The Annals of Probability
2000, Vol. 28, No. 4, 1884–1907

A SYMMETRIZATION–DESYMMETRIZATION PROCEDURE FOR UNIFORMLY GOOD APPROXIMATION OF EXPECTATIONS INVOLVING ARBITRARY SUMS OF GENERALIZED U-STATISTICS

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Let Φ be a symmetric function, nondecreasing on $[0,\infty)$ and satisfying a Δ_2 growth condition, $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be independent random vectors such that (for each $1\le i\le n$) either $Y_i=X_i$ or Y_i is independent of all the other variates, and the marginal distributions of $\{X_i\}$ and $\{Y_j\}$ are otherwise arbitrary. Let $\{f_{ij}(x,y)\}_{1\le i,\,j\le n}$ be any array of real valued measurable functions. We present a method of obtaining the order of magnitude of

$$E\Phi\biggl(\sum_{1\le i,\,j\le n} f_{ij}(X_i,Y_j)\biggr).$$

The proof employs a double symmetrization, introducing independent copies $\{\widetilde X_i,\widetilde Y_j\}$ of $\{X_i,\ Y_j\}$, and moving from summands of the form $f_{ij}(X_i,\ Y_j)$ to what we call $f_{ij}^{(s)}(X_i,\ Y_j,\ \widetilde X_i,\ \widetilde Y_j)$. Substitution of fixed constants $\tilde x_i$ and $\tilde y_j$ for $\widetilde X_i$ and $\widetilde Y_j$ results in $f_{ij}^{(s)}(X_i,Y_j,\tilde x_i,\tilde y_j)$, which equals $f_{ij}(X_i,Y_j)$ adjusted by a sum of quantities of first order separately in X_i and Y_j. Introducing further explicit first-order adjustments, call them $g_{1ij}(X_i,\tilde{\mathbf x},\tilde{\mathbf y})$ and $g_{2ij}(Y_j,\tilde{\mathbf x},\tilde{\mathbf y})$, it is proved that

$$E\Phi\biggl(\sum_{1\le i,\,j\le n}\Bigl(f_{ij}^{(s)}\bigl(X_i,Y_j,\tilde x_i,\tilde y_j\bigr)-g_{1ij}\bigl(X_i,\tilde{\mathbf x},\tilde{\mathbf y}\bigr)-g_{2ij}\bigl(Y_j,\tilde{\mathbf x},\tilde{\mathbf y}\bigr)\Bigr)\biggr)$$

$$\le_\alpha E\Phi\biggl(\sqrt{\sum_{1\le i,\,j\le n}\Bigl(f_{ij}^{(s)}\bigl(X_i,Y_j,\tilde x_i,\tilde y_j\bigr)\Bigr)^2}\biggr)\approx_\alpha \Phi\bigl(\mathbf f^{(s)},\mathbf X,\mathbf Y,\tilde{\mathbf x},\tilde{\mathbf y}\bigr)$$

where the latter is an explicitly computable quantity. For any $\tilde{\mathbf x}^0$ and $\tilde{\mathbf y}^0$ which come within a factor of two of minimizing $\Phi(\mathbf f^{(s)},\mathbf X,\mathbf Y,\tilde{\mathbf x},\tilde{\mathbf y})$ it is shown that

$$E\Phi\biggl(\sum_{1\le i,\,j\le n} f_{ij}(X_i,Y_j)\biggr)$$

$$\approx_\alpha \max\biggl\{\Phi\bigl(\mathbf f^{(s)},\mathbf X,\mathbf Y,\tilde{\mathbf x}^0,\tilde{\mathbf y}^0\bigr), E\Phi\biggl(\sum_{1\le i,\,j\le n}\Bigl(f_{ij}\bigl(X_i,\tilde y_j^0\bigr)+f_{ij}\bigl(\tilde x_i^0,Y_j\bigr)$$

$$-\,f_{ij}\bigl(\tilde x_i^0,\tilde y_j^0\bigr)+g_{1ij}\bigl(X_i,\tilde x_i^0,\tilde y_j^0\bigr)+g_{2ij}\bigl(Y_j,\tilde x_i^0,\tilde y_j^0\bigr)\Bigr)\biggr)\biggr\},$$

Received March 1998; revised November 1999.

[1]Supported by NSF Grants: DMS-96-26236 and DMS-99-72417.

[2]Supported by the Swedish Council for Research in the Humanities and Social Sciences (HSFR).

AMS 1991 *subject classifications*. 60E15, 60F25, 60G50.

Key words and phrases. Generalized U-statistics, symmetrization–desymmetrization, expectations of functions of second-order sums.

which is computable (approximable) in terms of the underlying random variables. These results extend to the expectation of Φ of a sum of functions of k-components.

1. Introduction. Let $\{X_i, Y_i\}_{i=1}^n$ be a collection of $2n$ independent random variables. Set

$$\Delta_2 \equiv \{\text{symmetric functions } \Phi(\cdot), \text{ nondecreasing on } [0,\infty) \text{ with } \Phi(0)=0 \text{ and such that for some } \alpha>0, \Phi(cx)\le|c|^\alpha\Phi(x) \text{ for all } |c|\ge 2 \text{ and all } x\}.$$

Such a $\Phi \in \Delta_2$ is said to have parameter α (and hence it has parameter β for all $\beta \ge \alpha$).

We are interested in approximating

$$\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y}) \equiv E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\right) \tag{1.1}$$

for arbitrary real-valued measurable functions $\mathbf{f} = \{f_{ij}(x, y)\}_{1\le i,\, j\le n}$.

First results in this direction were obtained by Giné and Zinn (1992). Specifically, they showed that for any independent r.v.'s $X_1, Y_1, \dots, X_n, Y_n$ such that $\mathscr{L}(X_i) = \mathscr{L}(Y_i)$ for $1 \le i \le n$ and any function $f(x, y)$ satisfying $f(x, y) = f(y, x)$ for all x, y with the further property that $Ef(X_i, y) = 0$ for all y and i,

$$E\left|\sum_{1\le i,\, j\le n} f(X_i, Y_j)\right|^p \le_p E\left[\max_{1\le i\le n}\left|\sum_{j=1}^n f(X_i, Y_j)\right|\right]^p + \left[E\left|\sum_{1\le i,\, j\le n} f(X_i, Y_j)\right|\right]^p \quad \text{for } p \ge 1,$$

where $A \le_p B (A \ge_p B)$ means that there is a universal constant $\bar{c}_p < \infty (c_p > 0)$ depending only on p such that $A \le \bar{c}_p B (A \ge c_p B)$ and $A \approx_p B$ means that $A \le_p B$ and $A \ge_p B$.

The nonnegative case of $f_{ij}(x, y) \ge 0$ and $\Phi \in \Delta_2$ was treated in Klass and Nowicki (1997). The goal of this paper is to convert the general problem of (1.1) into this nonnegative case, with the possible adjoining of a sum of first-order variates. To do so, we employ the use of conditionally symmetric variables, thereby obtaining a lower bound. Thus, just as symmetrization of Z entails $E\Phi(Z) \ge 2^{-\alpha-1}E\Phi(Z - \widetilde{Z})$, where $\widetilde{Z}$ is an independent copy of r.v. Z, applying this idea twice (first on the set of $\{X_i\}_{i=1}^n$ and then on the set of $\{Y_j\}_{j=1}^n$) gives

$$E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\right) \ge 4^{-\alpha-1}E\Phi\left(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\right). \tag{1.2}$$

where

$$f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j) = f_{ij}(X_i, Y_j) - f_{ij}(\widetilde{X}_i, Y_j) - f_{ij}(X_i, \widetilde{Y}_j) + f_{ij}(\widetilde{X}_i, \widetilde{Y}_j) \tag{1.3}$$

and $\{\widetilde{X}_i, \widetilde{Y}_i\}_{i=1}^n$ are independent copies of $\{X_i, Y_i\}_{i=1}^n$.

Take any fixed $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$ and $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_n)$. Observe that $\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)$ is just our original sum $\sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j)$ adjusted by a sum of first-order terms, that is,

$$\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) = \sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j) - \sum_{i=1}^n f_{1i}(X_i) - \sum_{j=1}^n f_{2j}(Y_j), \tag{1.4}$$

where

$$f_{1i}(X_i) = \sum_{j=1}^n f_{ij}(X_i, \tilde{y}_j) - \tfrac{1}{2} \sum_{j=1}^n f_{ij}(\tilde{x}_i, \tilde{y}_j) \tag{1.5}$$

and

$$f_{2j}(Y_j) = \sum_{i=1}^n f_{ij}(\tilde{x}_i, Y_j) - \tfrac{1}{2} \sum_{i=1}^n f_{ij}(\tilde{x}_i, \tilde{y}_j). \tag{1.6}$$

As explained further in Lemma 2.4 (below), since the RHS of (1.7) below is an average there must exist instances $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$ and $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_n)$ such that

$$E\Phi\biggl(\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\biggr) \le E\Phi\biggl(\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\biggr), \tag{1.7}$$

The idea of using the average of a nonconstant function over a set to produce the existence of an element of the set whose functional value is either less or greater (as desired) than that of the average was used by de Acosta (1980) and probably dates back hundreds of years. For a long time Erdős championed its use in probabilistic combinatorics.

