinitely many ends acts with finite edge stabilizers and with one or two orbits on a tree (see, e.g., [11]), but the vertex stabilizers will be finite only when the group is virtually free.

We fix a reference point $o \in X$ (its choice is irrelevant by the transitivity assumption) and denote by $K = G_o$ its stabilizer (another choice of o gives a conjugate of K). The group K is open and compact, so that we may normalize the left-invariant Haar measure m_G on G in such a way that $m_G(K) = 1$. In particular, the restriction of m_G to K is the normalized Haar measure m_K on K. Denote by $\widehat{m}_G$ the *involution* of m_G, that is, its image under the map $g \mapsto g^{-1}$, and by

$$\Delta(g) = \Delta_G(g) = \frac{dm_G}{d\widehat{m}_G}(g), \qquad g \in G,$$

the *modular function* of the group G.

REMARK. In some situations K is a maximal compact subgroup of G, but this is by no means the case in general. For example, if a countable group acts on itself by translations, then the point stabilizers are trivial, but the group may well have finite ($\equiv$ compact) subgroups.

For any two points $x, y \in X$, the subgroup G_x decomposes into a disjoint union of $|G_x y|$ left cosets of the subgroup $G_{x,y} = G_x \cap G_y$ (here $|A|$ denotes the cardinality of a finite set A). Therefore,

$$m_G(G_x) = m_G\big(\{g : gx = y\}\big) = |G_x y|\, m_G(G_{x,y}).$$

Comparing the measures m_G of the set $\{g : gx = y\}$ and its inverse $\{g : gy = x\}$, one arrives at the following formula for the modular function on G ([39] and [43]):

$$\Delta(g) = \Delta[x, gx] \qquad \forall x \in X,\ g \in G, \tag{2.2}$$

where

$$\Delta[x, y] = \frac{|G_x y|}{|G_y x|} = \frac{m_G(G_x)}{m_G(G_y)}, \qquad x, y \in X. \tag{2.3}$$

[We use the same notation Δ for both the modular function on G and the function (2.3) which should not lead to confusion; cf. the discussion of the modular function on double-coset hypergroups in Section 2.4 and the discussion of general multiplicative cocycles on G in Section 2.5.] One also immediately checks the following result.

PROPOSITION 2.4. *The function* (2.3) *is a G-invariant multiplicative cocycle on X, that is, $\Delta[x, y]\,\Delta[y, z] = \Delta[x, z]$ and $\Delta[gx, gy] = \Delta[x, y]$ for any $x, y, z \in X$ and $g \in G$.*

2.2. *Homogeneous Markov operators.* A Markov chain on a countable set X is determined by its *transition probabilities*

$$p(x, y) \geq 0, \qquad x, y \in X, \qquad \sum_y p(x, y) = 1,$$

or, equivalently, by the associated *Markov operator*

$$Pf(x) = \sum_y p(x, y) f(y)$$

on the space $\ell^\infty(X)$, whose adjoint acts on the space of measures θ on X by $\theta P(y) = \sum \theta(x) p(x, y)$. We shall always assume *irreducibility*; that is, for every $x, y \in X$, there is n such that $p^{(n)}(x, y) = \delta_x P^n(y) > 0$.

Given a Markov operator $P = (p(x, y))_{x,y \in X}$ and an initial distribution θ on X, we denote by $\mathbf{P}_\theta$ the corresponding probability measure in the space $\Omega_X = X^{\mathbb{Z}_+}$ of paths $\mathbf{x} = (x_0, x_1, \dots)$ of the associated Markov chain, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Thus, the projections x_n are X-valued random variables with $\mathbf{P}_\theta[x_0 = x] = \theta(x)$ and $\mathbf{P}_\theta[x_{n+1} = y | x_n = x] = p(x, y)$, where $x, y \in X$. As usual, when $\theta = \delta_x$ we write $\mathbf{P}_x$ instead of $\mathbf{P}_\theta$. The $\mathbf{P}_x$-distribution of x_n is the n-step transition probability $\pi_n^x = p^{(n)}(x, \cdot)$.

We denote by

$$\operatorname{Aut}(X, P) = \big\{ g \in \mathfrak{S}(X) : p(x, y) = p(gx, gy)\ \forall x, y \in X \big\}$$

the *group of symmetries* of a Markov operator P. In other words, $\operatorname{Aut}(X, P)$ consists of those elements from $\mathfrak{S}(X)$ whose action on the space $\ell^\infty(X)$ commutes with P.

LEMMA 2.5. *If the Markov operator P is irreducible, then the point stabilizers of $G = \operatorname{Aut}(X, P)$ have finite orbits.*

PROOF. Let $x, y \in X$. There is n such that $p^{(n)}(x, y) > 0$. We have $p^{(n)}(x, y) = p^{(n)}(x, gy)$ for any $g \in G_x$. Therefore, $|G_x y| \leq 1/p^{(n)}(x, y)$. $\square$

LEMMA 2.6. *If the Markov operator P is irreducible, then the group $G = \operatorname{Aut}(X, P)$ is closed.*

PROOF. Clearly, we only have to prove the closedness of all point stabilizers G_x. Let g_n be a pointwise convergent sequence in G_x; that is, there exists $\varphi \in \mathfrak{F}(x)$ such that $g_n y \to \varphi y$ for all $y \in X$. Then φ preserves the transition probabilities and is injective, so that we just have to check that φ is surjective. The latter follows from the fact that X splits into a disjoint union of *finite* orbits $\mathcal{O}_i$ of the group G_x. Since $G_x \mathcal{O}_i = \mathcal{O}_i$, we have that $\varphi \mathcal{O}_i \subset \mathcal{O}_i$ for any orbit $\mathcal{O}_i$. Injectivity of φ on the whole space X implies injectivity of its restriction to any $\mathcal{O}_i$. However, since $\mathcal{O}_i$ is finite, any self-injection of $\mathcal{O}_i$ must be a bijection. $\square$

A Markov operator P is called *homogeneous* if the group $\mathrm{Aut}(X, P)$ acts on X transitively. Then Lemmas 2.5 and 2.6 imply the following.

PROPOSITION 2.7. *If a Markov operator P on a countable set X is homogeneous and irreducible, then the group* $\mathrm{Aut}(X, P)$ *satisfies conditions* (2.1).

In the remainder of this article, we shall only consider operators satisfying the conditions of Proposition 2.7.

EXAMPLES 2.8. (i) A *graph structure* on X is determined by a symmetric subset $E \subset X \times X$. The latter is the set of edges of a graph with vertex set X. We write $\mathrm{Aut}(X, E) \subset \mathfrak{S}(X)$ for the group of isomorphisms of the graph (X, E). Suppose that (X, E) is locally finite; that is, for any vertex $x \in X$, its degree (i.e., the number of neighbors) $\deg(x)$ is finite. Then one can define the Markov operator P_E with the transition probabilities

$$p(x, y) = \begin{cases} 1/\deg(x), & (x, y) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The associated Markov chain on X is called the *simple random walk* on X. It is irreducible if and only if the graph is connected, and $\mathrm{Aut}(X, P_E) = \mathrm{Aut}(X, E)$.

(ii) Obviously, the graph structure E can be uniquely recovered from the operator P_E. However, not every Markov operator can be obtained in this way. Given a Markov operator $P\colon\ \ell^\infty(X) \hookleftarrow$, let $E_P = \{(x, y) : p(x, y) > 0 \text{ or } p(y, x) > 0\}$. Then $\mathrm{Aut}(X, P) \subset \mathrm{Aut}(X, E_P)$, but equality does not hold in general.

(iii) Let G be a countable group, and μ a probability measure on G. Then the Markov operator P_μ on G with transition probabilities $p(x, y) = \mu(x^{-1}y)$ is called the (right) random walk on G with law μ. Obviously, in this situation $p(gx, gy) = p(x, y)$ for any $g \in G$, so that $\mathrm{Aut}(G, P_\mu) \supset G$ and the operator P is homogeneous. It is also clear that $\mathrm{Aut}(G, P_\mu)$ contains the group $\mathrm{Aut}(G)_\mu$ of all automorphisms of G which preserve the measure μ. However, $\mathrm{Aut}(G, P_\mu)$ may be much bigger than the group generated by G and $\mathrm{Aut}(G)_\mu$. For an example take the simple random walk on the free group $\mathcal{F}_2$ with two generators; that is, the measure μ is equidistributed on the set of generators and their inverses. In this case $\mathrm{Aut}(G)_\mu$ consists just of the homomorphisms induced by the permutations of the set of generators; on the other hand, the group $\mathrm{Aut}(G, P_\mu)$ coincides with the group of automorphisms of the homogeneous tree of degree 4 (in particular, it is uncountable).

More specific examples will be studied in Section 6.

2.3. *The correspondence with random walks on G.* For what follows we do not really need to consider the full automorphism group of a homogeneous irreducible Markov operator, but any closed transitive subgroup that suits our purposes. Moreover, we may even take a more abstract point of view based on the following elementary observation.

PROPOSITION 2.9. *Let G be a totally disconnected, locally compact group with countable base of the topology, K a compact open subgroup, and $X = G/K$. Denote by $\Pi\colon\ G \to \mathfrak{S}(X)$ the group homomorphism defined as*

$$\Pi(g_1) \cdot g_2 K = g_1 g_2 K.$$

Then the group $\Pi(G) \subset \mathfrak{S}(X)$ satisfies conditions (2.1).

We shall always assume that the group G satisfies the conditions of Proposition 2.9 and will be interested in G-invariant Markov operators on the countable set $X = G/K$. Put $o = K$ (the image of the group identity under the natural projection $g \mapsto gK$) as the reference point in X. For any $x \in X$, choose $g_x \in G$ such that $x = g_x o$ and let

$$A_x = \{g \in G : go = x\} = g_x K.$$

As in Section 2.1, normalize the left Haar measure m_G by putting $m_G(K) = 1$ and denote by m_x the restrictions of the measure m_G onto the sets A_x. Then m_o coincides with the normalized Haar measure on the compact group $K = A_o$, so that it is bi-K-invariant. Therefore, the measures $m_x = g_x m_o$ are right-K-invariant probability measures; moreover, by the uniqueness of the Haar measure on K, the measures m_x can be characterized as the unique right-K-invariant probability measures on the sets A_x.

Recall that any probability measure μ on G determines the *right random walk* (G, μ) [cf. Example 2.8 (iii)]: the probability that the random walk goes from a point $g \in G$ to a measurable subset $A \subset G$ is $g\mu(A) = \mu(g^{-1}A)$. We denote by Ω_G the space of *increments* $\mathbf{g} = (g_1, g_2, \ldots) \in G^{\mathbb{N}}$ of the random walk (G, μ) endowed with the Bernoulli product measure $\mathbf{Q} = \mu^{\mathbb{N}}$.

PROPOSITION 2.10. *Let μ be a probability measure on G. Then the image of the right random walk (G, μ) under the projection $g \mapsto go$ is a Markov chain on X if and only if*

$$\mu(gK) = \mu(kgK) \qquad \forall g \in G,\ k \in K. \tag{2.11}$$

In this case the transition probabilities of the associated Markov operator $P = P_\mu$ on X are

$$p(go, ho) = \mu(g^{-1}hK). \tag{2.12}$$

PROOF. This is a special case of the general factorization theorem for Markov chains. Namely, if $P\colon L^\infty(S,m) \hookleftarrow$ is a Markov operator and α is a measurable partition of the state space (S,m), then P determines a Markov operator on the quotient space $(S,m)/\alpha$ if and only if P preserves the space of α-measurable functions considered as a subspace of $L^\infty(S,m)$. In our case the partition α consists of the sets A_x, and the latter condition means that

$$\mu(g_1^{-1}A_y) = \mu(g_2^{-1}A_y) \qquad \forall x, y \in X,\ g_1, g_2 \in A_x,$$

which is equivalent to (2.11) and readily implies (2.12). $\square$

COROLLARY 2.13. *If condition* (2.11) *is satisfied, then for any $x \in X$ the measure $\mathbf{P}_x$ on the path space Ω_X of the quotient Markov operator P is the image of the measure $\mathbf{Q}$ under the map*

$$\Phi\colon \mathbf{g} \mapsto \mathbf{x}, \qquad x_n = g_x g_1 g_2 \cdots g_n o. \tag{2.14}$$

Any homogeneous Markov operator P can be presented as a quotient of a certain random walk (G,μ). Moreover, the measure μ becomes uniquely determined if we subject it to a natural additional condition.

PROPOSITION 2.15. *The formulas*

$$d\mu(g) = p(o, go)\, dm_G(g) \tag{2.16}$$

and (2.12) *establish a one-to-one correspondence between G-invariant Markov operators $P = (p(x,y))_{x,y\in X}$ on X and bi-K-invariant probability measures μ on G. This correspondence is convex, and*

$$\mu_1 * \mu_2 \longleftrightarrow P_1 P_2,$$

where $*$ denotes the convolution over G, and P_1P_2 is the product of the Markov operators P_1 and P_2 corresponding to the measures μ_1 and μ_2, respectively.

PROOF. Formula (2.16) can be rewritten as

$$\mu = \sum_{x\in X} p(o,x) m_x. \tag{2.17}$$

The right-K-invariance of μ follows from the fact that all measures μ_x are right-K-invariant. As for the left-K-invariance,

$$k\mu = \sum_{x\in X} p(o,x) k m_x = \sum_{x\in X} p(o,x) m_{kx} = \sum_{x\in X} p(o,kx) m_{kx} = \mu$$

for any $k \in K$ by K-invariance of the operator P.

Conversely, if μ is a bi-K-invariant probability measure on G, then its restriction on any set A_x is proportional to m_x (because m_x is the unique right-K-invariant probability measure on A_x). The measure μ satisfies condition (2.11), and by (2.12) the transition probabilities of the corresponding quotient Markov operator are such that $p(o,x) = \mu(A_x)$, which means that applying formula (2.17) we recover the measure μ from the probabilities $p(o,x)$. Convexity of the correspondence $P \longleftrightarrow \mu$ is obvious.

Now let μ_1, μ_2 and μ be the bi-K-invariant measures on G corresponding to the G-invariant Markov operators P_1, P_2 and $P = P_1 P_2$, respectively. Since the projection $g \mapsto go$ maps the measure m_G onto the counting measure m_X on X, by formulas (2.16) and (2.17) we have, for any $x \in X$,

$$\begin{aligned}
\mu(A_x) = p(o,x) &= \sum_{y \in X} p_1(o,y) p_2(y,x) = \int p_1(o,y) p_2(y,x)\, dm_X(y) \\
&= \int_G p_1(o,go) p_2(go,x)\, dm_G(g) = \int_G \mu_2(g^{-1}A_x)\, d\mu_1(g) \\
&= \mu_1 * \mu_2(A_x).
\end{aligned}$$

Since the measures μ and $\mu_1 * \mu_2$ are both right-K-invariant, they are uniquely determined by their values on the sets A_x. Therefore, $\mu = \mu_1 * \mu_2$. $\square$

If μ is associated with P as in (2.16), then we denote by $\widehat{\mu}$ its involution, that is, the image of μ under the map $g \mapsto g^{-1}$. The probability measure $\widehat{\mu}$ is also bi-K-invariant and corresponds to a homogeneous Markov operator $\widehat{P}$ on X (the *involution* of P). The operator $\widehat{P}$ is the reversal of the operator P with respect to the stationary measure $\widehat{m}_X(x) = m_G(G_x)$. See Proposition 2.23 for a proof; also compare with [48] or [49], Lemma 3.25.

