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EXPONENTIAL INEQUALITIES AND CONVERGENCE OF MOMENTS IN THE REPLICA-SYMMETRIC REGIME OF THE HOPFIELD MODEL[1]

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In a previous work, we proved the validity of the replica-symmetric solution for the Hopfield model in a nontrivial domain of parameters. This was done at the accuracy of the LLN. In a somewhat larger domain, we obtain a description at the level of the CLT, or, in the terminology of physics, we calculate the fluctuations around the mean field. This is obtained by refining the tools we developed for a rigorous use of the cavity method and proving new a priori estimates about the "localization" of the Gibbs measure.

1. Introduction. The Hopfield model of memory was recently the object of much attention, and, in particular, this author devoted a long paper to the study of its many aspects [7]. In order to make the present paper reasonably self-contained, we will repeat the main definitions.

The Hopfield model centers on a certain random function defined on the space $\Sigma_N = \{-1, 1\}^N$. An element $\boldsymbol{\varepsilon}$ of Σ_N is called a configuration (because physically it describes a configuration of N spins). The randomness is brought in by an independent sequence $(\eta_{i,k})_{i\le N,\, k\le M}$ such that $P(\eta_{i,k} = 1) = P(\eta_{i,k} = -1) = 1/2$. The configurations $\boldsymbol{\eta}_k = (\eta_{i,k})_{i\le N}$, called the *prototypes*, play a special role, and so do the quantities

$$m_k(\boldsymbol{\varepsilon}) = \frac{1}{N}\sum_{i\le N} \eta_{i,k}\epsilon_i, \tag{1.1}$$

which are called the *overlaps*. The random function of interest (the Hamiltonian) is given by

$$H(\boldsymbol{\varepsilon}) = -\frac{N}{2}\sum_{k\le M} m_k(\boldsymbol{\varepsilon})^2 - hNm_1(\boldsymbol{\varepsilon}) \tag{1.2}$$

(where $h > 0$ is a parameter). When $h = 0$, this is the most natural function that takes large negatives values at each of the configurations $\boldsymbol{\eta}_k$. The purpose of the extra term $-hNm_1(\boldsymbol{\varepsilon})$ is to distinguish one of the overlaps. It is natural in statistical mechanics to study (1.1) through the introduction of an inverse temperature β, and to introduce the Gibbs measure given by

$$G(\boldsymbol{\varepsilon}) = \frac{2^{-N}}{Z}\exp\bigl(-\beta H(\boldsymbol{\varepsilon})\bigr), \tag{1.3}$$

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where Z is the normalization factor

$$Z = 2^{-N} \sum_{\boldsymbol{\varepsilon} \in \Sigma_N} \exp\bigl(-\beta H(\boldsymbol{\varepsilon})\bigr).$$

All the quantities we write then depend on N, M, β and the randomness $(\eta_{i,k})$. The dependence on the randomness is always kept implicit. The dependence on the other parameters will be made explicit whenever there is a risk of ambiguity. The model is studied as $N \to \infty$, for the "typical configuration" of the randomness. We are interested here only in the so-called case of the Hopfield model with many patterns, where M is a (small) proportion $M = \lfloor \alpha N \rfloor$ of N. (This is by far a more challenging situation than the case $M/N \to 0$.) The parameters of the model are then α, β, h (and $N \to \infty$). The Hopfield model (as well as other models for spin glasses) is, despite the apparent simplicity of its definition, an object of enormous complexity. All the results of the present paper concern the range of the parameters "where the system is in a pure state," in which physicists have discovered very beautiful formulas [A-G-S] by methods that the mathematically inclined reader will likely find of the utmost interest. The most important of these formulas, that is, the value of the average free energy per site,

$$F_N(\alpha, \beta, h) = \lim_{N \to \infty} \frac{1}{N} E \log Z_N, \tag{1.4}$$

was calculated rigorously in [7] in a domain that, although smaller than the region where physicists predict that the formula holds, contains an apparently very nontrivial region. This was done by developing the "cavity method," that is, computations of quantities relevant to the N-spin model in function of quantities relevant to the $(N-1)$-spin model. It is absolutely not trivial to see how to do these computations, and for that purpose we had to develop rather involved methods. Soon after reading our proof Bovier and Gayrard (who previously had authored a string of important papers on the Hopfield model) found a very different proof, with considerably greater geometric appeal than ours [3]. Despite the beauty of their argument, we see two reasons not to be fully satisfied by it. First, it seems that this argument uses very specific properties of the Hopfield model. On the other hand, the cavity method is very versatile and has been successful on a variety of spin-glass models [8, 9] (although with computational tools different from those we will use here). The second reason is that the geometric property on which the Bovier–Gayrard approach is based [the convexity of the function (6.2)] appears to be valid only in a subregion of the correct domain of parameters [as will be explained after (6.8)] so that it can be feared that (unless its formulation can be weakened) this geometric property is an accidental rather than an essential feature.

As far as aesthetics is concerned, it must also be said that [7] greatly suffers from the fact that we attempt there to make systematic use of "thermodynamical arguments." These require us to add a "perturbating term" to the Hamiltonian (1.2). This perturbating term creates a number of unessential but unpleasant complications. Moreover, while it does not change the value of (1.4), it somewhat changes the problem. The choice of this approach was the result of the fascinating work of Guerra [5] that appears to produce crucial information literally out of nothing. It turns out in retrospect that this was not the way to follow. For our purposes, perturbated Hamiltonians are a hindrance rather than a help, and we will not use them at all here. The primary motivation behind the present work is a recent paper [10] (itself motivated by physicists' questions) where (in a subregion of the "high temperature" region) the famous Sherrington–Kirkpatrick (SK) model is solved "at the accuracy of the CLT" (in a sense that will soon be explained). There are (at least in our mind) close connections between the SK and the Hopfield model. The validity of the physicists' solution for the SK model was first proved [6] (at nonzero external field) in a nontrivial region using a somewhat rough argument. A much more detailed picture was obtained in [10]; the main tools for this are an adaptation of arguments that were developed in [7] for the Hopfield model. The SK model is, however, much simpler than the Hopfield model, and the rather nontrivial task of extending the results of [10] to the Hopfield model will be carried out here. Specific motivation is also provided by the recent CLT proved by Bovier and Gayrard in [4] for the case where $M = M(N)$ satisfies $\lim_{M\to\infty} M(N)/N = 0$. While this is certainly nontrivial, it is, in our opinion, significantly easier than the case we consider here. In particular, our proofs seem to indicate that (even at high temperature) there is some intrinsic exquisite complication in the structure of the Hopfield model when $M = \lfloor \alpha N \rfloor$. This complication is of an algebraic nature; possibly a simple underlying structure remains to be discovered.

Let us now state our results. These are stated in terms of overlaps and replicas. We will consider p-replicas, which are simply the product Σ_N^p provided with the product measure $(G_N)^{\otimes p}$ (for the same realization of randomness). A point in a p-replica is a sequence $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^p)$ of p configurations. The overlap of two configurations is defined as

$$\boldsymbol{\varepsilon}^1 \cdot \boldsymbol{\varepsilon}^2 = \frac{1}{N} \sum_{i\le N} \epsilon_i^1 \epsilon_i^2. \tag{1.5}$$

We will also define

$$\boldsymbol{m}^l = \big(m_k(\boldsymbol{\varepsilon}^l)\big)_{2\le k\le M}$$

and, quite reasonably,

$$\boldsymbol{m}^l \cdot \boldsymbol{m}^{l'} = \sum_{2\le k\le M} m_k(\boldsymbol{\varepsilon}^l) m_k(\boldsymbol{\varepsilon}^{l'}). \tag{1.6}$$

A quantity such as (1.6) will also be called an overlap. The reader should observe that the summations do not include $k = 1$; the terms $m_1(\boldsymbol{\varepsilon}^l)$ have to be handled separately [their special role being obvious from (1.2)]. The reason for the different normalizations in (1.5) and (1.6) is that $\sum_{i\le N} \epsilon_i^2 = N$ while $\sum_{2\le k\le M} m_k^2(\boldsymbol{\varepsilon})$ is of order 1. The fact that the overlaps either as in (1.5) or as in (1.6) are "nearly constant" (in fact have fluctuations of order $N^{-1/2}$) is the central feature of the high-temperature phase. This is explained in [7], Section 4, and made even more blatant in [8, 9] and [11]. The fact that the overlaps are "nearly constant" will be expressed in a strong way by exponential inequalities that are interesting for their own sake. Before we state these, we must explain in which region we prove them. We define the *accessible region* as the region of the parameters for which either

$$1 \le \beta \le 2 \quad \text{and} \quad \alpha \le \frac{1}{L}(\beta - 1)^2 \tag{1.7}$$

or

$$\beta \ge 2 \quad \text{and} \quad \alpha \le \frac{1}{L \log \beta}. \tag{1.8}$$

There, as well as in the rest of the paper, L is a number, not necessarily the same at each occurrence. In the case of (1.7), (1.8), it helps to think of the constant L in (1.7), (1.8) as a parameter that we choose as large as convenient (as our methods are not appropriate to obtain reasonable numerical values, we make no attempt in this direction). The term "accessible" simply refers to the fact that we could prove something there. The parameter h can take any positive value, but we will provide complete details only when h is very small [i.e., for α, β as in (1.7), (1.8), our results will proved for $0 < h < h(\alpha, \beta)$ where $h(\alpha, \beta) > 0$]. For large h, condition (1.7) (in particular) can be improved upon. This is, however, a source of (real but unessential) complications and the reader is referred to Section 3 of [7] to enjoy these.

In contrast with (1.8) (which gives the correct behavior as $\beta \to \infty$) Bovier and Gayrard obtain only the smaller region $L\alpha\beta \le 1$ when $\beta \ge 1$, and their approach (as it stands now) does not extend to the correct region (1.8). [For later discussion, the subregion of the accessible region where (1.8) is replaced by $L\alpha\beta \le 1$ will be called the BG region.] The results of [7] are obtained only in the BG region, although the condition $L\alpha\beta \le 1$ occurred there for very different reasons than in the geometric method of Bovier and Gayrard.

THEOREM 1.1. *For each value of the parameters* (α, β, h) *inside the accessible region, there is a number* K (*possibly depending on* α, β, h) *such that for all* N *large enough, and all* $t \in \mathbb{R}$ *we have*

$$E\langle \exp t(\boldsymbol{\varepsilon}^1 \cdot \boldsymbol{\varepsilon}^2 - E\langle \boldsymbol{\varepsilon}^1 \cdot \boldsymbol{\varepsilon}^2 \rangle)\rangle \le \exp KNt^2, \tag{1.9}$$

$$E\langle \exp tN(\mathbf{m}^1 \cdot \mathbf{m}^2 - E\langle \mathbf{m}^1 \cdot \mathbf{m}^2 \rangle)\rangle \le \exp KNt^2. \tag{1.10}$$

There, as well as in the rest of the paper, the bracket $\langle \cdot \rangle$ denotes thermal average, that is, integration with respect to the Gibbs measure, and E denotes expectation in the "quenched variables" $(\eta_{i,k})$. There is an interesting interpretation of (1.9) as a strong way to express "lack of symmetry breaking"; see [10].

In order to express our convergence results in a general fashion, and considering now p-replicas, let us define an *admissible function* on Σ_N^p as a (finite) product of expressions of the following four possible types:

$$N^{-1/2}(\boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} - E\langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} \rangle), \tag{1.11}$$

$$N^{1/2}(\mathbf{m}^l \cdot \mathbf{m}^{l'} - E\langle \mathbf{m}^l \cdot \mathbf{m}^{l'} \rangle), \tag{1.12}$$

$$N^{1/2}(\|\mathbf{m}^l\|^2 - E\langle \|\mathbf{m}^l\|^2 \rangle), \tag{1.13}$$

$$N^{1/2}\Big(m_k(\boldsymbol{\varepsilon}^l) - E\langle m_k(\boldsymbol{\varepsilon}^l)\rangle\Big), \qquad 1 \le k \le M. \tag{1.14}$$

THEOREM 1.2. *If W is an admissible function on Σ_N^p, then*

$$\lim_{N\to\infty} E\langle W\rangle \qquad \textit{exists}. \tag{1.15}$$

Our proof of (1.15) is constructive and provides (in principle) a way to compute explicitly the limit for every W. The limits (1.15) contain all the information about the joint law of finitely many admissible functions $W_1, \dots, W_R$. Indeed, the joint law of $W_1, \dots, W_R$ under the Gibbs measure is determined by the (random) quantities $\langle W_1^{n_1}, \dots, W_R^{n_R}\rangle$. The joint h law of these under the disorder is determined by their moments, which (using replicas) are themselves of the type $E\langle W\rangle$ for an admissible function W. The only problem remaining is to find explicit formulas. This is a problem of a somewhat algebraic nature, and, as we feel that the challenge of the model (in the accessible region!) is largely gone, we could not find the energy to do this. It seems almost certain that the finite family of functions on Σ_N^p of the type

$$\begin{aligned} &N^{-1/2}(\boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} - \langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} \rangle),\\ &N^{1/2}(\mathbf{m}^l \cdot \mathbf{m}^{l'} - \langle \mathbf{m}^l \cdot \mathbf{m}^{l'} \rangle),\\ &N^{1/2}\big(m_k(\boldsymbol{\varepsilon}^l) - \langle m_k(\boldsymbol{\varepsilon}^l)\rangle\big) \end{aligned} \tag{1.16}$$

is under the Gibbs measure asymptotically Gaussian, perhaps even with a covariance structure independent of the disorder (the reader should have no problem to make such a statement precise!). Moreover, the families

$$\begin{aligned} &N^{-1/2}\big(\langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} \rangle - E\langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} \rangle\big),\\ &N^{1/2}(\langle \mathbf{m}^l \cdot \mathbf{m}^{l'} \rangle - E\langle \mathbf{m}^l \cdot \mathbf{m}^{l'} \rangle),\\ &N^{1/2}\big(\langle m_k(\boldsymbol{\varepsilon}^l)\rangle - E\langle m_k(\boldsymbol{\varepsilon}^l)\rangle\big) \end{aligned} \tag{1.17}$$

should also be asymptotically jointly Gaussian. We did check this in the simplest case of the family (1.16), $l = 1$, $k \ge 2$ varying (which is asymptotically i.i.d. with a variance independent of the disorder) and (1.17) (which is asymptotically i.i.d.).

To conclude, let us say a few words on how to read the present paper. As already mentioned, the methods we will use are closely related to these of [7], Sections 4–8. However, since we take a different route (not using "perturbated Hamiltonians"), we have to redo all the work, and familiarity with [7] is not required and even probably not helpful to penetrate the present work. The present work is also in theory rather independent of [10]. But, since [10] performs the same program as we do here, but in the technically much simpler case of the SK model, studying [10] first should provide invaluable help to the serious reader. We should also mention that one of the main contributions of the paper, the proof of new "a priori" estimates on Gibbs measures (Proposition 2.3), which is given in Section 6, can be read independently of the cavity method arguments of the other sections.

2. The tools. To simplify the notation, we observe that we obtain an equivalent model if we replace $\eta_{i,k}$ by $\eta_{i,k}\eta_{i,1}^{-1}$. That is, we can assume that $\eta_{1,k} = 1$ for all k and that $(\eta_{i,k})_{i\le N, 2\le k\le M}$ are i.i.d. Bernoulli.

The starting point is to relate a situation with $N+1$ spins to a situation with N spins. This is done basically through simple algebra. It will unfortunately look complicated because we have at the same time to introduce our basic notation, which will remain in force throughout the paper. Before we do this, we should mention that our point of view will be slightly different from that of [7] and [10]. When we relate a situation with $N+1$ spins to a situation with N spins, this introduces a slight change of β and of h, which, in the present case has the unpleasant tendency to push the parameter value (α, β, h) outside the accessible region. This is an obstacle toward the use of induction upon N as in [10]. This obstacle possibly could be passed using enough force; but rather than doing this, we find that it is more instructive to explore a slightly different point of view. In this point of view β, h are given once and for all.

Throughout the paper, we set

$$N' = N+1, \qquad \beta' = \frac{N'}{N}\beta, \qquad h' = \frac{N}{N'}h. \tag{2.1}$$

We consider a new independent Bernoulli sequence $(\eta_k)_{2<k\le M}$, we set $\eta_1 = 1$ and we set $\eta_{N',k} = \eta_k$. For $\boldsymbol{\sigma} \in \Sigma_{N+1}$, we consider the Hamiltonian

$$H_{N'}(\boldsymbol{\sigma}) = -\frac{1}{2N'}\sum_{k\le M}\biggl(\sum_{i\le N'}\eta_{i,k}\sigma_i\biggr)^2 - h'\sum_{i\le N'}\epsilon_i, \tag{2.2}$$

which corresponds to (1.2) for $N+1$ rather than N, with a small change of h.

To relate conveniently sequences in Σ_N and in $\Sigma_{N'}$, we make the following convention. Given $\boldsymbol{\varepsilon}$ in Σ_N, and $\sigma \in \{-1, 1\}$, we set $\boldsymbol{\sigma} = (\boldsymbol{\varepsilon}, \sigma) \in \Sigma_{N+1}$. Given $\boldsymbol{\sigma}$ in $\Sigma_{N'}$, we set $\sigma = \sigma_{N+1}$, $\boldsymbol{\varepsilon} = (\sigma_1, \dots, \sigma_N)$. With this notation elementary algebra shows the basic identity

$$-\beta' H_{N'}(\boldsymbol{\sigma}) = -\beta H_N(\boldsymbol{\varepsilon}) + \beta M/2N + \sigma\beta\biggl(\sum_{1\le k\le M}\eta_k m_k(\boldsymbol{\varepsilon}) + h\biggr). \tag{2.3}$$

We will now work with p-replicas. The generic point in a p-replica will be denoted $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^p)$, where $\boldsymbol{\varepsilon}^l \in \Sigma_N$. For simplicity, we will write $m_k^l = m_k(\boldsymbol{\varepsilon}^l)$,

$$\boldsymbol{\eta} = (\eta_k)_{2 \le k \le M}, \qquad \mathbf{m}^l = (m_k^l)_{2 \le k \le M}, \qquad \boldsymbol{\eta} \cdot \mathbf{m}^l = \sum_{2 \le k \le M} \eta_k m_k^l.$$

The last component of $\boldsymbol{\sigma}^l$ will be written as σ_l rather than σ^l, to avoid confusing the replica index l with a power. (We hope that the notation m_k^l will not create confusion in this respect.) Then (2.3) becomes

$$-\beta' H_{N'}(\boldsymbol{\sigma}^l) = -\beta H_N(\boldsymbol{\varepsilon}^l) + \beta' M/2N + \sigma_l \beta(\boldsymbol{\eta} \cdot \mathbf{m}^l + h + m_1^l). \tag{2.4}$$

We will denote by $\langle \cdot \rangle'$ average with respect to the Gibbs measure on $\Sigma_{N'}^p$ relative to the Hamiltonian (2.2) at inverse temperature β', and its products on $\Sigma_{N'}^p$, while $\langle \cdot \rangle$ denotes average of the Gibbs measure relative to the Hamiltonian (1.2) at inverse temperature β.

To simplify the notation, we write

$$\mathscr{E} = \mathscr{E}(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^p) = \exp \beta \sum_{l \le p} \sigma_l (\boldsymbol{\eta} \cdot \mathbf{m}^l + h + m_1^l). \tag{2.5}$$

A direct consequence of (2.4) is as follows.

Proposition 2.1. *Given a function f on $\Sigma_{N'}^p$, we have*

$$\langle f \rangle' = \frac{\langle \operatorname{Av} f \mathscr{E} \rangle}{\langle \operatorname{Av} \mathscr{E} \rangle}. \tag{2.6}$$

In this formula, Av means average over all the values of $\sigma_1, \ldots, \sigma_p = \pm 1$. Thus Av $f\mathscr{E}$ is a function of $\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^p$ only, and $\langle \operatorname{Av} f\mathscr{E} \rangle$ is its average with respect to (the pth power of) the Gibbs measure relative to the Hamiltonian (1.2). We should also note that in (2.6) f might possibly depend on $(\eta_{i,k})$.

We will use (2.6) to estimate $E\langle f \rangle'$, so that we want to estimate

$$E \frac{\langle \operatorname{Av} f\mathscr{E} \rangle}{\langle \operatorname{Av} \mathscr{E} \rangle}. \tag{2.7}$$

We will first integrate in the variables $\boldsymbol{\eta} = (\eta_k)$. The idea for handling the denominator is that (in the region of parameters under consideration) this denominator depends on $\boldsymbol{\eta}$ essentially only through $\boldsymbol{\eta} \cdot \mathbf{b}$, where $b_k = \langle m_k(\boldsymbol{\varepsilon}) \rangle$. (That this is indeed the case is not obvious at this stage.) The natural idea then is to try a conditioning argument upon $\boldsymbol{\eta} \cdot \mathbf{b}$. Unfortunately, there are essentially no techniques to work with the variables η_k. However, we would succeed if we could replace the variables (η_k) by i.i.d. $N(0, 1)$ variables. We could then appeal to the rich theory of Gaussian processes. But this can be done with a small error only if $\sum_{2 \le k \le M} m_k^4$ is small in average (of course, the reader has guessed that here the superscript is a power not a replica index!). Our first task will be to prove this. In [7], this was done through the cavity method, because one of our primary goals there was to check the stability of this property against addition of a new spin. Since here we are interested only in the accessible region, we will directly prove an exponential inequality.

THEOREM 2.2. *If the parameters α, β, h belong to the accessible region, then, for $k \geq 2$,*

$$E\left\langle \exp \frac{N}{K} m_k^2 \right\rangle \leq K. \tag{2.8}$$

Here, as in the rest of the paper, $m_k = m_k(\boldsymbol{\varepsilon})$, and K denotes a number depending on α, β, h only, but not on N. It is understood that such numbers (which are not necessarily the same at each occurrence) remain bounded on each compact subset of the accessible region. In the BG region, Theorem 2.2 is proved in [3].