Combining (1.2) and (1.7), Lemma 2.3 (below) shows that

$$E\Phi\biggl(\sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j)\biggr) \approx_\alpha \max\biggl\{E\Phi\biggl(\sum_{1 \le i,\, j \le n} f_{ij}(X_i, Y_j) - \sum_{i=1}^n f_{1i}(X_i) - \sum_{j=1}^n f_{2j}(Y_i)\biggr), E\Phi\biggl(\sum_{i=1}^n f_{1i}(X_i) + \sum_{j=1}^n f_{2j}(Y_i)\biggr)\biggr\} \tag{1.8}$$

and also that

$$
(1.9)\quad E\Phi\Big(\sum_{1\le i,\,j\le n} f_{ij}(X_i, Y_j)\Big) \approx_\alpha \max\Big\{E\Phi\Big(\sum_{1\le i,\,j\le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\Big),\ E\Phi\Big(\sum_{i=1}^n f_{1i}(X_i) + \sum_{j=1}^n f_{2j}(Y_j)\Big)\Big\}.
$$

If we were given $f_{1i}(\cdot)$ and $f_{2i}(\cdot)$, each of the components in the maximum above would now be computable. The expectation involving the sum of $2n$ independent r.v.'s could be approximated using results in Klass (1981), given as Theorem A.3 (below). As to the other quantity, let $\widehat{X}_i = (X_i, \widetilde{X}_i)$ and $\widehat{Y}_j = (Y_j, \widehat{Y}_j)$ and define $\hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_j)$ as $f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)$. The function $\hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_i)$ is (separately) conditionally symmetric in $\widehat{X}_i$ and in $\widehat{Y}_j$. Since Theorem 3.2 in Klass and Nowicki (1998), given as Theorem A.4 (below), also applies to random elements [see Lemma 2.2 (below)],

$$
(1.10)\quad E\Phi\Big(\sum_{1\le i,\,j\le n} \hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_j)\Big) \approx_\alpha E\Phi\left(\sqrt{\sum_{1\le i,\,j\le n}\Big[\hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_j)\Big]^2}\right).
$$

The latter involves the expectation of a Δ_2-function of a sum of nonnegative generalized U-statistics. This is approximable by Theorem 4.3 in Klass and Nowicki (1997) extended from r.v.'s X_i and Y_j to random elements $\widehat{X}_i$ and $\widehat{Y}_i$. To present this approximation we need to introduce the following quantities.

For $\hat{x}_i = (x_i, \tilde{x}_i)$ and $\hat{y}_i = (y_j, \tilde{y}_j)$ let $\hat{v}_{1i}(\hat{x}_i)$, $\hat{v}_{2j}(\hat{y}_j)$, $\hat{v}_{1*}$, $\hat{v}_{2*}$, and $\hat{w}_*$ be defined as

$$
(1.11)\quad \hat{v}_{1i}(\hat{x}_i) = \sup\Big\{v \ge 0,\ \sum_{j=1}^n E\Big(\Big(f_{ij}^{(s)}(x_i, Y_j, \tilde{x}_i, \widetilde{Y}_j)\Big)^2 \wedge v^2\Big) \ge v^2\Big\},
$$

$$
(1.12)\quad \hat{v}_{2j}(\hat{y}_j) = \sup\Big\{v \ge 0,\ \sum_{i=1}^n E\Big(\Big(f_{ij}^{(s)}(X_i, y_j, \widetilde{X}_i, \tilde{y}_j)\Big)^2 \wedge v^2\Big) \ge v^2\Big\},
$$

$$
(1.13)\quad \hat{v}_{1*} = \sup\Big\{v \ge 0,\ \sum_{i=1}^n E\big(v_{1i}^2(\widehat{X}_i) \wedge v^2\big) \ge v^2\Big\},
$$

$$
(1.14)\quad \hat{v}_{2*} = \sup\Big\{v \ge 0,\ \sum_{j=1}^n E\big(v_{2j}^2(\widehat{Y}_j) \wedge v^2\big) \ge v^2\Big\}
$$

and

$$
(1.15)\quad \hat{w}_* = \sup\Big\{w \ge 0 : \sum_{1\le i,\,j\le n} E\Big(\Big[(\hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_j))^2 \wedge w^2\Big] \times I\Big(\big|\hat{f}_{ij}^{(s)}(\widehat{X}_i, \widehat{Y}_j)\big| > \big(\hat{v}_{1i}(\widehat{X}_i) \vee \hat{v}_{2j}(\widehat{Y}_j)\big)\Big)\Big) \ge w^2\Big\}.
$$

Then,

$$E\Phi\left(\sqrt{\sum_{1\le i,\ j\le n}\left[\hat{f}_{ij}^{(s)}(\widehat{X}_i,\widehat{Y}_j)\right]^2}\right)\approx_\alpha \Phi(\hat{\mathbf{f}}^{(s)},\widehat{\mathbf{X}},\widehat{\mathbf{Y}}), \tag{1.16}$$

where

$$\begin{aligned}\Phi(\hat{\mathbf{f}}^{(s)},\ \widehat{\mathbf{X}},\ \widehat{\mathbf{Y}})\equiv\max\Big\{&E\max\nolimits_{1\le i,\ j\le n}\Phi\Big(\hat{f}_{ij}^{(s)}(\widehat{X}_i,\widehat{Y}_j)\Big),\\ &E\max\nolimits_{1\le i\le n}\Phi\big(\hat{v}_{1i}(\widehat{X}_i)\big),\\ &E\max_{1\le j\le n}\Phi\big(\hat{v}_{2j}(\widehat{Y}_j)\big),\Phi(\hat{v}_{1*}),\Phi(\hat{v}_{2*}),\Phi(\hat{w}_*)\Big\}.\end{aligned} \tag{1.17}$$

The above described approach shows that the order of magnitude of $\mathscr{E}(\Phi,\mathbf{f},\mathbf{X},\mathbf{Y})$ is governed by the maximum of $\Phi(\hat{\mathbf{f}}^{(s)},\widehat{\mathbf{X}},\widehat{\mathbf{Y}})$ and a quantity which involves the sum of first-order terms. Though we have not found a constructive method of producing such a first-order sum from a vector pair $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ satisfying (1.4) we now manage to overcome this hurdle by simply dropping the condition that $f_{1i}(\cdot)$ and $f_{2j}(\cdot)$ come from such a vector pair. We merely need to retain the key consequence of the vector pair assumption, namely that we can construct first-order terms $f_{1i}^0(\cdot)$ and $f_{2j}^0(\cdot)$ satisfying

$$\begin{aligned}&E\Phi\left(\sum_{1\le i,\ j\le n}f_{ij}(X_i,Y_j)-\sum_{i=1}^n f_{1i}^0(X_i)-\sum_{j=1}^n f_{2j}^0(Y_j)\right)\\ &\qquad\le_\alpha E\Phi\left(\sum_{1\le i,\ j\le n}f_{ij}^{(s)}(X_i,Y_j,\widetilde{X}_i,\widetilde{Y}_j)\right).\end{aligned} \tag{1.18}$$

Then, Lemma 2.3 together with (1.2), (1.10) and (1.16) ensure that

$$\begin{aligned}&E\Phi\left(\sum_{1\le i,\ j\le n}f_{ij}(X_i,Y_j)\right)\\ &\quad\approx_\alpha\max\left\{\Phi(\hat{\mathbf{f}}^{(s)},\widehat{\mathbf{X}},\widehat{\mathbf{Y}}),\ E\Phi\left(\sum_{i=1}^n f_{1i}^0(X_i)+\sum_{j=1}^n f_{2j}^0(Y_j)\right)\right\}.\end{aligned} \tag{1.19}$$

In Section 3 we prove that this leads to an explicit construction of $f_{1i}^0(\cdot)$ and $f_{2j}^0(\cdot)$.