2.4. *The double-coset hypergroup.* The description of homogeneous Markov operators in terms of bi-K-invariant probability measures on G is not completely invariant as it depends on the choice of the reference point $o \in X$ (or, equivalently, of the compact subgroup $K = G_o$). The hypergroup framework provides a more invariant description. Moreover, this is also the most "economical" description as the associated hypergroup automatically subsumes all the symmetries of a homogeneous Markov operator.

Denote by $H = K \setminus G / K = \{KgK, g \in G\}$ the set of *double cosets* of the group K in G. For any double coset $\xi = KgK \in H$, there exists a unique bi-K-invariant probability measure m_ξ concentrated on ξ, which is the image of the product measure $m_K \otimes m_K$ under the map $(g_1, g_2) \mapsto (g_1 g g_2)$. Equivalently, the measure m_ξ is the normalized restriction of the left Haar measure m_G onto the set KgK. Therefore, the map

$$\mu \mapsto \bar{\mu} = \sum \mu(\xi)\xi$$

establishes an isomorphism between the space of finite signed bi-K-invariant measures on G and the space $\ell^1(H)$. Since the former space has a structure of a convolution algebra (the convolution of any two bi-K-invariant measures on G is obviously also bi-K-invariant), we obtain an algebra structure on $\ell^1(H)$. Actually, the latter algebra is the *hypergroup algebra* of a hypergroup on H.

Roughly speaking, a *hypergroup* is a locally compact space with a "product" which assigns to any two points a probability measure on that space (rather than a point as in the group case) which still allows one to define associative convolution of measures (see [21] and [4] for precise definitions). The set H is naturally given a structure of a hypergroup (called the *double-coset hypergroup*) by putting

$$\bar{\mu}_1 \star \bar{\mu}_2 = \overline{(\mu_1 * \mu_2)} \tag{2.18}$$

for any two $\bar{\mu}_1, \bar{\mu}_2 \in \ell^1(H)$. Here μ_1, μ_2 are the corresponding bi-K-invariant measures on G, and $\mu_1 * \mu_2$ is their convolution in G. In other words, the "product" of $\xi_1, \xi_2 \in H$ is the measure obtained by projecting onto H the group convolutions of the bi-K-invariant probability measures corresponding to ξ_1, ξ_2. The identity of the hypergroup H is the class $\varepsilon = K$, and the involution is $(KgK)^\wedge = Kg^{-1}K$.

The maps $KgK \longleftrightarrow Kgo \longleftrightarrow G(o, go)$ are easily seen to establish one-to-one correspondences between H and the sets $K\backslash X$ of K-orbits in X and $G\setminus(X \times X)$ of G-orbits in $X \times X$. The latter correspondence gives a more invariant description of the hypergroup H independent of the choice of the reference point o (or, equivalently, of the subgroup $K = G_o$). Namely, for any pair $(x, y) \in X \times X$, denote by

$$[x, y] = G(x, y) \in H \cong G \setminus (X \times X)$$

its G-orbit considered as an element of the hypergroup H. In these terms the identity in H is the diagonal orbit $[x, x]$, $x \in X$, and the involution is just $[x, y]^\wedge = [y, x]$,

Applying the projections

$$G \to X \to H, \qquad g \mapsto go \mapsto [o, go] \tag{2.19}$$

to the left Haar measure m_G gives the counting measure

$$m_X(x) = m_G\{g : go = x\} = m_G(G_o) = 1$$

on X and the measure

$$m_H[x, y] = \big|\{z : [x, z] = [x, y]\}\big| = |G_x y|, \tag{2.20}$$

on H, respectively. By definition, $m_H = \overline{m}_G$, so that by (2.18) $\theta \star m_H = m_H$ for any probability measure θ on H, and m_H is a *left Haar measure* on H (cf. [4] and [30]).

Doing the same with the right Haar measure $\widehat{m}_G = \Delta^{-1} m_G$, we obtain the measure

$$\widehat{m}_X(x) = \Delta^{-1}(g)\big|_{go=x} m_X(x) = m_G(G_x) \tag{2.21}$$

on X and the (right Haar) measure

$$\widehat{m}_H[x,y] = m_H[y,x] = |G_y x|$$

on H. Therefore, by (2.3),

$$\Delta[x,y] = \frac{m_H[x,y]}{\widehat{m}_H[x,y]} = \frac{\widehat{m}_X(x)}{\widehat{m}_X(y)} \tag{2.22}$$

is the modular function of the hypergroup H (since the cocycle Δ is G-invariant by Proposition 2.4, it may be considered as a function on H, also denoted by Δ). Note that, as it follows from (2.21), the modular function of H is the projection of the modular function of G under the map (2.19).

We may now reformulate Proposition 2.15 in the following way.

PROPOSITION 2.23. *The formula*

$$p(x,y) = \frac{\bar{\mu}[x,y]}{m_H[x,y]}$$

establishes a one-to-one correspondence between the set of probability measures $\bar{\mu}$ on the hypergroup H and the set of G-invariant Markov operators P on X. This correspondence is convex, and

$$\bar{\mu}_1 \star \bar{\mu}_2 \longleftrightarrow P_1 P_2,$$

where $\bar{\mu}_1 \star \bar{\mu}_2$ is the convolution in H, and $P_1 P_2$ is the product of Markov operators. Moreover, $\widehat{\bar{\mu}} \longleftrightarrow \widehat{P}$, where $\widehat{\bar{\mu}}$ is the involution in the space of probability measures on H, and $\widehat{P}$ is the reversal of the operator P with respect to the measure $\widehat{m}_X$.

PROOF. We only have to prove the claim concerning the involutions. All the rest follows from Proposition 2.15 in view of the above discussion. Since the Markov operator of the random walk (G,μ) acts on the right, it preserves the *right* Haar measure $\widehat{m}_G$. Applying the projection $g \mapsto go$, we obtain that *the measure $\widehat{m}_X$ on X is preserved by all G-invariant Markov operators on X*. (The latter fact is basically the content of the "mass-transport principle" of Benjamini, Lyons, Peres and Schramm [2].)

If P and $\widehat{P}$ are two G-invariant Markov operators on X such that the corresponding probability measures on H are in the involution, then their transition probabilities are related by the formula

$$m_H[x,y]p(x,y) = m_H[y,x]\widehat{p}(y,x),$$

whence, by (2.22),

$$\widehat{p}(y,x) = p(x,y)\frac{m_H[x,y]}{m_H[y,x]} = p(x,y)\Delta[x,y] = p(x,y)\frac{\widehat{m}_X(x)}{\widehat{m}_X(y)}. \qquad \square$$

2.5. *Stationary measures, reversibility and symmetry.* As we have proved in Proposition 2.23, the reversal of any G-invariant Markov operator on X with respect to the measure $\widehat{m}_X$ is also G-invariant. We shall now describe all measures on X with this property. More generally, we shall consider λ-*stationary* measures θ, that is, such that $\theta P = \lambda P$ for a certain eigenvalue $\lambda > 0$. Denote by $P^\star_\theta$ the *reversal* of the operator P with respect to the measure θ. Its transition probabilities are given by the formula

$$\lambda\theta(y)p^\star_\theta(y,x) = \theta(x)p(x,y). \tag{2.24}$$

If $P^\star_\theta = P$, then the operator P is called *reversible* with respect to the measure θ (in this case necessarily $\lambda = 1$).

PROPOSITION 2.25. *Let P be an irreducible G-invariant Markov operator on X and let θ be a λ-stationary measure of P. Then the operator $P^\star_\theta$ is G-invariant if and only if all translations $g\theta$, $g \in G$, of the measure θ are pairwise proportional, that is, if there exists a multiplicative character χ of the group G such that*

$$g\theta = \chi(g)\theta \qquad \forall g \in G. \tag{2.26}$$

PROOF. Let $P^\star_\theta$ be G-invariant. Take $x, y \in X$. Since P is irreducible, we may assume without loss of generality that $p(x,y) > 0$. Then, by (2.24),

$$\frac{\theta(gy)}{\theta(gx)} = \frac{\theta(y)}{\theta(x)} \quad \forall g \in G,$$

which yields (2.26). Conversely, by looking at formula (2.24) one immediately sees that (2.26) implies G-invariance of $P^\star_\theta$. □

COROLLARY 2.27. *A G-invariant Markov operator P on X is reversible with respect to some stationary measure if and only if there exists a character χ such that*

$$p(gx,x) = \chi(g)p(x,gx) \qquad \forall x \in X,\ g \in G.$$

Any character $\chi\colon G \to \mathbb{R}^*_+$ determines a unique (up to a multiplier) measure θ_χ on X satisfying (2.26) by the formula $\theta_\chi(g^{-1}o) = g\theta_\chi(o) = \chi(g)$. Since $\chi \equiv 1$ on the compact group K, the character χ determines a G-invariant cocycle on X and descends to H as

$$\chi[x,gx] = \chi(g) \qquad \forall x \in X,\ g \in G. \tag{2.28}$$

Then

$$\chi[x,y] = \frac{\theta_\chi(x)}{\theta_\chi(y)} \qquad \forall x, y \in X. \tag{2.29}$$

PROPOSITION 2.30. *Let P be a G-invariant Markov operator on X represented by a probability measure $\bar{\mu}$ on the hypergroup H, and let χ be a multiplicative character of G. Then the measure θ_χ is λ-stationary with respect to P with the (possibly infinite) eigenvalue*

$$\lambda = \lambda(P, \chi) = \sum_{\xi \in H} \frac{\chi(\xi)}{\Delta(\xi)} \bar{\mu}(\xi).$$

PROOF. By (2.29), for any $y \in X$,

$$\begin{aligned}\theta_\chi P(y) &= \sum_{x \in X} p(x, y)\theta_\chi(x) = \sum_{\xi \in H} \sum_{x : [x,y]=\xi} p(x, y)\theta_\chi(x) \\ &= \theta_\chi(y) \sum_{\xi \in H} \chi(\xi) \sum_{x : [x,y]=\xi} p(x, y).\end{aligned}$$

On the other hand, by formula (2.20) and Proposition 2.23,

$$\sum_{x : [x,y]=\xi} p(x, y) = m_H(\widehat{\xi}) \frac{\bar{\mu}(\xi)}{m_H(\xi)} = \frac{\bar{\mu}(\xi)}{\Delta(\xi)}. \qquad \square$$

Below we shall use the simplified notation $P_\chi^\star$ for the reversal of the operator P with respect to the measure θ_χ [which is well defined if $\lambda(P, \chi) < \infty$]. We shall omit the subscript for the trivial character $\mathbf{1}(g) \equiv 1$ which determines the counting measure m_X. Formula (2.24) in combination with (2.22) and Proposition 2.23 implies the following result.

PROPOSITION 2.31. *If $\lambda = \lambda(P, \chi) < \infty$, then the measures $\bar{\mu}, \bar{\mu}_\chi^\star$ on the hypergroup H representing the operators P and $P_\chi^\star$, respectively, are connected by the formula*

$$\bar{\mu}_\chi^\star(\widehat{\xi}) = \frac{1}{\lambda} \cdot \frac{\chi(\xi)}{\Delta(\xi)} \cdot \bar{\mu}(\xi) \qquad \forall \xi \in H.$$

COROLLARY 2.32. *If $P = P_\chi^\star$, then*

$$\bar{\mu}(\widehat{\xi}) = \frac{\chi(\xi)}{\Delta(\xi)} \cdot \bar{\mu}(\xi) \qquad \forall \xi \in H.$$

REMARKS.

1. As it follows from (2.21), the measure $\widehat{m}_X$ on X corresponds to the modular character Δ. Proposition 2.30 shows the special role of the modular character in our context: $\lambda(P, \Delta) = 1$ and $P_\Delta^\star = \widehat{P}$ for all G-invariant operators P. On the other hand, $\lambda(P, \chi)$ a priori may take an arbitrary positive (and even infinite) value for any other character χ (including the trivial character $\mathbf{1}$, if $\Delta \neq \mathbf{1}$).

2. A Markov operator P on X is *involutive* if $\widehat{P} = P$ and *symmetric* if $p(x, y) = p(y, x)$ for all $x, y \in X$. Equivalently, the former means that P is reversible with respect to the stationary measure $\widehat{m}_X$, and the latter means that the counting measure m_X is P-stationary [i.e., $\lambda(P, \mathbf{1}) = 1$], and P is reversible with respect to this measure (i.e, $P = P^\star$). An example of a symmetric homogeneous Markov operator is provided by the simple random walk on a homogeneous graph. The above discussion shows that the operators $\widehat{P} = P^\star_\Delta$ and $P^\star$ are always different in the nonunimodular case (i.e., when $\Delta \neq \mathbf{1}$). Therefore, if $\Delta \neq \mathbf{1}$, then $\widehat{P} \neq P$ for any symmetric operator, and $P^\star \neq P$ for any involutive operator.
3. Corollary 2.32 implies that an operator P is reversible with respect to the measure θ_χ if and only if its involution $\widehat{P}$ is reversible with respect to the measure $\theta_{\widehat{\chi}}$, where $\widehat{\chi} = \Delta^2/\chi$.

3. The Poisson boundary. *From now on we shall fix an irreducible G-invariant Markov operator P on the countable set $X = G/K$, where G is a totally disconnected, second countable, locally compact group and K is its compact open subgroup. Fix $o = K$ as a reference point in X. By μ (resp., $\bar{\mu}$) we denote the bi-K-invariant probability measure on G (resp., the probability measure on the hypergroup $H = K\backslash G/K$) associated with the operator P by Proposition* 2.15 (*resp*., *Proposition* 2.23).

3.1. *Measures in the path space and the Poisson boundary.* Having chosen $o \in X$, we put $\mathbf{P} = \mathbf{P}_o$ and $\pi_n = \pi_n^o$. The measure $\mathbf{P}$ is concentrated on the space $\Omega_{X,o}$ of all sample paths starting from o.