The proof of this result, and in fact the entire paper, depends crucially on certain a priori estimates of the Gibbs measure. For $\beta > 1$, $h > 0$, we consider the unique root m^* of the equation

$$m^* = \operatorname{th} \beta(m^* + h). \tag{2.9}$$

The basic fact is that, if α is small, the image G' of the Gibbs measure on $\mathbb{R}^N$ by the map $\boldsymbol{\varepsilon} \to (m_k(\boldsymbol{\varepsilon}))_{1 \leq k \leq M}$ is, with high probability, almost supported by a small ball centered at the point $(m^*, 0, \ldots, 0)$ of $\mathbb{R}^M$. More precisely, if $\beta > 1$, we have, $\forall \rho > 0$, $\exists \alpha(\rho, \beta) > 0$,

$$\alpha < \alpha(\rho, \beta) \Rightarrow EG\left(\left\{\boldsymbol{\varepsilon}; (m_1 - m^*)^2 + \sum_{2 \leq k \leq M} m_k^2 \geq \rho^2\right\}\right) \leq \exp\left(-\frac{N}{K}\right). \tag{2.10}$$

This statement, which is not deep, is sufficient to get the main story; that is, at given $\beta > 1$, there is $\alpha(\beta) > 0$ such that our results hold for $\alpha \leq \alpha(\beta)$. The correct dependence of $\alpha(\beta)$ on β as $\beta \to 1$ or $\beta \to \infty$ is a side story, albeit an important one, that requires more work.

Before we state the a priori estimates, we should mention that we have not attempted to make these sharp, but rather to state what will be required in our proofs. Once the replica-symmetric equations have been proved, together with our exponential inequalities, we will have an extremely precise picture. The image G' of the Gibbs measure under the map $\boldsymbol{\varepsilon} \to (m_k(\boldsymbol{\varepsilon}))_{k \leq M}$ is sharply concentrated close to a sphere of random center and nonrandom radius, the value of which is explicitly known.

Throughout Sections 2 to 6, we will use the following notation:

$$\begin{aligned} &\text{If } 1 < \beta \leq 2, \qquad \rho_0 = \frac{L\alpha}{(\beta - 1)^{3/2}}, \qquad \rho = \rho_1 = L\left(\frac{\alpha}{\beta - 1}\right)^{1/2}, \\ &\text{If } \beta \geq 2, \qquad \rho_0 = \rho = \frac{L}{\beta^2}, \qquad \rho_1 = L\sqrt{\alpha}. \end{aligned} \tag{2.11}$$

Thus, in the admissible region, we have $\rho_0 \leq \rho \leq \rho_1$.

PROPOSITION 2.3. *If the constant* L *of* (1.7), (1.8) *is large enough, then the following occurs:*

$$EG\biggl(\biggl\{\boldsymbol{\varepsilon} : (m_1 - m^*)^2 + \sum_{2\le k\le M} m_k^2 \ge \rho_1^2\biggr\}\biggr) \le K\exp\biggl(-\frac{N}{K}\biggr). \tag{2.12}$$

Moreover, there is a random point c *of* $\mathbb{R}^M$ *with* $c_1 = m^*$ *such that*

$$EG\biggl(\biggl\{\boldsymbol{\varepsilon} : \sum_{1\le k\le M} (m_k - c_k)^2 \ge \rho^2\biggr\}\biggr) \le K\exp\biggl(-\frac{N}{K}\biggr). \tag{2.13}$$

Moreover, we have

$$EG\bigl(\{\boldsymbol{\varepsilon} : |m_1 - m^*| \ge \rho_0\}\bigr) \le \exp\biggl(-\frac{N}{K}\biggr). \tag{2.14}$$

The meaning of (2.13) is that G' is sharply concentrated on a ball of radius ρ_1 and of center $(m^*, 0, \dots)$. The meaning of (2.13) is that the radius ρ_1 can be decreased to ρ if we allow a random center.

Inequality (2.12) is proved in [2] and [7]. In the case $\beta \ge 2$, (2.13) is proved in Section 6, and is the new ingredient that allows us to prove our results in the accessible region rather than only in the BG region. There is nothing specific about the power 2 in the expression of ρ for $\beta \ge 2$ in (2.13) that could be replaced by a larger power. As strange as it may seem, the order of ρ_0 for $\beta \le 2$ is optimal in (2.14), because (5.4) below shows that $E\langle m_1\rangle - m^*$ is already of order ρ_0.

PROOF OF THEOREM 2.2. By symmetry we can assume that $k = M$. We consider the set U of configurations $\boldsymbol{\varepsilon}$ given by

$$U = \bigl\{\boldsymbol{\varepsilon} : |m_1 - m^*| \le \rho_0\bigr\}$$

so that by (2.14) we have

$$E\langle 1_{U^c}\rangle \le \exp\biggl(-\frac{N}{K_1}\biggr).$$

(Of course, K_1, K_2 denote specific numbers independent of N.) We immediately run into a minor recurring problem. A statement such as Proposition 2.3 does not control certain small sets of exponentially small measure. This was not an obstacle in [7] because there we were integrating functions that grew only polynomially with N. However, in the present case, the function $\exp Nm_k^2/K$ takes exponentially large values, up to $\exp(N/K)$. There is a simple way around this difficulty, which will have to be used in many occurrences. It is simply to observe that, since $|m_M(\boldsymbol{\varepsilon})| \le 1$, we have

$$E\biggl\langle 1_{U^c}\exp\frac{Nm_M^2}{K}\biggr\rangle \le \exp\frac{N}{K}E\langle 1_{U^c}\rangle \le \exp N\biggl(\frac{1}{K_1} - \frac{1}{K}\biggr)$$

and this is less than or equal to 1 if $K \geq K_1$. Thus, it is enough to prove

$$E\left\langle 1_U \exp \frac{N}{K} m_M^2 \right\rangle \leq K. \tag{2.15}$$

In other words, the solution to the obstacle of uncontrolled exponentially small sets is simply to prove exponential inequalities with a sufficiently small coefficient (here $1/K$ for large K) in the exponent. From now on, we will completely ignore any exponentially small set we wish, and we will use expressions such as "we can pretend that $|m_1 - m^*| < \rho_0$" to mean that we know that the fact that this condition sometimes fails is not an obstacle (as proved above) and that it suffices to prove (2.15) to have (2.8).

Let us denote by $\langle \cdot \rangle_1$ the Gibbs measure corresponding to the Hamiltonian

$$H_{N,\,M-1} = -\frac{N}{2} \sum_{1 \leq k \leq M-1} m_k(\boldsymbol{\varepsilon})^2 - hNm_1(\boldsymbol{\varepsilon}).$$

Considering a parameter A, in the spirit of Proposition 2.1 we have the (much easier) identity

$$\left\langle 1_U \exp \frac{N}{A} m_M^2 \right\rangle = \frac{\langle 1_U \exp \frac{1}{2}(\beta + 2/A) N m_M^2 \rangle_1}{\langle \exp \frac{1}{2} \beta N m_M^2 \rangle_1}. \tag{2.16}$$

Considering a parameter $x > 0$ to be determined later, we bound the right-hand side of (2.16) by

$$\exp \frac{Nx^2}{A} + \left\langle W \exp \frac{\bar{\beta}}{2} N m_M^2 \right\rangle_1 \tag{2.17}$$

where $\bar{\beta} = \beta + 2/A$ and where $W = 1_{U \cap \{|m_M| \geq x\}}$. Consider the parameters $s_1, s_2 > 0$, with $1/s_1 + 1/s_2 = 1$, so that, by Hölder's inequality, and since $W^{s_1} = W$, we have

$$\left\langle W \exp \frac{\bar{\beta}}{2} N m_M^2 \right\rangle_1 \leq \langle W \rangle_1^{1/s_1} \left\langle W \exp \frac{\bar{\beta}}{2} s_2 N m_M^2 \right\rangle_1^{1/s_2}.$$

To bound the first term on the right, we use the Chebyshev inequality:

$$\langle W \rangle_1 \leq \exp\left(-\frac{\bar{\beta}}{2} s_2 N x^2\right) \left\langle 1_U \exp \frac{\bar{\beta}}{2} s_2 N m_M^2 \right\rangle_1$$

and we get that the quantity (2.17) is bounded by

$$\exp \frac{Nx^2}{A} + \exp\left(-\frac{\bar{\beta} s_2}{2 s_1} N x^2\right) \left\langle 1_U \exp \frac{\bar{\beta}}{2} s_2 N m_M^2 \right\rangle_1. \tag{2.18}$$

We choose x the smallest possible to make the last term less than or equal to 1, and we see from (2.16) that

$$\left\langle 1_U \exp \frac{N}{A} m_M^2 \right\rangle \leq 2 \left\langle 1_U \exp \frac{\bar{\beta}}{2} s_2 N m_M^2 \right\rangle_1^{2s_1/A\bar{\beta}s_2}. \tag{2.19}$$

We now claim that to finish the proof it suffices to show that we can find $\beta_1 > \beta$ and $s > 0$ such that

$$E\left(\left\langle 1_U \exp\frac{\beta_1}{2} N m_M^2\right\rangle_1^s\right) \le K. \tag{2.20}$$

Indeed, we take s_2 close enough to 1 that $\beta s_2 < \beta_1$, and then A large enough that $\bar{\beta} s_2 < \beta_1$, $2s_1/A\bar{\beta}s_2 < s$.

To prove (2.20), we write

$$m_M(\boldsymbol{\varepsilon}) = \frac{1}{N}\left(\sum_{i\le N}\eta_{i,M}m^* + \sum_{i\le N}\eta_{i,M}(\epsilon_i - m^*)\right) \tag{2.21}$$

so that, for $t > 0$, using that $(a+b)^2 \le (1+t)a^2 + (1+1/t)b^2$, we have

$$\begin{aligned} N m_M^2(\boldsymbol{\varepsilon}) &\le (1+t)\frac{1}{N}\left(\sum_{i\le N}\eta_{i,M}m^*\right)^2 \\ &\quad + \left(1+\frac{1}{t}\right)\frac{1}{N}\left(\sum_{i\le N}\eta_{i,M}(\epsilon_i - m^*)\right)^2. \end{aligned} \tag{2.22}$$

Thus,

$$\begin{aligned} \left\langle 1_U \exp\frac{\beta_1}{2} N m_M^2\right\rangle_1^s &\le \exp\frac{(m^*)^2}{2N} s\beta_1(1+t)\left(\sum_{i\le N}\eta_{i,M}\right)^2 \\ &\quad \times \left\langle 1_U \exp\frac{\beta_1}{2N}\left(1+\frac{1}{t}\right)\left(\sum_{i\le N}\eta_{i,M}(\epsilon_i - m^*)\right)^2\right\rangle_1^s. \end{aligned}$$

and thus, using the Cauchy–Schwarz and Hölder inequalities (assuming, as we may, that $2s \le 1$),

$$\begin{aligned} E\left(\left\langle 1_U \exp\frac{\beta_1}{2} N m_N^2\right\rangle_1^s\right)^2 &\le E\exp\frac{s(m^*)^2}{N}\beta_1(1+t)\left(\sum_{i\le N}\eta_{i,M}\right)^2 \\ &\quad \times \left(E\left\langle \exp 1_U \exp\frac{\beta_1}{2N}\left(1+\frac{1}{t}\right)\right.\right. \\ &\quad \left.\left. \times \left(\sum_{i\le N}\eta_{i,M}(\epsilon_i - m^*)\right)^2\right\rangle_1\right)^{2s}. \end{aligned} \tag{2.23}$$

We note that, since $N m_1(\boldsymbol{\varepsilon}) = \sum_{i\le N}\epsilon_i$, we have

$$\begin{aligned} \sum_{i\le N}(\epsilon_i - m^*)^2 &= N(1 - 2m_1(\boldsymbol{\varepsilon})m^* + m^{*2}) \\ &= N\big((1-m^{*2}) - 2(m_1(\boldsymbol{\varepsilon}) - m^*)m^*\big) \end{aligned}$$

so that

$$\boldsymbol{\varepsilon} \in U \Rightarrow \sum_{i \le N} (\epsilon_i - m^*)^2 \le N(1 - m^{*2} + 2\rho_0 m^*). \tag{2.24}$$

We now use the simple fact (see [7], proof of Lemma 2.1) that

$$E \exp\left(\sum_{i \le N} \eta_{i,M} x_i\right)^2 \le \left(\frac{1}{1 - 2\sum_{i \le N} x_i^2}\right)^{1/2}. \tag{2.25}$$

Computing the expectation E inside the bracket $\langle \cdot \rangle_1$ on the right-hand side of (2.23), using (2.24) and (2.25), we see that all that is required is that

$$s(m^*)^2(1+t)\beta_1 < \frac{1}{2},$$

$$\beta_1\left(1 + \frac{1}{t}\right)(1 - m^{*2} + 2\rho_0 m^*) < 1.$$

Since s is arbitrarily small, all that is required is that, in the accessible region, we have

$$\beta(1 - m^{*2} + 2\rho_0 m^*) < 1.$$

For $1 < \beta \le 2$, this follows (at least for h very small) from the elementary fact that $1 - \beta(1 - m^{*2})$ is of order $\beta - 1$ and m^* of order $\sqrt{\beta - 1}$; while if $\beta \ge 2$, $\beta(1 - m^*)$ is less than or equal to L/β [and, in fact, less than or equal to $\exp(-\beta/L)$]. □

Theorem 2.3 gives us a very strong control of the functions m_k, but still we have to be careful because we integrate large functions f.

PROPOSITION 2.4. *Consider a function f on Σ_{N+1}^p. This function might depend on the r.v. $(\eta_{i,k})_{i \le N, k \le M}$, but may not depend on the variables $\eta_k = \eta_{N+1,k}$. If to compute $E\langle f \rangle'$ we replace in* (2.6) *the variables $(\eta_k)_{2 \le k \le M}$ by i.i.d $N(0, 1)$ variables, we make an error of at most*

$$KE\left\langle \mathrm{Av}|f| \left(\sum_{l \le p+1} \sum_{2 \le k \le M} (m_k^l)^4\right)\right\rangle. \tag{2.26}$$

COMMENTS. (1) In this formula, the thermal integral is over a $p+1$ replica Σ_N^{p+1}, and Av $|f|$ is identified to a function on Σ_N^{p+1} in the obvious manner,

$$\mathrm{Av}\,|f|(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^{p+1}) = \mathop{\mathrm{Av}}_{\sigma_1, \dots, \sigma_p \pm 1} |f|(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p),$$

where $\boldsymbol{\varepsilon}^l$ and $\boldsymbol{\sigma}^l$ are related as usual, that is, $\boldsymbol{\sigma}^l = (\boldsymbol{\varepsilon}^l, \sigma_l)$.

(2) The constant K in (2.26) depends on p.

(3) The true formula would involve an extra error term

$$\exp K\left(\sum_{l \le p+1} \|\mathbf{m}^l\|^2\right).$$

However, we have (see the Appendix of [7]) an easy inequality

$$P\Big(\sup_{\boldsymbol{\varepsilon}} \|\mathbf{m}^l(\boldsymbol{\varepsilon})\|^2 \geq Lt\Big) \leq \exp(-Nt) \tag{2.27}$$

for $t \geq 1$ (here and throughout the paper we assume without loss of generality that $\alpha \leq 1$), so that this term does not matter.

Even though Proposition 2.4 is proved in [7], it is worthwhile to repeat the proof in the present case, which is simpler because we do not have a perturbated Hamiltonian.

PROOF OF PROPOSITION 2.4. We replace the variables $(\eta_k)_{2\leq k\leq M}$ by i.i.d. $N(0, 1)$ variables g_k one at a time. Assuming that $\eta_2, \dots, \eta_k$ have already been replaced, we show that the error made when replacing η_{k+1} by g_{k+1} is at most

$$KE\Big\langle \mathrm{Av}|f| \sum_{l\leq p+1} (m_{k+1}^l)^4 \Big\rangle.$$

To see this, let us consider the function $\varphi(t)$ obtained by replacing η_{k+1} by t in each of its occurrences in $\langle \mathrm{Av}\, f\mathscr{E}\rangle / \langle \mathrm{Av}\, \mathscr{E}\rangle$. Thus, φ depends on $g_1, \dots, g_k$, $\eta_{k+2}, \dots, \eta_M$ as well as on the $(\eta_{i,\,M})_{i\leq N,\, 2\leq k\leq M}$. We want to bound

$$E\varphi(\eta_{k+1}) - E\varphi(g_{k+1}). \tag{2.28}$$

The basic fact is that the first three moments of η_{k+1} and g_{k+1} coincide, so that the quantity (2.28) is at most

$$E|\psi(\eta_{k+1})| + E|\psi(g_{k+1})|, \tag{2.29}$$

where

$$\psi(t) = \varphi(t) - \varphi(0) - t\varphi'(0) - \frac{t^2}{2}\varphi''(0) - \frac{t^3}{3!}\varphi^{(3)}(0).$$

A convenient (and somewhat crude) way to bound ψ is

$$|\psi(t)| \leq |t|^3 \int_{-\infty}^{\infty} |\varphi^{(4)}(x)| 1_{\{|t|\geq|x|\}}\, dx. \tag{2.30}$$

Now, we calculate $\varphi^{(4)}(x)$, and use replicas to express products of brackets as a single bracket; thus, $\varphi^{(4)}(x)$ is a sum of terms

$$c(l_1, \dots, l_4) \frac{\langle \mathrm{Av}\ f m_{k+1}^{l_1} m_{k+1}^{l_2} m_{k+1}^{l_3} m_{k+1}^{l_4} \mathscr{E}'\rangle}{\langle \mathrm{Av}\ \mathscr{E}\rangle^5}.$$

There, the numerator is a thermal integral on Σ_N^{5p}, $1 \leq l_1, \dots, l_4 \leq 5p$, $c(l_1, \dots, l_4)$ is a number, $\mathscr{E}'$ is defined as in (3.5) replacing p by $5p$, and it is understood that occurrences of η_{k+1} are replaced by x. To bound (2.29) using (2.30), only trivial bounds are needed. We use that $\langle \mathrm{Av}\, \mathscr{E}\rangle \geq 1$, we integrate first in $\eta_{k'}$, $g_{k'}$, $2 \leq k' \leq M$, and we use Hölder's inequality for

$\langle\cdot\rangle$ to replace terms $m_{k+1}^{l_1}\cdots m_{k+1}^{l_4}$ by terms $(m_{k+1}^{l})^4$. We then obtain (2.26), except that we have $5p$ rather than $p+1$. But, for $l \ge p+1$, all the terms $\langle \mathrm{Av}\,|f|(m_{k+1}^{l})^4\rangle$ are equal to $\langle \mathrm{Av}\,|f|\rangle\langle(m_{k+1}^{l})^4\rangle$ so it is enough to introduce only one of these terms in our error bound. □

The main effort is the evaluation of $E\langle f\rangle'$ in (2.6). Before we state the result, we set up the notation, which will remain in force through the paper. We define

$$\mathbf{b} = \langle \mathbf{m}\rangle, \tag{2.31}$$

$$\dot{\mathbf{m}}^l = \mathbf{m}^l - \mathbf{b} = \mathbf{m}^l - \langle \mathbf{m}^l\rangle, \tag{2.32}$$

$$a_l = \|\dot{\mathbf{m}}^l\|^2 - \langle \|\dot{\mathbf{m}}^l\|^2\rangle, \tag{2.33}$$

$$c_l = m_1^l - \langle m_1\rangle, \tag{2.34}$$

$$Y = \beta(\mathbf{g}\cdot\mathbf{b} + \langle m_1\rangle + h), \tag{2.35}$$

where, of course, $\mathbf{g}\cdot\mathbf{b} = \sum_{2\le k\le M} g_k b_k$, (g_k) is an i.i.d. $N(0,\,1)$ sequence, and

$$\begin{aligned}\mathscr{E} &= \exp\beta\sum_{l\le p}\sigma_l(\mathbf{g}\cdot\mathbf{m}^l + m_1 + h),\\ \mathscr{E}_0 &= \exp\beta\sum_{l\le p}\sigma_l(\mathbf{g}\cdot\mathbf{b} + \langle m_1\rangle + h).\end{aligned} \tag{2.36}$$

THEOREM 2.5. *Consider a number $R > 0$, and assume that*

$$EG(U) = E\langle 1_U\rangle \le \exp\left(-\frac{N}{K}\right),$$

where

$$U = \{\boldsymbol{\varepsilon};\ \|\dot{\mathbf{m}}^1\| \le R,\ |c_1| \le R,\ |a_1| \le R^2\}. \tag{2.37}$$

Then we can pretend that, for a function f on Σ_{N+1}^p, we have

$$E\frac{\langle Av\ f\mathscr{E}\rangle}{\langle Av\ \mathscr{E}\rangle} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + S, \tag{2.38}$$

where

$$\mathrm{I} = E\frac{1}{\mathrm{ch}^p Y}\langle \mathrm{Av}\, f\mathscr{E}_0\rangle, \tag{2.39}$$

$$\mathrm{II} = E\frac{\beta^2}{\mathrm{ch}^p Y}\left\langle \mathrm{Av}\, f\mathscr{E}_0\sum_{l\le l'}\sigma_l\sigma_{l'}\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}\right\rangle, \tag{2.40}$$

$$\mathrm{III} = \frac{1}{2}E\frac{\beta^2}{\mathrm{ch}^p Y}\left\langle \mathrm{Av}\, f\mathscr{E}_0\left(\sum_{l\le p} a_l\right)\right\rangle, \tag{2.41}$$

$$\mathrm{IV} = E\frac{\beta^2}{\mathrm{ch}^p Y}\left\langle \mathrm{Av}\, f\mathscr{E}_0\left(\sum_{l\le p}\sigma_l\right)\left(\sum_{l\le p}\sigma_l\dot{\mathbf{m}}^l\cdot\mathbf{b}\right)\right\rangle, \tag{2.42}$$

$$\text{(2.43)} \qquad \mathrm{V} = -pE\frac{\beta^2 \operatorname{th} Y}{\operatorname{ch}^p Y}\Big\langle \operatorname{Av} f\mathscr{E}_0\Big(\sum_{l\le p}\sigma_l \dot{\mathbf{m}}^l\cdot\mathbf{b}\Big)\Big\rangle,$$

$$\text{(2.44)} \qquad \mathrm{VI} = E\frac{\beta}{\operatorname{ch}^p Y}\Big\langle \operatorname{Av} f\mathscr{E}_0\Big(\sum_{l\le p}\sigma_l c_l\Big)\Big\rangle,$$

$$\text{(2.45)} \qquad |S| \le K(p)(\beta^2+\beta^4)\exp K(p)(\beta R+\beta^2R^2) \times \Big\langle Av|f|\Big(\sum_{l<l'\le p+2}(\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'})^2 + \sum_{l\le p+1}a_l^2 + \sum_{l\le p+1}c_l^2\Big)\Big\rangle.$$

Moreover, if

$$\text{(2.46)} \qquad f\ne 0 \Rightarrow \Big|\sum_{l\le p}\sigma_l\Big| \le p-1,$$

then the remainder term S can be improved by a factor

$$\exp\Big(-\frac{1}{L\alpha}\Big)+\frac{1}{\operatorname{ch}(\beta/2)}.$$

COMMENTS. (1) By the expression "we can pretend that" we mean that there is an extra error term besides S, which takes into account the fact that it is not always true that $\|\dot{\mathbf{m}}^l\|\le R$, $|c_l|\le R$, $|a_l|\le R^2$. This term can, however, be shown (as in the proof of Theorem 2.2) to be unimportant for the functions f we will consider.