In the approximation of $\mathscr{E}(\Phi,\mathbf{f},\mathbf{X},\mathbf{Y})$ discussed above we initially passed from $f_{ij}(X_i,Y_j)$ to

$$\begin{aligned}f_{ij}^{(s)}(X_i,Y_j,\tilde{x}_i,\tilde{y}_j)\equiv f_{ij}(X_i,Y_j)&-f_{ij}(\tilde{x}_i,Y_j)\\ &-f_{ij}(X_i,\tilde{y}_j)+f_{ij}(\tilde{x}_i,\tilde{y}_j).\end{aligned} \tag{1.20}$$

The reason for such an unexpected transformation seems to be explained roughly as follows: we would like to construct quantities to approximate

$\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y})$ based on the local behavior of f_{ij}. If such a method is to work, the local truncation levels must become zero whenever the global sum

$$\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)$$

is zero. Thus, for example, suppose $f_{ij}(X_i, Y_j) = (-1)^{i+j}(X_i + Y_j)$ and n is even. Then the global sum $\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j) \equiv 0$ (even if X_i and Y_j lack every positive moment), so

$$E\Phi\biggl(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\biggr) = E\Phi(0) = \Phi(0) = 0.$$

However, for any local approximation method based on $f_{ij}(X_i, Y_j)$ alone [i.e., based on quantities such as $E\Phi(\max_{1\le i,\, j\le n} |f_{ij}(X_i, Y_j)|)$, $E\Phi(\max_{1\le i\le n} |\sum_{j=1}^n f_{ij}(X_i, Y_j)|)$ or $E\Phi(\max_{1\le j\le n} |\sum_{i=1}^n f_{ij}(X_i, Y_j)|)$] our approximation method will produce nonzero quantities and truncation levels whenever $P(X_i + Y_j = 0) < 1$ for some i and j. Hence any such approximation would produce Φ of a positive number (which is positive) and thus fail to be proportional to $\Phi(0)$. Therefore, using f_{ij} alone to generate our approximation quantities cannot be uniformly valid. For this reason we use $f_{ij}^{(s)}$ and construct the quantities given in (1.11)–(1.15).

We believe that the general form of the pathology described above is characterized by the fact that for any functions $g_{ij}(X_i)$ and $h_{ij}(Y_j)$ such that $\sum_{j=1}^n g_{ij}(X_i) \equiv 0$ and $\sum_{i=1}^n h_{ij}(Y_j) \equiv 0$ and any functions $f_{ij}(X_i, Y_j)$,

$$\begin{aligned} E\Phi\biggl(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\biggr) \\ \equiv E\Phi\biggl(\sum_{1\le i,\, j\le n} \bigl(f_{ij}(X_i, Y_j) - g_{ij}(X_i) - h_{ij}(Y_j)\bigr)\biggr). \end{aligned} \tag{1.21}$$

The compensate for the very real possibility that $f_{ij}(X_i, Y_j)$ in the LHS of (1.21) has been locally distorted by quantities such as $g_{ij}(X_i)$ and $h_{ij}(Y_j)$ as in the RHS of (1.21), we rewrite $f_{ij}(X_i, Y_j)$ as a sum of $f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)$ and first-order terms, thereby creating quasi-canonical second-order terms which cancel the effect of any $g_{ij}(X_i)$ and $h_{ij}(Y_j)$ which may be present in the original formulation, plus some remaining (quasi-canonically determined) first-order terms.

An analogous complication but in simpler form already occurs in the problem of approximation of the expectation of a sum of independent r.v.'s. For that problem,

$$E\Phi\biggl(\sum_{i=1}^n X_i\biggr) = E\Phi\biggl(\sum_{i=1}^n (X_i + c_i)\biggr),$$

for any real $\{c_i\}_{i=1}^n$ such that $\sum_{i=1}^n c_i = 0$. To approximate this expectation, Klass (1981) used a method relying on precentering by medians and then subtracting the resulting truncated expectations to produce canonical independent variates together with the augmentation of an $(n+1)$th constant term to compensate for the constants added to (or, rather, subtructed from) each individual term. Here, we introduce a different method which extends more readily to the case of k-component generalized U-statistics.

Our results can be generalized to the case in which $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ are independent random vectors such that either $Y_i = X_i$ or Y_i is independent of all the other variates, and the marginal distributions $\{X_i\}$ and $\{Y_j\}$ are otherwise arbitrary as in Klass and Nowicki (1998); see Lemma 4.12, Remark 4.13, Theorem 4.14 and Remark 4.15 and then further extend to the k-component case.

The paper is organized as follows: Section 2 introduces some lemmas used to derive bounds for quantities related to $\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y})$. Section 3 develops two-sided bounds for $\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y})$. Section 4 shows that the previous results are generalizable to the k-component case

$$E\Phi\bigg(\sum_{1\le i_1,\dots,i_k\le n} f_{i_1\cdots i_k}\Big(X_{i_1}^{(1)}, \dots, X_{i_k}^{(k)}\Big)\bigg),$$

where, for each $1 \le j \le k$ and $1 \le i \le n$, $X_i^{(j)}$ are independent random elements. Finally, the Appendix provides the reader with some supplementary results which he may want to have on hand.

2. Preliminaries. Unless augmented or stated to the contrary, the subsequent lemmas and theorems of the paper will be based on the following assumptions: $\{X_i, \widetilde{X}_i, Y_i, \widetilde{Y}_i\}_{i=1}^n$ is a collection of $4n$ independent random elements such that $\mathscr{L}(X_i, Y_j) = \mathscr{L}(\widetilde{X}_i, \widetilde{Y}_j)$, for $1 \le i, j \le n$, $\{f_{ij}(x, y)\}_{1\le i,\, j\le n}$ is any array of real valued functions and $\Phi \in \Delta_2$ has some parameter $\alpha > 0$.

Double-symmetrizing, we introduce $f_{ij}^{(s)}$ and obtain a lower bound for $\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y})$.

LEMMA 2.1. *Let* $f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)$ *be defined as in* (1.3). *Then*

$$E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\bigg) \ge 4^{-\alpha-1} E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\bigg). \tag{2.1}$$

PROOF. First, note that

$$\begin{aligned}
&\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\bigg)\\
&\quad\le \Phi\bigg(4\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)\bigg) + \Phi\bigg(4\sum_{1\le i,\, j\le n} f_{ij}(\widetilde{X}_i, Y_j)\bigg)\\
&\qquad + \Phi\bigg(4\sum_{1\le i,\, j\le n} f_{ij}(X_i, \widetilde{Y}_j)\bigg) + \Phi\bigg(4\sum_{1\le i,\, j\le n} f_{ij}(\widetilde{X}_i, \widetilde{Y}_j)\bigg)
\end{aligned}$$

$$\le 4^{\alpha}\Bigg(\Phi\Bigg(\sum_{1\le i,\,j\le n} f_{ij}(X_i, Y_j)\Bigg) + \Phi\Bigg(\sum_{1\le i,\,j\le n} f_{ij}(\widetilde{X}_i, Y_j)\Bigg) + \Phi\Bigg(\sum_{1\le i,\,j\le n} f_{ij}(X_i, \widetilde{Y}_j)\Bigg) + \Phi\Bigg(\sum_{1\le i,\,j\le n} f_{ij}(\widetilde{X}_i, \widetilde{Y}_j)\Bigg)\Bigg).$$

Observing that $\mathscr{L}(f_{ij}(X_i, Y_j)) = \mathscr{L}(f_{ij}(\widetilde{X}_i, Y_j)) = \mathscr{L}(f_{ij}(X_i, \widetilde{Y}_j)) = \mathscr{L}(f_{ij}(\widetilde{X}_i, \widetilde{Y}_j))$ and taking expectations

$$E\Phi\Bigg(\sum_{1\le i,\,j\le n} f^{(s)}_{ij}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\Bigg) \le 4^{\alpha+1} E\Phi\Bigg(\sum_{1\le i,\,j\le n} f_{ij}(X_i, Y_j)\Bigg). \qquad \square$$

As already discussed in Section 1, the next lemma follows from Theorem 3.2, in Klass and Nowicki (1998).

LEMMA 2.2. *Let* $f^{(s)}_{ij}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)$ *be defined as in* (1.3). *Then*

$$E\Phi\Bigg(\sum_{1\le i,\,j\le n} f^{(s)}_{ij}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\Bigg) \approx_{\alpha} E\Phi\Bigg(\sqrt{\sum_{1\le i,\,j\le n}\Big(f^{(s)}_{ij}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\Big)^2}\Bigg). \tag{2.2}$$

The next lemma gives a sufficient condition for approximating $E\Phi(S_1+S_2)$ in terms of the simpler quantities $E\Phi(S_1)$ and $E\Phi(S_2)$.

LEMMA 2.3. *Let* $S = S_1 + S_2$. *If* $E\Phi(S_1) \le_{\alpha} E\Phi(S)$ *then*

$$E\Phi(S) \approx_{\alpha} \max\{E\Phi(S_1), E\Phi(S_2)\}.$$

PROOF.

$$\Phi(S) = \Phi(S_1 + S_2) \le \Phi(2S_1) + \Phi(2S_2) \le 2^{\alpha}\Phi(S_1) + 2^{\alpha}\Phi(S_2).$$

Taking expectations,

$$E\Phi(S) \le 2^{\alpha+1} \max_{1\le j\le 2} E\Phi(S_j).$$

Since $S_2 = S + (-S_1)$, and since $\Phi(x) = \Phi(|x|)$ the same argument gives

$$E\Phi(S_2) \le 2^{\alpha+1} \max\{E\Phi(S), E\Phi(S_1)\} \le_{\alpha} E\Phi(S). \qquad \square$$

Double symmetrization begins with $2n$ independent r.v.'s (or random elements) and ends with $4n$ independent r.v.'s (or random elements). The following lemma introduces a substitution principle by which one can reduce back to $2n$ independent r.v.'s (or random elements).