Recall that the *Poisson boundary* Γ of the operator P is defined as the space of ergodic components of the time shift in the path space. Let $\mathbf{bnd}$ be the associated projection from Ω_X to Γ. The image $\mathbf{bnd}\,\mathbf{P}_x = \nu_x$ is the *harmonic measure* corresponding to the starting point x. For an arbitrary initial distribution θ, its harmonic measure is $\nu_\theta = \mathbf{bnd}\,\mathbf{P}_\theta = \sum \theta(x)\nu_x$. The harmonic measures satisfy the stationarity condition

$$\nu_x = \sum_y p(x, y)\nu_y \qquad \forall x \in X.$$

In view of the irreducibility assumption, this implies that all harmonic measures ν_x are equivalent to the measure $\nu = \nu_o$.

In our situation $g\mathbf{P}_\theta = \mathbf{P}_{g\theta}$ for all $g \in G$ and any initial distribution θ, where $g\mathbf{P}_\theta$ denotes the image of the measure $\mathbf{P}_\theta$ under the coordinate-wise action of G on the path space Ω_X. Since the action of G on the path space commutes with the time shift, it descends from Ω_X to an action on Γ which preserves the type of the harmonic measure ν. Since G acts on Γ, we can convolve measures on G with measures on Γ. In particular, writing $g\nu_x = \delta_g * \nu_x$, we have $g\nu_x = \nu_{gx}$ for any $g \in G$, $x \in X$.

A function $f \in \ell^\infty(X)$ is called *harmonic* if $Pf = f$. The *Poisson formula*

$$f(x) = \langle F, \nu_x \rangle = \int_\Gamma F(\gamma)\, d\nu_x(\gamma)$$

establishes an isometry between the space $H^\infty(X, P) \subset \ell^\infty(X)$ of bounded harmonic functions f and the space $L^\infty(\Gamma, \nu)$ of bounded measurable functions F on the Poisson boundary.

PROPOSITION 3.1. *The Poisson boundaries of the operator P and of the random walk (G, μ) are isomorphic.*

PROOF. In view of Proposition 2.15, the formula $f(go) = \widetilde{f}(g)$ establishes a one-to-one correspondence between the space of bounded harmonic functions f of the operator P and the space of bounded harmonic functions $\widetilde{f}$ of the operator P_μ of the right random walk (G, μ). The latter space is isometric to the Poisson boundary $(\widetilde{\Gamma}, \widetilde{\nu})$ of the random walk (G, μ) via the Poisson formula $\widetilde{f}(g) = \langle \widetilde{F}, g\widetilde{\nu} \rangle$. Since μ is bi-K-invariant, the measure $\widetilde{\nu}$ is K-invariant, so that if we put $\nu'_{go} = g\widetilde{\nu}$, then the space $\widetilde{\Gamma}$ equipped with the family of measures ν'_x, $x \in G$, gives a Poisson representation of the space of bounded P-harmonic functions on X. Therefore, $\widetilde{\Gamma}$ must be isomorphic (as a measure G-space equipped with the family of harmonic measures) to the Poisson boundary Γ of the operator P. $\square$

REMARK. The measure $\bar{\mu}$ (see Proposition 2.23) determines a random walk on the hypergroup H with the transition probabilities $\overline{p}(\xi_1, \xi_2) = \xi_1 \star \bar{\mu}(\xi_2)$. The bounded harmonic functions of this random walk are in one-to-one correspondence with bi-K-invariant bounded harmonic functions on G, that is, via the Poisson formula, with K-invariant bounded functions on the Poisson boundary $(\widetilde{\Gamma}, \widetilde{\nu})$. Therefore, the Poisson boundary of the random walk on H coincides with the space of ergodic components of the K-action on $\widetilde{\Gamma}$. We shall return to the characterization of the Poisson boundary of the hypergroup H elsewhere.

Since triviality of the Poisson boundary of the random walk (G, μ) implies amenability of G [16], we have the following result.

COROLLARY 3.2. *If the group G is nonamenable, then the Poisson boundary of the operator P is necessarily nontrivial.*

REMARK. By reproducing the arguments from [25], one can show that either the Poisson boundary is trivial or the harmonic measure ν is purely nonatomic.

3.2. *The tail boundary.* Another measure-theoretic boundary associated with a Markov operator is the *tail boundary*. It is obtained by taking the quotient of the path space with respect to the *tail partition* α^∞ which is the limit of the decreasing sequence of the coordinate partitions α_n^∞, where two paths $\mathbf{x}$ and $\mathbf{x}'$ belong to the same element of α_n^∞ if and only if $x_i = x_i'$ for all $i \geq n$. Denote by d the *period* of the operator P, that is, the greatest common divisor of the set $\{k \in \mathbb{Z}_+ : p^{(k)}(o,o) > 0\}$ (it is finite by the irreducibility assumption), and let $o \in D_0, D_1, \dots, D_{d-1}$ be the *periodic classes*; that is, if $p(x,y) > 0$ and $x \in D_i$, then necessarily $y \in D_{i+1}$ (addition mod d). Set $\varkappa(x) = i$ if $x \in D_i$, $i \in \mathbb{Z}_d$. Take a reference probability distribution θ on X equivalent to the counting measure.

THEOREM 3.3. *The tail boundary of an irreducible homogeneous Markov operator is* $\mathbf{P}_\theta$-mod 0 *isomorphic to the product of* $\mathbb{Z}_d$ *and of the Poisson boundary* Γ, *and it is the image of the path space under the map* $\mathbf{x} \mapsto (\mathbf{bnd}\,\mathbf{x}, \varkappa(x_0))$.

PROOF. Let us first show that for $d = 1$ the tail and the Poisson boundaries coincide. Indeed, $d = 1$ means that there exists an $N > 0$ such that $p_N(o,o)$ and $p_{N+1}(o,o)$ are both positive. Denote by δ their minimum. Then, for any initial distribution θ and any $n > N$,

$$\|\theta P^n - \theta P^{n+1}\| \leq \|\theta P^N - \theta P^{N+1}\| \leq 2 - 2\delta,$$

which by the corresponding 0–2 law of Derriennic [8] (see also [24]) implies coincidence of the tail and the Poisson boundaries.

Now let $d > 1$. By the definition of the period, $\varkappa(x_n) = \varkappa(x_0) + n \bmod d$ for almost every sample path $\mathbf{x}$, so that the map $\mathbf{x} \mapsto \varkappa(x_0)$ is measurable with respect to the tail partition α^∞. On the other hand, the preceding argument applied to the power P^d shows that the tail and the Poisson boundaries coincide for the initial distribution concentrated on a single periodicity class. $\square$

COROLLARY 3.4. *The tail and the Poisson boundaries coincide* $\mathbf{P}_x$-mod 0 *for any starting point* $x \in X$.

4. The entropy. The coincidence of the tail and the Poisson boundaries and the fact that the former one is obtained by taking the quotient of the path space with respect to a decreasing sequence of coordinate partitions allow one to use the *entropy theory of measurable partitions* due to Rokhlin [36] for studying the Poisson boundary.

4.1. *The asymptotic entropy.* Recall that the *entropy* of a discrete probability distribution $p = (p_i)$ is defined as

$$H(p) = -\sum p_i \log p_i.$$

We shall say that a probability measure Λ on Ω_X has *asymptotic entropy* $\mathbf{h}(\Lambda)$ [27] if it has the following Shannon–Breiman–McMillan type of *equidistribution property*:

$$-\frac{1}{n}\log\lambda_n(x_n)\to\mathbf{h}(\Lambda) \tag{4.1}$$

for Λ-almost every sequence $\mathbf{x}=\{x_n\}\in\Omega_X$ and in the space $L^1(\Omega_X,\Lambda)$, where λ_n is the distribution of the nth projection x_n with respect to Λ.

THEOREM 4.2. *Let* $P\colon \ell^\infty(X)\hookleftarrow$ *be a homogeneous Markov operator. If the entropy of its one-step transition probabilities* $H(\pi_1)$ *is finite, then the asymptotic entropy* $\mathbf{h}(\mathbf{P})$ *of the measure* $\mathbf{P}$ *exists.*

PROOF. Recall the space $(\Omega_G,\mathbf{Q})$ of "increments" defined in Section 2.3. It will be more convenient to deal with $(\Omega_G,\mathbf{Q})$ instead of the path space $(\Omega_{X,o},\mathbf{P})$ because the former one is endowed with the measure-preserving action of the Bernoulli shift T.

Given a measurable partition α of the path space $(\Omega_{X,o},\mathbf{P})$, denote by $\overline{\alpha}$ the partition of the space $(\Omega_G,\mathbf{Q})$ consisting of the preimages of the elements of α with respect to the mapping Φ of (2.14). Then the entropies and the conditional entropies of all partitions of the space $(\Omega_{X,o},\mathbf{P})$ coincide with the entropies and the conditional entropies of the corresponding preimage partitions of the space $(\Omega_G,\mathbf{Q})$.

Define on the space $(\Omega_G,\mathbf{Q})$ the sequence of nonnegative functions

$$\varphi_n(\mathbf{g})=-\log p^{(n)}(o,g_1g_2\cdots g_no).$$

Obviously, for any two elements $g,h\in G$ and for any two integers n,m,

$$p^{(n+m)}(o,gho)\ge p^{(n)}(o,go)p^{(m)}(go,gho)=p^{(n)}(o,go)p^{(m)}(o,ho).$$

Therefore, the sequence φ_n is subadditive with respect to the transformation T:

$$\varphi_{n+m}(\mathbf{g})\le\varphi_n(\mathbf{g})+\varphi_m(T^n\mathbf{g})\qquad\forall n,m\ge 0.$$

Moreover,

$$\int\varphi_1(\mathbf{g})\,d\mathbf{Q}(\mathbf{g})=H(\pi_1)<\infty$$

by the assumption of the theorem. Thus, we can apply to the sequence φ_m the subadditive ergodic theorem of Kingman [31] (see also [9]), which immediately yields the claim. $\square$

We shall call the asymptotic entropy $\mathbf{h}(\mathbf{P})$ the *entropy of the operator* P and denote it by $\mathfrak{h}=\mathfrak{h}(P)$.

COROLLARY 4.3. *The entropy $\mathfrak{h}$ is the linear rate of growth of the entropies of the n-step transition probabilities of the operator P*:

$$\mathfrak{h} = \lim_{n\to\infty} \frac{H(\pi_n)}{n}.$$

4.2. *Boundary triviality.* For $m \geq k$, denote by α_k^m the partition of the path space $(\Omega_{X,o}, \mathbf{P})$ determined by the positions of the Markov chain at times k, $k+1, \dots, m$ (i.e., two sample paths $\mathbf{x}$, $\mathbf{x}'$ belong to the same class of α_k^m if and only if $x_i = x_i'$ for all $i = k, k+1, \dots, m$).

LEMMA 4.4. *For any $k \geq 1$,*

$$H(\alpha_1^k) = kH(\pi_1).$$

PROOF. The entropy $H(\alpha_1^k)$ coincides with the entropy $H(\overline{\alpha}_1^k)$ of the corresponding preimage partition of the space $(\Omega_G, \mathbf{Q})$. Therefore,

$$\begin{aligned} H(\overline{\alpha}_1^k) &= -\int \big[\log p(o, g_1 o) + \log p(g_1 o, g_1 g_2 o) \\ &\qquad + \cdots + \log p(g_1 g_2 \dots g_{k-1} o, g_1 g_2 \dots g_{k-1} g_k o)\big]\, d\mathbf{Q}(\mathbf{g}) \\ &= -k \int \log p(o, g_1 o)\, d\mathbf{Q}(\mathbf{g}) = kH(\pi_1). \end{aligned}$$ □

LEMMA 4.5. *The conditional entropy of the partition α_1^k, $k \geq 1$, with respect to the tail partition α^∞ is*

$$H(\alpha_1^k | \alpha^\infty) = kH(\alpha_1^1 | \alpha^\infty) = k[H(\pi_1) - \mathfrak{h}].$$

PROOF. We shall use the fact that the tail partition α^∞ is the decreasing limit of the coordinate partitions α_n^∞. By the Markov property, the conditional measures on the elements of the partition α_1^k with respect to the partition α_n^∞ for $n > k$ are the same as the conditional measures with respect to the partition α_n^n. Therefore, the value of this conditional measure on the element of the partition α_1^k containing a sample path $\mathbf{x}$ is

$$\frac{p(o, x_1) p(x_1, x_2) \cdots p(x_{k-1}, x_k) p^{(n-k)}(x_k, x_n)}{p^{(n)}(o, x_n)}.$$

Integrating the logarithms of these conditional probabilities, we get that

$$H(\alpha_1^k | \alpha_n^\infty) = kH(\pi_1) + H(\pi_{n-k}) - H(\pi_n), \qquad n > k. \tag{4.6}$$

As α_n^∞ is a decreasing sequence of measurable partitions, there exists a limit of (4.6) as n tends to ∞, so that the difference $H(\pi_n) - H(\pi_{n-k})$ must converge. By Corollary 4.3, the limit of this difference is then $k\mathfrak{h}$. □

THEOREM 4.7. *Suppose that the entropy $H(\pi_1)$ of the one-step transition probabilities of the irreducible homogeneous Markov operator P is finite. Then its Poisson boundary is trivial if and only if* $\mathfrak{h}(P) = 0$.

PROOF. If $\mathfrak{h}(P) = 0$, then, by Lemmas 4.4 and 4.5, the tail partition α^∞ is independent of all coordinate partitions α_1^k, which by the Kolmogorov 0–1 law is only possible if α^∞ is trivial. Conversely, if α^∞ is trivial, then $H(\alpha_1^k|\alpha^\infty) = H(\alpha_1^k) = kH(\pi)$, whence $\mathfrak{h}(P) = 0$. Now, by Corollary 3.4, triviality of the Poisson boundary of the operator P is equivalent to triviality of the tail partition α^∞ of the space $(\Omega_{X,o}, \mathbf{P})$. $\square$

Theorem 4.2 now implies the following result.

COROLLARY 4.8. *The Poisson boundary is trivial if and only if there exist* $\varepsilon > 0$ *and a sequence of sets* A_n *such that* $\pi_n(A_n) > \varepsilon$ *and* $\log|A_n| = o(n)$.

4.3. *Conditional chains.* Almost every point γ of the Poisson boundary determines the *conditional chain* on X whose transition probabilities (no longer G-invariant) are

$$p^\gamma(x, y) = p(x, y)\frac{d\nu_y}{d\nu_x}(\gamma). \tag{4.9}$$

By $\mathbf{P}^\gamma$ we denote the corresponding probability measure in the path space $\Omega_{X,o}$.