(2) In (2.45), it is essential that $K(p)$ depends on p only, not on α, β, h.

Before we start the proof we explain why when $\beta>2$ the condition $L\alpha\beta\le 1$ was required in [7], and why is now can be improved into (1.8). In the case that (2.46) holds and, say, $p=4$, it is essential that in S the coefficient of the bracket in (2.45) be bounded independently of α, β. This coefficient is

$$\text{(2.47)} \qquad L(\beta^2+\beta^4)\Big(\exp-\frac{1}{L\alpha}+\frac{1}{\operatorname{ch}(\beta/2)}\Big)\exp L(\beta R+\beta^2R^2).$$

In [7] we used (2.12), and $R=L\sqrt{\alpha}$, and to make (2.47) bounded we needed $L\alpha\beta\le 1$. Now, we will be able to use (2.13) with $R=L\beta^{-1}$, and $\alpha\le 1/L\log\beta$ is sufficient. This is the *only* place in the entire paper where (2.13) will be needed.

PROOF OF THEOREM 2.5. A statement of the same nature appears in [7], Proposition 6.4. The proof is thoroughly rewritten in the simplest possible case, of the SK model in [10], to which the reader is referred for a first understanding of the method of proof (which is essentially a second-order expansion). After reading this, the reader should be able to read the proof of Proposition 6.4 of [7] without undue effort, and we will simply point out the rather simple changes that are necessary to obtain the present statement. There are two main issues, the algebraic form of the main terms and bounds for the remainder. To simplify the notation, we will write β rather than β' (and we will explain later why such a change is irrelevant). In the course of the proof, one has to handle terms of the type

$$\exp \frac{\beta^2}{2} \|\dot{\mathbf{m}}^l\|^2.$$

In [7] [equation after (6.23), and taking into account that what we write $\mathbf{m}^l$ here is denoted $\mathbf{u}^l$ there] these are written as

$$\exp \frac{\beta^2}{2} \langle \|\dot{\mathbf{m}}^l\|^2 \rangle \exp \frac{\beta^2}{2} (\|\mathbf{m}^l\|^2 - \langle \|\mathbf{m}^l\|^2 \rangle) \exp -\beta^2 \dot{\mathbf{m}}^l \cdot \mathbf{b}$$

and it is argued by general arguments that $\|\mathbf{m}^l\|^2 - \langle \|\mathbf{m}^l\|^2 \rangle$ gives a vanishing contribution. Now, we use that

$$|e^x - (1+x)| \le x^2 e^{|x|} \tag{2.48}$$

for $x = (\beta^2/2)(\|\mathbf{m}^l\|^2 - \langle \|\mathbf{m}\|^2 \rangle)$. The corresponding contribution of the term x in (2.45) combined with term III of [7], Proposition 6.4, yields the term (2.41).

We also have to deal with terms

$$\exp \beta \sigma_l m_1^l = \exp \beta \sigma_l \langle m_1^l \rangle \exp \beta \sigma_l c_l.$$

In [7] it is argued through general principles that the contribution of c_l is vanishingly small. Rather, we now use (2.48) for $x = \beta \sigma_l c_l$, and this creates term VI.

The error terms arise from replacing at places e^x by $1 + x$, and the error in doing this is at most $x^2 e^{|x|}$ by (2.48). This is used in the proof for the value

$$x = \sum_{l \le p'} \frac{\beta^2}{2} a_l + \sum_{l \le p'} \beta \sigma_l c_l + \sum_{l < l' \le p'} \beta^2 \sigma_l \sigma_{l'} \dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'},$$

where $p' \le 3p$. Thus,

$$x^2 \le K(p)(\beta^2 + \beta^4)\Big(\sum a_l^2 + \sum c_l^2 + \sum (\dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'})^2\Big),$$

while

$$|x| \le K(p)(\beta^2 R^2 + \beta R),$$

when $\boldsymbol{\varepsilon}^l \in U$ for each l. This is what explains the error term S. As for the improvement of the error term under (2.46), the reader is better referred to the argument in the case of the SK model [10].

Terms I to VI in Theorem 2.5 will be called the *main terms* and S the *error term*. The terms III to VI are of the same nature. It saves energy to perform once and for all their computation in the special situation that we will use. These elementary computations are left to the reader.

PROPOSITION 2.6. *Assume that for a certain subset* I *of* $\{1, \dots, p\}$ *of cardinal* n, *we have*

$$f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p) = \left(\prod_{l \in I} \sigma_l\right) \bar{f}(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^p),$$

where f *is a function on* Σ_N^p. *Then [writing* $\bar{f}$ *for* $\bar{f}(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^p)$*], the contributions of the terms of Theorem* 2.5 *are as follows. The contribution of term* II *is*

$$\begin{aligned}&\beta^2 E\mathrm{th}^{n-2} Y \Big\langle \bar{f} \sum_{l<l',\, l,\, l' \in I} \dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'} \Big\rangle \\ &\quad + \beta^2 E\mathrm{th}^{n+2} Y \Big\langle \bar{f} \sum_{l<l',\, l,\, l' \notin I} \dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'} \Big\rangle \\ &\quad + \beta^2 E\mathrm{th}^{n} Y \Big\langle \bar{f} \sum_{l<l'} \dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'} \Big\rangle,\end{aligned}$$

where the last summation is over the pairs l, l' *such that exactly one of the indexes* l, l' *belongs to* I *and the other to its complement. The contributions of terms* IV *and* V *are*

$$(n-1)\beta^2 E \frac{\mathrm{th}^{n-2} Y}{\mathrm{ch}^2 Y} \Big\langle f \sum_{l \in I} \dot{\mathbf{m}}^l \cdot \mathbf{b} \Big\rangle - (n+1)\beta^2 E \frac{\mathrm{th}^{n} Y}{\mathrm{ch}^2 Y} \Big\langle f \sum_{l \notin I} \dot{\mathbf{m}}^l \cdot \mathbf{b} \Big\rangle.$$

The contribution of term III *is*

$$\frac{1}{2} \beta^2 E\mathrm{th}^n Y \Big\langle f \sum_{l \le p} a_l \Big\rangle.$$

The contribution of term VI *is*

$$\beta E\mathrm{th}^{n-1} Y \Big\langle f \sum_{l \in I} c_l \Big\rangle + \beta E\mathrm{th}^{n+1} Y \Big\langle f \sum_{l \notin I} c_l \Big\rangle.$$

Throughout the paper, we write $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$, $\tilde{\mathbf{m}} = \mathbf{m}^1 - \mathbf{m}^2$, $\tilde{\sigma} = \sigma_1 - \sigma_2$ (where σ_l is the last component of $\boldsymbol{\sigma}^l$) and so forth.

COROLLARY 2.7. *If*

$$f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p) = \tilde{\sigma} \sigma_3 \bar{f}(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^p),$$

then the main terms of Theorem 2.5 *reduce to*

$$\begin{aligned}&\beta^2 E \frac{1}{\mathrm{ch}^2 Y} \langle \bar{f} \tilde{\mathbf{m}} \cdot \dot{\mathbf{m}}^3 \rangle + \beta^2 E \left(\frac{1 - 3\mathrm{th}^2 Y}{\mathrm{ch}^2 Y} \right) \langle \bar{f} \tilde{\mathbf{m}} \cdot \mathbf{b} \rangle \\ &\quad + \beta E \frac{\mathrm{th}\, Y}{\mathrm{ch}^2 Y} \langle \bar{f} (c_1 - c_2) \rangle + \beta^2 E \frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y} \Big\langle \bar{f} \sum_{l \ge 4} \tilde{\mathbf{m}} \cdot \dot{\mathbf{m}}^l \Big\rangle.\end{aligned}$$

Of course, the terms $\mathrm{ch}^{-2}Y$ arise as $1-\mathrm{th}^2 Y$.

COROLLARY 2.8. *If*

$$f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p)=\tilde{\sigma}(\sigma_3-\sigma_4)\bar{f}(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^p),$$

then the main terms of Theorem 2.5 *reduce to*

$$\beta^2 E\,\frac{1}{\mathrm{ch}^4 Y}\,\langle \bar{f}\tilde{\mathbf{m}}\cdot(\mathbf{m}^3-\mathbf{m}^4)\rangle.$$

COROLLARY 2.9. *If*

$$f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p)=\tilde{\sigma}\bar{f}(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^p),$$

then the main terms of Theorem 2.5 *reduce to*

$$\beta^2 E\,\frac{\mathrm{th}\,Y}{\mathrm{ch}^2 Y}\Big\langle \bar{f}\sum_{l\ge 3}\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}_3\Big\rangle-2\beta^2 E\,\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}\langle\bar{f}\tilde{\mathbf{m}}\cdot\mathbf{b}\rangle$$
$$+\beta E\,\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}(c_1-c_2)\rangle.$$

COROLLARY 2.10. *If*

$$f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^p)=\tfrac{1}{2}(\tilde{\sigma})^2\,\bar{f}(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^p),$$

then the main terms of Theorem 2.5 *reduce to*

$$\begin{aligned}
&E\,\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}\rangle-\beta^2 E\,\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}\dot{\mathbf{m}}^1\cdot\dot{\mathbf{m}}^2\rangle\\
&\quad+\beta^2\sum_{3\le l<l'}E\,\frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\langle\bar{f}\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}\rangle+\beta^2 E\Big(\frac{3\mathrm{th}^2 Y-1}{\mathrm{ch}^2 Y}\Big)\Big\langle\bar{f}\sum_{l\ge 2}\dot{\mathbf{m}}^l\cdot\mathbf{b}\Big\rangle\\
&\quad-2\beta^2 E\,\frac{1}{\mathrm{ch}^2 Y}\Big\langle\bar{f}\sum_{l=1,2}\dot{\mathbf{m}}^l\cdot\mathbf{b}\Big\rangle+\frac{\beta^2}{2}E\,\frac{1}{\mathrm{ch}^2 Y}\Big\langle\bar{f}\sum_{l\ge 1}a_l\Big\rangle\\
&\quad+\beta E\,\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}\Big\langle\bar{f}\sum_{l\ge 3}c_l\Big\rangle.
\end{aligned}$$

When ξ is a smooth function and g is $N(0,1)$, integration by parts yields the very useful formula

$$Eg\xi(g)=E\xi'(g). \tag{2.49}$$

We would like to find a substitute for this when instead of g we use a variable η with $P(\eta=1)=P(\eta=-1)=1/2$. The formula

$$\xi(1)-\xi(-1)=\xi'(1)+\xi'(-1)+\int_{-1}^{1}\frac{1}{2}(t^2-1)\xi^{(3)}(t)\,dt \tag{2.50}$$

shows that

$$|E\eta\xi(\eta) - E\xi'(\eta)| \le \int_{-1}^{1} |\xi^{(3)}(t)|\,dt. \tag{2.51}$$

We will estimate $E\eta_k\langle f\rangle'$, where the thermal integral is on Σ_{N+1}^p, and where f is a function of $\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p$, which will also depend on η_k. In all the cases we will consider, this function will be defined when we replace $\eta_k \in \{-1, 1\}$ by any real number, and it will be clear what is meant by $\partial f/\partial\eta_k$. We denote by E_η integration in the r.v. (η_k) only.

PROPOSITION 2.11. (*We can pretend that*) *we have*

$$E\eta_k\langle f\rangle' = E\left\langle \frac{\partial f}{\partial \eta_k}\right\rangle' + \beta\sum_{l\le p} E\langle m_k^l \sigma_l f\rangle' - p\beta E\langle m_k^{p+1}\sigma_{p+1} f\rangle' + S', \tag{2.52}$$

where

$$|S'| \le KE\left\langle \sum_{s=0}^{3}\left(\operatorname{Av} E_\eta \left|\frac{\partial^s f}{\partial \eta_k^s}\right|^2\right)^{1/2}\left(\sum_{l\le p+1} |m_k^l|^{3-s}\right)\right\rangle. \tag{2.53}$$

COMMENTS. (1) The "we can pretend" refers to the fact that we have not taken into account the fact that it is not exactly true that $\|\mathbf{m}^l(\boldsymbol{\varepsilon})\| \le L$, but that rather we have (2.27), a distinction that, as explained, is irrelevant for our purposes.

(2) As the reader correctly guessed, the last bracket in (2.52) is on a $p+1$-replica, as well as the bracket in (2.53).

PROOF. We use Proposition 2.1 to write

$$E\eta_k\langle f\rangle' = E\eta_k \frac{\langle \operatorname{Av} f\mathscr{E}\rangle}{\langle \operatorname{Av}\mathscr{E}\rangle}.$$

There is no longer dependence on η_k in the bracket $\langle\cdot\rangle$. We have

$$\frac{\partial \mathscr{E}}{\partial \eta_k} = \beta\left(\sum_{l\le p}\sigma_l m_k^l\right)\mathscr{E}.$$

We then apply (2.51) at all other r.v. fixed. We have

$$\begin{aligned}\frac{\partial}{\partial\eta_k}\frac{\langle \operatorname{Av} f\mathscr{E}\rangle}{\langle \operatorname{Av}\mathscr{E}\rangle} &= \frac{\langle \operatorname{Av}(\partial f/\partial\eta_k)\mathscr{E}\rangle}{\langle \operatorname{Av}\mathscr{E}\rangle} + \frac{\beta\langle \operatorname{Av} f(\sum_{l\le p}\sigma_l m_k^l)\mathscr{E}\rangle}{\langle \operatorname{Av}\mathscr{E}\rangle}\\ &\quad - \frac{\beta\langle \operatorname{Av} f\mathscr{E}\rangle\langle \operatorname{Av}(\sum_{l\le p}\sigma_l m_k^l)\mathscr{E}\rangle}{\langle \operatorname{Av}\mathscr{E}\rangle^2}.\end{aligned} \tag{2.54}$$

It should be obvious using Proposition 2.1 that the first two terms on the right-hand side of (2.54) contribute to the first two terms on the right-hand side of (2.52). As for the last term, we observe that, when taking expectations, each term $\langle \operatorname{Av}\sigma_l m_k^l\mathscr{E}\rangle$ contributes as $\langle \operatorname{Av}\sigma_{p+1} m_k^{p+1}\mathscr{E}\rangle$ and one then sees that this gives the last bracket of (2.52). Only crude bounds are necessary for the error term [the right-hand side of (2.51)]. As in the proof of Proposition 2.4, one uses that $\langle \mathrm{Av}\,\mathscr{E} \rangle \geq 1$. The details are left to the reader. What makes everything easy is that we allow the constant K of (2.53) to depend on α, β.

3. Exponential inequalities, I.

The aim of this section is to prove the following result.

THEOREM 3.1. *At values of the parameters α, β in the accessible region, we have, for each $0 \leq t \leq N/L_0$,*

$$E\langle \exp tN\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \rangle \leq \exp Kt^2 N, \tag{3.1}$$

$$E\langle \exp tN(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2) \rangle \leq \exp Kt^2 N, \tag{3.2}$$

$$E\langle \exp tN\tilde{m}_1 \rangle \leq \exp Kt^2 N. \tag{3.3}$$

There K depends on α, β, h, but not on N, $\tilde{\mathbf{m}} = \mathbf{m}^1 - \mathbf{m}^2$, $\mathbf{m}^* = \mathbf{m}^3 - \mathbf{m}^4$. [Please do not confuse $\mathbf{m}^*$ and m^* of (2.9).] The quantity $\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2$, is a symmetrized version of $a_1 = \|\dot{\mathbf{m}}^1\|^2 - \langle \|\dot{\mathbf{m}}^1\|^2 \rangle$, and the quantity $\tilde{m}_1 = m_1^1 - m_1^2$ is a symmetrization of $c_1 = m_1 - \langle m_1 \rangle$. The reason why the three inequalities (3.1) to (3.3) go hand in hand is that Theorem 2.5 introduces three types of error terms, so that there is no other choice than controlling them all together.

The overall idea of the proof is as follows. We consider the best possible constants for which (a variant of) (3.1) to (3.3) hold. We then estimate certain quantities $E\langle f \rangle'$ in two different ways using Theorem 2.5 and using (3.1) to (3.3) to control the error terms. Comparing the (equal) results of the two computations, we then derive relations from which it will follow that the best possible constants in (3.1) to (3.3) are bounded independently of N.

Let us consider the functions

$$A(t) = E\langle \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \exp tN\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \rangle, \tag{3.4}$$

$$B(t) = E\langle (\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2) \exp tN(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2) \rangle, \tag{3.5}$$

$$C(t) = E\langle \tilde{m}_1 \exp tN\tilde{m}_1 \rangle, \tag{3.6}$$

which are the derivatives of the left-hand sides of (3.1) to (3.3), respectively. Let us recall the quantity ρ_1 of Proposition 2.2.

LEMMA 3.2. *There exist L_0 and K_0 such that if for a number $H_1 \geq K_0$ we have*

$$\forall t \geq 0, \qquad NH_1 t^2 \geq 1, \qquad t \leq L_0 \rho_1^2 / H_1 \Rightarrow A(t) \leq 2tH_1 \exp H_1 N t^2, \tag{3.7}$$

then

$$0 \leq t \leq \frac{1}{L_0} \Rightarrow E\langle \exp Nt\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \rangle \leq 8 \exp NH_1 t^2. \tag{3.8}$$

COMMENT. In (3.7) the restriction $NH_1t^2 \geq 1$ is an inessential detail. What is important is that in (3.7) it is enough to control $A(t)$ for $t \leq L_0\rho_1^2/H_1$. Then we automatically get control of $E\langle \exp Nt\tilde{\mathbf{m}} \cdot \mathbf{m}^3\rangle$ for much larger values of t, control that will be crucial in bounding error terms.

PROOF OF LEMMA 3.2. If $\xi(t) = E\langle \exp Nt(\tilde{\mathbf{m}} \cdot \mathbf{m}^3)\rangle$, then $NA(t) = \xi'(t)$. It is obvious that $A'(t) \geq 0$, so that if $0 \leq t \leq t_0 = 1/\sqrt{NH_1}$, then $A(t) \leq A(t_0)$, so that [using (3.7) to bound $A(t_0)$]

$$\xi(t) = 1 + \int_0^t NA(u)\,du \leq 1 + Nt_0A(t_0) = 1 + 2e \leq 8.$$

It is then obvious from (3.7) that $\xi(t) \leq 8\,\exp H_1Nt^2$ if $t \leq L_0\rho_1^2/H_1$. To control the larger values of t, we write

$$\begin{aligned} E\langle \exp tN(\tilde{\mathbf{m}} \cdot \mathbf{m}^3)\rangle &\leq \exp 2N\rho_1^2 t + E\langle 1_{\{|\tilde{\mathbf{m}}\cdot\mathbf{m}^3| \geq 2\rho_1^2\}}\rangle \exp tNL_1 \\ &\qquad + E\langle 1_{\{|\tilde{\mathbf{m}}\cdot\mathbf{m}^3| \geq L_1\}} \exp tN\tilde{\mathbf{m}} \cdot \mathbf{m}^3\rangle, \end{aligned} \tag{3.9}$$

where L_1 is a constant such that

$$x \geq L_1 \Rightarrow P\Bigl(\sup_{\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2, \boldsymbol{\varepsilon}^3} |\tilde{\mathbf{m}} \cdot \mathbf{m}^3| \geq x\Bigr) \leq \exp\Bigl(-\frac{Nx}{L_1}\Bigr)$$

[see (2.27)]. Thus, for $t \leq 1/2L_1$, using (2.12) to see that $EG(\{|\tilde{\mathbf{m}} \cdot \mathbf{m}^3| \geq 2\rho_1^2\}) \leq 3\,\exp -N/K_1$, the right-hand side of (3.9) is at most

$$\exp 2N\rho_1^2 t + 3\,\exp\Bigl(-\frac{N}{K_1} + tNL_1\Bigr) + L. \tag{3.10}$$

If $tH_1 \geq 2\rho_1^2$, then $2N\rho_1^2 t \leq H_1Nt^2$. If $tL_1 \leq 1/K_1$, then

$$\exp\Bigl(-\frac{N}{K_1} + tNL_1\Bigr) \leq 1 \leq \exp H_1Nt^2,$$

while if $tL_1 \geq 1/K_1$, then

$$\exp\Bigl(-\frac{N}{K_1} + tNL_1\Bigr) \leq \exp H_1Nt^2,$$

provided $H_1 \geq L_1^2K_1$. In conclusion, if $H_1 \geq L_1^2K_1$ and $tH_1 \geq 2\rho_1^2$, $t \leq 1/2L_1$, then term (3.10) is less than or equal to $8\,\exp H_1Nt^2$. Thus, the lemma holds for $K_0 = L_1^2K_1$, $L_0 = 2L_1$. □

The reader must certainly be wondering why we use (2.12), and do not take advantage of (2.13), from which we know that (essentially) $|\tilde{\mathbf{m}} \cdot \mathbf{m}^3| \leq 2\rho\rho_1$ (much smaller than ρ_1^2 for large β). This is simply because this better result never helps.