LEMMA 2.4. *Let*

$$\mathscr{A}_c = \left\{ (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \colon E\Phi\left(\sqrt{\sum_{1\le i,\, j\le n} \left(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\right)^2}\right) \le cE\Phi\left(\sqrt{\sum_{1\le i,\, j\le n} \left(f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\right)^2}\right)\right\}. \tag{2.3}$$

Then for all $c \ge 1$,

$$\mathscr{A}_c \ne \varnothing.$$

PROOF. It suffices to observe that $P((\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) \notin \mathscr{A}_c) < 1$. This holds since not all values assumed by a r.v.,

$$H(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) = E\left(\Phi\left(\sqrt{\sum_{1\le i,\, j\le n} \left(f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\right)^2}\right) \middle| \{\widetilde{X}_i, \widetilde{Y}_j\}\right)$$

can exceed its expectation

$$EH(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) = E\Phi\left(\sqrt{\sum_{1\le i,\, j\le n} \left(f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\right)^2}\right). \qquad \square$$

Used to approximate the RHS of (2.3), Theorem 4.3 in Klass and Nowicki (1997) can also be applied to approximate the LHS of (2.3). To do so, let $v_{1i}(x, \tilde{x}_i, \tilde{\mathbf{y}})$, $v_{2j}(\tilde{y}, \tilde{\mathbf{x}}, \tilde{y}_j)$, $v_{1*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, $v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, and $w_*(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ be defined as

$$v_{1i}(x, \tilde{x}_i, \tilde{\mathbf{y}}) = \sup\left\{v \ge 0,\ \sum_{j=1}^{n} E\left(\left(f_{ij}^{(s)}(x, Y_j, \tilde{x}_i, \tilde{y}_j)\right)^2 \wedge v^2\right) \ge v^2\right\}, \tag{2.4}$$

$$v_{2j}(y, \tilde{\mathbf{x}}, \tilde{y}_j) = \sup\left\{v \ge 0,\ \sum_{i=1}^{n} E\left(\left(f_{ij}^{(s)}(X_i, y, \tilde{x}_i, \tilde{y}_j)\right)^2 \wedge v^2\right) \ge v^2\right\}, \tag{2.5}$$

$$v_{1*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \sup\left\{v \ge 0,\ \sum_{i=1}^{n} E\left(v_{1i}^2(X_i, x_i, \tilde{\mathbf{y}}) \wedge v^2\right) \ge v^2\right\}, \tag{2.6}$$

$$v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \sup\left\{v \ge 0,\ \sum_{j=1}^{n} E\left(v_{2j}^2(Y_j, \tilde{\mathbf{x}}, y_j) \wedge v^2\right) \ge v^2\right\} \tag{2.7}$$

and

$$w_*(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \sup\Bigg\{w \ge 0 : \sum_{1\le i,\, j\le n} E\left(\left(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\right)^2 \wedge w^2\right) \times I\Big(|f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)| > \left(v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}}) \vee v_{2j}(Y_j, \tilde{\mathbf{x}}, y_j)\right)\Big) \ge w^2\Bigg\}. \tag{2.8}$$

LEMMA 2.5.

$$E\Phi\left(\sqrt{\sum_{1\le i,\, j\le n}\left(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\right)^2}\right) \approx_\alpha \Phi(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \tag{2.9}$$

where

$$\begin{aligned}\Phi(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \equiv \max\Big\{&E \max_{1\le i,\, j\le n} \Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big),\\ &E\max_{1\le i\le n}\Phi\Big(v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}})\Big),\\ &E\max_{1\le i\le n}\Phi\Big(v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j)\Big),\\ &\Phi\big(v_{1*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big), \Phi\big(v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big), \Phi\big(w_*(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)\Big\}.\end{aligned} \tag{2.10}$$

3. Two-sided uniform bounds for generalized U-statistics. We begin Section 3 by introducing an explicit method of adjusting $\sum_{1\ge i,\, j\ge n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)$ by a sum of first-order terms so that the relevant functional expectation of the adjusted quantity is of order no larger than $\Phi(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. To expedite the derivation of this fact we define the following sets of events:

$$A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}}) = \Big\{\Big|f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big| > v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}})\Big\}, \tag{3.1}$$

$$A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j) = \Big\{\Big|f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big| > v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j)\Big\}, \tag{3.2}$$

$$B_{ij}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \Big\{\Big|f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big| \le w_*\Big\}, \tag{3.3}$$

$$C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \Big\{v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}}) \le v_{1*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big\}, \tag{3.4}$$

$$C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \Big\{v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) \le v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big\}, \tag{3.5}$$

and note, for the further reference, that, for all $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$,

$$\sum_{j=1}^{n} P\Big(A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}})|X_i\Big) \le 1, \tag{3.6}$$

$$\sum_{i=1}^{n} P\Big(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)|Y_j\Big) \le 1, \tag{3.7}$$

$$\sum_{i=1}^{n} P\Big(C_{1i}^{c}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big) \le 1, \tag{3.8}$$

$$\sum_{j=1}^{n} P\Big(C_{2j}^{c}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big) \le 1 \tag{3.9}$$

and

$$\sum_{1\le i,\, j\le n} P(B^c_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})A_{1ij}(\tilde{x}_i,\tilde{\mathbf{y}})A_{2ij}(\tilde{\mathbf{x}},\tilde{y}_j)) \le 1. \tag{3.10}$$

THEOREM 3.1. *Let*

$$G_{ij} = f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)I(C_{1i}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})(C_{2j}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})) \times I(B_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \cup A^c_{1ij}(\tilde{x}_i,\tilde{\mathbf{y}}) \cup A^c_{2ij}(\tilde{\mathbf{x}},\tilde{y}_j)).$$

Then, for any $\tilde{\mathbf{x}}$ *and* $\tilde{\mathbf{y}}$,

$$E\Phi\biggl(\sum_{1\le i,\, j\le n} f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) - \sum_{i=1}^n \bar{f}_{1i}(X_i;\tilde{\mathbf{x}},\tilde{\mathbf{y}}) - \sum_{j=1}^n \bar{f}_{2j}(Y_j;\tilde{\mathbf{x}},\tilde{\mathbf{y}})\biggr) \le_\alpha \Phi\bigl(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}\bigr),$$

where

$$\bar{f}_{1i}(X_i;\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = I\bigl(C^c_{1i}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\bigr)\sum_{j=1}^n E\Bigl[f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)I\bigl(A^c_{1ij}(\tilde{x}_i,\tilde{\mathbf{y}})\bigr)|X_i\Bigr] + \sum_{j=1}^n\bigl(E(G_{ij}|X_i) - \tfrac{1}{2}E(G_{ij})\bigr) \tag{3.11}$$

and

$$\bar{f}_{2j}(Y_j;\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = I\bigl(C^c_{2j}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\bigr)\sum_{i=1}^n E\Bigl[f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)I\bigl(A^c_{2ij}(\tilde{\mathbf{x}},\tilde{y}_j)\bigr)|Y_j\Bigr] + \sum_{i=1}^n\bigl(E(G_{ij}|Y_j) - \tfrac{1}{2}E(G_{ij})\bigr). \tag{3.12}$$

If, in addition, $(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathscr{A}_2$ *(or* $\mathscr{A}_c$ *for some bounded* $c \ge 1$*), then*

$$E\Phi\biggl(\sum_{1\le i,\, j\le n} f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) - \sum_{i=1}^n \bar{f}_{1i}\bigl(X_i;\tilde{\mathbf{x}},\tilde{\mathbf{y}}\bigr) - \sum_{j=1}^n \bar{f}_{2j}\bigl(Y_j;\tilde{\mathbf{x}},\tilde{\mathbf{y}}\bigr)\biggr) \approx_\alpha \Phi\bigl(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}\bigr). \tag{3.13}$$

PROOF. We begin the proof of Theorem 3.1 by introducing the decomposition

$$\sum_{1\le i,\, j\le n} f^{(s)}_{ij}\bigl(X_i, Y_j, \tilde{x}_i, \tilde{y}_j\bigr) - \sum_{i=1}^n \bar{f}_{1i}\bigl(X_i;\tilde{\mathbf{x}},\tilde{\mathbf{y}}\bigr) - \sum_{j=1}^n \bar{f}_{2j}\bigl(Y_j;\tilde{\mathbf{x}},\tilde{\mathbf{y}}\bigr) = \sum_{i=1}^5 U_i, \tag{3.14}$$

where

$$
\begin{aligned}
U_1 &= \sum_{i=1}^{n} I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \sum_{j=1}^{n} \Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) \\
&\qquad - E\Big[f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\Big(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}})\Big) | X_i \Big]\Big),\\
U_2 &= \sum_{j=1}^{n} I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \sum_{i=1}^{n} \Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) \\
&\qquad - E\Big[f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\Big(A_{2ij}^c(\tilde{\mathbf{x}}, \tilde{y}_j)\Big) | Y_j \Big]\Big),\\
U_3 &= - \sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big),\\
U_4 &= \sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) \\
&\qquad \times I\Big(C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}}) A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j) B_{ij}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big),\\
U_5 &= \sum_{1 \le i,\, j \le n} \Big(G_{ij} - E(G_{ij} | X_i) - E(G_{ij} | Y_j) + E(G_{ij}) \Big).
\end{aligned}
$$