Given a G-invariant partition ξ of the Poisson boundary Γ, denote by Γ_ξ the associated quotient space, and by ν_x^ξ, $x \in X$ (resp., $\nu^\xi = \nu_o^\xi$), the images of the harmonic measures ν_x (resp., $\nu = \nu_o$). By α_ξ^∞ we denote the corresponding partition of the path space $(\Omega_X, \mathbf{P})$, and by $\mathbf{bnd}_\xi$ the projection from the path space to Γ_ξ, so that $\nu_x^\xi = \mathbf{bnd}_\xi \mathbf{P}_x$, $x \in X$. Since the partition ξ is G-invariant, the action of G descends from Γ to Γ_ξ, and $g\nu_x^\xi = \nu_{gx}^\xi$ for any $g \in G$ and $x \in X$. In terms of the random walk on the group G determined by the measure μ, we may say that the space (Γ_ξ, ν_o^ξ) is a *μ-boundary* (see [16]).

The transition probabilities of the conditional chains determined by the points from Γ_ξ are

$$p^{\gamma_\xi}(x, y) = p(x, y)\frac{d\nu_y^\xi}{d\nu_x^\xi}(\gamma_\xi) \tag{4.10}$$

(cf. [28], Theorem 3.3). Denote by $\mathbf{P}^{\gamma_\xi}$ the corresponding probability measures in the space Ω_X. In other words, $\mathbf{P}^{\gamma_\xi}$ are the conditional measures of the measure $\mathbf{P}$ with respect to the partition α_ξ^∞.

4.4. *The asymptotic entropy of conditional chains.*

LEMMA 4.11. *For any $k \ge 1$,*

$$H(\alpha_1^k|\alpha_\xi^\infty) = kH(\alpha_1|\alpha_\xi^\infty) = k\left[H(\pi_1) - \int \log \frac{d\nu_{x_1}^\xi}{d\nu^\xi}(\mathbf{bnd}_\xi \mathbf{x})\, d\mathbf{P}(\mathbf{x})\right].$$

PROOF. Given a sample path $\mathbf{x} \in \Omega_{X,o}$, the conditional probability of the element of the partition α_1^k containing $\mathbf{x}$ with respect to the partition α_ξ^∞ is

$$p(o, x_1)p(x_1, x_2)\cdots p(x_{k-1}, x_k)\frac{d\nu_{x_k}^\xi}{d\nu^\xi}(\mathbf{bnd}_\xi \mathbf{x}),$$

whence, integrating, we obtain

$$H(\alpha_1^k|\alpha_\xi^\infty) = kH(\pi_1) - \int \log \frac{d\nu_{x_k}^\xi}{d\nu_\xi}(\mathbf{bnd}_\xi \mathbf{x})\, d\mathbf{P}(\mathbf{x}).$$

Passing to the space $(\Omega_G, \mathbf{Q})$, the last term on the right-hand side can be rewritten as

$$\int \log \frac{dg_1 g_2 \cdots g_k \nu^\xi}{d\nu^\xi}\big[\mathbf{bnd}_\xi\big(\Phi(\mathbf{g})\big)\big]\, d\mathbf{Q}(\mathbf{g}). \tag{4.12}$$

Since the Bernoulli shift T preserves the measure $\mathbf{Q}$ and

$$g_1 \mathbf{bnd}_\xi\big(\Phi(T\mathbf{g})\big) = \mathbf{bnd}_\xi\big(\Phi(\mathbf{g})\big), \tag{4.13}$$

telescoping (4.12) we get the claim. □

LEMMA 4.14. *Let ξ and ξ' be two G-invariant measurable partitions of the Poisson boundary (Γ, ν) such that ξ' is a refinement of ξ. Then $H(\alpha_1^1|\alpha_\xi^\infty) \ge H(\alpha_1^1|\alpha_{\xi'}^\infty)$, and equality holds if and only if $\xi = \xi'$.*

PROOF. Obviously, if ξ' is a refinement of ξ, then $\alpha_{\xi'}^\infty$ is a refinement of α_ξ^∞, so that the inequality follows from the general properties of the conditional entropy. If $H(\alpha_1^1|\alpha_\xi^\infty) = H(\alpha_1^1|\alpha_{\xi'}^\infty)$, then, by Lemma 4.11, $H(\alpha_1^k|\alpha_\xi^\infty) = H(\alpha_1^k|\alpha_{\xi'}^\infty)$ for any $k \ge 1$, which by the general properties of the conditional entropy means that for ν-almost every point $\gamma \in \Gamma$ the measures $\mathbf{P}^{\gamma_\xi}$ and $\mathbf{P}^{\gamma_{\xi'}}$ are the same, which is only possible when $\xi = \xi'$. □

THEOREM 4.15. *Let ξ be a measurable G-invariant partition of the Poisson boundary (Γ, ν). Then, for ν_ξ-almost every point $\gamma_\xi \in \Gamma_\xi$, the asymptotic entropy (in the sense of Definition 4.1) of the conditional measure $\mathbf{P}^{\gamma_\xi}$ exists and is equal to*

$$\mathbf{h}(\mathbf{P}^{\gamma_\xi}) = H(\alpha_1^1|\alpha_\xi^\infty) - H(\alpha_1^1|\alpha^\infty).$$

PROOF. We have to check that, for ν_ξ-almost every point $\gamma_\xi \in \Gamma_\xi$,

$$-\frac{1}{n}\log \pi_n^{\gamma_\xi}(x_n) \to H(\alpha_1^1|\alpha_\xi^\infty) - H(\alpha_1^1|\alpha^\infty)$$

for $\mathbf{P}^{\gamma_\xi}$-almost every sample path $\mathbf{x} = \{x_n\}$ and in the space $L^1(\Omega_{X,o}, \mathbf{P}^{\gamma_\xi})$, where $\pi_n^{\gamma_\xi}$ are the one-dimensional distributions of the measure $\mathbf{P}^{\gamma_\xi}$. Since the measures $\mathbf{P}^{\gamma_\xi}$ are conditional measures of the measure $\mathbf{P}$, it amounts to proving that

$$-\frac{1}{n}\log \pi_n^{\mathbf{bnd}_\xi \mathbf{x}}(x_n) \to H(\alpha_1^1|\alpha_\xi^\infty) - H(\alpha_1^1|\alpha^\infty)$$

$\mathbf{P}$-almost everywhere and in the space $L^1(\Omega_{X,o}, \mathbf{P})$. Passing to the space $(\Omega_G, \mathbf{Q})$, we may rewrite formula (4.10) as

$$\pi_n^{\mathbf{bnd}_\xi \mathbf{x}}(g_1 g_2 \cdots g_n o) = \pi_n(g_1 g_2 \cdots g_n o)\frac{d g_1 g_2 \cdots g_n \nu^\xi}{d\nu^\xi}\big[\mathbf{bnd}_\xi(\Phi(\mathbf{g}))\big],$$

whence we get the claim by applying Theorem 4.2, Lemma 4.11 and the Birkhoff ergodic theorem for the Bernoulli shift T. □

Now, combining Lemma 4.14 with Theorem 4.15, we get the following generalization of Theorem 4.7.

THEOREM 4.16. *A μ-boundary (Γ_ξ, ν_ξ) is the Poisson boundary if and only if the asymptotic entropy $\mathbf{h}(\mathbf{P}^{\gamma_\xi})$ of almost all conditional measures of the measure $\mathbf{P}$ with respect to Γ_ξ vanishes.*

COROLLARY 4.17. *A μ-boundary (Γ_ξ, ν_ξ) is the Poisson boundary if and only if, for ν_ξ-almost every point $\gamma_\xi \in \Gamma_\xi$, there exist $\varepsilon > 0$ and a sequence of sets $A_n = A_n(\gamma_\xi) \subset X$ such that $\log|A_n| = o(n)$ and $\pi_n^{\gamma_\xi}(A_n) > \varepsilon$ for all sufficiently large n.*

5. Applications of the entropy theory.

5.1. *Entropy and growth.* Throughout this section we shall assume that the space X is endowed with a G-invariant connected graph structure and that the number of neighbors (which is the same for any vertex) is finite. Denote by $d(\cdot,\cdot)$ the resulting G-invariant graph metric on X, and by $|[x,y]| = d(x,y)$ its projection onto the hypergroup H. Put $|x| = d(o,x)$. By

$$B(x,n) = \{y \in X : d(x,y) \le n\}$$

we denote the balls of the metric d. Note that, for every $x \in X$,

$$|B(x,n)| = |B(o,n)| = m_G\big(\{g \in G : |go| \le n\}\big)$$

and that $B(o, m+n) = \bigcup_{x \in B(o,m)} B(x, n)$. Therefore, $|B(o, m+n)| \le |B(o, m)|\,|B(o, n)|$, and the limit

$$\mathrm{gr}(X) = \lim_{n\to\infty} \frac{\log |B(o, n)|}{n}$$

exists. It is called the (exponential) *growth rate* of X. If $\mathrm{gr}(X) > 0$, one says that the graph X has *exponential growth*; otherwise, one speaks of *subexponential growth.* For details regarding the growth of finitely generated groups (i.e., their Cayley graphs) and transitive graphs, see, for example, the surveys of Grigorchuk and de la Harpe [18] or Imrich and Seifter [20]. In particular, we note that transitive graphs with polynomial growth [i.e., $|B(o, n)| \le Cn^d$] are "very similar" to Cayley graphs of nilpotent groups (see [42]).

We shall say that the operator P has a *finite first moment* if the first moment of its one-step transition probabilities is finite:

$$\sum |x|\, p(o, x) = \sum_{\xi} |\xi|\, \bar{\mu}(\xi) < \infty.$$

The subadditive ergodic theorem immediately implies the following (compare with the proof of Theorem 4.2; see [49], Theorem 8.14).

PROPOSITION 5.1. *If P has a finite first moment, then there exists a finite number $\ell = \ell(P)$ such that $|x_n|/n \to \ell$ for $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ and in the space $L^1(\Omega_{X,o}, \mathbf{P})$.*

We shall call the number $\ell(P)$ the (*linear*) *rate of escape* of the operator P. The following is easy to prove (compare with [10]).

LEMMA 5.2. *If the operator P has a finite first moment, then the entropy $H(\pi_1)$ of its one-step transition probabilities is finite.*

THEOREM 5.3. *If P has a finite first moment, then $\mathfrak{h}(P) \le \ell(P)\mathrm{gr}(X)$.*

PROOF. Fix a number $\varepsilon > 0$. Let $\mathfrak{h} = \mathfrak{h}(P)$. By Theorem 4.2 and Proposition 5.1, there exists an integer N such that

$$\pi_n\big(\{x \in X : -\log \pi_n(x) \ge (\mathfrak{h} - \varepsilon)n, |x| \le (\ell + \varepsilon)n\}\big) \ge 1 - \varepsilon$$

for all $n \ge N$. Therefore, $|B(o, (\ell + \varepsilon)n)| e^{-(\mathfrak{h}-\varepsilon)n} \ge 1 - \varepsilon$. Taking logarithms, dividing by n and making ε arbitrarily small, we get the claim. $\square$

COROLLARY 5.4. *Suppose that P has a finite first moment. Then the Poisson boundary of P is trivial if*

(a) *X has subexponential growth*

or if

(b) *the rate of escape $\ell(P)$ vanishes.*

REMARK. As we shall see below (Corollaries 5.14 and 5.15), under some additional conditions triviality of the Poisson boundary in fact implies (b).

5.2. *The modular drift.* Recall that any multiplicative character χ descends to the hypergroup H by formula (2.28).

LEMMA 5.5. *If the operator P has a finite first moment, then the number*

$$\delta(P,\chi) = \sum_x \log \chi[o,x] p(o,x) = \int_G \log \chi(g)\, d\mu(g) = \sum_\xi \log \chi(\xi) \bar{\mu}(\xi)$$

is finite.

PROOF. Since the number of neighbors of o is finite, there is a constant $C > 0$ such that $|\log \chi[o,x]| \le C$ for any $x \in B_1$. By G-invariance this means that $|\log \chi[x,y]| \le C$ whenever $d(x,y) \le 1$. Therefore, $|\log \chi(\xi)| \le C|\xi|$ for any $\xi \in H$. $\square$

We call the number $\delta(P,\chi)$ the *drift* of the operator P with respect to the character χ. In particular, the drift $\delta(P,\Delta)$ with respect to the modular character Δ is called the *modular drift* of P. By Proposition 2.15, $\delta(P_1P_2,\chi) = \delta(P_1,\chi) + \delta(P_2,\chi)$ for any two G-invariant Markov operators P_1, P_2 on X. In particular, $\delta(P^n,\chi) = n\delta(P,\chi)$. By Corollary 2.13, $\log \chi[o,x_n]$ is a sum of i.i.d. random variables. Therefore, the law of large numbers immediately implies the following.

PROPOSITION 5.6. *If the operator P has a finite first moment, then*

$$\frac{\log \chi[o,x_n]}{n} \to \delta(P,\chi)$$

P-*almost everywhere and in the space $L^1(\Omega_{X,o}, \mathbf{P})$.*

PROPOSITION 5.7. *If the operator P has a finite first moment and $\lambda(P,\chi) < \infty$, then*

$$\delta(P,\chi) - \delta(P,\Delta) \le \log \lambda(P,\chi),$$

and equality holds if and only if $\chi = \Delta$.

PROOF. By Jensen's inequality applied to the formula for $\lambda(P,\chi)$, from Proposition 2.30,

$$\log \lambda(P,\chi) = \log \sum_\xi \frac{\chi(\xi)}{\Delta(\xi)} \bar{\mu}(\xi) \ge \sum_\xi \log \frac{\chi(\xi)}{\Delta(\xi)} \bar{\mu}(\xi) = \delta(P,\chi) - \delta(P,\Delta),$$

and the equality holds if and only if the ratio character $\chi' = \chi/\Delta$ is equal to a constant $C > 0$ on $\operatorname{supp} \bar{\mu}$. The latter implies that $\chi' = C^n$ on the support of the

n-fold convolution $\bar{\mu}^n$ of $\bar{\mu}$. By irreducibility, $\bigcup_n \operatorname{supp} \bar{\mu}^n = H$. In particular, for any $\xi \in \operatorname{supp} \bar{\mu}$, the involution $\widehat{\xi}$ belongs to $\operatorname{supp} \bar{\mu}^n$ for a certain $n > 0$. Then simultaneously $\chi'(\xi) = C$ and $\chi'(\xi) = [\chi'(\widehat{\xi})]^{-1} = C^{-n}$, so that $C = 1$. □

COROLLARY 5.8. *If $P = P_\chi^\star$, then $\lambda(P, \chi) = 1$, so that $\delta(P, \chi) < \delta(P, \Delta)$ unless $\chi = \Delta$. In particular* (*taking $\chi = \mathbf{1}$*), *in the nonunimodular case the modular drift is strictly positive for any symmetric operator P.*

THEOREM 5.9. *If the operator P has a finite first moment*, *then*

$$\mathfrak{h}(P) = \mathfrak{h}(\widehat{P}) + \delta(P, \Delta).$$

PROOF. By Proposition 2.23, the entropy of the n-step transition probabilities of the operator P is

$$H(\pi_n) = -\sum_\xi \log \frac{\bar{\mu}^n(\xi)}{m_H(\xi)}\, \bar{\mu}^n(\xi) = H(\bar{\mu}^n) + \sum_\xi \log m_H(\xi)\, \bar{\mu}^n(\xi).$$

Using the same formula for the operator $\widehat{P}$, we obtain

$$\begin{aligned} H(\pi_n) - H(\overline{\pi}_n) &= \sum_\xi \log \frac{m_H(\xi)}{m_H(\widehat{\xi})} \bar{\mu}^n(\xi) \\ &= \sum_\xi \log \Delta(\xi) \bar{\mu}^n(\xi) = \delta(P^n, \Delta) = n\delta(P, \Delta), \end{aligned}$$

whence the claim. □

REMARKS.