With the same proof we have the following results.

LEMMA 3.3. *If for a number $H_2 \geq K_0$ we have*

$$\forall t > 0, \qquad NH_2t^2 \geq 1, \qquad t \leq L_0\rho_1^2/H_2 \Rightarrow B(t) \leq 2tH_2 \exp H_2Nt^2, \tag{3.11}$$

then

$$0 \leq t \leq \frac{1}{L_0} \Rightarrow E\langle \exp Nt(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2)\rangle \leq 8 \exp H_2Nt^2. \tag{3.12}$$

LEMMA 3.4. *If for a number $H_3 \geq K_0$ we have*

$$\forall t > 0, \qquad NH_3t^2 \geq 1, \qquad t \leq L_0\rho_1/H_3 \Rightarrow C(t) \leq 2tH_1 \exp H_3Nt^2, \tag{3.13}$$

then

$$0 \leq t \leq \frac{1}{L_0} \Rightarrow E\langle \exp Nt\tilde{m}_1\rangle \leq 8 \exp H_3Nt^2. \tag{3.14}$$

To obtain the condition $t \leq L_0\rho_1/H_3$ of (3.13), we use that we can pretend from (2.12) that $|m_1 - m^*| \leq \rho_1$. We could also use (2.14) and ρ_0 instead of ρ_1 (an improvement that will be required later on).

We now choose numbers $H_1,\ H_2,\ H_3$ that satisfy (3.7), (3.11), (3.13), respectively. These numbers might well depend on N and be very large. It is the purpose of the proof to show that they can be taken bounded independently of N. We will use (3.8), (3.12), (3.14) to bound our error terms.

We will now start the main line of arguments, whcih is a method of computation based on the tools of Section 2. The same method will essentially be used in Sections 4 and 5, but each time we will be able to say more because we will build on the understanding reached in the previous sections. In the present section there are six occurrences of the same (somewhat complicated) computation. The only reasonable way to proceed is to detail the first occurrence of this computation, and to let the readers convince themselves that the other cases are handled (exactly) the same way.

We will consider the quantity $A'(t')$, which is the same as $A(t)$ when one replaces N by $N' = N + 1$, β by $\beta' = \beta N'/N$, h by $h' = hN/N'$, t by $t' = tN'/N$, and one does not change M. Thus,

$$A'(t') = E\langle \tilde{\mathbf{m}}' \cdot \mathbf{m}'^3 \exp(t'N'\tilde{\mathbf{m}}' \cdot \mathbf{m}'^3)\rangle', \tag{3.15}$$

with the notation of Proposition 2.1.

We replace $\tilde{m}'_k$ by its value

$$\tilde{m}'_k = \frac{1}{N'} \sum_{i \leq N'} \eta_{i,k}\tilde{\sigma}_i$$

so that

$$\tilde{\mathbf{m}}' \cdot \mathbf{m}'^3 = \frac{1}{N'} \sum_{i \leq N'} \sum_{k \leq M} \eta_{i,k}\tilde{\sigma}_i m_k'^3.$$

(It is important to *resist* the temptation of also replacing $m_k'^3$ by its value of $N'^{-1}\sum_{i\le N'}\eta_{i,k}\sigma_i^3$.) Thus, by symmetry between the sites,

$$A'(t') = E\sum_k \eta_k\langle\tilde{\sigma} m_k'^3 f\rangle', \tag{3.16}$$

where $f = \exp t'N'\tilde{\mathbf{m}}'\cdot\mathbf{m}'^3$. (Let us recall that $\eta_{N',k} = \eta_k$ and that $\tilde{\sigma}$ is the last component of $\tilde{\boldsymbol{\sigma}}$.) The basic computation consists of three steps:

Step 1. Compute the dependence on η_k due to the fact that

$$m_k'^3 = \frac{N}{N'}m_k^3 + \frac{1}{N'}\eta_k\sigma_3, \tag{3.17}$$

where σ_3 is the last component of $\boldsymbol{\sigma}^3$ and where

$$m_k^3 = \frac{1}{N}\sum_{i\le N}\eta_{i,k}\sigma_i.$$

Step 2. Use Proposition 2.11 to integrate by parts, using Theorem 2.2 to control the error terms.

Step 3. Use Theorem 2.5 and its corollaries to estimate the various terms.

Once this is done, we will use symmetry between sites differently, writing now

$$A'(t') = E\sum_k\langle\tilde{m}_k m_k'^3 f\rangle', \tag{3.18}$$

where $\tilde{m}_k = (1/N)\sum_{i\le N}\eta_{i,k}\tilde{\sigma}_k$. We will then perform the same steps. Information will be gained by comparing the results of the two previous computations. The same program will then be accomplished for $B(t)$ and $C(t)$. Fortunately, once the method is understood, each computation reduces to a half page of simple algebra. The difficulty is, of course, to control the error terms, but it is always done in the same manner.

Step 1 is trivial. One substitutes (3.17) into (3.16) to get

$$A'(t') = \frac{M}{N'}E\langle\tilde{\sigma}\sigma_3 f\rangle' + \frac{N}{N'}E\sum_{k\le M}\eta_k\langle\tilde{\sigma}m_k^3 f\rangle'. \tag{3.19}$$

Both terms are important, but, of course, the first looks smaller because of the coefficient $\alpha' = M/N'$, which is small. In both terms we face the problem that f depends on η_k. To make explicit this dependence, we write, by simple algebra,

$$N't'\tilde{\mathbf{m}}'\cdot\mathbf{m}'^3 = Nt\tilde{\mathbf{m}}\cdot\mathbf{m}^3 + t\sum_{2\le k\le M}\eta_k(\tilde{m}_k\sigma_3 + m_k^3\tilde{\sigma}) + \frac{M-1}{N}t\tilde{\sigma}\sigma_3. \tag{3.20}$$

Thus, with the notation of Proposition 2.11, we have

$$\frac{\partial^s f}{\partial \eta_k^s} = t^s(\tilde{m}_k \sigma_3 + m_k^3 \tilde{\sigma})^s f.$$

We now apply Proposition 2.11 (with $p = 3$, $\tilde{\sigma} m_k f$ rather than f) for each k, and sum the results, to find

$$\begin{aligned} E \sum_{2 \le k \le M} \eta_k \langle \tilde{\sigma} m_k^3 f \rangle' = {} & tE\big\langle\big(\tilde{\sigma}\sigma_3 \tilde{\mathbf{m}} \cdot \mathbf{m}^3 + (\tilde{\sigma})^2 \|\mathbf{m}^3\|^2\big) f\big\rangle' \\ & + \beta E\langle(\tilde{\sigma}\sigma_1 \mathbf{m}^1 \cdot \mathbf{m}^3 + \tilde{\sigma}\sigma_2 \mathbf{m}^2 \cdot \mathbf{m}^3 + \tilde{\sigma}\sigma_3 \|\mathbf{m}^3\|^2) f\rangle' \\ & - 3\beta E\langle \tilde{\sigma}\sigma_4 \mathbf{m}^3 \cdot \mathbf{m}^4 f\rangle' + S', \end{aligned} \tag{3.21}$$

where $|S'|$ is bounded by (2.53), with an extra summation over k. (Of course, we have written sums such as $\sum_{2 \le k \le M} \tilde{m}_k m_k^3$ as $\tilde{\mathbf{m}} \cdot \mathbf{m}^3$.) To work out the bound (2.53), we use trivial bounds:

$$\left|\frac{\partial^s f}{\partial \eta_k^s}\right| \le (|\tilde{m}_k| + 2|m_k^3|)^s f$$

and

$$(E_\eta f^2)^{1/2} \le K\bar{f}, \tag{3.22}$$

where

$$\bar{f} = \exp Nt\tilde{\mathbf{m}} \cdot \mathbf{m}^3. \tag{3.23}$$

[To write (3.22), we pretend that $\|\mathbf{m}\|$ is bounded. The reader can write a complete argument in the spirit of Lemma 3.1. Writing these unimportant details makes the proof too hard to read.] Using Hölder's inequality in $\langle\cdot\rangle$, we see that

$$|S'| \le KE\bigg\langle \sum_{1 \le l \le 4} \sum_{2 \le k \le M} |m_k^l|^4 \bar{f} \bigg\rangle. \tag{3.24}$$

This is the first of many error terms, which will all be handled through the same general principle toward which we turn now.

LEMMA 3.6. *Consider two random variables U_1, U_2. Assume that for $j = 1, 2$ we have*

$$|t| \le t_0 \Rightarrow E \exp tU_j \le 8 \exp A_j t^2. \tag{3.25}$$

Assume that $2 \le t_0\sqrt{A_1}$. Then for each integer $n \le 4$, each t with

$$|t| \le t_0/2, \qquad t^2 \le \frac{t_0^2 A_1}{16 A_2} \tag{3.26}$$

for

$$x = 8|t|\sqrt{A_1 A_2},$$

we have

$$E(|U_1|^n 1_{\{|U_1|\geq x\}} \exp tU_2) \leq LA_1^{n/2} \exp(-6t^2 A_2). \tag{3.27}$$

If, moreover, $1 \leq t^2 A_2$, *we have*

$$E(|U_1|^n \exp tU_2) \leq L|t|^n (A_1 A_2)^{n/2} \exp A_2 t^2. \tag{3.28}$$

COMMENTS. (1) It is because of the restriction $1 \leq t^2 A_2$ that we assume that $NH_1 t^2 \geq 1$ in (3.7).

(2) We start with information for $|t| \leq t_0$, and end up (at best) with information for $|t| \leq t_0/2$. This potential disaster is fortunately compensated for by the opposite phenomenon when going from (3.7) to (3.8).

PROOF. We use Hölder's inequality to bound the right-hand side of (3.27) by

$$(EU_1^{4n})^{1/4} P(|U_1| \geq x)^{1/4} (E \exp 2tU_2)^{1/2}. \tag{3.29}$$

Since we assume that $|t| \leq t_0/2$, we can use (3.25) for $j = 2, 2t$ instead of t to get

$$(E \exp 2tU_2)^{1/2} \leq 8 \exp 2A_2 t^2.$$

Also,

$$EU_1^{16} \leq \frac{L}{t^{16}} E \operatorname{ch} tU_1 \leq \frac{L}{t^{16}} \exp A_1 t^2$$

whenever $|t| \leq t_0/2$. Taking $t = A_1^{-1/2}$, we have $EU_1^{16} \leq LA_1^8$ (this requires $2 \leq t_0\sqrt{A_1}$), and thus

$$E|U_1|^n \leq LA_1^{n/2} \quad \text{for } n \leq 16.$$

Also, we have for each y with $|y| \leq t_0$ that

$$P(U_1 \geq x) \leq 8 \exp(-yx + y^2 A_1)$$

so that if $x \leq 2t_0 A_1$ we can take $y = x/2A_1$ to get

$$P(|U_1| \geq x) \leq 16 \exp\left(-\frac{x^2}{2A_1}\right).$$

Combining these estimates yields a bound for the term (3.29) of

$$LA_1^{n/2} \exp\left(-\frac{x^2}{8A_1}\right) \exp 2A_2 t^2$$

and taking $x = 8|t|\sqrt{A_1 A_2}$ [which requires $|t| \leq (t_0/4)\sqrt{A_1/A_2}$], we have proved (3.27). To prove (3.28), we write, for the same value of x,

$$\begin{aligned} E|U_1^n| \exp tU_2 &\leq x^n E \exp tU_2 + E|U_1^n| 1_{\{|U_1|\geq x\}} \exp tU_2 \\ &\leq Lt^n (A_1 A_2)^{n/2} \exp A_2 t^2 + LA_1^{n/2} \exp(-6A_2 t^2) \end{aligned}$$

using (3.25), (3.27). The first term dominates for $t^2 A_2 \geq 1$. □

We will use that the operator $E\langle\cdot\rangle$ has the formal properties of an expectation to apply Lemma 3.5. We go back to the study of the bound (3.24).

LEMMA 3.7. *Under* (3.7), *and if the constant* K_0 *of Lemma* 3.2 *is large enough, we have the following:*

$$NH_1t^2 \geq 1, \qquad t \leq L_0\rho_1^2/H_1 \Rightarrow |S'| \leq Kt \exp NtH_1. \tag{3.30}$$

PROOF. We will use (3.27) for $t_0 = 1/L_0$, $U_2 = N\tilde{\mathbf{m}} \cdot \mathbf{m}^3$, $A_2 = NH_1$ [so that (3.25) holds for $j = 2$ by (3.8); because of symmetry (3.8) is valid for $-t$ when it is valid for t] and for $U_1 = Nm_k^l$, $A_1 = NK$, so that (3.25) holds for $j = 1$ by Theorem 2.2. Then $x = KN|t|\sqrt{H_1}$ and, by (3.27),

$$E\langle (m_k^l)^4 1_{\{Nm_k^l|\geq x\}} \exp t\tilde{\mathbf{m}} \cdot \mathbf{m}^3\rangle \leq \frac{K}{N^2}\exp(-6NH_1t^2). \tag{3.31}$$

We can do this because if $0 \leq t \leq L_0\rho_1^2/H_1$ then $t \leq 1/L_0$ and

$$t^2 \leq \frac{K}{H_1^2} \leq \frac{t_0^2 A_1}{16A_2} = \frac{K}{H_1},$$

provided the constant K_0 of Lemma 3.2 is large enough (the restriction $2 \leq t_0\sqrt{A_1}$ is trivially satisfied).

Thus, from (3.31), we get

$$E\left\langle \sum_{\substack{l\leq p\\ k\leq M}} (m_k^l)^4 1_{\{|Nm_k^l|\geq x\}} \exp t\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \right\rangle \leq \frac{K}{N}\exp(-6NH_1t^2). \tag{3.32}$$

On the other hand, pretending as usual that $\|\mathbf{m}^l\|$ is bounded [or otherwise using (2.27)], we have

$$\begin{aligned} &E\left\langle \sum_{\substack{l\leq p\\ k\leq M}} (m_k^l)^4 1_{\{|Nm_k^l|\leq x\}} \exp t\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \right\rangle \\ &\quad \leq K\left(\frac{x}{N}\right)^2 E\left\langle \sum_{l\leq p} \|\mathbf{m}^l\|^2 \exp t\tilde{\mathbf{m}} \cdot \mathbf{m}^3 \right\rangle \\ &\quad \leq Kt^2 H_1 \exp Nt^2 H_1. \end{aligned} \tag{3.33}$$

Combining (3.32), (3.33) yields (3.30), since, for $t^2H_1N \geq 1$, the right-hand side of (3.33) dominates the right-hand side of (3.32), and since $t \leq \rho_1^2/H_1 \leq 1/H_1$. $\square$

Before we can use Theorem 2.5 (the third step in our program), there remains the obstacle that in (3.21) f depends on the r.v. η_k. We now show, by most brutal bounds, that we can replace f by $\bar{f}$ with an error at most

$Kt \exp NtH_1$, not worse than the error we already made when integrating by parts. We simply write

$$|\bar{f} - f| \le f \exp\left(\frac{M-1}{N} t\tilde{\sigma}\sigma_3\right) |\exp W - 1| + f\left|\exp \frac{M-1}{N} t\tilde{\sigma}\sigma_3 - 1\right|, \tag{3.34}$$

where $W = t\sum_{2\le k\le M} \eta_k(\tilde{m}_k\sigma_3 + m_k^3\tilde{\sigma})$. Now

$$\begin{aligned} E_\eta(\exp W - 1)^2 &= E_\eta(\exp 2W - 2\exp W + 1) \\ &\le E_\eta(\exp 2W - 1) \le Kt^2 \end{aligned}$$

by pretending as usual that $\|\tilde{\mathbf{m}}\|$, $\|\mathbf{m}^3\|$ remain bounded, and since we consider only $t \le 1$. Thus, by (3.34), we have

$$E_\eta |\bar{f} - f|^2 \le Kt^2 f^2. \tag{3.35}$$

To apply this, we write (using Proposition 2.1)

$$\begin{aligned} |E\langle\tilde{\sigma}\sigma^3 f\rangle' - E\langle\tilde{\sigma}\sigma^3\bar{f}\rangle'| &\le E\langle|\bar{f} - f|\rangle' \\ &\le E\langle \mathrm{Av}\ |\bar{f} - f|\mathscr{E}\rangle \\ &\le E\langle \mathrm{Av}\ (E_\eta|\bar{f} - f|^2)^{1/2}(E_\eta\mathscr{E}^2)^{1/2}\rangle \\ &\le KtE\langle f\rangle \le Kt \exp NH_1 t^2. \end{aligned} \tag{3.36}$$

There, we have pretended again that $\|\mathbf{m}\|$ remains bounded, and we have used (3.8) in the last inequality.

Let us now summarize the situation:

LEMMA 3.8. *If $NH_1t^2 \ge 1$ and $t \le L_0\rho_1^2/H_1$, then*

$$\begin{aligned} A'(t') = \alpha' E\langle\tilde{\sigma}\sigma_3\bar{f}\rangle' + \beta E\langle(\tilde{\sigma}\sigma_1\mathbf{m}^1\cdot\mathbf{m}^3 + \tilde{\sigma}\sigma_2\mathbf{m}^2\cdot\mathbf{m}^3 + \tilde{\sigma}\sigma_3\|\mathbf{m}^3\|^2)\bar{f}\rangle' \\ -3\beta E\langle\tilde{\sigma}\sigma_4\mathbf{m}^3\cdot\mathbf{m}^4\bar{f}\rangle' + S'', \end{aligned} \tag{3.37}$$

where $\bar{f} = \exp N\tilde{\mathbf{m}}\cdot\mathbf{m}^3$ and where $|S''| \le Kt \exp NH_1t$.

PROOF. We apply the method of (3.36) to the terms on the right-hand side of (3.21); we absorb the first of these terms into the error term by obvious bounds. □

We are ready for the main computation, namely, the application of Theorem 2.5 (and its corollaries) to the five first terms on the right-hand side of (3.37) and control of the error terms. A first observation is that, if $\tilde{\sigma} = \sigma_1 - \sigma_2$ is not 0, since $\sigma_1, \sigma_2 \in \{-1, 1\}$, then $\sigma_1 = -\sigma_2 = \tilde{\sigma}/2$, so that

$$\tilde{\sigma}\sigma_1\mathbf{m}^1\cdot\mathbf{m}^3 + \tilde{\sigma}\sigma_2\mathbf{m}^2\cdot\mathbf{m}^3 = \tfrac{1}{2}(\tilde{\sigma})^2\tilde{\mathbf{m}}\cdot\mathbf{m}^3.$$

Thus, we have to apply Theorem 2.5 to the following four terms:

$$\alpha' E\langle \tilde{\sigma}\sigma_3 \bar{f}\rangle'; \quad \frac{\beta}{2} E\langle (\tilde{\sigma})^2 \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \bar{f}\rangle';$$
$$\beta E\langle \tilde{\sigma}\sigma_3 \|\mathbf{m}^3\|^2 \bar{f}\rangle'; \quad -3\beta E\langle \tilde{\sigma}\sigma_4 \mathbf{m}^3 \cdot \mathbf{m}^4 \bar{f}\rangle'. \tag{3.38}$$

LEMMA 3.9. *If $NH_1 t^2 \geq 1$ and $t \leq L_0 \rho_1^2 / H_1$, then under* (3.7), (3.11), (3.13) *the error terms when Theorem* 2.5 *is used to compute the terms* (3.38) *have a total contribution at most*

$$Lt(\alpha + \rho_1^2)\rho_1^2 (H_1 + H_2 + H_3).$$

PROOF. We have to apply Theorem 2.5 when the function denoted f there is now, respectively, $\tilde{\sigma}\sigma_3 \bar{f}$, $(\tilde{\sigma})^2 \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \bar{f}$, $\tilde{\sigma}\sigma_3 \|\mathbf{m}^3\|^2 \bar{f}$, $\tilde{\sigma}\sigma_4 \mathbf{m}^3 \cdot \mathbf{m}^4 \bar{f}$. These all satisfy (2.46), since $\tilde{\sigma} \neq 0 \Rightarrow \sigma_1 + \sigma_2 = 0$. As explained after the statement of this theorem, and owing to (2.13), the coefficient of the bracket in (2.45) remains uniformly bounded in the accessible region, even after multiplication by an extraneous factor β. The first term of (3.38) has a factor α; in all the other ones, there is a term such as $\tilde{\mathbf{m}} \cdot \mathbf{m}^3$ in front of $\bar{f}$, a term that we can pretend is bounded by $L\rho_1^2$. Thus, the error terms are bounded by

$$L(\alpha + \rho_1^2) E\left\langle \bar{f} \left(\sum_{l < l' \leq 5} (\dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'})^2 + \sum_{l \leq 4} a_l^2 + \sum_{l \leq 4} c_l^2 \right)^2 \right\rangle \tag{3.39}$$

and we will control these through the case $n = 2$ of (3.28). Using (3.12), for $0 \leq t \leq 1/L_0$, we have

$$E\langle \exp t(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2)\rangle \leq 8 \exp NH_2 t^2.$$

If we integrate in $\mathbf{m}^2$, $\mathbf{m}^3$, $\mathbf{m}^4$ for the Gibbs measure in the exponent rather than outside, Jensen's inequality shows that we decrease the bracket. This yields

$$E\big\langle \exp t(\|\dot{\mathbf{m}}^1\|^2 - \langle \|\dot{\mathbf{m}}^1\|^2 \rangle)\big\rangle \leq 8 \exp H_2 t^2. \tag{3.40}$$

Thus, we can use (3.28) for $A_1 = NH_2$, $A_2 = NH_1$ to obtain

$$E\langle \bar{f} a_l^2 \rangle \leq Lt^2 H_1 H_2 \exp NH_1 t^2$$

provided

$$t \leq \frac{1}{L_0}, \qquad t^2 \leq \frac{t_0^2 A_1}{16 A_2} = \frac{H_2}{16 L_0^2 H_1},$$

which is brutally satisfied in the range we consider. We then use that $t^2 \leq L_0 t \rho_1^2 / H_1$. The other terms are handled similarly. □

We now turn to the main terms obtained by applying Theorem 2.5 to each of the terms (3.38). What will keep the situation manageable is that all but one of these terms [given by (3.44) below] will be treated as error terms (because they are much smaller).