The first part of Theorem 3.1 and the bound $\le_\alpha$ in formula (3.13) are proved by means of the next two lemmas. To then show the inequality $\ge_\alpha$ in formula (3.13), take $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ in $\mathscr{A}_2$ as in Lemma 2.4 and write

$$
\begin{aligned}
&E\Phi\Big(\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) - \sum_{i=1}^{n} \bar{f}_{1i}(X_i; \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) - \sum_{j=1}^{n} \bar{f}_{2j}(Y_j; \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \Big)\\
&\quad \ge_\alpha E\Phi\Big(\sum_{1 \le i,\, j \le n} f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j) \Big) \qquad \text{(by double symmetrization)}\\
&\quad \approx_\alpha E\Phi\Bigg(\sqrt{\sum_{1 \le i,\, j \le n} \big(f_{ij}^{(s)}(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j)\big)^2} \Bigg) \qquad \text{(by Lemma 2.2)}\\
&\quad \ge_\alpha E\Phi\Bigg(\sqrt{\sum_{1 \le i,\, j \le n} \big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\big)^2} \Bigg)\\
&\qquad\qquad \text{(by assumptions and Lemma 2.4)}\\
&\quad \approx_\alpha \Phi(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \quad \text{(by Lemma 2.5).} \qquad \square
\end{aligned}
$$

LEMMA 3.2.

$$E\Phi(U_1) \leq_\alpha \max\Big\{E \max_{1\leq i,\, j\leq n} \Phi\big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\big), \qquad E\max_{1\leq i\leq n} \Phi\big(v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}})\big)\Big\}.$$

Analogously,

$$E\Phi(U_2) \leq_\alpha \max\Big\{E \max_{1\leq i,\, j\leq n} \Phi\big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\big), \qquad E\max_{1\leq j\leq n} \Phi\big(v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j)\big)\Big\}.$$

Moreover,

$$E\Phi(U_3) \leq_\alpha E \max_{1\leq i,\, j\leq n} \Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) I\Big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big) I\Big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big)$$

and

$$E\Phi(U_4) \leq_\alpha E \max_{1\leq i,\, j\leq n} \phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) I\Big(C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big) \times I\Big(A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}}) A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j) B_{ij}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big).$$

PROOF. To prove the first inequality, write

$$\begin{aligned}
E\Phi(U_1) &\leq_\alpha \sum_{i=1}^n E\Phi\Big(\sum_{j=1}^n \Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) \\
&\qquad - E\big[f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\big(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}})\big) | X_i\big]\Big)\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \\
&\qquad \text{[conditioning on } \{Y_j\} \text{ and using (3.8) and Lemma A.1 below]} \\
&\leq_\alpha \sum_{i=1}^n E\Phi\Big\{\sum_{j=1}^n \Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\big(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}})\big) \\
&\qquad - E\big[f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\big(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}})\big) | X_i\big]\Big)\Big\} I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \\
&\qquad + \sum_{i=1}^n E\Phi\Big(\sum_{j=1}^n f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\Big(A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}})\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)\Big) \\
&\qquad \text{[since } \Phi(a+b) \leq 2^\alpha(\Phi(a) + \Phi(b))\text{]} \\
&\leq_\alpha \sum_{i=1}^n E\Phi\Big(v_{1i}(X_i, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \\
&\qquad + \sum_{i=1}^n E \max_{1\leq j\leq n} \Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)
\end{aligned}$$

[conditioning on $\{X_i\}$, using Lemma A.2 below for the first sum and applying (3.6) to Lemma A.1 below for the second sum]

$$\leq_\alpha E \max_{1\leq i\leq n} \Phi\Big(v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}})\Big) + E \max_{1\leq i,\, j\leq n} \Phi\big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\big)$$

[by (3.8) applied to Lemma A.1 below twice].

The second inequality is proved analogously.

To prove the third,

$$E\Phi(U_3) = E\Phi\Bigg(\sum_{1\leq i,\, j\leq n} f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)\Bigg)$$

$$\leq_\alpha \sum_{i=1}^n E\Phi\Bigg(\sum_{j=1}^n f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Bigg) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)$$

[by conditioning on $\{Y_j\}$ and applying (3.8) to Lemma A.1 below]

$$\leq_\alpha \sum_{1\leq i,\, j\leq n} E\Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)$$

[as above but employing (3.9)].

Furthermore, let $N_1 = \sum_{i=1}^n I(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}))$, $N_2 = \sum_{j=1}^n I(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}))$, $N_{1i} = \sum_{i'\neq i} I(C_{1i'}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}))$ and $N_{2j} = \sum_{j'\neq j} I(C_{2j'}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}))$. We have

$$\begin{aligned}
&E \max_{1\leq i,\, j\leq n} \Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \\
&\quad \geq \sum_{1\leq i,\, j\leq n} \tfrac{1}{4} E\Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) \\
&\qquad\qquad \times I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I(N_1 \leq 2) I(N_2 \leq 2) \\
&\quad = \sum_{1\leq i,\, j\leq n} \tfrac{1}{4} E\Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) \\
&\qquad\qquad \times I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) P(N_{1i} \leq 1) P(N_{2j} \leq 1) \\
&\qquad \text{(by independence)} \\
&\quad \geq \tfrac{1}{16} \sum_{1\leq i,\, j\leq n} E\Phi\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big) \\
&\qquad \times I\big(C_{1i}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) I\big(C_{2j}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big)
\end{aligned}$$

[since $P(N_{1i} \leq 1) = 1 - P(N_{1i} \geq 2) \geq$ $1 - \frac{1}{2} EN_{1i} \geq \frac{1}{2}$ and similarly for $P(N_{2j} \leq 1)$].

To prove the fourth inequality we let

$$D_{ij}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \{C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\} \cap \{C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\} \cap \{A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}})\} \cap \{A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)\} \cap \{B_{ij}^c(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\},$$

$$D^*_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \{A_{1ij}(\tilde{x}_i,\tilde{\mathbf{y}})\}\cap\{A_{2ij}(\tilde{\mathbf{x}},\tilde{y}_j)\}\cap\{B^c_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\}$$

and

$$W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = f^{(s)}_{ij}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j) I(D_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})).$$

Further, we set

$$N'_{ij} = \sum_{1\le i',\, j'\le n:\, i'\ne i \text{ and } j'\ne j} I(D^*_{i'j'}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})),$$

$$N'_{i\cdot}(j) = \sum_{1\le j'\le n:\, j'\ne j} I(A_{1ij'}(\tilde{x}_i,\tilde{\mathbf{y}}))$$

and

$$N'_{\cdot j}(i) = \sum_{1\le i'\le n:\, i'\ne i} I(A_{2i'j}(\tilde{\mathbf{x}},\tilde{y}_j)).$$

Then,

$$\begin{aligned}
&E \max_{1\le i,\, j\le n} \Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\\
&\quad\ge E \max_{1\le i,\, j\le n} \Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})) I(N'_{ij}\le 3, N'_{i\cdot}(j)\le 3, N'_{\cdot j}(i)\le 3)\\
&\quad\ge E \tfrac{1}{10} \sum_{1\le i,\, j\le n} \Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})) I(N'_{ij}\le 3, N'_{i\cdot}(j)\le 3, N'_{\cdot j}(i)\le 3)\\
&\quad\ge E \tfrac{1}{10} \sum_{1\le i,\, j\le n} \Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))(1 - I(N'_{ij}\ge 4) - I(N'_{i\cdot}(j)\ge 4) - I(N'_{\cdot j}(i)\ge 4))\\
&\quad\ge E \tfrac{1}{10} \sum_{1\le i,\, j\le n} \Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))(1 - P(N'_{ij}\ge 4) - P(N'_{i\cdot}(j)\ge 4|X_i)\\
&\qquad\qquad\qquad\qquad\qquad\qquad - P(N'_{\cdot j}(i)\ge 4|Y_j))\\
&\quad\ge \tfrac{1}{40} \sum_{1\le i,\, j\le n} E\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})),
\end{aligned}$$

since

$$P(N'_{ij}\ge 4) \le \tfrac{1}{4} E \sum_{1\le i',\, j'\le n} I(D^*_{i'j'}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})) \le \tfrac{1}{4} \quad \text{by (3.10)},$$

$$P(N'_{i\cdot}(j)\ge 4|X_i) \le \tfrac{1}{4} E \sum_{1\le j'\le n} I(A_{1ij'}(\tilde{x}_i,\tilde{\mathbf{y}})|X_i) \le \tfrac{1}{4} \quad \text{by (3.6)}$$

and similarly,

$$P(N'_{\cdot j}(i)\ge 4|Y_j) \le \tfrac{1}{4} \quad \text{by (3.7)}.$$

Finally,

$$
\begin{aligned}
E\Phi\Big(\sum_{1\le i,\, j\le n} W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\Big) &\le E\sum_{1\le i,\, j\le n}\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})(1+N'_{ij}+N'_{\cdot j}(i)+N'_{i\cdot}(j)))\\
&\le \sum_{1\le i,\, j\le n} E\Big(1+N'_{ij}+N'_{\cdot j}(i)+N'_{i\cdot}(j)\Big)^{\alpha}\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\\
&\le \sum_{1\le i,\, j\le n} E\Big(3^{(\alpha-1)^+}(1+N'_{ij})^{\alpha}+3^{(\alpha-1)^+}(N'_{\cdot j}(i))^{\alpha}\\
&\qquad\qquad + 3^{(\alpha-1)^+}(N'_{i\cdot}(j))^{\alpha}\Big)\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\\
&\le 3^{(\alpha-1)^+}\sum_{1\le i,\, j\le n} E\Big(\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\Big)\Big(E(1+N'_{ij})^{\alpha}+E\big((N'_{\cdot j}(i))^{\alpha}|Y_j\big)\\
&\qquad\qquad + E\big((N'_{i\cdot}(j))^{\alpha}|X_i\big)\Big).
\end{aligned}
$$