1. In the context of Brownian motion on foliations, an analogous formula was first obtained in [22].
2. Proposition 2.31 shows that there is no analogous general formula connecting the entropies of an operator P and its reversal $P_\chi^\star$ unless $\chi = \Delta$.

COROLLARY 5.10. (a) *If the modular drift of P is positive*, *then the Poisson boundary of P is nontrivial.*

(b) *If the Poisson boundaries of both operators P and $\widehat{P}$ are trivial*, *then the modular drift of P vanishes.*

COROLLARY 5.11. *If the group G is nonunimodular*, *then the Poisson boundary of the simple random walk on X is nontrivial.*

REMARK. Triviality of the Poisson boundaries of both P and $\widehat{P}$ does *not* imply that the group G has to be unimodular; see the class of examples provided by Theorem 6.6(i).

5.3. *Entropy and the rate of escape.* Recall that the *range* of a Markov operator P is defined as

$$R = R(P) = \max\{d(x, y) : p(x, y) > 0\};$$

that is, $R(P)$ is the minimal number such that $p(x, y) = 0$ whenever $d(x, y) > R$. The following is a specialization of a general estimate of Carne [5] to our situation.

LEMMA 5.12. *If the operator P has bounded range $R = R(P)$ and is reversible with respect to a measure θ_χ, then, for any $x, y \in X$ and any $n > 0$,*

$$p^{(n)}(x, y) \le 2\chi[x, y]^{-1/2} \exp\left(-\frac{d^2}{2R^2 n}\right),$$

where $d = d(x, y)$.

THEOREM 5.13. *If the homogeneous G-invariant Markov operator P is reversible with respect to a measure θ_χ (where χ is a multiplicative character of G) and has a bounded range, then*

$$\mathfrak{h}(P) \ge \frac{1}{2}\delta(P, \chi) + \frac{\ell^2(P)}{2R^2(P)}.$$

PROOF. Apply Lemma 5.12 to the transition probabilities $p^{(n)}(o, x_n)$ along **P**-a.e. sample path of the Markov chain (X, P) and use Theorem 4.2 and Propositions 5.1 and 5.6. $\square$

COROLLARY 5.14. *If $\mathfrak{h}(P) = 0$ and $\delta(P, \chi) \ge 0$, then $\ell(P) = 0$.*

In the last corollary, observe the particular case $\chi = \mathbf{1}$. In the specific case of random walks on discrete groups, the following was proved by Varopoulos [44].

COROLLARY 5.15. *For the simple random walk on a homogeneous graph, the entropy $\mathfrak{h}(P)$ and the rate of escape $\ell(P)$ are zero or nonzero simultaneously.*

5.4. *Entropy and the spectral radius.* The number

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x, y)^{1/n}$$

is called the spectral radius of the operator P. It is independent of the choice of $x, y \in X$, and if P is aperiodic, then the “lim sup” in the above is a “lim.” If the operator P is reversible, that is, $P = P_\chi^\star$ for a certain character χ (see Section 2.5), then $\rho(P)$ is the norm of P in the space $\ell^2(X, \theta_\chi)$.

If P is the simple random walk, then $\rho(P) = 1$ if and only if the graph is *amenable*, which means that $\inf |\partial A|/|A| = 0$, where the infimum is taken over finite subsets of X and ∂A denotes the set of edges from A to $X \setminus A$. This equivalence was proved by Dodziuk [12] and does not require transitivity of the group action. Under the assumption of transitivity, the graph X is amenable $\iff$ $\rho(P) = 1$ for some (equivalently, every) symmetric, irreducible, homogeneous transition operator P $\iff$ the group G is both amenable and unimodular; see [37], [38] and [40].

THEOREM 5.16. *If the operator P is reversible with respect to a measure θ_χ (where χ is a multiplicative character of G) and has a finite first moment, then*

$$-2\log\rho(P) \le \mathfrak{h}(P) - \delta(P,\chi).$$

PROOF. By (2.24) and (2.29), we have

$$\begin{aligned} p^{(2n)}(o,o) &= \sum_x p^{(n)}(o,x)p^{(n)}(x,o) \\ &= \sum_x p^{(n)}(o,x)p^{(n)}(o,x)\frac{\theta_\chi(o)}{\theta_\chi(x)} \\ &= \sum_x p^{(n)}(o,x)p^{(n)}(o,x)\chi[o,x], \end{aligned}$$

whence, by concavity of the logarithm,

$$\begin{aligned} \log p^{(2n)}(o,o) &\ge \sum_x p^{(n)}(o,x)\log p^{(n)}(o,x) + \sum_x p^{(n)}(o,x)\log\chi[o,x] \\ &= -H(\pi_n) + n\delta(P,\chi), \end{aligned}$$

which after dividing by n and passing to the limit on n implies the claim. □

COROLLARY 5.17. *If the graph X is nonamenable, then the Poisson boundary of the simple random walk is nontrivial.*

REMARKS.

1. The inequality $p^{(2n)}(o,o) \ge p^{(n)}(o,x_n)p^{(n)}(x_n,o)$ along **P**-a.e. sample path of the Markov chain (X,P) would give just $-\log\rho(P) \le \mathfrak{h}(P) - \delta(P,\chi)$. Our proof is based on the trick used by Avez for random walks on groups [1]. A better estimate of $\rho(P)$ in terms of $\mathfrak{h}(P)$ can be obtained by using the methods of Ledrappier [33].
2. Theorem 5.16 applied to the operator $\widehat{P}$ (see Remark 3 at the end of Section 2) gives the inequality

$$\begin{aligned} -2\log\rho(P) &= -2\log\rho(\widehat{P}) \le \mathfrak{h}(\widehat{P}) - \delta(\widehat{P},\widehat{\chi}) \\ &= \mathfrak{h}(P) - \delta(P,\Delta) + \delta(P,\widehat{\chi}) = \mathfrak{h}(P) + \delta(P,\Delta) - \delta(P,\chi). \end{aligned}$$

5.5. *Criteria for identification of the Poisson boundary.* Combinatorial or geometric considerations often provide us with a certain "pattern" of behavior at infinity of sample paths of the Markov chain determined by our operator P. In the measure-theoretic language, this pattern determines a measure space (B, λ) which a priori is just the quotient (Γ_ξ, ν_ξ) of the whole Poisson boundary (Γ, ν) with respect to a certain G-invariant partition ξ (we may assume that this partition is G-invariant because all "natural" combinatorial or geometric constructions must, by definition, be G-invariant). In other words, (B, λ) is a μ-boundary (recall that μ is the bi-K-invariant probability measure on G which determines the operator P). If we want to identify the Poisson boundary, we have to prove that, in fact, the μ-boundary $(B, \lambda) \cong (\Gamma_\xi, \nu_\xi)$ coincides with the whole Poisson boundary, that is, prove that the partition ξ is the point partition.

We shall give two geometric criteria for identification of the Poisson boundary based on Theorem 4.16. For simplicity, we shall always assume that the operator P has a finite first moment, although this assumption could, in fact, be relaxed; see [28]. Denote by $\mathbf{x} \mapsto x_\infty$ the projection from the path space $(\Omega_{X,o}, \mathbf{P})$ onto the μ-boundary $(B, \lambda) \cong (\Gamma_\xi, \nu_\xi)$ under consideration. The first criterion ("ray approximation") immediately follows from Theorem 4.16.

THEOREM 5.18. *Let P be a homogeneous Markov operator with a finite first moment and let (B, λ) be a μ-boundary. If there exists a sequence of measurable maps $R_n \colon B \to X$ such that*

$$d\big(x_n, R_n(x_\infty)\big) = o(n)$$

for $\mathbf{P}$-almost every sample path $\mathbf{x} \in \Omega_{X,o}$, then (B, λ) is the whole Poisson boundary of the operator P.

In the second criterion ("strip approximation"), we shall assume that simultaneously with a μ-boundary (B_+, λ_+) we are also given a $\widehat{\mu}$-boundary (B_-, λ_-) which is a G-equivariant quotient of the Poisson boundary of the reverse Markov operator $\widehat{P}$. This criterion is symmetric with respect to the time reversal and leads to a simultaneous identification of the Poisson boundaries of the operators P and $\widehat{P}$. The strip approximation criterion for homogeneous Markov operators is very similar to the analogous criterion for random walks on countable groups (see [28]); however, for the sake of completeness, we shall present below an outline of the proof.

THEOREM 5.19. *Let P be a homogeneous Markov operator with a finite first moment and let (B_+, λ_+), (B_-, λ_-) be a μ- and a $\widehat{\mu}$-boundary, respectively. If there exists a measurable G-equivariant map S assigning to pairs of points $(b_-, b_+) \in B_- \times B_+$ nonempty "strips" $S(b_-, b_+) \subset X$ such that*

$$\frac{1}{n} \log\big|S(b_-, b_+) \cap B(o, n)\big| \underset{n\to\infty}{\longrightarrow} 0$$

for $\lambda_- \otimes \lambda_+$-almost every $(b_-, b_+) \in B_- \times B_+$, then (B_+, λ_+) and (B_-, λ_-) are the Poisson boundaries of the operators P and $\widehat{P}$, respectively.

PROOF. Replacing, if necessary, the strips $S(b_-, b_+)$ with their d-neighborhoods

$$S'(b_-, b_+) = \{x \in X : d(x, S(b_-, b_+)) \le d\}$$

for a sufficiently large d, we may assume that

$$\lambda_- \otimes \lambda_+(\{(b_-, b_+) \in B_- \times B_+ : o \in S(b_-, b_+)\}) = \kappa > 0.$$

Let us now consider the space $(\overline{\Omega}_G, \overline{\mathbf{Q}})$ of bilateral sequences $\overline{\mathbf{g}}$ of independent μ-distributed increments (g_i), $i \in \mathbb{Z}$. The image of the measure $\overline{\mathbf{Q}}$ under the map

$$\overline{\mathbf{g}} \mapsto (\widehat{\mathbf{x}}, \mathbf{x}), \qquad \mathbf{x} = (o, g_1 o, g_1 g_2 o, \ldots), \qquad \widehat{\mathbf{x}} = (o, g_0^{-1} o, g_0^{-1} g_{-1}^{-1} o, \ldots)$$

from $\overline{\Omega}_G$ to the product $\Omega_{X,o} \times \Omega_{X,o}$ is the product of the measure $\mathbf{P}$ and the measure $\widehat{\mathbf{P}}$ determined by the operator $\widehat{P}$. Denote by $\overline{T}$ the Bernoulli shift in the space $(\overline{\Omega}_G, \overline{\mathbf{Q}})$ and let $x_\infty \in B_+$ and $x_{-\infty} \in B_-$ be the boundary points corresponding to the sample paths $\mathbf{x}$ and $\widehat{\mathbf{x}}$, respectively. Then, by formula (4.13), the boundary points $x'_\infty \in B_+$, $x'_{-\infty} \in B_-$ corresponding to the shifted sequence $\overline{\mathbf{g}}' = T^n \overline{\mathbf{g}}$, $n > 0$, are

$$x'_{\pm\infty} = g_1^{-1} g_2^{-1} \cdots g_n^{-1} x_{\pm\infty}.$$

Since $\overline{T}$ preserves the measure $\overline{\mathbf{Q}}$, we obtain that, for any $n > 0$,

$$\begin{aligned}
&\overline{\mathbf{Q}}(\{\overline{g} : x_n \in S(x_{-\infty}, x_\infty)\}) \\
&\quad = \overline{\mathbf{Q}}(\{\overline{g} : g_1 g_2 \cdots g_n o \in S(x_{-\infty}, x_\infty)\}) \\
&\quad = \overline{\mathbf{Q}}(\{\overline{g} : o \in S(x'_{-\infty}, x'_\infty)\}) \\
&\quad = \overline{\mathbf{Q}}(\{\overline{g} : o \in S(x_{-\infty}, x_\infty)\}) \\
&\quad = \lambda_- \otimes \lambda_+(\{(b_-, b_+) \in B_- \times B_+ : o \in S(b_-, b_+)\}) = \kappa > 0,
\end{aligned}$$

or, in a slightly different form,

$$\int \mathbf{P}(\{\mathbf{x} : x_n \in S(b_-, x_\infty)\}) \, d\lambda_-(b_-) = \kappa.$$

Therefore, there exists $b_- \in B_-$ such that

$$\mathbf{P}(\{\mathbf{x} : x_n \in S(b_-, x_\infty)\}) = \int \pi_n^{b_+}[S(b_-, b_+)] \, d\lambda_+(b_+) \ge \kappa,$$

where $\pi_n^{b_+}$ is the one-dimensional distribution at time n of the conditional measure $\mathbf{P}^{b_+}$ determined by a point $b_+ \in B_+$. The latter inequality implies that

$$\lambda_+\left(\left\{b_+ \in B_+ : \pi_n^{b_+}[S(b_-, b_+)] \ge \frac{\kappa}{2}\right\}\right) \ge \frac{\kappa}{2}.$$

As it follows from Proposition 5.1, if n is sufficiently large, then

$$\lambda_+\left(\left\{b_+ \in B_+ : \pi_n^{b_+}\left[S(b_-, b_+) \cap B_{2\ell n}\right] \geq \frac{\kappa}{4}\right\}\right) \geq \frac{\kappa}{4},$$

where $\ell = \ell(P)$ is the rate of escape of the operator P. [We may exclude the case $\ell(P) = 0$ since then also $\ell(\widehat{P}) = 0$ and both Poisson boundaries are trivial by Corollary 5.4.] The last inequality means that for any sufficiently large n there exists a subset A_n of B with $\lambda_+(A_n) \geq \kappa/4$ such that, for any $b_+ \in A_n$, the one-dimensional distribution at time n of the conditional measure $\mathbf{P}^{b_+}$ is concentrated on a subset of X of asymptotically subexponential size. Since the measures $\mathbf{P}^{b_+}$ all have the same asymptotic entropy, this is only possible when this asymptotic entropy is 0, that is, when (B_+, λ_+) is the whole Poisson boundary (Theorem 4.16). $\square$

6. Applications and examples.

6.1. *The homogeneous tree and its affine group.* This example is well understood from Cartwright, Kaimanovich and Woess [6], but as it is a key example that will be used subsequently, we reconsider it briefly.