The main terms arising from $E\langle\tilde{\sigma}\sigma_3\bar{f}\rangle'$ are provided by Corollary 2.7, and are

$$\beta^2 E\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}\tilde{\mathbf{m}}\cdot\mathbf{m}^3\rangle + \beta^2 E\frac{1-3\,\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\langle\bar{f}\tilde{\mathbf{m}}\cdot\mathbf{b}\rangle + \beta E\frac{\mathrm{th}\ Y}{\mathrm{ch}^2 Y}\langle\bar{f}(c_1-c_2)\rangle$$

We bound this by

$$\beta^2 E\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}|\tilde{\mathbf{m}}\cdot\mathbf{m}^3|\rangle + 3\beta^2 E\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}|\tilde{\mathbf{m}}\cdot\mathbf{b}|\rangle + \beta E\frac{1}{\mathrm{ch}^2 Y}\langle\bar{f}(|c_1|+|c_2|)\rangle. \tag{3.41}$$

Before we can proceed, we have to learn how to handle factors such as $\mathrm{ch}^{-2}Y$. We recall that

$$Y = \beta(\mathbf{g}\cdot\mathbf{b} + \langle m_1\rangle + h),$$

where $\mathbf{g}$ is independent of all the other random variables. Thus, when integrating, we can integrate first in $\mathbf{g}$ to replace $\mathrm{ch}^{-2}Y$ by $E_g(\mathrm{ch}^{-2}Y)$. Moreover, (2.12) shows that with probability greater than or equal to $1-\exp(-N/K)$ (in the quenched variables $\eta_{i,k}$) we have

$$\|\mathbf{b}\|^2 + (m^* - \langle m_1\rangle)^2 \le 2\rho_1^2.$$

To simplify the notation, we will set

$$\frac{1}{\mathrm{ch}} = \sup\left\{E\frac{1}{\mathrm{ch}^2\beta(\mathbf{g}\cdot\mathbf{v}+x+h)};\ \|\mathbf{v}\|^2 + (x-m^*)^2 \le 2\rho_1^2\right\}. \tag{3.42}$$

We have already explained why events with exponentially small probability [$\le \exp(-N/K)$] can be ignored for our purposes, so that we can pretend, if B is a bracket that is greater than or equal to 0, that

$$E\frac{1}{\mathrm{ch}^2 Y}B \le \frac{1}{\mathrm{ch}^2}E\ B.$$

We now observe that β^2/ch^2 remains bounded in the accessible region; in fact,

$$\frac{\beta}{\mathrm{ch}^2} \le \frac{L}{\mathrm{ch}^2\beta/4} + L\ \exp\left(-\frac{1}{L\alpha}\right)$$

[which is the reason of the improvement under (2.46) of the error term in Theorem 2.5]. This is simply because $m^* \ge 3/4$ (for β large) and thus $x \ge 1/4$ in (3.42), so that $\mathrm{ch}^{-2}\beta(\mathbf{g}\cdot\mathbf{v}+x+h) \le \mathrm{ch}^{-2}\beta/4$ unless $|\mathbf{g}\cdot\mathbf{v}| \ge 1/4$; but $\|\mathbf{v}\|^2 \le L\alpha$.

We bound in this manner the terms (3.41) by

$$LE\langle\bar{f}|\tilde{\mathbf{m}}\cdot\mathbf{m}^3|\rangle + LE\langle\bar{f}|\tilde{\mathbf{m}}\cdot\mathbf{b}|\rangle + LE\langle\bar{f}(|c_1|+|c_2|)\rangle.$$

We now apply (3.28), for $n = 1$, to get a bound

$$\alpha' Lt\Big(H_1 + \sqrt{H_1 H_3}\Big) \exp NH_1 t^2 \tag{3.43}$$

for the main terms of Theorem 2.5 corresponding to $\alpha' E\langle \tilde{\sigma}\sigma_3 \bar{f}\rangle$.

For $(\beta/2)E\langle(\tilde{\sigma})^2 \tilde{\mathbf{m}}\cdot\mathbf{m}^3 \bar{f}\rangle$, the main terms are provided by Corollary 2.10, with $p = 3$, and where $\bar{f}$ has to be replaced by $\tilde{\mathbf{m}}\cdot\mathbf{m}^3\bar{f}$. The first, and most crucial, of these terms is

$$\beta E \frac{1}{\mathrm{ch}^2 Y}\langle \tilde{\mathbf{m}}\cdot\mathbf{m}^3 \bar{f}\rangle. \tag{3.44}$$

We observe the important fact that $\langle \tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle \geq 0$ so that (we can pretend that) we can write

$$\beta E \frac{1}{\mathrm{ch}^2 Y}\langle \tilde{\mathbf{m}}\cdot\mathbf{m}^3 \bar{f}\rangle \leq \frac{\beta}{\mathrm{ch}^2} A(t).$$

In the other terms provided by Corollary 2.10, we use that $|\tilde{\mathbf{m}}\cdot\mathbf{m}^3| \leq L\rho_1^2$, $|\mathrm{th}\, Y| \leq 1$ and $\beta^2/\mathrm{ch}^2 \leq L$. We then appeal to (3.28), with $n = 1$, to bound these terms by

$$L\rho_1^2 t\Big(H_1 + \sqrt{H_1 H_2} + \sqrt{H_1 H_3}\Big) \exp NH_1 t^2.$$

Thus, the contribution of the main terms of Theorem 2.5 corresponding to $(\beta/2)E\langle(\tilde{\sigma})^2 \tilde{\mathbf{m}}\cdot\mathbf{m}^3 \bar{f}\rangle'$ is at most

$$\frac{\beta}{\mathrm{ch}^2} A(t) + L\rho_1^2 t\Big(H_1 + \sqrt{H_1 H_2} + \sqrt{H_1 H_3}\Big) \exp NH_1 t^2. \tag{3.45}$$

The main terms of $\beta E\langle \tilde{\sigma}\sigma_3 \|\mathbf{m}^3\|^2 \bar{f}\rangle$ are provided by Corollary 2.7, with $p = 3$, replacing $\bar{f}$ by $\|\mathbf{m}^3\|^2 \bar{f}$. In each of these terms we use that $\|\mathbf{m}^3\|^2 \leq \rho_1^2$, $|\mathrm{th}\, Y| \leq 1$, $\beta^2/\mathrm{ch} \leq L$, and then (3.28), with $n = 1$, to obtain a bound

$$L\rho_1^2 t\Big(H_1 + \sqrt{H_1 H_2} + \sqrt{H_1 H_3}\Big).$$

If we use that $\sqrt{H_1 H_2} \leq H_1 + H_2$, $\sqrt{H_1 H_3} \leq H_1 + H_3$ and $\alpha \leq \rho_1^2 \leq L$, we see that we have proved the following result.

PROPOSITION 3.11. *Under* (3.7), (3.11), (3.13) *and if t is as in* (3.7), *we have*

$$\frac{N'}{N} A'(t') \leq \frac{\beta}{\mathrm{ch}^2} A(t) + (t \exp NH_1 t^2)[L\rho_1^2 (H_1 + H_2 + H_3) + K]. \tag{3.46}$$

Surely the reader will take comfort in the fact that we have done one sixth of the job. We begin the second sixth, which is to find a lower bound for $A'(t')$, starting from (3.19),

$$\begin{aligned} A'(t') &= E \sum_{2\leq k\leq M} \langle \tilde{m}_k m'^3 f\rangle' \\ &= \frac{N}{N'} E\langle \tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle' + \frac{1}{N'} E \sum_{2\leq k\leq M} \eta_k \langle \tilde{m}_k \sigma^3 f\rangle'. \end{aligned} \tag{3.47}$$

To study the first term, we replace f by $\bar{f}$ and apply Proposition 2.4. We then apply Theorem 2.5. It should be obvious that the error terms at each step are of the same type as previously. The main terms are computed through the case $I = \varnothing$ of Proposition 2.6, and are

$$E\langle \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \bar{f} \rangle + \beta^2 E \ \mathrm{th}^n Y \Big\langle \bar{f} \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \sum_{l<l'} \dot{\mathbf{m}}^l \cdot \mathbf{m}^{l'} \Big\rangle$$
$$+ \tfrac{1}{2}\beta^2 E \ \mathrm{th}^2 Y \Big\langle \bar{f} \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \sum_{l \le 3} a_l \Big\rangle + \beta E \ \mathrm{th} Y \Big\langle \bar{f} \tilde{\mathbf{m}} \cdot \mathbf{m}^3 \sum_{l \le 3} c_l \Big\rangle.$$

The first term is $A(t)$. We bound the others as previously by

$$L\rho_1^2 t\Big(H_1 + \sqrt{H_1 H_2} + \sqrt{H_1 H_3}\Big) \exp NH_1 t^2.$$

We will let the reader deal with the last term of (3.47), where the factor $1/N'$ created a huge safety margin, to see that (after integration by parts) it does not create new error terms. We now combine with (3.46) to get our first basic estimate.

PROPOSITION 3.12. *Under* (3.7)*,* (3.11)*,* (3.13) *and if t is as in* (3.7)*, we have*

$$\left(1 - \frac{\beta}{\mathrm{ch}^2}\right) A(t) \le (t \exp NH_1 t^2)[L\rho_1^2 (H_1 + H_2 + H_3) + K]. \tag{3.48}$$

So that the reader can see where we are heading, let us assume for a moment that we already know that H_2, H_3 are bounded depending only on α, β, h. Then (3.48) implies

$$\left(1 - \frac{\beta}{\mathrm{ch}^2}\right) A(t) \le (t \ \exp NH_1 t^2)[L\rho_1^2 H_1 + K]. \tag{3.49}$$

If we take for H_1 the smallest number for which (3.7) holds, then an obvious compactness argument shows that there must be some t satisfying (3.7) for which equality holds, that is,

$$A(t) = 2H_1 t \ \exp NH_1 t^2. \tag{3.50}$$

Now (3.49) was established under the hypothesis $H_1 \ge K_0$, but, of course, there is nothing to show if $H_1 \le K_0$. Substituting (3.50) into (3.49) yields

$$\left(1 - \frac{\beta}{\mathrm{ch}^2}\right) H_1 \le L\rho_1^2 H_1 + K$$

and yields $H_1 \le K$ provided $\beta/\mathrm{ch}^2 + L\rho_1^2 < 1$ in the admissible region, a critical (but easy) fact that is proved in detail in [7].

Our program is simply to prove a version of Proposition 3.11 for $B(t)$ and $C(t)$ and proceed as above, but simultaneously for H_1, H_2, H_3. We turn to the case of $B(t)$ which fortunately is very similar to the case of $A(t)$. We consider the quantity

$$B'(t') = E\langle(\|\tilde{\mathbf{m}}'\|^2 - \|\mathbf{m}'^*\|^2)f\rangle',$$

where now

$$f = \exp N't'(\|\tilde{\mathbf{m}}'\|^2 - \|\mathbf{m}'^*\|^2). \tag{3.51}$$

We separate $B'(t')$ in two pieces

$$B'(t') = E\langle\|\tilde{\mathbf{m}}'\|^2 f\rangle' - E\langle\|\mathbf{m}'^*\|^2 f\rangle' \tag{3.52}$$

and we show how to deal with the first term on the right-hand side. We write

$$\tilde{m}'_k = \frac{1}{N'}\sum_{i\le N'} \eta_{i,k}\tilde{\sigma}_i$$

so that by symmetry between sites we get the two basic relations

$$E\langle\|\tilde{\mathbf{m}}'\|^2 f\rangle' = E\sum_{2\le k\le M} \eta_k\langle\tilde{\sigma}\tilde{m}'_k f\rangle', \tag{3.53}$$

$$E\langle\|\tilde{\mathbf{m}}'\|^2 f\rangle' = E\sum_{2\le k\le M} \langle\tilde{m}_k\tilde{m}'_k f\rangle'. \tag{3.54}$$

Using that

$$\tilde{m}'_k = \frac{N}{N'}\tilde{m}_k + \frac{1}{N'}\eta_k\tilde{\sigma}, \tag{3.55}$$

we get from (3.53) that

$$E\langle\|\tilde{\mathbf{m}}'\|^2 f\rangle' = \frac{M-1}{N'}E\langle(\tilde{\sigma})^2 f\rangle' + \frac{N}{N'}E\sum_{2\le k\le M}\eta_k\langle\tilde{\sigma}\tilde{m}_k f\rangle'. \tag{3.56}$$

To make the dependence on η_k explicit in f, we write

$$\begin{aligned} N't'(\|\tilde{\mathbf{m}}'\|^2 - \|\mathbf{m}'^*\|^2) &= Nt(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2) + 2t\sum_{2\le k\le M}\eta_k(\tilde{\sigma}\tilde{m}_k + \sigma^* m_k^*) \\ &\quad + t\frac{M-1}{N}(\tilde{\sigma}^2 + \sigma^{*2}) \end{aligned} \tag{3.57}$$

and we set

$$\bar{f} = \exp Nt(\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2).$$

The errors made while integrating by parts, replacing f by $\bar{f}$ and treating the term $E\langle\tilde{\sigma}\tilde{m}_k(\partial f/\partial\eta_k)\rangle'$ of (2.52) as an error term are controlled exactly as in the case of $A(t)$. The other terms resulting from the integration by parts of $E\sum_{2\le k\le M}\eta_k\langle\tilde{\sigma}\tilde{m}_k f\rangle'$ are (after replacing f by $\bar{f}$)

$$\begin{aligned} &\beta\langle(\tilde{\sigma}\sigma_1\tilde{\mathbf{m}}\cdot\mathbf{m}^1 + \tilde{\sigma}\sigma_2\tilde{\mathbf{m}}\cdot\mathbf{m}^2 + \tilde{\sigma}\sigma_3\tilde{\mathbf{m}}\cdot\mathbf{m}^3 + \tilde{\sigma}\sigma_4\tilde{\mathbf{m}}\cdot\mathbf{m}^4)\bar{f}\rangle' \\ &\quad -4\beta\langle\tilde{\sigma}\sigma_5\tilde{\mathbf{m}}\cdot\mathbf{m}^5\bar{f}\rangle'. \end{aligned}$$

As in the case of $A(t)$, we have

$$\tilde{\sigma}\sigma_1\tilde{\mathbf{m}}\cdot\mathbf{m}^1+\tilde{\sigma}\sigma_2\tilde{\mathbf{m}}\cdot\mathbf{m}^2=\tfrac{1}{2}(\tilde{\sigma})^2\|\tilde{\mathbf{m}}\|^2$$

so we are left with four terms

$$\tfrac{1}{2}\beta\langle(\tilde{\sigma})^2\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle';\ \beta\langle\tilde{\sigma}\sigma_3\tilde{\mathbf{m}}\cdot\mathbf{m}^3\bar{f}\rangle';\ \beta\langle\tilde{\sigma}\sigma_4\tilde{\mathbf{m}}\cdot\mathbf{m}^4\bar{f}\rangle';-4\beta\langle\tilde{\sigma}\sigma_5\tilde{\mathbf{m}}\cdot\mathbf{m}^5\bar{f}\rangle'. \tag{3.58}$$

Applying Theorem 2.5 to the first term of (3.58), the main terms are given by Corollary 2.10 as

$$\beta E\frac{1}{\mathrm{ch}^2Y}\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle-\beta^3E\frac{1}{\mathrm{ch}^2Y}\langle\bar{f}\|\tilde{\mathbf{m}}\|^2\dot{\mathbf{m}}^1\cdot\dot{\mathbf{m}}^2\rangle.$$

We can bound the last term by

$$Lt\sqrt{H_1H_2}\,\exp NH_2t^2.$$

The main terms arising from the last three terms of (3.58) can be bound by

$$(Lt\,\exp NH_2t^2)\Big[\sqrt{H_1H_2}+H_2+\sqrt{H_2H_3}\Big].$$

When we regroup the term $\beta E(1/\mathrm{ch}^2Y)\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle$ with the corresponding term $-\beta E(1/\mathrm{ch}^2Y)\langle\|\mathbf{m}^*\|^2\bar{f}\rangle$ arising from $-E\langle\|\mathbf{m}^*\|^2\bar{f}\rangle$, we get

$$\beta E\frac{1}{\mathrm{ch}^2Y}\langle(\|\tilde{\mathbf{m}}\|^2-\|\mathbf{m}^*\|^2)\bar{f}\rangle$$

and, as the bracket is positive, this is at most $(\beta/\mathrm{ch}^2)B(t)$. Applying now Corollary 2.10 to the term $E\langle(\tilde{\sigma})^2\bar{f}\rangle'$ of (3.56), the main terms give a contribution

$$2E\frac{1}{\mathrm{ch}^2Y}\langle\bar{f}\rangle-\beta^2E\frac{1}{\mathrm{ch}^2Y}\langle\bar{f}\dot{\mathbf{m}}^1\cdot\dot{\mathbf{m}}^2\rangle+\beta^2E\frac{\mathrm{th}^2Y}{\mathrm{ch}^2Y}\langle\bar{f}\dot{\mathbf{m}}^3\cdot\dot{\mathbf{m}}^4\rangle.$$

The first term would be devastating, but fortunately it cancels with the corresponding contribution from $-E\langle(\sigma^*)^2\bar{f}\rangle'$. The other two terms can be bounded by $Lt\sqrt{H_1H_2}\exp NH_2t^2$.

If we were now to start from (3.54) to get a lower bound for $B(t)$, we would find that, with the same error terms, $N'/NB'(t')\geq B(t)$. In conclusion, we have obtained for $B(t)$ exactly the same result as for $A(t)$.

PROPOSITION 3.13. *Under* (3.7), (3.11), (3.13) *and if t is as in* (3.11), *then*

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)B(t)\leq(t\,\exp NH_2t^2)[L\rho_1^2(H_1+H_2+H_3)+K]. \tag{3.59}$$

It remains to handle the case of $C(t)$, which, while technically simpler, is somewhat different, and does not give as good control. We consider the quantity

$$C'(t)=E\langle\tilde{m}_1'\exp N't\tilde{m}_1'\rangle'$$

(we have t rather than t' in the exponent). Since $N'\tilde{m}'_1 = \sum_{i\le N'} \tilde{\sigma}_i$, we have by symmetry between sites the two basic relations

$$C'(t) = E\langle \tilde{\sigma} \exp N' t \tilde{m}'_1 \rangle', \tag{3.60}$$

$$C'(t) = E\langle \tilde{m}_1 \exp N' t \tilde{m}'_1 \rangle'. \tag{3.61}$$

Now replacing $N'\tilde{m}'_1 = N\tilde{m}_1 + \tilde{\sigma}$ by $N\tilde{m}_1$ in the exponent creates an error at most $Kt \exp NH_3 t^2$, so that we have to evaluate

$$E\langle \tilde{\sigma} \bar{f} \rangle' \tag{3.62}$$

for $\bar{f} = \exp Nt\tilde{m}_1$. The error terms when applying Proposition 2.4 are bounded by $Kt \exp NH_3 t^2$. The error terms of Theorem 2.5 are bounded by

$$E\left\langle \bar{f}\left(\sum_{l<l'\le 3} (\dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}^{l'})^2 + \sum_{l\le 2} a_l^2 + \sum_{l\le 2} c_l^2 \right) \right\rangle,$$

which, through (3.28), are bounded by

$$\begin{aligned} &Lt^2(H_1H_3 + H_2H_3 + H_3^2) \exp NH_3 t^2 \\ &\quad \le Lt\rho_1(H_1 + H_2 + H_3) \exp NH_3 t^2 \end{aligned}$$

since now we consider t as in (3.13). The main terms are provided by Corollary 2.9 as

$$-2\beta^2 E \frac{\operatorname{th} Y}{\operatorname{ch}^2 Y} \langle \bar{f} \tilde{\mathbf{m}} \cdot \mathbf{b} \rangle + \beta E \frac{1}{\operatorname{ch}^2 Y} \langle \bar{f}(c_1 - c_2) \rangle. \tag{3.63}$$

We observe that $c_1 - c_2 = \tilde{m}$, so that, since $\langle \tilde{m} \bar{f} \rangle \ge 0$, the last term is at most $\beta C(t)/\operatorname{ch}^2$, and (3.63) is at most

$$Lt\sqrt{H_1 H_3} + \frac{\beta}{\operatorname{ch}^2} C(t). \tag{3.64}$$

When getting a lower bound for $C'(t)$ using (3.61), there is no new contribution to the error terms, but use of Proposition 2.6 in the case $I = \varnothing$ gives the following for the main terms of $E\langle \tilde{m}_1 \bar{f} \rangle'$:

$$\begin{aligned} &E\langle \tilde{m}_1 \bar{f} \rangle + \beta^2 E \operatorname{th}^2 Y \langle \tilde{m}_1 \bar{f} \dot{\mathbf{m}}^1 \cdot \dot{\mathbf{m}}^2 \rangle \\ &\quad - \beta^2 E \frac{1}{\operatorname{ch}^2 Y} \langle \tilde{m}_1 \bar{f} (\dot{\mathbf{m}}^1 \cdot \mathbf{b} + \dot{\mathbf{m}}^2 \cdot \mathbf{b}) \rangle \\ &\quad + \frac{\beta^2}{2} E\langle \tilde{m}_1 \bar{f}(a_1 + a_2) \rangle + \beta E \operatorname{th} Y \langle \tilde{m}_1 \bar{f}(c_1 + c_2) \rangle. \end{aligned}$$

The first term is $C(t)$. As we can pretend that $|\dot{\mathbf{m}}^1 \cdot \dot{\mathbf{m}}^2| \le L\rho_1^2$, $|\dot{\mathbf{m}}^l \cdot \mathbf{b}| \le L\rho_1^2$, $|a_1 + a_2| \le L\rho_1^2$, $|c_1 + c_2| \le L\rho_1$, the sum of the second and third terms is at least $-Lt\rho_1 H_3$, and introduces no new error terms. Thus, we have the following result.