Note that

$$E\Big((N'_{\cdot j}(i))^{\alpha}|Y_j\Big)\le_{\alpha} 1$$

by applying (3.7) to Corollary 2.4 of Klass and Nowicki (1997), and similarly,

$$E\Big((N'_{i\cdot}(j))^{\alpha}|X_i\Big)\le_{\alpha} 1$$

by applying (3.6) to Corollary 2.4 of Klass and Nowicki (1997). Moreover,

$$E(1+N'_{ij})^{\alpha}\le E\Big(1+\sum_{1\le i,\, j\le n} I\Big(D^*_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\Big)\Big)^{\alpha}\le_{\alpha} 1$$

by applying (3.6), (3.7) and (3.10) to Lemma 2.5 of Klass and Nowicki (1997). Consequently,

$$E\Phi\Big(\sum_{1\le i,\, j\le n} W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\Big)\le_{\alpha}\sum_{1\le i,\, j\le n} E\big(\Phi(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\big)\le_{\alpha} E\max_{1\le i,\, j\le n}\Phi\big(W_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)$$

which completes the proof. □

LEMMA 3.3.

$$
\begin{aligned}
E\Phi(U_5) &\equiv E\Phi\Big(\sum_{1\le i,\, j\le n}\Big(G_{ij}-E(G_{ij}|X_i)-E(G_{ij}|Y_j)+E(G_{ij})\Big)\Big)\\
&\le_{\alpha}\max\big\{\Phi(v_{1*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})),\Phi(v_{2*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})),\Phi(w_*(\tilde{\mathbf{x}},\tilde{\mathbf{y}}))\big\}\equiv\Phi(q_*(\tilde{\mathbf{x}},\tilde{\mathbf{y}})),
\end{aligned}
$$

where

$$
\begin{aligned}
G_{ij} &= f^{(s)}_{ij}(X_i,Y_j,\tilde{x}_i,\tilde{y}_j)I\big(C_{1i}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})C_{2j}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)\\
&\quad\times I\big(B_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\cup A^c_{1ij}(\tilde{x}_i,\tilde{\mathbf{y}})\cup A^c_{2ij}(\tilde{\mathbf{x}},\tilde{y}_j)\big).
\end{aligned}
$$

PROOF. To use Lemma A.5 (below) we verify its conditions. First,

$$\text{ess} \sup_{1\le i,\, j\le n} |G_{ij}| \le q_*(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).$$

Second,

$$\begin{aligned}
&\text{ess} \sup_{1\le j\le n} \sum_{i=1}^{n} E\big(G_{ij}^2 | Y_j\big) \\
&\quad \le \text{ess} \sup_{1\le j\le n} \sum_{i=1}^{n} E\Big[\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big)^2 I\big(A_{2ij}^c(\tilde{\mathbf{x}}, \tilde{y}_j)\big) I\big(C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\big) \big| Y_j\Big] \\
&\qquad + \text{ess} \sup_{1\le j\le n} \sum_{i=1}^{n} E\Big[\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big)^2 I(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}})) I(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)) \\
&\qquad\qquad \times I(C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) | Y_j\Big] \\
&\qquad + \text{ess} \sup_{1\le j\le n} \sum_{i=1}^{n} E\Big[\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Big)^2 I(B_{ij}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) \\
&\qquad\qquad I(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)) | Y_j\Big] \\
&\quad \equiv \text{ess} \sup_{1\le j\le n} t_{1j} + \text{ess} \sup_{1\le j\le n} t_{2j} + \text{ess} \sup_{1\le j\le n} t_{3j}.
\end{aligned}$$

Now,

$$\begin{aligned}
t_{1j} &= \sum_{i=1}^{n} E\Big[\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, y_j)\Big)^2 I(|f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, y_j)| \\
&\quad \le v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j)) I(v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) \le v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) \mid Y_j\Big] \\
&\le \left(\sum_{i=1}^{n} E\Big[\Big(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, y_j)\Big)^2 \wedge v_{2j}^2(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) \mid Y_j\Big]\right) \\
&\quad \times I(v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) \le v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) \\
&\le v_{2j}^2(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) I(v_{2j}(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) \le v_{2*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) \quad \text{by (2.5)} \\
&\le v_{2*}^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).
\end{aligned}$$

For t_{2j} we have

$$\begin{aligned}
t_{2j} &\le \sum_{i=1}^{n} E\Big[v_{1i}^2(X_i, \tilde{x}_i, \tilde{\mathbf{y}}) I(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)) I(v_{1i}(X_i, \tilde{x}_i, \tilde{\mathbf{y}}) \le v_{1*}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) \mid Y_j\Big] \\
&\le v_{1*}^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \sum_{i=1}^{n} P(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j) \mid Y_j) \le v_{1*}^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \quad \text{by (3.7)}.
\end{aligned}$$

Finally,

$$t_{3j} \le w_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \sum_{i=1}^n P(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j) \mid Y_j) \le w_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \quad \text{by (3.7)}.$$

Hence, $\operatorname{ess\,sup}_{1\le j\le n} \sum_{i=1}^n E(G_{ij}^2|Y_j) \le 3q_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$.

Analogously,

$$\operatorname{ess} \sup_{1\le i\le n} \sum_{j=1}^n E(G_{ij}^2|X_i) \le 3q_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).$$

Finally, to verify the last condition of Lemma A.5 we write

$$\begin{aligned}
\sum_{1\le i,\, j\le n} EG_{ij}^2 &\le \sum_{j=1}^n \biggl(\sum_{i=1}^n E\Bigl(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Bigr)^2 I\Bigl(A_{2ij}^c(\tilde{\mathbf{x}}, \tilde{y}_j) C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Bigr)\biggr) \\
&\quad + \sum_{i=1}^n \biggl(\sum_{j=1}^n E\Bigl(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Bigr)^2 I\Bigl(A_{1ij}^c(\tilde{x}_i, \tilde{\mathbf{y}}) C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Bigr)\biggr) \\
&\quad + \sum_{i=1}^n \biggl(\sum_{j=1}^n E\Bigl(\Bigl(\Bigl(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\Bigr)^2 \wedge w_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Bigr) \\
&\qquad\qquad \times I\bigl(A_{1ij}(\tilde{x}_i, \tilde{\mathbf{y}})\bigr) I\bigl(A_{2ij}(\tilde{\mathbf{x}}, \tilde{y}_j)\bigr)\Bigr)\biggr) \\
&\le \sum_{j=1}^n E\Bigl(v_{2j}^2(Y_j, \tilde{\mathbf{x}}, \tilde{y}_j) I\Bigl(C_{2j}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Bigr)\Bigr) \\
&\quad + \sum_{i=1}^n E\Bigl(v_{1i}^2(X_i, \tilde{x}_i, \tilde{\mathbf{y}}) I\Bigl(C_{1i}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\Bigr)\Bigr) + w_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \\
&\quad \text{[by (2.5) and (3.2), (2.4) and (3.1), and (3.1)} \\
&\qquad \text{and (3.2) applied to (2.8)]} \\
&\le v_{2*}^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + v_{1*}^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + w_*^2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \quad \text{[by (2.6) and (2.7)]}.
\end{aligned}$$

Hence our lemma follows from Lemma A.5. □

By virtue of Theorem 3.1 and the following series of inequalities we obtain a formula for a suitable first-order adjustment of $\sum_{1\le i,\, j\le n} f_{ij}(X_i, Y_j)$, thereby enabling us to exhibit a fully constructive method of identifying the order of magnitude of $\mathscr{E}(\Phi, \mathbf{f}, \mathbf{X}, \mathbf{Y})$, stated below as Theorem 3.4.