Let $\mathbf{T} = \mathbf{T}_q$ be the homogeneous tree with degree $q + 1$, where $q \geq 2$, and let $\partial\mathbf{T}$ be its boundary (space of ends). We omit repeating the description of $\partial\mathbf{T}$ and the topology of $\widehat{\mathbf{T}} = \mathbf{T} \cup \partial\mathbf{T}$, which should be well known and can be found, for example, in [6]. We choose and fix an end $\omega \in \partial\mathbf{T}$ and consider the *affine group* $\mathrm{Aff}(\mathbf{T})$ of $\mathbf{T}$, that is, the group of all automorphisms (self-isometries) of $\mathbf{T}$ that leave ω fixed. This is the simplest example of a group that acts transitively on a graph and is nondiscrete, amenable and nonunimodular (cf. [40] and [43]).

Now let P be a homogeneous Markov operator with finite first moment on $\mathbf{T}$ such that $G = \mathrm{Aut}(\mathbf{T}, P)$ is a transitive subgroup of $\mathrm{Aff}(\mathbf{T})$. It is closed and again nonunimodular. Using formula (2.2), one can compute the modular function $\Delta(g) = \Delta[o, go]$. The level sets of $x \mapsto \Delta[o, x]$ are the *horocycles* H_k, $k \in Z$; see [40] and Figure 1. If $\mathrm{hor}(x)$ denotes the Busemann function of x [i.e., $\mathrm{hor}(x) = k$ for $x \in H_k$; see Figure 1], then $\Delta[o, x] = q^{\mathrm{hor}(x)}$. The horocycles may be thought of as successive "generations," so that each vertex in H_k has q "sons" in H_{k+1} and a unique "father" in H_{k-1}.

The following is known from [6], Theorems 2 and 4.

THEOREM 6.1 [6]. (a) *Let* $\delta(P, \Delta) = (\log q) \sum_x \mathrm{hor}(x) p(o, x)$ *be the modular drift of the Markov chain on* $\mathbf{T}$. *Then the rate of escape is* $\ell(P) = \delta(P, \Delta)/\log q$.

(b) *If* $\delta(P, \Delta) > 0$, *then the Markov chain* (x_n) *converges almost surely to a random point* $x_\infty \in \partial^*\mathbf{T} = \partial\mathbf{T} \setminus \{\omega\}$. *Denoting by* ν *the* $\mathbf{P}_o$*-distribution of* x_∞ *on* $\partial^*\mathbf{T}$, *we have that* $\mathrm{supp}(\nu) = \partial^*\mathbf{T}$.

(c) *If* $\delta(P, \Delta) < 0$, *then* $x_n \to \omega$ *almost surely.*

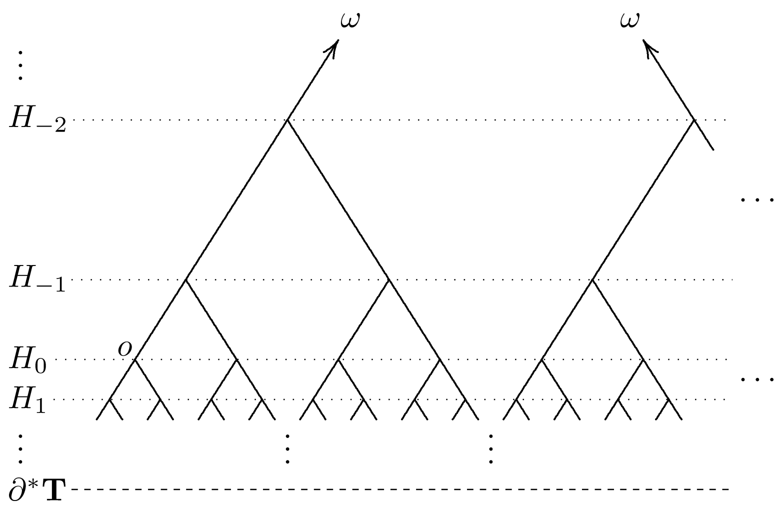

FIG. 1.

Here, convergence refers to the topology of $\widehat{\mathbf{T}}$. If $\delta(P, \Delta) = 0$, then almost sure convergence to ω is known only under an exponential moment condition. However, for our present purpose this is irrelevant.

THEOREM 6.2. (i) *If* $\delta(P, \Delta) \le 0$, *then the Poisson boundary is trivial.*

(ii) *If* $\delta(P, \Delta) > 0$, *then the Poisson boundary coincides with the measure space* $(\partial^*\mathbf{T}, \nu)$, *where* ν *is the* $\mathbf{P}_o$-*distribution of* x_∞.

PROOF. If $\delta(P, \Delta) = 0$, then we can apply Corollary 5.4(c).

If $\delta(P, \Delta) > 0$, then $\delta(\widehat{P}, \Delta) < 0$, both operators have finite first moment and we can use Theorem 5.19: by Theorem 6.1(b), the pair $(\partial^*\mathbf{T}, \nu)$ is a μ-boundary, where—as usual—μ is associated with P by (2.16). By Theorem 6.1(a), the pair $(\{\omega\}, \delta_\omega)$ is a $\widehat{\mu}$-boundary. Thus, we can choose the geodesic lines between $b \in \partial^*\mathbf{T}$ and ω as the strips $S(b, \omega)$ required by the criterion of Theorem 5.19. Measurability of the map $b \mapsto S(b, \omega)$ is obvious, and $gS(b, \omega) = S(gb, \omega)$ for every $g \in G$ and $b \in \partial^*\mathbf{T}$. This proves (ii).

Exchanging the roles of P and $\widehat{P}$, the same argument proves (i) in the case $\delta(P, \Delta) < 0$. $\square$

In [6], this identification of the Poisson boundary is indicated (under more general assumptions) via the criterion of "ray approximation" in its variant for locally compact groups.

COROLLARY 6.3. *If* $\delta(P,\Delta) > 0$, *then* $\mathfrak{h}(P) = \delta(P,\Delta)$. *Otherwise*, $\mathfrak{h}(P) = 0$.

6.2. *The Diestel–Leader graphs.* We start with two trees $\mathbf{T}^1 = \mathbf{T}_q$ and $\mathbf{T}^2 = \mathbf{T}_r$, where $q \geq r \geq 2$. As in the previous subsection, we choose and fix reference vertices o^1 and o^2 and ends ω^1 and ω^2 of $\mathbf{T}_q$ and $\mathbf{T}_r$, respectively. The corresponding *Diestel–Leader graph* is the following subgraph of the direct product of the two trees:

$$\mathrm{DL}_{q,r} = \{x^1x^2 \in \mathbf{T}_q \times \mathbf{T}_r : \mathrm{hor}(x^1) + \mathrm{hor}(x^2) = 0\}.$$

Here, $\mathrm{hor}(\cdot)$ stands for the Busemann function with respect to the chosen end in the respective tree. Thus, the $\sim$ neighborhood of a point x^1x^2 in the graph $\mathrm{DL}_{p,q}$ is given by

$$x^1x^2 \sim y^1y^2 \iff x^1 \sim y^1, x^2 \sim y^2.$$

To visualize this graph, draw $\mathbf{T}_q$ as in Figure 1 and draw (on the right) $\mathbf{T}_r$ in the same way, but upside down, with the respective horocycles $H_k(\mathbf{T}_q)$ and $H_{-k}(\mathbf{T}_r)$ on the same level. Connect the two origins o^1, o^2 by an elastic spring. It can move along each of the two trees and may expand infinitely, but must always remain in a horizontal position. The vertex set of $\mathrm{DL}_{q,r}$ consists of all admissible positions of the spring. From a position x^1x^2 with $\mathrm{hor}(x^1) + \mathrm{hor}(x^2) = 0$, the spring may move downward to one of the "sons" of x^2 and at the same time to the "father" of x^1, or upward in an analogous way. Such a move corresponds to going to a neighbor of x^1x^2. See Figure 2, which corresponds to $\mathrm{DL}_{2,2}$.

As our reference point in $\mathrm{DL}_{q,r}$, we choose $o = o^1o^2$. As was shown by Bertacchi [3], the distance in $\mathrm{DL}_{q,r}$ is given by

$$d(x^1x^2, y^1y^2) = d(x^1, y^1) + d(x^2, y^2) - |\mathrm{hor}(y^1) - \mathrm{hor}(x^1)|.$$

The group of automorphisms

$$G = \{g = g^1g^2 \in \mathrm{Aff}(\mathbf{T}_q) \times \mathrm{Aff}(\mathbf{T}_r) : \mathrm{hor}(g^1o^1) + \mathrm{hor}(g^2o^2) = 0\} \tag{6.4}$$

acts transitively on $\mathrm{DL}_{q,r}$ and is amenable. Here, we mean, of course, that for a pair g^1g^2 the action of g^1 is on $\mathbf{T}_q$ and the action of g^2 on $\mathbf{T}_r$. If $q \neq r$, then G is the whole automorphism group (Schramm, personal communication), it is nonunimodular and $\mathrm{DL}_{q,r}$ is a nonamenable graph by [40]. On the other hand, when $q = r$, this group is unimodular and the graph $\mathrm{DL}_{q,q}$ is amenable. Indeed, we have $\Delta_G(g) = \Delta[o, go]$, where, for $x = x^1x^2$ and $y = y^1y^2 \in \mathrm{DL}_{q,r}$,

$$\Delta[x, y] = \Delta_{\mathbf{T}_q}[x^1, y^1]\Delta_{\mathbf{T}_r}[x^2, y^2] = (q/r)^{\mathrm{hor}(y^1) - \mathrm{hor}(x^1)}.$$

The same remains true when in (6.4) we replace the affine subgroups of the two trees by arbitrary closed subgroups that act transitively on the respective tree. In the remainder of this article, we may assume that G is of this more general form.

REMARK. We did not study the question whether *every* transitive subgroup of the group of (6.4) must factorize in this way, and if this is not the case, whether the modular function may look differently.

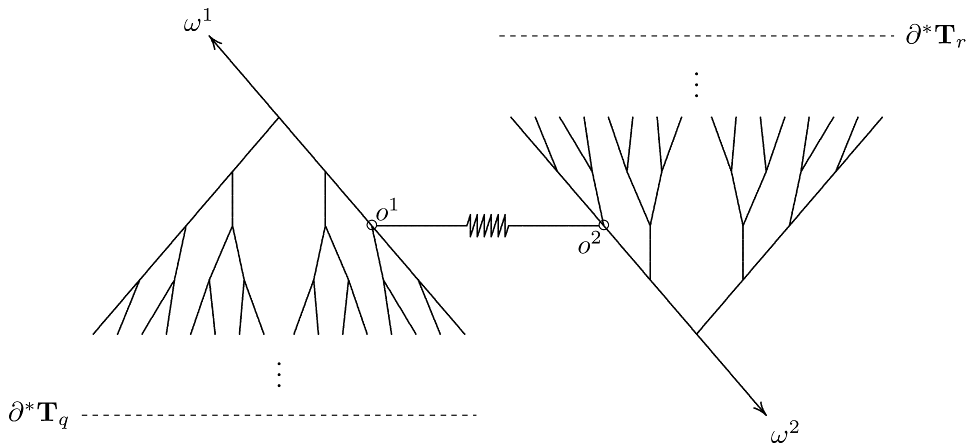

FIG. 2.

The graph $\mathrm{DL}_{q,r}$ has a natural compactification $\widehat{\mathrm{DL}}_{q,r}$, namely, its closure in $\widehat{\mathbf{T}}_q \times \widehat{\mathbf{T}}_r$. Thus,

$$\widehat{\mathrm{DL}}_{q,r} = \big(\widehat{T}_q \times \{\omega^2\}\big) \cup \big(\{\omega^1\} \times \widehat{T}_r\big).$$

We split the boundary into five disjoint pieces:

$$\begin{aligned}\partial DL_{q,r} = {}& \big(\partial^*\mathbf{T}_q \times \{\omega^2\}\big) \cup \big(\{\omega^1\} \times \partial^*\mathbf{T}_r\big)\\ & \cup \big(\{\omega^1\} \times \{\omega^2\}\big) \cup \big(\mathbf{T}_q \times \{\omega^2\}\big) \cup \big(\{\omega^1\} \times \mathbf{T}_r\big);\end{aligned}$$

compare with [3]. The first piece consists of the limits of sequences $x(n) = x_n^1 x_n^2$ in $\mathrm{DL}_{q,r}$ for which $x_n^1 \to \xi^1 \in \partial^*\mathbf{T}_q$ and $x_n^2 \to \omega^2$ in the topologies of $\widehat{\mathbf{T}}_q$ and $\widehat{\mathbf{T}}_r$, respectively. The second piece is analogous, by exchanging roles. The third piece consists of the limits of sequences where $|\mathrm{hor}(x_n^1)| = |\mathrm{hor}(x_n^2)|$ remains bounded, while $|x_n^1|$ and $|x_n^2|$ tend to ∞. The fourth piece corresponds to sequences where almost all x_n^1 coincide, while $|x_n^2| \to \infty$, and the fifth piece is again analogous, by exchanging roles.

Now let P be a G-invariant, irreducible transition operator on $\mathrm{DL}_{q,r}$ with finite first moment. We define the *vertical drift* of P as

$$\mathrm{vd}(P) = \sum_x p(o, x)\,\mathrm{hor}\big(x^1\big).$$

Note that the modular drift is $\delta(P, \Delta) = \mathrm{vd}(P) \log(q/r)$. Then the following is known in analogy with Theorem 6.1.

THEOREM 6.5 [3]. (a) *The rate of escape of the Markov chain generated by P is $\ell(P) = |\mathrm{vd}(P)|$.*

(b) *If* $\mathrm{vd}(P) > 0$, *then the Markov chain* (x_n) *converges almost surely to a random point* $x_\infty = x^1_\infty \omega^2 \in \partial^* \mathbf{T}_q \times \{\omega^2\}$. *Denoting by* ν *the* $\mathbf{P}_o$*-distribution of* x^1_∞ *on* $\partial^* \mathbf{T}_q$, *we have that* $\mathrm{supp}(\nu) = \partial^* \mathbf{T}_q$.

(c) *If* $\mathrm{vd}(P) < 0$, *then* (x_n) *converges almost surely to a random point* $x_\infty = \omega^1 x^2_\infty \in \{\omega^1\} \times \partial^* \mathbf{T}_q$. *Denoting by* ν *the* $\mathbf{P}_o$*-distribution of* x^2_∞ *on* $\partial^* \mathbf{T}_r$, *we have that* $\mathrm{supp}(\nu) = \partial^* \mathbf{T}_r$.