PROPOSITION 3.13. *Under* (3.7), (3.11), (3.13) *and if t is as in* (3.13), *then*

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)C(t) \le (t\exp NH_3t^2)[L\rho_1(H_1+H_2+H_3)+L\sqrt{H_1H_3}]. \tag{3.65}$$

COMMENT. A significant difference with (3.46), (3.59) is that now we do not have a small coefficient in front of the term $\sqrt{H_1H_3}$.

PROOF OF THEOREM 3.1. We recall that Propositions 3.11 to 3.13 assume that H_1, H_2, $H_3 \ge K_0$. The (easier) case where one of these relations fails is left to the reader. (This case is easier because we already have part of the result.) Proceeding as after Proposition 3.11 by taking H_1, H_2, H_3 optimal (i.e., as small as possible) and considering a number t with equality in (3.7) [resp. (3.11), (3.13)], we get the relations

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)H_1 \le L\rho_1^2(H_1+H_2+H_3)+K,$$

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)H_2 \le L\rho_1^2(H_1+H_2+H_3)+K,$$

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)H_3 \le L\rho_1(H_1+H_2+H_3)+L\sqrt{H_1H_3}+K.$$

We add the first two relations to get, setting $H = H_1+H_2$,

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)H \le L\rho_1^2(H+H_3)+K, \tag{3.66}$$

$$\left(1-\frac{\beta}{\mathrm{ch}^2}\right)H_3 \le L\rho_1(H+H_3)+L\sqrt{HH_3}+K. \tag{3.67}$$

Let us now consider the case of large β. Then β/ch^2 is small, and $1-\beta/\mathrm{ch}^2-L\rho_1^2 \ge 1/2$, so that (3.66) yields

$$H \le L\rho_1^2 H_3 + K$$

and plugging into (3.67) yields

$$\frac{1}{2}H_3 \le \left(1-\frac{\beta}{\mathrm{ch}^2}\right)H_3 \le L\rho_1 H_3 + K$$

so that $H_3 \le K$ if ρ_1 (i.e., α) is small enough. The same argument yields that the inequalities in Theorem 3.1 hold at each β if α is small enough [$\alpha \le \alpha_0(\beta)$], but, unfortunately, (3.67) cannot give the correct dependence as $\beta \to 1$ because, in particular, it is not true in the accessible region that $1-\beta/\mathrm{ch}^2 + L\rho_1 < 1$ since $1-\beta/\mathrm{ch}^2 \sim \beta - 1$ as $\beta \to 1$ and $\rho_1 = L\sqrt{\alpha/(\beta-1)}$. One reason for this failure is that the a priori estimate

$$EG(|m_1 - m^*| \ge \rho_1) \le \exp(-N/K) \tag{3.68}$$

used in (3.13) is not sufficient and has to be replaced by (2.14) instead. But getting the correct dependence as $\beta \to 1$ requires significant other work, and rather than doing this now, we will observe that, quite conveniently, Bovier and Gayrard have proved through the Brascamp–Lieb inequalities that $H_3 \le K$ in the BG region (which coincides with the accessible region when one restricts β to be ≤ 2) so that then Theorem 3.1 follows from (3.66). □

COMMENTS. (1) In the BG region, Bovier and Gayrard also prove that $H_1 < \infty$; however, I do not see how their approach could yield $H_2 < \infty$.

(2) In the next section, the difficulty linked to the several insufficient estimates used in this section will resurface, and this time we will have no other choice than to improve upon them. So, in fact, we could have proved Theorem 3.1 in all the accessible region, without lengthening the paper, not using the results of Bovier and Gayrard; but this is not the best way to make friends.

4. Exponential inequalities, II. In this section we will prove the following improvement of Theorem 3.1.

THEOREM 4.1. *For each value of the parameters in the accessible region, we have, for* $|t| \le N/L_0$,

$$E\big\langle \exp tN(\mathbf{m}^1 \cdot \mathbf{m}^2 - E\langle \mathbf{m}^1 \cdot \mathbf{m}^2 \rangle)\big\rangle \le \exp KNt^2, \tag{4.1}$$

$$E\big\langle \exp tN(\|\mathbf{m}\|^2 - E\langle \|\mathbf{m}\|^2 \rangle)\big\rangle \le \exp KNt^2, \tag{4.2}$$

$$E\big\langle \exp tN(m_1 - E\langle m_1 \rangle)\big\rangle \le \exp KNt^2. \tag{4.3}$$

[*There is only one replica in* (4.3).]

The reason that we can take up this greater challenge is that Theorem 3.1 gives us excellent control of the error terms of Theorem 2.5, which now makes it a rather irresistible tool.

We set

$$a = E\langle \mathbf{m}^1 \cdot \mathbf{m}^2 \rangle = E\|\langle \mathbf{m} \rangle\|^2, \qquad b = E\langle \|\mathbf{m}^1\|^2 \rangle, \qquad c = E\langle m_1 \rangle$$

(hoping that this will not create confusion with the notation a_l, c_l) and we define

$$A(t) = E\langle (\mathbf{m}^1 \cdot \mathbf{m}^2 - a) \exp Nt(\mathbf{m}^1 \cdot \mathbf{m}^2 - a) \rangle,$$

$$B(t) = E\langle (\|\mathbf{m}\|^2 - b) \exp Nt(\|\mathbf{m}\|^2 - b) \rangle,$$

$$C(t) = E\langle (m_1 - c) \exp Nt(m_1 - c) \rangle.$$

[Thus, $A(t)$, $B(t)$ etc. are not the same as in Section 3.] We consider numbers H_1, H_2, H_3 with the following properties:

$$\begin{aligned} Nt^2H_1 \ge 1, \qquad 0 \le t \le L_1\rho_1^2/H_1 \Rightarrow A(t) &\le 2tH_1 \exp NH_1t^2, \\ -A(-t) &\le 2tH_1 \exp NH_1t^2 \end{aligned} \tag{4.4}$$

$$\begin{aligned} Nt^2H_2 \ge 1, \qquad 0 \le t \le L_1\rho_1^2/H_2 \Rightarrow B(t) &\le 2tH_2 \exp NH_2t^2, \\ -B(-t) &\le 2tH_2 \exp NH_2t^2 \end{aligned} \tag{4.5}$$

$$\begin{aligned} Nt^2H_3 \ge 1, \qquad 0 \le t \le L_1\rho_0/H_3 \Rightarrow C(t) &\le 2tH_3 \exp NH_3t^2, \\ -C(-t) &\le 2tH_3 \exp NH_3t^2. \end{aligned} \tag{4.6}$$

The reason we need to consider $-A(-t)$ is that we can no longer invoke symmetry to say that $-A(-t) = A(t)$. Thus, unfortunately, all the estimates we have to do for $A(t)$ are also required for $-A(-t)$. Fortunately, this is done exactly in the same way and will simply be ignored. As in Section 3, we have the following, provided L_1 has been chosen large enough,

LEMMA 4.2. *If H_1, H_2, H_3 are greater than or equal to K_0, then, under* (4.4) *to* (4.6)*, we have*

$$\begin{aligned} 0 \le |t| \le 1/L_1 \Rightarrow E\langle \exp t(\mathbf{m}^1 \cdot \mathbf{m}^2 - a)\rangle &\le 8 \exp NH_1t^2, \\ E\langle \exp t(\|\mathbf{m}\|^2 - b)\rangle &\le 8 \exp NH_2t^2, \\ E\langle \exp t(m_1 - c)\rangle &\le 8 \exp NH_3t^2. \end{aligned}$$

It is to prove that $E\langle \exp t(m_1 - c)\rangle \le 8 \exp NH_3t^3$ for $|t| \le 1/L_1$ when we know (4.6) only for $t \le L_1\rho_0/H_2$ that (2.14) is needed.

Lemma 4.2, together with (3.28), will be essential in producing bounds. The principle of the proof of Theorem 4.1 is very similar to the principle of the proof of Theorem 3.1. The a priori estimates (4.4) to (4.6) are used to produce bounds, that will imply that the best possible choices of H_1, H_2, H_3 will be bounded independently of N. The main difference with Theorem 3.1 will be in the different behavior of the main terms arising from Theorem 2.5 through Proposition 2.6.

We consider t such that $Nt^2H_1 \ge 1$, $0 \le t \le L_1\rho_1^2/H_1$ and

$$A'(t') = E\langle(\mathbf{m}'^1 \cdot \mathbf{m}'^2 - a')\exp(N't'\mathbf{m}'^1 \cdot \mathbf{m}'^2 - Nta)\rangle',$$

where $a' = E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2\rangle'$, $t' = N't/N$, so that $N't'a' = Nta$. We write

$$\mathbf{m}'^1 = \frac{1}{N'}\sum_{i \le N'} \eta_{i,k}\sigma_i^1$$

so that, by symmetry, setting $f = \exp(N't'\mathbf{m}'^1 \cdot \mathbf{m}'^2 - Nta)$

$$E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2 f\rangle' = E\sum_{2 \le k \le M} \eta_k \langle \sigma_1 m'^2_k f\rangle' \tag{4.7}$$

and also

$$E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2 f\rangle' = E \sum_{2\le k\le M} \langle m_k^1 m'^2_k f\rangle'. \tag{4.8}$$

From (4.7) we write

$$E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2 f\rangle' = \frac{M-1}{N'} E\langle \sigma_1\sigma_2 f\rangle' + \frac{N}{N'} E \sum_{2\le k\le M} \eta_k \langle \sigma_2 m_k^2 f\rangle'. \tag{4.9}$$

To make explicit the dependence of f on η_k, we write

$$N't'\mathbf{m}'^1 \cdot \mathbf{m}'^2 = Nt\mathbf{m}^1 \cdot \mathbf{m}^2 + t \sum_{2\le k\le M} \eta_k(\sigma_1 m_k^2 + \sigma_2 m_k^1) + t\frac{M-1}{N}\sigma_1\sigma_2. \tag{4.10}$$

Setting $\bar f = \exp Nt(\mathbf{m}^1 \cdot \mathbf{m}^2 - a)$, we see through integration by parts of the last summation in (4.9) as in Section 3 that we make an error at most

$$Kt \exp NH_1 t^2 \tag{4.11}$$

if we estimate $E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2 f\rangle'$ by

$$\begin{aligned}\frac{M-1}{N'} E\langle \sigma_2\sigma_2 \bar f\rangle' + \beta \frac{N}{N'} \big(\, & E\langle \mathbf{m}^1 \cdot \mathbf{m}^2 \bar f\rangle' + E\langle \sigma_1\sigma_2 \|\mathbf{m}^2\|^2 \bar f\rangle' \\ & - 2E\langle \sigma_1\sigma_3 \mathbf{m}^2 \cdot \mathbf{m}^3 \bar f\rangle'\big).\end{aligned}$$

We will use Proposition 2.4 [which creates an error of the type (4.11)] and Theorem 2.5. The error terms created by Theorem 2.5 are of the type

$$LE\langle \bar f((\dot{\mathbf{m}}^l \cdot \dot{\mathbf{m}}'^l)^2 + c_l^2 + a_l^2\rangle$$

(bounding $\mathbf{m}^1 \cdot \mathbf{m}^2$ by L and losing a factor ρ_1^2), and use of Theorem 3.1 and of (3.28) (case $n = 2$) provides a bound $KH_1 t^2 \exp NH_1 t^2$; but, as we are interested only in $t \le L_1\rho_1^2/H_1$, this is of the type (4.10).

What makes the use of Theorem 2.5 possible without extreme complication is that (3.28) and Theorem 3.1 imply that all the terms that do not arise from the term I of Theorem 2.5 are bounded by

$$Kt\sqrt{H_1} \exp NH_1 t^2 \tag{4.12}$$

and are not dangerous as we will see because $\sqrt{H_1} << H_1$. Thus, we see that, modulo terms (4.12), $E(\mathbf{m}'^1 \cdot \mathbf{m}'^2 f\rangle'$ is equal to

$$\begin{aligned}\frac{M-1}{N'} E \text{ th}^2 Y\langle \bar f\rangle + \beta \frac{N}{N'} \ \big(& E\langle \mathbf{m}^1 \cdot \mathbf{m}^2 \bar f\rangle + E \text{ th}^2 Y\langle \|\mathbf{m}^2\|^2 \bar f\rangle \\ & - 2E \text{ th}^2 Y\langle \mathbf{m}^2 \cdot \mathbf{m}^3 \bar f\rangle\big).\end{aligned} \tag{4.13}$$

We now turn to the evaluation of $a'E\langle f\rangle'$. In order to obtain terms that look like those of (4.13), we write

$$\begin{aligned} a' = E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2\rangle' &= E\sum_k \eta_k \langle \sigma_1 m'^2{}_k\rangle' \\ &= \frac{M-1}{N'} E\langle \sigma_1\sigma_2\rangle + \frac{N}{N'} E\sum_k \eta_k \langle \sigma_1 m_k^2\rangle. \end{aligned}$$

We integrate by parts, apply Proposition 2.4 and Theorem 2.5 and obtain

$$\begin{aligned}\Big|a' - \Big(\frac{M-1}{N'} E\,\mathrm{th}^2 Y + \beta\frac{N}{N'}\Big(E\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle \\ + E\,\mathrm{th}^2 Y\langle \|\mathbf{m}^2\|^2\rangle - 2E\,\mathrm{th}^2 Y\langle \mathbf{m}^2\cdot\mathbf{m}^3\rangle\Big)\Big)\Big| \le \frac{K}{\sqrt{N}}.\end{aligned} \tag{4.14}$$

This is because from Theorem 3.1 we know that $E\langle(\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'})^n\rangle \le K(n)\,N^{-n/2}$ (etc.) so that the error terms can be controlled via the Cauchy–Schwarz inequality.

We also obtain

$$|E\langle f\rangle' - E\langle\bar f\rangle'| \le Kt\sqrt{H_1}\exp NH_1t^2. \tag{4.15}$$

We observe (using 2.14) that

$$\frac{K}{\sqrt{N}}E\langle\bar f\rangle \le \frac{K}{\sqrt{N}}\exp NH_1t^2 \le Kt\sqrt{H_1}\exp NH_1t^2$$

as we deal only with $Nt^2H_1 \ge 1$. Combining this observation with (4.14) and (4.15), we see that, modulo an error (4.12), we can estimate $a'E\langle f\rangle'$ by

$$\begin{aligned}E\langle\bar f\rangle\Big[\frac{M-1}{N'}E\mathrm{th}^2Y + \beta\frac{N}{N'}\big(E\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle + E\mathrm{th}^2Y\langle\|\mathbf{m}^2\|^2\rangle \\ - 2E\mathrm{th}^2Y\langle\mathbf{m}^2\cdot\mathbf{m}^3\rangle\big)\Big].\end{aligned}$$

It is apparent by symmetry that one could replace $\langle\mathbf{m}^2\cdot\mathbf{m}^3\rangle$ in the last term by $\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle$ and also $\|\mathbf{m}^2\|^2$ by $\frac{1}{2}(\|\mathbf{m}^1\|^2 + \|\mathbf{m}^2\|^2)$, and (4.16) becomes

$$E\langle\bar f\rangle\Big[\frac{M-1}{N'}E\mathrm{th}^2Y + \beta\frac{N}{N'}\Big(E\frac{1}{\mathrm{ch}^2Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle + E\mathrm{th}^2Y\langle\|\tilde{\mathbf{m}}\|^2\rangle\Big)\Big].$$

Of course, we would want to perform the same simplifications in (4.13), and this leads to the basic observation that will keep computations reasonable. Modulo error terms (4.12), we can change the replica indexes in $\langle\mathbf{m}^l\cdot\mathbf{m}^{l'}\bar f\rangle$, $l\ne l'$. Specifically, here, we can replace

$$E\mathrm{th}^2Y\langle\mathbf{m}^2\cdot\mathbf{m}^3\bar f\rangle \quad \text{by } E\mathrm{th}^2Y\langle\mathbf{m}^1\cdot\mathbf{m}^2\bar f\rangle$$

because the difference is at most

$$E\langle|(\mathbf{m}^1-\mathbf{m}^3)\cdot\mathbf{m}^2|\bar f\rangle,$$

which is bounded by (4.12) due to Theorem 3.1 and to (3.28).

Thus, we have obtained that, modulo an error of type (4.12), $A'(t')$ is

$$\begin{aligned}&\frac{M-1}{N'}(E\mathrm{th}^2Y\langle\bar f\rangle - E\mathrm{th}^2YE\langle\bar f\rangle) \\ &\quad +\beta\frac{N}{N'}\Big(E\frac{1}{\mathrm{ch}^2Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\bar f\rangle - E\frac{1}{\mathrm{ch}^2Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle E\langle\bar f\rangle\Big)\end{aligned}$$

$$+E\operatorname{th}^2 Y\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle - E\operatorname{th}^2 Y\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle\bar{f}\rangle).$$

The reader can check the much easier fact that within the same error (4.12) $A'(t') = NA(t')/N'$ so that we have the following result.

PROPOSITION 4.3. *If t is as in* (4.4), *then, under* (4.4) *to* (4.6),

$$\begin{aligned}A(t) \le{}& \frac{M-1}{N}(E\operatorname{th}^2 Y\langle\bar{f}\rangle - E\operatorname{th}^2 YE\langle\bar{f}\rangle)\\ &+\beta\left(E\frac{1}{\operatorname{ch}^2 Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\bar{f}\rangle - E\frac{1}{\operatorname{ch}^2 Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle E\langle\bar{f}\rangle\right)\\ &+\beta(E\operatorname{th}^2 Y\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle - E\operatorname{th}^2 Y\langle\|\tilde{\mathbf{m}}\|^2\rangle E(\bar{f}\rangle)\\ &+Kt\sqrt{H_1}\exp Nt^2 H_1\end{aligned}\tag{4.16}$$

for $\bar{f} = \exp tN(\mathbf{m}^1\cdot\mathbf{m}^2 - a)$.

Before we attempt to extract information from (4.16), let us perform the same computation for $B(t)$. It is nearly identical, so we will give only the "algebraic" part and leave the control of the error terms to the reader. We start with

$$B'(t') = E\sum_k \eta_k\langle\tilde{\sigma}\tilde{m}'_k f\rangle'$$

[where, of course, here $f = \exp(Nt'\|\tilde{\mathbf{m}}'\|^2 - Ntb)$]

$$= \frac{M-1}{N'}E\langle(\tilde{\sigma})^2 f\rangle' + \frac{N}{N'}E\sum_k \eta_k\langle\tilde{\sigma}\tilde{m}_k f\rangle'.$$

Integration by parts of the last term yields that this is

$$[\text{for } \bar{f} = \exp Nt(\|\tilde{\mathbf{m}}\|^2 - b)]$$

$$\beta(E\langle\tilde{\sigma}\sigma_1\mathbf{m}^1\cdot\tilde{\mathbf{m}}\bar{f}\rangle' + E\langle\tilde{\sigma}\sigma_1\mathbf{m}^2\cdot\tilde{\mathbf{m}}\bar{f}\rangle' - 2E\langle\tilde{\sigma}\sigma^3\tilde{\mathbf{m}}\cdot\mathbf{m}^3\bar{f}\rangle').$$

The first two terms regroup as $(\beta/2)E\langle(\tilde{\sigma})^2\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle'$; the last term will give *no* contribution because of the factor $\tilde{\mathbf{m}}\cdot\mathbf{m}^3$. Thus, we get the following result.

PROPOSITION 4.4. *If t is as in* (4.5), *then under* (4.4) *to* (4.8) *we have*

$$\begin{aligned}B(t) \le{}& 2\frac{M-1}{N}\left(E\frac{1}{\operatorname{ch}^2 Y}\langle\bar{f}\rangle - E\frac{1}{\operatorname{ch}^2 Y}E\langle\bar{f}\rangle\right)\\ &+\beta\frac{N}{N'}\left(E\frac{1}{\operatorname{ch}^2 Y}\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle - E\left(\frac{1}{\operatorname{ch}^2 Y}\langle\|\tilde{\mathbf{m}}\|^2\rangle\right)E\langle\bar{f}\rangle\right)\\ &+Kt\sqrt{H_2}\exp Nt^2 H_2.\end{aligned}\tag{4.17}$$

The case of $C(t)$ is much simpler, and left to the reader.