Since $\mathscr{A}_1 \ne \varnothing$ there exists $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ such that

$$\begin{aligned}
&E\Phi\Biggl(\sqrt{\sum_{1\le i,\, j\le n} \bigl(f_{ij}^{(s)}(X_i, Y_j, \tilde{X}_i, \tilde{Y}_j)\bigr)^2}\Biggr) \\
&\quad \ge E\Phi\Biggl(\sqrt{\sum_{1\le i,\, j\le n} \bigl(f_{ij}^{(s)}(X_i, Y_j, \tilde{x}_i, \tilde{y}_j)\bigr)^2}\Biggr) \quad \text{(as in Lemma 2.4)}
\end{aligned}$$

$$
\begin{aligned}
&\approx_\alpha \Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}\big) \text{ (as in Lemma 2.5)}\\
&\geq \Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big) \text{ [by (3.15) (below)]}\\
&\geq_\alpha E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}\big(X_i, Y_j, \tilde{x}_i^0, \tilde{y}_j^0\big) - \sum_{i=1}^n \bar{f}_{1i}\big(X_i; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big)\\
&\qquad\qquad - \sum_{j=1}^n \bar{f}_{2j}\big(Y_j; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big)\bigg)\\
&\qquad \text{[by Theorem 3.1 where } \bar{f}_{1i} \text{ and } \bar{f}_{2j} \text{ are defined}\\
&\qquad\quad \text{as in (3.11) and (3.12)]}\\
&\geq_\alpha E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}\big(X_i, Y_j, \tilde{X}_i, \tilde{Y}_j\big)\bigg) \text{ (by a variant of Lemma 2.1)}\\
&\approx_\alpha E\Phi\Bigg(\sqrt{\sum_{1\le i,\, j\le n} \Big(f_{ij}^{(s)}\big(X_i, Y_j, \tilde{X}_i, \tilde{Y}_j\big)\Big)^2}\Bigg) \text{ (by Lemma 2.2),}
\end{aligned}
$$

where

$$
\Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big) = \inf_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}} \Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}\big). \tag{3.15}
$$

REMARK. It is possible that no vector pair $(\tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0)$ will exist which achieves the infimum in (3.15). In this case (and even in general) we can use any vector pair $(\tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0)$ for which the LHS of (3.15) is at most bounded by a known multiple of the RHS of (3.15) (e.g., by a factor of 2). For any such $(\tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0)$,

$$
\begin{aligned}
&E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}\big(X_i, Y_j, \tilde{x}_i^0, \tilde{y}_j^0\big) - \sum_{i=1}^n \bar{f}_{1i}\big(X_i; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big) - \sum_{j=1}^n \bar{f}_{2j}\big(Y_j; \tilde{\mathbf{x}}^0; \tilde{\mathbf{y}}^0\big)\bigg)\\
&\qquad \approx_\alpha E\Phi\bigg(\sum_{1\le i,\, j\le n} f_{ij}^{(s)}\big(X_i, Y_j, \widetilde{X}_i, \widetilde{Y}_j\big)\bigg).
\end{aligned} \tag{3.16}
$$

THEOREM 3.4. *Let* $(\tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0)$ *satisfy*

$$
\Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big) \le 2 \inf_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}} \Phi\big(\mathbf{f}^{(s)}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}\big) \tag{3.17}
$$

and set

$$
f_{1i}^0(X_i) = \sum_{j=1}^n f_{ij}\big(X_i, \tilde{y}_j^0\big) - \tfrac{1}{2}\sum_{j=1}^n f_{ij}\big(\tilde{x}_i^0, \tilde{y}_j^0\big) + \bar{f}_{1i}\big(X_i; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big) \tag{3.18}
$$

and

$$
f_{2j}^0(Y_j) = \sum_{i=1}^n f_{ij}\big(\tilde{x}_i^0, Y_j\big) - \tfrac{1}{2}\sum_{i=1}^n f_{ij}\big(\tilde{x}_i^0, \tilde{y}_j^0\big) + \bar{f}_{2j}\big(Y_j; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0\big), \tag{3.19}
$$

where $\bar f_{1i}(\cdot)$, $\bar f_{2j}(\cdot)$ *and* $\Phi(\hat{\mathbf{f}}^{(s)}, \widehat{\mathbf{X}}, \widehat{\mathbf{Y}})$ *are defined in* (3.11), (3.12) *and* (1.17), *respectively*.

Then

$$E\Phi\biggl(\sum_{1\le i,\ j\le n} f_{ij}(X_i, Y_j)\biggr) \approx_\alpha \max\biggl\{\Phi\Bigl(\hat{\mathbf{f}}^{(s)}, \widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\Bigr), E\Phi\biggl(\sum_{i=1}^n f_{1i}^0(X_i) + \sum_{j=1}^n f_{2j}^0(Y_j)\biggr)\biggr\}. \tag{3.20}$$

PROOF. We can rewrite

$$\sum_{1\le i,\ j\le n} f_{ij}^{(s)}(X_i, Y_j, \tilde x_i^0, \tilde y_j^0) - \sum_{i=1}^n \bar f_{1i}(X_i; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0) - \sum_{j=1}^n \bar f_{2j}(Y_j; \tilde{\mathbf{x}}^0, \tilde{\mathbf{y}}^0)$$

as

$$\sum_{1\le i,\ j\le n} f_{ij}(X_i, Y_j) - \sum_{i=1}^n f_{1i}^0(X_i) - \sum_{j=1}^n f_{2j}^0(Y_j).$$

The result then follows from (3.16) and (1.19). □

4. The *k*-dimensional case. In this section we show in principle how to construct an approximation for the expectation of the k-component case given the existence of an approximation method for the nonnegative k-component case and the general $(k-1)$-component case. For each $1 \le j \le k$ and $1 \le i \le n$ let $X_i^{(j)}$ be independent random elements and, for each $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$, let $f_{i_1 i_2 \cdots i_k}(X_{i_1}^{(1)}, X_{i_2}^{(2)}, \dots, X_{i_k}^{(k)})$ be a real valued r.v. Let Φ be a Δ_2-function. We want to approximate

$$E\Phi\biggl(\sum_{1\le i_1,\dots,i_k\le n} f_{i_1\cdots i_k}\Bigl(X_{i_1}^{(1)}, \dots, X_{i_k}^{(k)}\Bigr)\biggr). \tag{4.1}$$

Introduce $X_i^{(j,l)}$, two independent copies of the random variables X_i^j for $l = 0, 1$, $j = 1, \dots, k$ and $i = 1, \dots, n$ such that $\mathscr{L}(X_i^{(j,l)}) = \mathscr{L}(X_i^j)$. Repeating our lower-bounding symmetrization procedure k times we obtain

$$\begin{aligned} &E\Phi\biggl(\sum_{1\le i_1,\dots,i_k\le n} f_{i_1\cdots i_k}\Bigl(X_{i_1}^{(1)}, \dots, X_{i_k}^{(k)}\Bigr)\biggr) \\ &\quad\ge 2^{-k(\alpha+1)} E\Phi\biggl(\sum_{j_k=0}^1 \cdots \sum_{j_1=0}^1 (-1)^{j_1+\cdots+j_k} \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n f_{i_1\cdots i_k}\Bigl(X_{i_1}^{(1,j_1)}, \dots, X_{i_k}^{(k,j_k)}\Bigr)\biggr) \\ &\quad\equiv 2^{-k(\alpha+1)} E\Phi\biggl(\sum_{1\le i_1,\dots,i_k\le n} f_{i_1\cdots i_k}^{(s)}\Bigl(\widehat{\mathbf{X}}_{i_1}^1, \dots, \widehat{\mathbf{X}}_{i_k}^k\Bigr)\biggr). \end{aligned} \tag{4.2}$$

where $\widehat{\mathbf{X}}_i^j = (X_i^{(j,0)}, X_i^{(j,1)})$.

Arguing as in the 2-dim case there must exist $\{(\tilde{x}_1^{(j)}, \dots, \tilde{x}_n^{(j)})\}_{1 \le j \le k}$ such that

$$E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f^{(s)}_{i_1 \cdots i_k}\Big(X^{(1)}_{i_1}, \dots, X^{(k)}_{i_k}, \tilde{x}^{(1)}_{i_1}, \dots, \tilde{x}^{(k)}_{i_k}\Big)\Big) \le E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f^{(s)}_{i_1 \cdots i_k}\Big(\widehat{\mathbf{X}}^{(1)}_{i_1}, \dots, \widehat{\mathbf{X}}^{(k)}_{i_k}\Big)\Big). \tag{4.3}$$

By the k-fold iterated symmetrization procedure applied to the sum of the LHS of (4.3) we also have

$$E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f^{(s)}_{i_1 \cdots i_k}\Big(X^{(1)}_{i_1}, \dots, X^{(k)}_{i_k}, \tilde{x}^{(1)}_{i_1}, \dots, \tilde{x}^{(k)}_{i_k}\Big)\Big) \ge 2^{-k(\alpha+1)} E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f^{(s)}_{i_1 \cdots i_k}\Big(\widehat{\mathbf{X}}^{(1)}_{i_1}, \dots, \widehat{\mathbf{X}}^{(k)}_{i_k}\Big)\Big). \tag{4.4}$$