THEOREM 6.6. (i) *If* $\mathrm{vd}(P) = 0$, *then the Poisson boundary is trivial.*

(ii) *If* $\mathrm{vd}(P) > 0$, *then the Poisson boundary coincides with the measure space* $(\partial^* \mathbf{T}_q, \nu)$, *where* ν *is the* $\mathbf{P}_o$*-distribution of* x^1_∞.

(iii) *If* $\mathrm{vd}(P) < 0$, *then the Poisson boundary coincides with the measure space* $(\partial^* \mathbf{T}_r, \nu)$, *where* ν *is the* $\mathbf{P}_o$*-distribution of* x^2_∞.

PROOF. This is very similar to the proof of Theorem 6.2. If $\mathrm{vd}(P) = 0$, then we can apply Corollary 5.4(c).

Let μ be the probability measure on G associated with P. If $\mathrm{vd}(P) > 0$, then $\mathrm{vd}(\widehat{P}) < 0$. By Theorem 6.5(b), the pair $(\partial^* \mathbf{T}_q, \nu)$ is a μ-boundary. By Theorem 6.5(c), the pair $(\partial^* \mathbf{T}_r, \widehat{\nu})$ is a $\widehat{\mu}$-boundary, where $\widehat{\nu}$ is the $\widehat{\mathbf{P}}_o$-distribution of x^2_∞.

We use Theorem 5.19. If $\xi^1 \in \partial^* \mathbf{T}_q$ and $\xi^2 \in \partial^* \mathbf{T}_r$, then we define the strip $S(\xi^1, \xi^2)$ as the set of all vertices $x = x^1 x^2$ in $\mathrm{DL}_{q,r}$ such that x^1 lies on the two-way-infinite geodesic from ω^1 to ξ^1 in $\mathbf{T}_q$ and x^2 on the analogous geodesic in $\mathbf{T}_r$. It is clear that all requirements of Theorem 5.19 are satisfied.

If $\mathrm{vd}(P) < 0$, then we just have to exchange the two "sides." $\square$

REMARKS.

1. The Diestel–Leader graphs are "relatives" of the Cayley graphs of the amenable Baumslag–Solitar groups $\mathrm{BS}_{1,p} = \langle a, b \mid ab = b^p a \rangle$. The latter has a representation as the group of all matrices $g = \begin{pmatrix} p^k & m/p^\ell \\ 0 & 1 \end{pmatrix}$, where $k, \ell, m \in \mathbb{Z}$. To understand the analogy, suppose for simplicity that p is a prime. Then $\mathrm{BS}_{1,p}$ can be considered as a subgroup of the affine group of the field $\mathbb{Q}_p$ of p-adic numbers and as such acts by graph automorphisms on $\mathbf{T}_q$. Analogously, $\mathrm{BS}_{1,p}$ can be seen as a subgroup of the affine group over $\mathbb{R}$ that acts by isometries (Möbius transformations) on the hyperbolic upper half plane $\mathbb{H}$. Both embeddings of $\mathrm{BS}_{1,p}$ are nonclosed and nondiscrete, but the diagonal embedding of $\mathrm{BS}_{1,p}$ into $\mathrm{Aff}(\mathbb{Q}_p) \times \mathrm{Aff}(\mathbb{R})$ is discrete. The joint action of $\mathrm{BS}_{1,p}$ on $\mathbf{T}_p \times \mathbb{H}$ is similar to the action of the group G of (6.4) on $\mathbf{T}_q \times \mathbf{T}_r$. (Think of the tree as a discrete analogue of the hyperbolic plane!) Random walks on the groups $\mathrm{BS}_{1,p}$ were first considered in [29]. The description of the Poisson boundary for the Diestel–Leader graphs is analogous to its description for the groups $\mathrm{BS}_{1,p}$, for which it is the real line, or the dyadic line, or is trivial, depending on the sign of the drift [23].

2. The Diestel–Leader graphs are a particular case of the following general construction. Let X^i, $i = 1, 2, \ldots, n$, be homogeneous graphs presented as G^i/K^i, where G^i are totally disconnected locally compact groups with compact open subgroups K_i. Suppose that the groups G^i are endowed with homomorphisms φ_i onto abelian groups A^i. Denote by φ: $G^1 \times G^2 \times \cdots \times G^n \to A$ the composition of $\varphi_1 \times \varphi_2 \times \cdots \times \varphi_n$ with a homomorphism from $A^1 \times A^2 \times \cdots \times A^n$ onto another abelian group A, and put $G = \ker\varphi$, $K = G \cap \prod K_i$. Then $X = G/K$ is endowed with a homogeneous graph structure coming from the graph structures of the X_i in the same way as for the Diestel–Leader graphs (for them $n = 2$ and $A^1 = A^2 = A = \mathbb{Z}$). It will be interesting to have a closer look at this generalization in future work.

6.3. *Graphs with infinitely many ends.* Let X be a connected, locally finite graph with edge set $E(X)$. We describe its *end compactification*, originally introduced by Freudenthal [15], in the context of random walks (Markov chains); see [46] and [49], Section 21, for more details.

For a general locally compact topological space $\mathcal{X}$, the *space of ends* $\partial\mathcal{X}$ is defined as the projective limit of the spaces $\partial_K\mathcal{X}$ of connected components of $X \setminus K$ when compacts K exhaust X. The corresponding compactification $\widehat{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$ obtained as the projective limit of the compactifications $\mathcal{X} \cup \partial_K\mathcal{X}$ is called the *end compactification* of $\mathcal{X}$. However, in the context of graphs, a more explicit description better suits our purposes.

An infinite *path* without self-intersections in a graph X is a sequence $\mathfrak{p} = [x_0, x_1, \ldots]$ of distinct vertices such that $x_i \sim x_{i-1}$ for all i. If F is a finite set of edges of X, then the (induced) graph $X \setminus F$ has finitely many connected components. Every path $\mathfrak{p}$ must have all but finitely many points in precisely one of them, and we say that $\mathfrak{p}$ *ends up* in that component. Two paths are called *equivalent* if, for any finite $F \subset E(X)$, they end up in the same component of $X \setminus F$. An *end* of X is an equivalence class of paths. In this subsection, we write ∂X for the space of ends of X, and $\widehat{X} = X \cup \partial X$. If C is a component of $X \setminus F$ [$F \subset E(X)$ finite], then we write ∂C for the set of those ends whose paths end up in C, and $\widehat{C} = C \cup \partial C$ for the resulting completion of C.

We now explain the topology of $\widehat{X}$. If $F \subset E(X)$ is finite and $w \in \widehat{X}$, then there is precisely one component of $X \setminus F$ whose completion contains w. We denote it by $\widehat{C}(w, F)$. Varying F, we obtain a neighborhood base of w. Then $\widehat{X}$ is compact, the topology is discrete on X, it has a countable base and it is Hausdorff. Each $\widehat{C}(w, F)$ is open and compact, and $\widehat{X}$ is totally disconnected.

For Cayley graphs of a finitely generated group, the end compactification is independent of the choice of the finite generating set that induces the Cayley graph, and one speaks of the ends of the group itself.

A transitive, infinite graph X has one, two or infinitely many ends. If it has one end, then the end compactification is not suitable for a good description of the structure of X at infinity. If it has two ends, then it is roughly isometric with the two-way-infinite path, and it is easily understood that the Poisson boundary of any homogeneous Markov chain with finite first moment is trivial (compare with [49], Theorem 25.4). Thus, we consider the case when X has infinitely many ends and $G = \mathrm{Aut}(X, P)$ acts transitively.

We use the powerful theory of cuts and structure trees developed by Dunwoody; see the book by Dicks and Dunwoody [11]. For a more detailed description with the same notation as used here, see [49] and also [41].

A *cut* of a connected graph X is a set F of edges whose deletion disconnects X. If it disconnects X into precisely two connected components $A = A(F)$ and $A^* = A^*(F) = X \setminus A$, then we call F *tight*, and A, A^* are the *sides* of F. Thomassen in [41] has given a clever proof of the following.

LEMMA 6.7 [41]. *For any $k \in \mathbb{N}$, there are only finitely many tight cuts F with $|F| = k$ that contain a given edge of X.*

Two cuts F, F' are said to *cross* if all four sets

$$A(F) \cap A(F'), \quad A(F) \cap A^*(F'), \quad A^*(F) \cap A(F'), \quad A^*(F) \cap A^*(F')$$

are nonempty. Dunwoody [13] has proved the following important theorem; see also [11].

THEOREM 6.8. *Every infinite, connected graph with more than one end has a finite tight cut F with infinite sides such that F crosses no gF, where $g \in \mathrm{Aut}(X)$.*

A cut with these properties will be called a *D-cut*. Owing to Lemma 6.7 and Theorem 6.8, one can construct Dunwoody's *structure tree* of X. Let F be a D-cut of X and define

$$\mathcal{E} = \{A(gF), A^*(gF) : g \in G\}.$$

This collection has the following properties.

1. All $A \in \mathcal{E}$ are infinite and connected.
2. If $A \in \mathcal{E}$ then $A^* = X \setminus A \in \mathcal{E}$.
3. If $A, B \in \mathcal{E}$ and $A \subset B$ then there are only finitely many $C \in \mathcal{E}$ such that $A \subset C \subset B$.
4. If $A, B \in \mathcal{E}$, then one of $A \subset B$, $A \subset B^*$, $A^* \subset B$ or $A^* \subset B^*$ holds.

To construct the structure tree $\mathcal{T}$ of X with respect to G and the D-cut F, it will be convenient to think of an unoriented edge of $\mathcal{T}$ as a pair of oriented edges, where the second edge points from the endpoint to the initial point of the first one. We define $\mathcal{T}$ in terms of its edges and their incidence. The oriented edge set of $\mathcal{T}$ is $\mathcal{E}$. If $A \in \mathcal{E}$, then (A, A^*) constitutes a pair of oppositely oriented edges between the same two vertices. If $A, B \in \mathcal{E}$ and $B \neq A^*$, then the endpoint of A is the initial point of B if $A \supset B$ and there is no $C \in \mathcal{E}$ such that $A \supset C \supset B$ properly.

The tree is countable, but not necessarily locally finite. We write $\partial\mathcal{T}$ for the set of ends of $\mathcal{T}$. The group G acts by automorphisms on $\mathcal{T}$ via $A \mapsto gA$, where $g \in G$ and $A \in \mathcal{E}$. The action has one or two orbits on $\mathcal{E}$ according to whether $gA(F) = A^*(F)$ for some $g \in G$ or not.

We introduce the *structure map* φ: $\widehat{X} \to \widehat{\mathcal{T}}$. Let $z \in \widehat{X}$. Then there is some $A_0 \in \mathcal{E}$ that contains z. If there is a minimal $A \in \mathcal{E}$ with this property, then we define φz as the end vertex of A as an edge of $\mathcal{T}$. If there is no minimal A with this property, then there must be a maximal strictly descending sequence $A_0 \supset A_1 \supset A_2 \supset \cdots$ in $\mathcal{E}$ such that $w \in A_n$ for all n. As edges of $\mathcal{T}$, the A_n constitute a path that defines an end in $\partial\mathcal{T}$. This end is φz. The image of z does not depend on the particular choice of the initial $A_0 \in \mathcal{E}$ containing z.

Via $(g, A) \mapsto gA$ for $A \in \mathcal{E}$, the group G acts on $\mathcal{T}$, and φ commutes with the actions of G on $\widehat{X}$ and on $\widehat{\mathcal{T}}$.

If x is a vertex of X, then φx is a vertex of $\mathcal{T}$. Given an end of $\mathcal{T}$, its preimage under φ consists of a single end of X. However, usually there are ends of X that are mapped to vertices of $\mathcal{T}$ under φ. We write $\partial^{(0)}X = \varphi^{-1}\partial\mathcal{T}$. This is a Borel subset of ∂X.

Now let P be a homogeneous Markov operator on X with $G = \mathrm{Aut}(X, P)$. We have to distinguish two substantially different cases.

CASE 1. No end of X is fixed by G. Then G is nonamenable by [45], and the following is known.

THEOREM 6.9 [46]. *The Markov chain (x_n) converges almost surely in the end topology to a random point $x_\infty \in \partial X$. Denoting by ν the $\mathbf{P}_o$-distribution of x_∞, we have that* (i) $\mathrm{supp}(\nu) = \partial X$, (ii) $\nu(\{\xi\}) = 0$ *for every* $\xi \in \partial X$ *and* (iii) $\nu(\partial X \setminus \partial^{(0)}X) = 0$.

In [46], $\partial^{(0)}X$ is denoted by $\Omega^{(0)}$, and the structure tree appears only implicitly. The main result of [46] is a proof, using a completely different method, of the following under the restriction that P has bounded range [i.e., there is M such that $p(x, y) > 0$ only when $d(x, y) \le M$]. Here, we need only the finite first moment assumption.

THEOREM 6.10. *If G does not fix an end of X and P has finite first moment, then the Poisson boundary of P coincides with the measure space $(\partial X, \nu)$.*

PROOF. By Theorem 6.9, the pair $(\partial X, \nu)$ is a μ-boundary for the probability measure μ of (2.16). Theorem 6.9 also applies to $\widehat{P}$, and if we denote the corresponding limit distribution on ∂X by $\widehat{\nu}$, then $(\partial X, \widehat{\nu})$ is a $\widehat{\mu}$-boundary.

We intend to apply the "strip criterion" (5.19) once more. Let F be the D-cut that we have used to define $\mathcal{E}$ and the structure tree. We write F^0 for the set of all end vertices of the edges of F. For ends $\xi, \eta \in \partial X$, we define

$$S(\xi, \eta) = \bigcup \{gF^0 : g \in G, \widehat{C}(\xi, gF) \neq \widehat{C}(\eta, gF)\}.$$

Clearly, $gS(\xi, \eta) = S(g\xi, g\eta)$ for every $g \in G$. The "strip" $S(\xi, \eta)$ is the union of all gF^0 such that the sides of gF, seen as edges of the structure tree $\mathcal{T}$, lie on the geodesic between $\varphi\xi$ and $\varphi\eta$. This geodesic can be empty (when $\varphi\xi = \varphi\eta$), finite, one way infinite or two way infinite. The latter holds precisely when $\xi, \eta \in \partial^{(0)} X$ are distinct, and in view of properties (i), (ii) and (iii) of ν and $\widehat{\nu}$, we have to check the condition of Theorem 5.19 only in this case. But by Lemma 6.7 and the fact that F is a D-cut, there is an integer $k > 0$ such that the following holds: if $A_0, A_1, \ldots, A_k \in \mathcal{E}$ and $A_0 \supset A_1 \supset \cdots \supset A_k$ properly, then $d(A_k, A_0^*) \geq 2$ (i.e., if A_k is one of the sides of gF, where $g \in G$, then gF^0 is entirely contained in A_0). Finiteness of F^0 now implies that there is a constant $c > 0$ such that

$$\left|S(\xi, \eta) \cap B(o, n)\right| \leq c\, n$$

for all n and for all distinct $\xi, \eta \in \partial^{(0)} X$. $\square$

CASE 2. G fixes an end ω of X. Then the following facts are known from [34], [45] and [46]; see [49] a for unified presentation.