PROPOSITION 4.5. *If t is as in* (4.6), *then under* (4.4) *to* (4.6) *we have*

$$\begin{aligned} C(t) \le\ & E\big(\operatorname{th} Y \langle \exp Nt(m_1 - c)\rangle\big) - E\operatorname{th} YE\langle \exp Nt(m_1 - c)\rangle \\ & + Kt\sqrt{H_3}\exp Nt^2 H_3. \end{aligned} \tag{4.18}$$

Quite obviously, this is the relation to discuss first. We recall that $Y = \beta(\mathbf{g}\cdot\mathbf{b} + \langle m_1\rangle + h)$ so that

$$E_g \operatorname{th} Y = \psi_1(\langle \mathbf{m}^1 \cdot \mathbf{m}^2\rangle, \langle m_1\rangle)$$

where

$$\psi_1(x, y) = E_g \operatorname{th} \beta\big(g\sqrt{x} + y + h\big),$$

where g is $N(0, 1)$. Now, since we can replace $\operatorname{th} Y$ by $E_g \operatorname{th} Y$ in (4.18), it is natural to use a first-order expansion

$$\begin{aligned} &|\psi_1(x, y) - \psi_1(a, c) - \partial_1\psi_1(a, c)(x - a) - \partial_2\psi_1(a, c)(y - c)| \\ &\quad \le R_1(x - a)^2 + R_3(y - c)^2 \end{aligned} \tag{4.19}$$

(where $\partial_1\psi_1 = \partial\psi_1/\partial x$, etc). The reader should observe that we allow the different coefficients R_1, R_3. The reason for this is that getting the correct shape of the accessible region at $\beta \to 1$ is an extremely unforgiving job, which essentially requires us to use the correct order in every estimate. We need to use R_1, R_3 such that (4.19) holds in the domain

$$\Delta = \{(x, y);\ \sqrt{x} \le 2\rho_1,\ |y - m^*| \le \rho_0\}$$

because we know that $(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle, \langle m_1\rangle)$ belongs to this domain with probability greater than or equal to $1 - \exp(-N/K)$. It is easy to compute the partial derivatives of ψ_1. In fact, for any smooth function φ if $\xi(x) = E\varphi(g\sqrt{x})$, then

$$\xi'(x) = \frac{1}{2\sqrt{x}} E g\varphi'(g\sqrt{x}) = \frac{1}{2} E_g \varphi''(g\sqrt{x}) \tag{4.20}$$

by integration by parts. Using Taylor's formula for the function

$$t \to \psi_1\big(a + t(x - a), c + t(y - c)\big),$$

we see that in (4.19) we can take

$$R_1 = \sup_{(x,\,y)\in D}\left|\frac{\partial^2\psi_1}{(\partial x)^2}\right|, \qquad R_3 = \sup_{(x,\,y)\in D}\left|\frac{\partial^2\psi_1}{(\partial y)^2}\right|. \tag{4.21}$$

We will see in due time what we need to know about these. (The reason for the index 3 in R_3 is that "R_1 goes with H_1 and R_3 with H_3.")

Using Jensen's inequality, we have

$$E\exp Nt(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a) \le E\langle \exp t(\mathbf{m}^1\cdot\mathbf{m}^2 - a)\rangle$$

so that by Lemma 4.2 and (3.28) we have, for $\bar{f} = \exp Nt(m_1 - c)$,

$$\begin{aligned} E(|\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a|\langle \bar{f}\rangle) &= E\langle |\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a|\bar{f}\rangle \\ &\le Lt\sqrt{H_1 H_3}\exp NH_3 t^2. \end{aligned}$$

Similarly, and using that $t \le L_1\rho_0/H_3$,

$$\begin{aligned} &E\big(R_1(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a)^2 + R_3(\langle m_1\rangle - c)^2)\langle \bar f\rangle\big)\\ &\quad\le Lt^2(R_1H_1H_3 + R_3H_3^2)\exp NH_3t^2\\ &\quad\le Lt\rho_0(R_1H_1 + R_3H_3)\exp NH_3t^2. \end{aligned}\tag{4.22}$$

Since

$$E\big(R_1(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a)^2 + R_3(\langle m_1\rangle - c)^2\big) \le L\left(\frac{R_1H_1}{N} + \frac{R_3H_3}{N}\right)$$

[easily, by (4.4) to (4.6)], and since we consider only $NH_3t^2 \ge 1$, so that $N^{-1} \le t^2H_3$, the bound of (4.22) is also good for

$$E(R_1(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle - a)^2 + R_3(\langle m_1\rangle - c)^2)E\langle \bar f\rangle.$$

Using (4.19) for $x = \langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle$, $y = \langle m_1\rangle$ to estimate $E_g \operatorname{th} Y$, substituting in (4.18) for $\operatorname{th} Y$ and using the previous estimates, we have proved the following result.

LEMMA 4.6. *Under the conditions of Proposition* 4.5, *we have*

$$\begin{aligned} &C(t)(1 - \partial_2\psi_1(a, c))\\ &\quad\le (Lt\exp NH_3t^2)\Big[|\partial_1\psi_1(a, c)|\sqrt{H_1H_3}\\ &\qquad\qquad + \rho_0(R_1H_1 + R_3H_3) + K\sqrt{H_3}\Big]. \end{aligned}\tag{4.23}$$

It remains to extract information by the same method from (4.16) and (4.17). We start with (4.17). We use the function

$$\psi_2(x, y) = E_g\frac{1}{\operatorname{ch}^2\beta(g\sqrt{x} + y + h)}$$

so that

$$E_g\frac{1}{\operatorname{ch}^2 Y} = \psi_2(\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle, \langle m_1\rangle)$$

and we make a first-order expansion as in (4.19). The reader will note that $\beta\psi_2 = \partial_1\psi_1$. We write

$$\begin{aligned} &|\psi_2(x, y)\psi_2(a, c) - (x - a)\partial_1\psi_2(a, c) - (y - c)\partial_2\psi_2(a, c)|\\ &\quad\le R_2\big((x - a)^2 + (y - c)^2\big), \end{aligned}$$

where R_2 is smallest possible that this holds for (x, y) in D, and we bound the error terms as previously, to obtain

$$\begin{aligned} &E\frac{1}{\operatorname{ch}^2 Y}\langle \bar f\rangle - E\frac{1}{\operatorname{ch}^2 Y}E\langle \bar f\rangle\\ &\quad\le \partial_1\psi_2(a, c)E\langle(\mathbf{m}^1\cdot\mathbf{m}^2 - a)\bar f\rangle + \partial_2\psi_2(a, c)E\langle(m_1 - c)\bar f\rangle \end{aligned}$$

$$
\begin{aligned}
&+LR_2t^2(H_2H_1+H_2H_3)\exp NH_3t^2\\
&\le (Lt\exp NH_3t^2)\Big[|\partial_1\psi_2(a,c)|\sqrt{H_1H_2}+|\partial_2\psi_2(a,c)|\sqrt{H_3H_1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad +R_2\rho_1^2(H_1+H_3)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&E\frac{1}{\mathrm{ch}^2Y}\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle-E\bigg(\frac{1}{\mathrm{ch}^2Y}\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle\bar{f}\rangle\bigg)\\
&\quad\le \psi_2(a,c)(E\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle-E\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle\bar{f}\rangle)\\
&\qquad+\partial_1\psi_2(a,c)(E\langle\mathbf{m}^1\cdot\mathbf{m}^2-a\rangle\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle)\\
&\qquad+\partial_2\psi_2(a,c)(E\langle m_1-c\rangle\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle)\\
&\qquad+R_2E\big(\langle(\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle-a)^2+(\langle m_1\rangle-c)^2\big)\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle)\\
&\qquad+R_2E\big(\langle(\mathbf{m}^1\cdot\mathbf{m}^2-a)^2+(m_1-c)^2\rangle\big)E(\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle).
\end{aligned}
$$

As in Section 3, we can pretend that $\|\tilde{\mathbf{m}}\|^2\le L\rho_1^2$, so that the above is bounded by

$$
\begin{aligned}
&\psi_2(a,c)B(t)+(Lt\rho_1^2\exp NH_2t^2)\\
&\quad\times\Big[|\partial_1\psi_2(a,c)|\sqrt{H_1H_2}+|\partial_2\psi_2(a,c)|\sqrt{H_3H_2}+R_2\rho_1^2(H_1+H_3)\Big].
\end{aligned}
$$

Remembering that $\alpha\le L\rho_1^2$, we have proved from (4.17) the following counterpart of Lemma 4.6 for $B(t)$.

LEMMA 4.7. *If t is as in* (4.5), *then under* (4.4) *to* (4.6) *we have*

$$
\begin{aligned}
&B(t)\big(1-\beta\psi_2(a,c)\big)\\
&\quad\le (Lt\rho_1^2\exp NH_2t^2)\Big[\beta|\partial_1\psi_2(a,c)|\sqrt{H_1H_2}\\
&\qquad\qquad+\beta|\partial_2\psi_2(a,c)|\sqrt{H_3H_2}\\
&\qquad\qquad+\beta R_2\rho_1^2(H_1+H_3+K\sqrt{H_2})\Big].
\end{aligned}
\tag{4.24}
$$

To handle the case of (4.16), we use that $\mathrm{th}^2Y=1-\mathrm{ch}^2Y$, so we fortunately do not need a new function, and we prove the following, exactly as in the case of Lemma 4.7.

LEMMA 4.8. *If t is as in* (4.4), *then, under* (4.4) *to* (4.6),

$$
\begin{aligned}
&A(t)\big(1-\beta\psi_2(a,c)\big)\\
&\quad\le (Lt\rho_1^2\exp NH_1t^2)\Big[\beta|\partial_1\psi_2(a,c)|H_1\\
&\qquad\qquad+\beta|\partial_2\psi_2(a,c)|\sqrt{H_3H_1}+\beta R_2\rho_1^2(H_1+H_2)\\
&\qquad\qquad\qquad+\beta|1-\psi_2(a,c)|\sqrt{H_1H_2}+K\sqrt{H_1}\Big].
\end{aligned}
\tag{4.25}
$$

The (very dangerous) term $\beta|1-\psi_2(a,c)|\sqrt{H_1H_2}$ is created by

$$\big(1-\psi_2(a,c)\big)\big(E\langle\|\tilde{\mathbf{m}}\|^2\bar{f}\rangle - E\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle\bar{f}\rangle\big).$$

PROOF OF THEOREM 4.1. We choose H_1, H_2, H_3 as small as possible. Using values of t that witness this optimality, we get from Lemmas 4.6 to 4.8 that

$$\begin{aligned}&H_3(1-\beta\psi_2(a,c))\\ &\quad\le L[|\partial_1\psi_1(a,c)|\sqrt{H_1H_3}+\rho_0(R_1H_1+R_3H_3)]+K\sqrt{H_3}].\end{aligned}\tag{4.26}$$

[On the left-hand side we have replaced $\partial_2\psi_1(a,c)$ by its value $\beta\psi_2(a,c)$.]

$$\begin{aligned}H_2\big(1-\beta\psi_2(a,c)\big)\le L\rho_1^2\Big[&\beta|\partial_1\psi_2(a,c)|\sqrt{H_1H_2}\\ &+\beta|\partial_2\psi_2(a,c)|\sqrt{H_3H_2}\\ &+\beta R_2\rho_1^2(H_1+H_3)+K\sqrt{H_2})\Big],\end{aligned}\tag{4.27}$$

$$\begin{aligned}H_1\big(1-\beta\psi_2(a,c)\big)\le L\rho_1^2\Big[&\beta|\partial_1\psi_2(a,c)|H_1+\beta|\partial_2\psi_2(a,c)|\sqrt{H_1H_3}\\ &+\beta R_2\rho_1^2(H_1+H_3)\\ &+\beta|1-\psi_2(a,c)|\sqrt{H_1H_2}+K\sqrt{H_1}\Big].\end{aligned}\tag{4.28}$$

To illustrate the ideas, let us start by proving that, for each $\beta>1$, there is a number $\alpha(\beta)>0$ such that, for $\alpha<\alpha(\beta)$, the previous relations imply $H_1, H_2, H_3 < K$. We will denote by $L(\beta)$ a number depending on β only. Bounding in (4.27) and (4.28) all the products such as $\sqrt{H_1H_2}$ by H_1+H_2 (etc.), and, since $|\partial_1\psi_2(a,c)|$ (etc.) remain bounded in function of β only, we get, since $\rho_1^2\le\alpha L(\beta)$,

$$H_2\big(1-\beta\psi_2(a,c)\big)\le\alpha L(\beta)\big(H_1+H_2+H_3+K\sqrt{H_2}\big),\tag{4.29}$$

$$H_1\big(1-\beta\psi_2(a,c)\big)\le\alpha L(\beta)\big(H_1+H_2+H_3+K\sqrt{H_1}\big).\tag{4.30}$$

To handle the dangerous term $\sqrt{H_1H_3}$ in (4.26), we will use the inequality

$$2xy\le\gamma x^2+\frac{1}{\gamma}y^2\tag{4.31}$$

so that, setting

$$\theta=1-\beta\psi_2(a,c),\tag{4.32}$$

we get, from (4.26),

$$H_3\theta\le\frac{\theta}{2}H_3+\frac{L(\beta)}{\theta}H_1+\alpha L(\beta)(H_1+H_3)+K\sqrt{H_3}$$

so that, for α small enough, as θ remains (at fixed β) bounded away from 0 we get

$$H_3\le L(\beta)H_1+K\sqrt{H_3}$$

so that

$$H_3 \le L(\beta)H_1 + K$$

and substitution in (4.29), (4.30) yields the result.

The rest of the proof is devoted to the more special task of getting the correct behavior of $\alpha(\beta)$ as $\beta \to \infty$, or $\beta \to 1$. The easiest case is $\beta \to \infty$. In that case (inside the accessible region) $\beta\psi_2(a, c)$ stays bounded away from 1, while βR_1, βR_2, βR_3, $\beta|\partial_1\psi_2(a, c)|$, $\beta|\partial_2\psi_2(a, c)|$ (etc.) remain bounded, so that (4.26) yields

$$H_3 \le LH_1 + K. \tag{4.33}$$

Substitution in (4.28) yields

$$H_1 \le L\alpha\big[H_1 + \beta\sqrt{H_1H_2} + K\sqrt{H_1}\big]$$

so that, for α small enough,

$$H_1 \le \beta^2 H_2 + K \tag{4.34}$$

so that, from (4.33),

$$H_3 \le \beta^2 H_2 + K. \tag{4.35}$$

Substituting in (4.27) yields

$$H_2 \le L\alpha\Big[\beta^2|\partial_1\psi_2(a, c)| + \beta^2|\partial_2\psi_2(a, c)| + \beta^2(R_1 + R_2)H_2 + K\sqrt{H_2}\Big].$$

The coefficient of H_2 in the brackets has the good taste to remain bounded over the accessible region, so that, if $\alpha \le \alpha_0$ (α_0 universal), we have $H_2 \le K$. (The reader should note that the condition $\alpha \le 1/L \log\beta$ is required only through the uniform control of $\beta^2 R_1$, etc.)

Finally, we turn to the case $1 \le \beta \le 2$. In that case it is easy to see (and is done in detail in [7]) that in the accessible region $1 - \beta\psi_2(a, c) \ge (\beta - 1)/L$. Also,

$$\partial_1\psi_1(x, y) = \frac{\beta^2}{2} E_g \varphi\Big(\beta(g\sqrt{x} + y + h)\Big),$$

where $\varphi(x) = (\operatorname{th} x)'' = \operatorname{th} x/\operatorname{ch}^2 x$. It is simple to show that in Δ, for $\beta < 2$, this behaves like $\operatorname{th}\beta m^* = m^*$ of order $\sqrt{\beta - 1}$ (for h small). Since $\beta\partial_2\psi_2 = 2\partial_1\psi_1$, we can also bound $|\partial_2\psi_2(a, c)|$ by $L\sqrt{\beta - 1}$. Also,

$$\frac{\partial^2\psi_1}{\partial y^2} = \frac{1}{2}\partial_1\psi_1$$

so that, on Δ, $|(\partial^2\psi_1/\partial y^2)(x, y)|$ is at most of order $\sqrt{\beta - 1}$; thus, we can take $R_3 \le L\sqrt{\beta - 1}$, we bound R_1, R_2, $\partial_1\psi_2(a, c)$ by L, and (4.26) becomes

$$H_3 \le \frac{L}{\sqrt{\beta - 1}}\sqrt{H_1H_3} + \frac{L\rho_0}{\beta - 1}\big(H_1 + \sqrt{\beta - 1}H_3\big) + K\sqrt{H_3}. \tag{4.36}$$

One then realizes the very remarkable fact that the value of ρ_0 exactly ensures that, if $\alpha/(\beta-1)^2$ is small enough, then $L\rho_0/\sqrt{\beta-1} \le 1/2$, so that (4.36) implies

$$H_3 \le \frac{LH_1}{\beta-1} + K. \tag{4.37}$$

We substitute this in (4.28), observing that

$$\begin{aligned} |\partial_2\psi_2(a,c)|\sqrt{H_1H_3} &\le L\sqrt{\beta-1}\sqrt{H_1H_3} \\ &\le LH_1 + K \end{aligned}$$

and, since $\rho_1^2 \le L\alpha/(\beta-1)$, (4.28) implies

$$H_1 \le \frac{L\alpha}{(\beta-1)^2}\Big[H_1 + H_2 + \sqrt{H_1H_2} + K\sqrt{H_1}\Big]$$

so that $H_1 \le LH_2$ in the accessible region. We substitute this in (4.27), and use (4.37) again; the dangerous factor $(\beta-1)^{-1}$ gets neutralized as in the case of (4.28), and the result follows. □

To end this section, we prove (1.9). We will consider

$$D(t) = E\langle(\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2 - d)\exp t(\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2 - d)\rangle, \tag{4.38}$$

where $d = E\langle\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2\rangle$. We consider the smallest constant H_4 such that

$$\begin{aligned} \forall t>0,\quad Nt^2H_4 \ge 1,\quad t \le 2/H_4 \Rightarrow D(t) &\le 2NtH_4\exp NH_4t^2; \\ -D(-t) &\le 2NtH_4\exp NH_4t^2. \end{aligned} \tag{4.39}$$

[As before, we will leave it to the reader to worry about $D(-t)$. The factor N on the right-hand side comes from a different normalization than previously.] To use Lemma 3.5, we observe that from (4.39), using $|\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2| \le N$, we have

$$\forall t,\qquad E\langle\exp t(\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2 - d)\rangle \le 8\exp Nt^2H_4. \tag{4.40}$$

Consider

$$D'(t) = E\langle(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - d')\exp t(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - d)\rangle',$$

where $d' = E\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle'$. By symmetry,

$$E\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\exp t(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - d)\rangle' = NE\langle\sigma_1\sigma_2\exp t(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - d)\rangle'. \tag{4.41}$$

For $t \le 2/H_4$ we make an error at most $KNt\exp Nt^2H_4$ if on the right-hand side of (4.41) we replace $t\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2$ by $t\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2$, since the difference is $t\sigma_1\sigma_2$. Using Theorem 2.5, we get

$$E\langle\sigma_1\sigma_2\bar{f}\rangle = E\mathrm{th}^2Y\langle\bar{f}\rangle,$$

where $\bar{f} = \exp t(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 - d)$, and with error at most $KtN\sqrt{H_4}\exp NH_3t^2$. Continuing in this very predictable manner, we get

$$D(t) \le E\mathrm{th}^2Y\langle\bar{f}\rangle - (E\mathrm{th}^2Y)E\langle\bar{f}\rangle + KtN\Big(\sqrt{H_4}+1\Big)\exp NH_4t^2.$$

Now we need only a trivial estimate

$$|\mathrm{th}^2 Y - E\mathrm{th}^2 Y| \le K(|\langle \mathbf{m}^1 \cdot \mathbf{m}^2\rangle - a| + |\langle m_1\rangle - b|)$$

to get, using Theorem 4.1, that

$$D(t) \le KtN\Big(\sqrt{H_4}+1\Big)\exp NH_4t^2$$

and to conclude as usual.

5. Convergence.

Consider the "replica-symmetric" equations

$$\mu = E\mathrm{th}\,\beta\big(g\sqrt{r}+\mu+h\big), \tag{5.1}$$

$$q = E\mathrm{th}^2\beta\big(g\sqrt{r}+\mu+h\big), \tag{5.2}$$

$$r = \frac{\alpha q}{(1-\beta(1-q))^{-2}}. \tag{5.3}$$

It is proved in [7] that when the parameters belong to the BG region these equations have a solution. The relations

$$\lim_{N\to\infty} E\langle m_1\rangle = \mu, \tag{5.4}$$

$$\lim_{N\to\infty} \frac{1}{N} E\langle \boldsymbol{\varepsilon}^1 \cdot \boldsymbol{\varepsilon}^2\rangle = q, \tag{5.5}$$

$$\lim_{N\to\infty} E\langle \mathbf{m}^1 \cdot \mathbf{m}^2\rangle = r, \tag{5.6}$$

$$\lim_{N\to\infty} E\langle \|\mathbf{m}\|^2\rangle = \alpha\frac{1-\beta(1-q)^2}{(1-\beta(1-q))^2} \tag{5.7}$$

are proved in [7] for the perturbated Hamiltonian (and later by Bovier and Gayrard for the standard Hamiltonian we use here). The key to these results is Theorem 3.1 and only trivial modifications are needed to prove them in all the accessible region.

In fact, it is a mere exercise to show that in (5.4) to (5.7) the speed of convergence is in $1/\sqrt{N}$, that is, $|E\langle m_1\rangle - \mu| \le KN^{-1/2}$ (etc.). This allows us to obtain the following inequalities (valid for $t \le 1/L_0$):

$$E\langle \exp tN(\mathbf{m}^1\cdot\mathbf{m}^2 - r)\rangle \le 2\exp KNt^2,$$

$$E\bigg\langle \exp tN\bigg(\|\mathbf{m}\|^2 - \alpha\frac{1-\beta(1-q)^2}{(1-\beta(1-q))^2}\bigg)\bigg\rangle \le 2\exp KNt^2,$$

$$E\langle \exp tN(m_1-\mu)\rangle \le 2\exp KNt^2,$$

$$E\langle \exp t(\boldsymbol{\varepsilon}^1\cdot\boldsymbol{\varepsilon}^2 - Nq)\rangle \le 2\exp KNt^2.$$

It follows from (5.4) to (5.6) that

$$\lim_{N\to\infty} E\mathrm{th}Y = \mu, \tag{5.8}$$

$$\lim_{N\to\infty} E\mathrm{th}^2 Y = q \tag{5.9}$$

and more generally that

$$\lim_{N\to\infty} E\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y} \tag{5.10}$$

exists for all $n, n' \geq 0$.