Hence the two quantities have essentially the same order of magnitude. Since $\mathscr{L}(f_{i_1 \cdots i_k}(\widehat{\mathbf{X}}^{(1)}_{i_1}, \dots, \widehat{\mathbf{X}}^{(k)}_{i_k}) \mid \widehat{\mathbf{X}}^{(j)}_{i_j})$ is symmetric for $1 \le j \le k$ and $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$ it follows by suitable further generalization of Khintchine's inequality that

$$E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f^{(s)}_{i_1 \cdots i_k}\Big(\widehat{\mathbf{X}}^{(1)}_{i_1}, \dots, \widehat{\mathbf{X}}^{(k)}_{i_k}\Big)\Big) \approx_\alpha E\Phi\Bigg(\sqrt{\sum_{1 \le i_1, \dots, i_k \le n} \Big[f^{(s)}_{\{i_1 \cdots i_k\}}\Big(\widehat{\mathbf{X}}^{(1)}_{i_1}, \dots, \widehat{\mathbf{X}}^{(k)}_{i_k}\Big)\Big]^2}\Bigg) \equiv \Phi\Big(\mathbf{f}^{(s)}_{\mathbf{i}_1 \cdots \mathbf{i}_k}, \widehat{\mathbf{X}}^{\{\mathbf{j}\}}_{\{\mathbf{i}\}}\Big) \tag{4.5}$$

and therefore

$$\begin{aligned} E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} f_{i_1 \cdots i_k}(X_{i_1}, \dots, X_{i_k})\Big) \approx_\alpha \max\Bigg\{ & \Phi\Big(\mathbf{f}^{(s)}_{\{\mathbf{i}_1 \cdots \mathbf{i}_k\}}, \widehat{\mathbf{X}}^{\{\mathbf{j}\}}_{\{\mathbf{i}\}}\Big), \\ & E\Phi\Big(\sum_{1 \le i_1, \dots, i_k \le n} \Big[f^{(s)}_{i_1 \cdots i_k}\Big(X^{(1)}_{i_1}, \dots, X^{(k)}_{i_k}, \tilde{x}^{(1)}_{i_1}, \dots, \tilde{x}^{(k)}_{i_k}\Big) \\ & \qquad - f_{i_1 \cdots i_k}\Big(X^{(1)}_{i_1}, \dots, X^{(k)}_{i_k}\Big)\Big]\Big)\Bigg\}. \end{aligned} \tag{4.6}$$

Though the construction of explicit vectors $\{(\tilde{x}_1^{0(j)}, \dots, \tilde{x}_n^{0(j)})\}_{1 \le j \le k}$ becomes increasingly involved it would presumably follow the lines suggested in Section 3.

APPENDIX

Supplementary results.

LEMMA A.1 [Klass and Nowicki (1997)]. *For $1 \le j \le n$, let the ordered pair (B_j, Z_j) be an event and a nonnegative random variable, respectively. Suppose there is a σ-field $\mathscr{F}$ (which could be trivial) such that*

$$\sum_{j=1}^{n} P(B_j|\mathscr{F}) \le 1 \quad a.s.$$

and such that each $1 \le j \le n$, $Z_j I(B_j)$ is conditionally independent of $N_j = \sum_{i=1, i\ne j}^{n} I(B_i)$ given $\mathscr{F}$ and that the $\{B_j\}$ are mutually independent given $\mathscr{F}$. Then, for $\Phi \in \Delta_2$ with parameter α,

$$E\Phi\biggl(\sum_{j=1}^{n} Z_j I(B_j)\biggr) \approx_\alpha \sum_{j=1}^{n} E\Phi(Z_j) I(B_j) \approx_\alpha E \max_{1\le j\le n} \Phi(Z_j) I(B_j). \tag{A.1}$$

Dropping the nonnegativity assumption on Z_j,

$$E\Phi\biggl(\sum_{j=1}^{n} Z_j I(B_j)\biggr) \le E\Phi\biggl(\sum_{j=1}^{n} |Z_j| I(B_j)\biggr)$$

and so by (A.1),

$$E\Phi\biggl(\sum_{j=1}^{n} Z_j I(B_j)\biggr) \le_\alpha \sum_{j=1}^{n} E\Phi(Z_j) I(B_j) \approx_\alpha E \max_{1\le j\le n} \Phi(Z_j) I(B_j). \tag{A.2}$$

In fact, the reverse inequality $\ge_\alpha$ also holds in (A.2).

LEMMA A.2 [Klass and Nowicki (1998)]. *Let $\{Y_j\}_{j=1}^n$ be independent, mean zero random variables. Let Φ be any Δ_2-function with parameter α. Suppose that $\sum_{j=1}^n EY_j^2 \le w_n^2$ and that $|Y_j| \le w_n$, for each $j = 1, \dots, n$. Then*

$$E\Phi\biggl(\sum_{j=1}^{n} Y_j\biggr) \le_\alpha \Phi(w_n).$$

If, for some $0 < c \le \lambda_* \le 1$, $\sum_{j=1}^n EY_j^2 = \lambda_* w_n^2$ then

$$E\Phi\biggl(\sum_{j=1}^{n} Y_j\biggr) \approx_{\alpha, c} \Phi(w_n).$$

THEOREM A.3 [Klass (1981)]. *Let $Y_1, Y_2, \dots, Y_n$ be independent random variables such that $\min_{1\le j\le n}\{P(Y_j \ge 0), P(Y_j \le 0)\} \ge \frac{1}{4}$. Let*

$$a_n = \sup\biggl\{a \ge 0; \sum_{j=1}^{n} E(Y_j^2 \wedge a^2) \ge a^2\biggr\}, \tag{A.3}$$

$$m_{n,b} = b + \sum_{j=1}^{n} EY_j I(|Y_j| \le a_n) \tag{A.4}$$

and

$$t_n = \sup\left\{t : \sum_{j=1}^{n} E\Phi(Y_j) I(|Y_j| > t) \geq \Phi(t)\right\}. \tag{A.5}$$

Let $\Phi \in \Delta_2$ *be of parameter* $\alpha > 0$. *Then*

$$E\Phi\left(b + \sum_{j=1}^{n} Y_j\right) \approx_\alpha \max\{\Phi(a_n), \Phi(m_{n,b}), \Phi(t_n)\}. \tag{A.6}$$

For easy reference we also note that

$$\tfrac{1}{2}\Phi(t_n) \leq E \max_{1\leq j\leq n} \Phi(Y_j) \leq 2\Phi(t_n). \tag{A.7}$$

THEOREM A.4 [Klass and Nowicki (1998)]. *Let* $\{X_i\}_{i=1}^n$ *and* $\{Y_j\}_{j=1}^n$ *be two independent sequences of independent random variables,* f_{ij} *a sequence of real valued functions and* Φ *a* Δ_2*-function with parameter* α. *Suppose that either*:

(i) $\mathscr{L}(f_{ij}(X_i, Y_j)|X_i)$ *and* $\mathscr{L}(f_{ij}(X_i, Y_j)|Y_j)$ *are symmetric a.s. for all* $1 \leq i, j \leq n$ *or*

(ii) Φ *is convex and* $E[f_{ij}(X_i, Y_j)|X_i] = E[f_{ij}(X_i, Y_j)|Y_j] = 0$, *for all* $1 \leq i, j \leq n$. *Then*

$$E\Phi\left(\sum_{1\leq i, j\leq n} f_{ij}(X_i, Y_j)\right) \approx_\alpha E\Phi\left(\sqrt{\sum_{1\leq i, j\leq n} f_{ij}^2(X_i, Y_j)}\right).$$

LEMMA A.5 [Klass and Nowicki (1998)]. *Let* $\{X_i\}_{i=1}^n$, $\{Y_j\}_{j=1}^n$ *be two independent sequences of independent random variables. Let* $\{W_{ij}\}_{1\leq i, j\leq n}$ *be random variables such that* W_{ij} *depends only on* X_i *and* Y_j. *Assume further the existence of a nonnegative real* z_* *such that*:

(i) ess $\sup_{1\leq i, j\leq n} |W_{ij}| \leq z_*$.

(ii) ess $\sup_{1\leq j\leq n} \sum_{i=1}^n E(W_{ij}^2|Y_j) \leq z_*^2$.

(iii) ess $\sup_{1\leq i\leq n} \sum_{j=1}^n E(W_{ij}^2|X_i) \leq z_*^2$.

(iv) $\sum_{1\leq i, j\leq n} EW_{ij}^2 \leq z_*^2$.

Then, for a Δ_2*-function* Φ *with parameter* α,

$$E\Phi\left(\sum_{1\leq i, j\leq n} (W_{ij} - E(W_{ij}|X_i) - E(W_{ij}|Y_j) + EW_{ij})\right) \leq_\alpha \Phi(z_*). \tag{A.8}$$

If, for some $0 < c \leq \lambda_* \leq 1$,

$$\sum_{1\leq i, j\leq n} E(W_{ij} - E(W_{ij}|X_i) - E(W_{ij}|Y_j) + EW_{ij})^2 = \lambda_* z_*^2 \tag{A.9}$$

then

$$E\Phi\left(\sum_{1\leq i, j\leq n} (W_{ij} - E(W_{ij}|X_i) - E(W_{ij}|Y_j) + EW_{ij})\right) \approx_{\alpha, c} \Phi(z_*). \tag{A.10}$$

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