THEOREM 6.11 ([34], [45] and [46]). (i) *The group G is amenable, and G acts transitively on $\partial^* X = \partial X \setminus \{o\}$.*

(ii) *The structure tree $\mathcal{T}$ is homogeneous with finite degree $q + 1 \geq 3$.*

(iii) *The structure map φ: $\widehat{X} \to \widehat{\mathcal{T}}$ is onto, and its restriction to ∂X is a homeomorphism $\partial X \to \partial \mathbf{T}$. There is an integer $a > 0$ such that*

$$d_{\mathcal{T}}(\varphi x, \varphi y) \leq d_X(x, y) \leq a\big(d(\varphi x, \varphi y) + 1\big).$$

This means that the picture is—up to the small "perturbation" described by the structure map—precisely the same as in the example of Section 6.1. Since G acts with compact stabilizers on the edges of $\mathcal{T}$ and $\mathcal{T}$ is locally finite, G also acts with compact stabilizers on the vertices of $\mathcal{T}$. Therefore, we may use in the formula (2.2) the action of G on $\mathcal{T}$ to determine the modular function of G. In other terms, $\Delta[x, y] = \Delta[\varphi x, \varphi y]$ for the cocycles $\Delta[\cdot, \cdot]$ on X and $\mathcal{T}$, respectively. Thus, one only has to make a few obvious modifications in the proofs of the results of [6] that we have subsumed above in Theorem 6.1 to obtain the following.

THEOREM 6.12. *Let P be a homogeneous Markov operator on X.*

(a) *If the modular drift $\delta(P, \Delta) = 0$, then the Poisson boundary is trivial.*

(b) *If* $\delta(P, \Delta) > 0$, *then the Markov chain* (x_n) *converges almost surely to a random point* $x_\infty \in \partial^* X$. *Denoting by* ν *the* $\mathbf{P}_o$*-distribution of* x_∞ *on* $\partial^* X$, *we have that* $\mathrm{supp}(\nu) = \partial^* \mathbf{T}$, *and the pair* $(\partial^* X, \nu)$ *is the Poisson boundary*.

(c) *If* $\delta(P, \Delta) < 0$, *then* $x_n \to \omega$ *almost surely*, *and the Poisson boundary is trivial*.

6.4. *Hyperbolic graphs.* A graph X with its discrete metric $d(\cdot, \cdot)$ is called hyperbolic (in the sense of Gromov) if there is a $\delta \geq 0$ such that every geodesic triangle (with vertices in X) is δ-thin. The latter means that for any point on one of the three sides there is another point at distance at most δ on one of the other two sides. We shall not lay out once more the basic features of hyperbolic graphs and their hyperbolic boundary and compactification, which (by an abuse of the notation of Section 6.3) we denote by ∂X and $\widehat{X}$, respectively. The reader is referred to the texts by Gromov [19], Ghys and de la Harpe [17], Coornaert, Delzant and Papadopoulos [7] or, for a presentation in the context of random walks on graphs, [49], Section 22.

The boundary of an infinite vertex-transitive hyperbolic graph is either infinite or has cardinality 2. In the latter case, it is again a graph with two ends that is roughly isometric with the two-way-infinite path, and the Poisson boundary of any homogeneous Markov operator with finite first moment is trivial (compare with Section 6.3). Thus, we assume that ∂X is infinite and—as usual—that $G = \mathrm{Aut}(X, P)$ acts transitively on X. In close analogy with Theorem 6.9, the following is known from Woess [47] (see also [49]) without assuming any moment condition.

THEOREM 6.13 [47]. *If* G *does not fix an element of* ∂X, *then* G *is nonamenable and the Markov chain* (x_n) *converges almost surely in the hyperbolic topology to a random point* $x_\infty \in \partial X$. *Denoting by* ν *the* $\mathbf{P}_o$*-distribution of* x_∞, *we have that* (i) $\mathrm{supp}(\nu) = \partial X$ *and* (ii) $\nu(\{\xi\}) = 0$ *for every* $\xi \in \partial X$.

In the special case of discrete groups that are Gromov-hyperbolic, the following result was proved by the same method by Kaimanovich [28].

THEOREM 6.14. *If* G *does not fix an element of* ∂X *and* P *has finite first moment*, *then the Poisson boundary of* P *coincides with the measure space* $(\partial X, \nu)$.

PROOF. The proof is as in the preceding examples. Theorem 6.13 applies both to P and to $\widehat{P}$, so that, analogously to the μ-boundary $(\partial X, \nu)$, we get the $\widehat{\mu}$-boundary $(\partial X, \widehat{\nu})$. We have to define the "strips" $S(\xi, \eta)$, where $\xi, \eta \in \partial X$. By property (ii) of ν and $\widehat{\nu}$, we have $\nu \otimes \widehat{\nu}(\{(\xi, \xi) : \xi \in \partial X\}) = 0$, so that it is sufficient to define $S(\xi, \eta)$ when $\xi \neq \eta$. We let

$$S(\xi, \eta) = \bigcup \{x \in X : x \text{ lies on a two way infinite geodesic between } \xi \text{ and } \eta\}.$$

Now, in a δ-hyperbolic graph as introduced above in terms of geodesic triangles, any two geodesics between the same pair of boundary points are at Hausdorff distance at most 2δ, and therefore there is a constant $c > 0$ such that

$$|S(\xi, \eta) \cap B(o, n)| \le c\, n$$

for all n and for all distinct $\xi, \eta \in \partial^{(0)} X$, just as in Theorem 6.10. $\square$

What one still has to consider is the "degenerate" case when G fixes a point in ∂X (which then has to be unique, given that ∂X is assumed to be infinite). The question is whether there exists an example of this type that goes beyond Theorem 6.11. In other words, if X is hyperbolic and has infinitely many ends and G fixes a point of the hyperbolic boundary, then it is easy to understand that the graph X is precisely as described in Theorem 6.11 (since the ends are the connected components of the hyperbolic boundary by Pavone [35]). So the question is whether there is a one-ended hyperbolic graph with a transitive group of automorphisms that fixes a boundary point. We have asked a few experts who believe that the answer is "no." (We gratefully acknowledge E-mail conversations with Nadia Benakli and Werner Ballmann on this question.)

REFERENCES

[1] AVEZ, A. (1972). Entropie des groupes de type fini. *C. R. Acad. Sci. Paris Sér. A–B* **275** 1363–1366.

[2] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999). Group-invariant percolation on graphs. *Geom. Funct. Anal.* **9** 29–66.

[3] BERTACCHI, D. (2001). Random walks on Diestel–Leader graphs. *Abh. Math. Sem. Univ. Hamburg* **71** 205–224.

[4] BLOOM, W. R. and HEYER, H. (1995). *Harmonic Analysis of Probability Measures on Hypergroups*. de Gruyter, Berlin.

[5] CARNE, T. K. (1985). A transmutation formula for Markov chains. *Bull. Sci. Math.* **109** 399–405.

[6] CARTWRIGHT, D. I., KAIMANOVICH, V. A. and WOESS, W. (1994). Random walks on the affine group of a homogeneous tree. *Ann. Inst. Fourier* (*Grenoble*) **44** 1243–1288.

[7] COORNAERT, M., DELZANT, T. and PAPADOPOULOS, A. (1990). *Géométrie et Théorie des Groupes: les Groupes Hyperboliques de Gromov. Lecture Notes in Math.* **1441**. Springer, Berlin.

[8] DERRIENNIC, Y. (1976). Lois "zéro ou deux" pour les processus de Markov, applications aux marches aléatoires. *Ann. Inst. H. Poincaré Sec. B* **12** 111–129.

[9] DERRIENNIC, Y. (1980). Quelques applications du théorème ergodique sous-additif. *Astérisque* **74** 183–201.

[10] DERRIENNIC, Y. (1986). Entropie, théorèmes limites et marches aléatoires. *Lecture Notes in Math.* **1210** 241–284. Springer, Berlin.

[11] DICKS, W. and DUNWOODY, M. J. (1989). *Groups Acting on Graphs*. Cambridge Univ. Press, Cambridge.

[12] DODZIUK, J. (1984). Difference equations, isoperimetric inequality, and transience of certain random walks. *Trans. Amer. Math. Soc.* **284** 787–794.

[13] DUNWOODY, M. J. (1982). Cutting up graphs. *Combinatorica* **2** 15–23.

[14] DYNKIN, E. B. (1982). *Markov Processes and Related Problems of Analysis*. Cambridge Univ. Press.

[15] FREUDENTHAL, H. (1944). Über die Enden diskreter Räume und Gruppen. *Comment. Math. Helv.* **17** 1–38.

[16] FURSTENBERG, H. (1973). Boundary theory and stochastic processes on homogeneous spaces. *Proc. Sympos. Pure Math.* **26** 193–229.

[17] GHYS, E. and DE LA HARPE, P. (eds.) (1990). *Sur les Groupes Hyperboliques d'après Mikhael Gromov*. Birkhäuser, Boston.

[18] GRIGORCHUK, R. I. and DE LA HARPE, P. (1997). On problems related to growth, entropy and spectrum in group theory. *J. Dynam. Control Systems* **3** 55–89.

[19] GROMOV, M. (1987). Hyperbolic groups. In *Essays in Group Theory* (S. M. Gersten, ed.) 75–263. Springer, New York.

[20] IMRICH, W. and SEIFTER, N. (1991). A survey on graphs with polynomial growth. *Discrete Math.* **95** 101–117.

[21] JEWETT, R. I. (1975). Spaces with an abstract convolution of measures. *Adv. Math.* **18** 1–101.

[22] KAIMANOVICH, V. A. (1988). Brownian motion on foliations: entropy, invariant measures, mixing. *Funct. Anal. Appl.* **22** 326–328.

[23] KAIMANOVICH, V. A. (1991). Poisson boundaries of random walks on discrete solvable groups. In *Proceedings of Conference on Probability Measures on Groups* (H. Heyer, ed.) 205–238. Plenum, New York.

[24] KAIMANOVICH, V. A. (1992). Measure-theoretic boundaries of Markov chains, 0–2 laws and entropy. In *Proceedings of the Conference on Harmonic Analysis and Discrete Potential Theory* (M. A. Picardello, ed.) 145–180. Plenum, New York.

[25] KAIMANOVICH, V. A. (1995). The Poisson boundary of covering Markov operators. *Israel J. Math.* **89** 77–134.

[26] KAIMANOVICH, V. A. (1996). Boundaries of invariant Markov operators: the identification problem. In *Ergodic Theory of $\mathbb{Z}^d$ Actions* (M. Pollicott and K. Schmidt, eds.) 127–176. Cambridge Univ. Press.

[27] KAIMANOVICH, V. A. (1998). Hausdorff dimension of the harmonic measure on trees. *Ergodic Theory Dynam. Systems* **18** 631–660.

[28] KAIMANOVICH, V. A. (2000). The Poisson formula for groups with hyperbolic properties. *Ann. Math.* **152** 659–692.

[29] KAIMANOVICH, V. A. and VERSHIK, A. M. (1983). Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** 457–490.

[30] KAIMANOVICH, V. A. and WOESS, W. (1995). Construction of discrete, non-unimodular hypergroups. In *Probability Measures on Groups and Related Structures, XI* (H. Heyer, ed.) 196–209. World Scientific, River Edge, NJ.

[31] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

[32] LEDRAPPIER, F. (1985). Poisson boundaries of discrete groups of matrices. *Israel J. Math.* **50** 319–336.

[33] LEDRAPPIER, F. (1992). Sharp estimates for the entropy. In *Proceedings of the Conference on Harmonic Analysis and Discrete Potential Theory* (M. A. Picardello, ed.) 281–288. Plenum, New York.

[34] MÖLLER, R. G. (1992). Ends of graphs II. *Math. Proc. Cambridge Philos. Soc.* **111** 455–460.

[35] PAVONE, M. (1989). Boundaries of discrete groups, Toeplitz operators and extensions of the reduced C^*-algebra. Ph.D. dissertation, Univ. California, Berkeley.

[36] ROKHLIN, V. A. (1996). Lectures on the entropy theory of measure preserving transformations. *Russian Math. Surveys* **22** 1–52.

[37] SALOFF-COSTE, L. and WOESS, W. (1996). Computing norms of group-invariant transition operators. *Combin. Probab. Comput.* **4** 419–442.

[38] SALOFF-COSTE, L. and WOESS, W. (1997). Transition operators, groups, norms, and spectral radii. *Pacific J. Math.* **180** 333–367.

[39] SCHLICHTING, G. (1979). Polynomidentitäten und Permutationsdarstellungen lokalkompakter Gruppen. *Invent. Math.* **55** 97–106.

[40] SOARDI, P. M. and WOESS, W. (1990). Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. *Math. Z.* **205** 471–486.

[41] THOMASSEN, C. and WOESS, W. (1993). Vertex-transitive graphs and accessibility. *J. Combin. Theory Ser. B* **58** 248–268.

[42] TROFIMOV, V. I. (1985). Graphs with polynomial growth. *Math. USSR Sb.* **51** 405–417.

[43] TROFIMOV, V. I. (1985). Automorphism groups of graphs as topological groups. *Math. Notes* **38** 717–720.

[44] VAROPOULOS, N. TH. (1985). Long range estimates for Markov chains. *Bull. Sci. Math.* **109** 225–252.

[45] WOESS, W. (1989). Amenable group actions on infinite graphs. *Math. Ann.* **284** 251–265.

[46] WOESS, W. (1989). Boundaries of random walks on graphs and groups with infinitely many ends. *Israel J. Math.* **68** 271–301.

[47] WOESS, W. (1993). Fixed sets and free subgroups of groups acting on metric spaces. *Math. Z.* **214** 425–440.

[48] WOESS, W. (1994). Topological groups and recurrence of quasi transitive graphs. *Rend. Sem. Mat. Fis. Milano* **64** 185–213.

[49] WOESS, W. (2000). *Random Walks on Infinite Graphs and Groups*. Cambridge Univ. Press.

CNRS UMR-6625
IRMAR, UNIVERSITÉ DE RENNES-1
35042 RENNES
FRANCE
E-MAIL: kaimanov@univ-rennes1.fr

INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT GRAZ
8010 GRAZ
AUSTRIA
E-MAIL: woess@tugraz.at