In this section we will consider p-replicas and functions on p-replicas. A function of one of the following types:

$$\mathbf{m}^l \cdot \mathbf{m}^{l'} - E\langle \mathbf{m}^l \cdot \mathbf{m}^{l'} \rangle, \qquad l \neq l', \tag{5.11}$$

$$\|\mathbf{m}^l\|^2 - E\langle \|\mathbf{m}^l\|^2 \rangle, \tag{5.12}$$

$$m_1^l - E\langle m_1^l \rangle, \tag{5.13}$$

$$m_k^l, \qquad k \geq 2, \tag{5.14}$$

$$\frac{1}{N}(\boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} - E\langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} \rangle) \tag{5.15}$$

will be called an *expression of order* 1.

The reason for which there is no centering term in (5.14) is that $|E\langle m_k \rangle| \leq K/N$ (integrating $E\eta_k\langle \epsilon_N \rangle$ by parts). For an integer τ, an *expression of order* τ is the product of τ expressions of order 1 (and a constant if $\tau = 0$). It will be convenient for the proof to consider certain linear combinations of the expressions (5.11) to (5.15), among which we will use

$$(\mathbf{m}^{l_1} - \mathbf{m}^{l_2}) \cdot (\mathbf{m}^{l_3} - \mathbf{m}^{l_4}), \tag{5.16}$$

$$\|\mathbf{m}^{l_1} - \mathbf{m}^{l_2}\|^2 - \|\mathbf{m}^{l_3} - \mathbf{m}^{l_4}\|^2, \tag{5.17}$$

$$\|\mathbf{m}^{l_1} - \mathbf{m}^{l_2}\|^2 - E\|\mathbf{m}^{l_1} - \mathbf{m}^{l_2}\|^2, \tag{5.18}$$

$$m_1^{l_1} - m_1^{l_2}, \tag{5.19}$$

$$(\mathbf{m}^{l_1} - \mathbf{m}^{l_2}) \cdot \mathbf{m}^{l_3}. \tag{5.20}$$

A product of τ quantities of the type (5.11) to (5.20) will be called an *extended expression of order* τ. Each of the quantities A [(5.11) to (5.20)] satisfies an exponential inequality $E\langle \exp NtA \rangle \leq \exp Kt^2N$ for $|t| \leq 1/L$, and, in particular, if f is an extended expression of order τ, we have $E\langle |f| \rangle \leq KN^{-\tau/2}$, a statement that we will shorten in "f is of order $N^{-\tau/2}$."

THEOREM 5.1. *If f is an expression of order τ, then $\lim_{N\to\infty} N^{\tau/2}E\langle f \rangle$ exists.*

The proof will be by induction over τ. Certainly the result holds for $\tau = 1$ or $\tau = 0$.

Before the proof starts, let us explain how the induction hypothesis will be used.

LEMMA 5.2. *Let us assume that* $\lim_{N\to\infty} N^{(\tau-2)/2}E\langle f\rangle$ *exists for each expression* f *of order less than or equal to* $\tau-2$. *Then, if* n, n' *are integers greater than or equal to* 0, *if* A *is a product of terms of one of the following types:*

$$\mathbf{m}^l\cdot\mathbf{m}^{l'};\ \frac{1}{N}\boldsymbol{\varepsilon}^l\cdot\boldsymbol{\varepsilon}^{l'};\ m_1^l, \tag{5.21}$$

then the limit

$$\lim_{N\to\infty} N^{(\tau-2)/2}E\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y}\langle Af\rangle \tag{5.22}$$

exists for each expression f *of order* $\tau-2$.

PROOF. It is worthwhile to spell out the general principle at work here, which will be used many times in the sequel, namely, that if A is a finite product, $A=\prod A_s$, where each A_s is of the type (5.21), and if f is an extended expression of order ρ, then

$$\left|E\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y}\langle Af\rangle - E\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y}E\langle f\rangle\prod_s E\langle A_s\rangle\right| \le KN^{-(\rho+1)/2}. \tag{5.23}$$

[The lemma then follows from (5.4) to (5.6) and (5.10) taking $\rho=\tau-2$.] To prove this, we successively replace each A_s by $E\langle A_s\rangle$, using that $(A_s-E\langle A_s\rangle)f$ is an extended expression of order $\rho+1$.

If we set

$$\varphi(x,y)=E_g\frac{\mathrm{th}^n\beta(g\sqrt{x}+y+h)}{\mathrm{ch}^{n'}\beta(g\sqrt{x}+y+h)}$$

we then bound

$$\left|\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y}-E\frac{\mathrm{th}^n Y}{\mathrm{ch}^{n'} Y}\right|$$

by

$$\sup\partial_1\varphi_2|\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle - E\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle| + \sup\partial_2\varphi|\langle m_1\rangle - E\langle m_1\rangle| + \frac{K}{\sqrt{N}}. \qquad \square$$

To prove Theorem 5.1, we will assume the following induction hypothesis:

$(H(\tau-1))$ For each expression f of order
$\rho\le\tau-1$ the limit $\lim_{N\to\infty} N^{\rho/2}E\langle f\rangle$ exists.

We will then prove that $H(\tau-1)\Rightarrow H(\tau)$.

Given one expression (5.11) to (5.20), say W, we will prove that $\lim_{N\to\infty} N^\tau E\langle Wf\rangle$ exists when f is an expression of order $\tau-1$. These expressions will be considered in a carefully chosen order, and the proof for each new expression will build on the previous ones. We will start with expressions (5.16); without loss of generality, we can assume that this is $\tilde{\mathbf{m}}\cdot\mathbf{m}^*=(\mathbf{m}^1-\mathbf{m}^2)\cdot(\mathbf{m}^3-\mathbf{m}^4)$.

PROPOSITION 5.3. *Under $H(\tau-1)$, for any expression f of order $\tau-1$,*

$$\lim_{N\to\infty} N^{\tau/2} E\langle \tilde{\mathbf{m}} \cdot \mathbf{m}^* f\rangle$$

exists.

PROOF. The principle of the proof is to show that

$$E\langle \tilde{\mathbf{m}} \cdot \mathbf{m}^* f\rangle = CE\langle \tilde{\mathbf{m}} \cdot \mathbf{m}^* f\rangle + R, \tag{5.24}$$

where $\lim_{N\to\infty} C$ exists and is less than 1 and $\lim_{N\to\infty} R$ exists. The method consists of repeating the proofs of Section 3 and 4 (looking at them differently). We will compute $E\langle \tilde{f}'\rangle'$ for a certain expression $\tilde{f}'$ in a situation with $N+1$ sites in two different ways, and comparing the results will yield (5.24). For $\tilde{f}'$, we could simply consider the expression that is to $N+1$ sites what $\tilde{\mathbf{m}} \cdot \mathbf{m}^* f$ is to N sites, but it will be easier to proceed a bit differently. We write $f = \prod_{s\le\tau-1} f_s$, where each f_s is an expression of order 1. For each s, we define "the" (somewhat canonical) quantity f'_s such that $f_s + f'_s$ is symmetric between sites when seen as a function on Σ^p_{N+1}. Thus,

$$\text{if } f_s = \frac{1}{N}(\boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'} - E\langle \boldsymbol{\varepsilon}^l \cdot \boldsymbol{\varepsilon}^{l'}\rangle), \text{ then } f'_s = \frac{\sigma_l \sigma_{l'}}{N}, \tag{5.25}$$

$$\text{if } f_s = m^l_k, \text{ then } f'_s = \frac{1}{N}\eta_k \sigma_l, \qquad k \ge 2, \tag{5.26}$$

$$\text{if } f_s = m^l_1 - E\langle m^l_1\rangle, \text{ then } f'_s = \frac{1}{N}\sigma_l, \tag{5.27}$$

$$\begin{aligned} &\text{if } f_s = \mathbf{m}^l \cdot \mathbf{m}^{l'} - E\langle \mathbf{m}^l \cdot \mathbf{m}^{l'}\rangle, \text{ then } f'_s = \frac{\sigma_l \sigma_{l'}}{N^2} \\ &\qquad + \frac{1}{N}\sum_{2\le k\le M} \eta_k(\sigma_l m^{l'}_k + \sigma_{l'} m^l_k). \end{aligned} \tag{5.28}$$

(The only reason that we distinguished between expressions of order τ and extended expressions of order τ is to shorten the previous enumeration.)

We consider the quantity

$$A' = E\langle (\tilde{\mathbf{m}}' \cdot \mathbf{m}'^*) f'\rangle', \tag{5.29}$$

where $f' = \prod_{s\le\tau-1}(f_s + f'_s)$. By symmetry between the sites, we have

$$\begin{aligned} A' &= E\sum_{2\le k\le M} \eta_k \langle (\tilde{\sigma} m'^*_k) f'\rangle' \\ &= \frac{M-1}{N'} E\langle \tilde{\sigma}\sigma^* f'\rangle' + \frac{N}{N'} E\sum_{2\le k\le M} \eta_k \langle \tilde{\sigma} m^*_k f'\rangle'. \end{aligned} \tag{5.30}$$

We will integrate by parts the last term. The error terms are of order $N^{-(\tau+1)/2}$ so are irrelevant for our purposes. To simplify the notation, throughout this section we will use the notation $A \overset{c}{=} B$ to mean that $\lim_{N\to\infty} N^{\tau/2}(A - B)$

exists, so that for the purpose of convergence, there is no need to distinguish A and B. Thus, integration by parts yields

$$E\sum_{2\le k\le M}\eta_k\langle\tilde\sigma m_k^* f'\rangle' \overset{c}{=} \sum_{l\le p}\beta E\langle\tilde\sigma\sigma_l \mathbf{m}^l\cdot\mathbf{m}^* f'\rangle' \tag{5.31}$$
$$-\beta p E\langle\tilde\sigma\sigma_{p+1}\mathbf{m}^{p+1}\cdot\mathbf{m}^* f'\rangle'$$
$$+\sum_{2\le k\le M}E\left\langle\tilde\sigma m_k^*\frac{\partial f'}{\partial\eta_k}\right\rangle'.$$

Our first goal is as follows.

LEMMA 5.4. *We have*

$$\sum_{2\le k\le M}E\left\langle\tilde\sigma m_k^*\frac{\partial f'}{\partial\eta_k}\right\rangle' \overset{c}{=} 0.$$

PROOF. Since f' is the product of $\tau-1$ terms $f_s+f'_s$, a look at the possible values of f'_s shows that it is enough to prove that, for each k,

$$\frac{1}{N}E\langle\tilde\sigma\sigma_l m_k^* f''\rangle' \overset{c}{=} 0 \tag{5.32}$$

and that

$$\frac{1}{N}E\langle\tilde\sigma\sigma_l\mathbf{m}^*\cdot\mathbf{m}^{l'} f''\rangle' \overset{c}{=} 0, \tag{5.33}$$

where f'' is the product of $\tau-2$ terms $f_s+f'_s$. We expand this product f'' to reduce to the case where f'' is a product of terms that are either f_s or f'_s. A basic idea is that "the terms f'_s are of order $1/N$, and the terms f_s of order $1/\sqrt{N}$." What this means is that $E\langle|\bar f|\rangle' \le KN^{-a-b/2}$ when $\bar f$ is the product of a terms f'_s and b terms f_s. To see this, we first observe that, for each integer n,

$$E_\eta\left|\sum_k\eta_k m_k^l\right|^n \le K(n).$$

We then use Proposition 3.1 to bound $E\langle|\bar f|\rangle'$ by $E\langle|\bar f|\mathscr{E}\rangle$, we integrate in $(\eta_1,\dots,\eta_k)$ first with Hölder's inequality, and we use Theorem 4.1.

To prove (5.32), we observe that the term m_k^* is itself of order $N^{-1/2}$, so that the left-hand side of (5.32) is of order $N^{-\tau/2-1/2}$. To prove (5.33), only the case where f'' is a product of $\tau-2$ terms f_s has to be considered, for if one of the factors is f'_s, the expression (5.33) is of order $N^{-(\tau+1)/2}$. When f'' is a product of $\tau-2$ terms f_s, use of Proposition 2.4 and Theorem 2.5 transform these into a sum of terms to which we can apply Lemma 5.3. □

LEMMA 5.5. *For $l\le p+1$, we have*

$$E\langle\tilde\sigma\sigma_l\mathbf{m}^l\cdot\mathbf{m}^* f'\rangle' \overset{c}{=} E\langle\tilde\sigma\sigma_l\mathbf{m}^l\cdot\mathbf{m}^* f\rangle'.$$

PROOF. We expand the product $f' = \prod_{s\le\tau-1}(f_s + f'_s)$. Thus, it is enough to prove the following. If f'' is a product of terms f_s, f'_s, $s \le \tau - 1$ (i.e., for each s we choose between f_s and f'_s), then $E\langle\tilde{\sigma}\sigma_l\mathbf{m}^l\cdot\mathbf{m}^* f''\rangle \overset{c}{=} 0$ when at least one of the factors is f'_s.

This is obvious if there are at least *two* factors f'_s, because then the term is of order $N^{-(\tau+1)/2}$, so we consider only the case where there is exactly one factor f'_s. But, in that case, integration by parts [if we are in case (5.26) or (5.28)], use of Proposition 2.4 and Theorem 2.5 reduce again to Lemma 5.3. □

Going back to (5.31), we have shown that

$$E\sum_{2\le k\le M}\eta_k\langle\tilde{\sigma}m_k^* f'\rangle' \overset{c}{=} \beta\sum_{l\le p}E\langle\tilde{\sigma}\sigma_l\mathbf{m}^l\cdot\mathbf{m}^* f\rangle' - \beta p E\langle\tilde{\sigma}\sigma_{p+1}\mathbf{m}^{p+1}\cdot\mathbf{m}^* f\rangle'. \tag{5.34}$$

We regroup as usual the terms for $l = 1$, $l = 2$ into

$$\frac{\beta}{2}E\langle(\tilde{\sigma})^2\tilde{\mathbf{m}}\cdot\mathbf{m}^* f\rangle' \tag{5.35}$$

and we apply Theorem 2.5. What greatly helps is that $\tilde{\mathbf{m}}\cdot\mathbf{m}^* f$ is an extended expression of order τ, so that many of the extended expressions we will find are of order greater than or equal to $\tau + 1$ and hence of order $N^{-(\tau+1)/2}$. This is the case for all terms where, besides f, at least *two* terms such as $\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}$, a_l, $a_{l'}$, $\tilde{\mathbf{m}}\cdot\mathbf{m}^l$, $l \ge 3$, $\tilde{\mathbf{m}}\cdot\mathbf{m}^*$, $\mathbf{m}^l\cdot\mathbf{m}^*$, $l \notin 3, 4$ occur. Thus, the only contribution from (5.35) is

$$\beta E\frac{1}{\mathrm{ch}^2 Y}\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^* f\rangle. \tag{5.36}$$

None of the terms $E\langle\tilde{\sigma}\sigma_l\mathbf{m}^l\cdot\mathbf{m}^* f\rangle'$ for $l \ge 5$ contributes. Using Corollary 2.7, the contributions of the terms for $l = 3$, $l = 4$ are (regrouping these terms)

$$\begin{aligned}
&\beta^2 E\frac{1}{\mathrm{ch}^2 Y}\langle(\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^3\mathbf{m}^3\cdot\mathbf{m}^* + \tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^4\mathbf{m}^4\cdot\mathbf{m}^*)f\rangle\\
&\quad+\beta^2 E\left(\frac{1-3\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\right)\langle(\mathbf{m}^3\cdot\mathbf{m}^* + \mathbf{m}^4\cdot\mathbf{m}^*)f\tilde{\mathbf{m}}\cdot\mathbf{b}\rangle\\
&\quad+\beta E\frac{\mathrm{th} Y}{\mathrm{ch}^3 Y}\langle(\mathbf{m}^3\cdot\mathbf{m}^* + \mathbf{m}^4\cdot\mathbf{m}^*)f(c_1 - c_2)\rangle\\
&\quad+\beta^2 E\frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\langle(\mathbf{m}^3\cdot\mathbf{m}^*\tilde{\mathbf{m}}\cdot\mathbf{m}^4 + \mathbf{m}^4\cdot\mathbf{m}^*\tilde{\mathbf{m}}\cdot\mathbf{m}^3)f\rangle.
\end{aligned} \tag{5.37}$$

It is fortunate that $\mathbf{m}^3\cdot\mathbf{m}^* + \mathbf{m}^4\cdot\mathbf{m}^* = \|\mathbf{m}^3\|^2 - \|\mathbf{m}^4\|^2$ so that only the first and the last term are not of order $(N^{-(\tau+1)/2})$. We observe that, using simple algebra,

$$\begin{aligned}&\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^3\mathbf{m}^3\cdot\mathbf{m}^*+\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^4\mathbf{m}^4\cdot\mathbf{m}^*\\ &\quad=\tilde{\mathbf{m}}\cdot\mathbf{m}^*(\|\mathbf{m}^3\|^2-\mathbf{m}^3\mathbf{m}^4)+\tilde{\mathbf{m}}\cdot\mathbf{m}^4(\|\mathbf{m}^3\|^2-\|\mathbf{m}^4\|^2)\\ &\qquad-\tilde{\mathbf{m}}\cdot\mathbf{b}(\|\mathbf{m}^3\|^2-\|\mathbf{m}^4\|^2).\end{aligned}\tag{5.38}$$

It follows that

$$\begin{aligned}&E\frac{1}{\mathrm{ch}^2Y}\langle(\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^3\mathbf{m}^3\cdot\mathbf{m}^4+\tilde{\mathbf{m}}\cdot\dot{\mathbf{m}}^4\mathbf{m}^4\cdot\mathbf{m}^*)f\rangle\\ &\quad\stackrel{c}{=}E\frac{1}{\mathrm{ch}^2Y}\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*(\|\mathbf{m}^3\|^2-\mathbf{m}^3\cdot\mathbf{m}^4)f\rangle\\ &\quad\stackrel{c}{=}\left(E\frac{1}{\mathrm{ch}^2Y}\right)(E\langle\|\mathbf{m}\|^2\rangle-E\langle\mathbf{m}^3\cdot\mathbf{m}^4\rangle)E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle\end{aligned}$$

(the last equality uses in an essential way that $\tilde{\mathbf{m}}\cdot\mathbf{m}^*f$ is of order τ).

For the last term of (5.37), a similar computation yields that (modulo converging terms) it is

$$-\beta^2E\frac{\mathrm{th}^2Y}{\mathrm{ch}^2Y}(E\langle\|\mathbf{m}\|^2\rangle-E\langle\mathbf{m}^3\cdot\mathbf{m}^4\rangle)E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle.$$

Estimating the contribution of $E\langle\tilde{\sigma}\sigma^*f'\rangle'$ in (5.30) is much easier. One shows as above that

$$E\langle\tilde{\sigma}\sigma^*f'\rangle'\stackrel{c}{=}E\langle\tilde{\sigma}\sigma^*f\rangle'$$

and Corollary 2.8 gives

$$E\langle\tilde{\sigma}\sigma^*f'\rangle\stackrel{c}{=}\beta^2\left(E\frac{1}{\mathrm{ch}^4Y}\right)E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle.$$

Finally, rather than (5.28) using now that

$$A'=\frac{N}{N'}E\sum_k\langle\tilde{m}_km_k'^*f\rangle$$

and using the previous arguments (in a much simpler situation), we get

$$A'\stackrel{c}{=}\frac{N}{N'}E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle.$$

Combining these estimates, we arrive at the relation promised at the beginning of the proof, namely,

$$E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle\stackrel{c}{=}CE\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^*f\rangle,$$

where

$$\begin{aligned}C&=\beta^2\left(E\frac{1}{\mathrm{ch}^4Y}\right)\left(\frac{M-1}{N}+\beta(E\|\mathbf{m}^1\|^2-E\langle\mathbf{m}^3\cdot\mathbf{m}^4\rangle)\right)\\ &\quad+\beta E\frac{1}{\mathrm{ch}^2Y}.\end{aligned}\tag{5.39}$$

In the accessible region, this is at most

$$\beta^2 E\frac{1}{\mathrm{ch}^2 Y}\left(\beta\alpha + L\frac{\alpha}{\beta-1}\right) + \beta E\frac{1}{\mathrm{ch}^2 Y} < 1$$

since $1-\beta E(1/\mathrm{ch}^2 Y)$ is of order $\beta-1$ for $\beta \le 2$. The proof is complete. □

REMARK. The previous computation is an identification of the A-T line. Indeed, using (5.6), (5.7), we see that (assuming the validity of the RS solution ...)

$$C_\infty =: \lim_{N\to\infty} C = \beta^2 E\frac{1}{\mathrm{ch}^4\beta(g\sqrt{r}+\mu+h)}\left(\alpha + \frac{\alpha\beta(1-q)}{(1-\beta(1-q))}\right) + \beta(1-q)$$

so that $C_\infty < 1$ if and only if

$$\alpha\beta^2 E\frac{1}{\mathrm{ch}^4\beta(g\sqrt{r}+\mu+h)} < (1-\beta(1-q))^2 \tag{5.40}$$

(the condition of the A-T line).

Proposition 5.3 is good step forward, because in the use of Theorem 2.5 we no longer have to worry about terms containing $\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}$, as these terms can be replaced by $(\mathbf{m}^l-\mathbf{m}^{p+1})\cdot(\mathbf{m}^{l'}-\mathbf{m}^{p+2})$ for two new replicas of rank $p+1$, $p+2$. Thus, if f is an expression of order $\tau-1$, we know that $\lim_{N\to\infty} N^{\tau/2}E\langle\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}f\rangle$ exists, and, since $\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}f$ is an extended expression of order τ, this is also the case for $\lim_{N\to\infty} N^{\tau/2}E\mathrm{th}^2 Y\langle\dot{\mathbf{m}}^l\cdot\dot{\mathbf{m}}^{l'}f\rangle$ and so on, as explained in Lemma 5.2.

PROPOSITION 5.6. *Under $H(\tau-1)$, if f is an expression of order $\tau-1$, then* $\lim_{N\to\infty} N^{\tau/2}E\langle\tilde{\mathbf{m}}\cdot\mathbf{b}f\rangle$ *and* $\lim_{N\to\infty} N^{\tau/2}E\langle\tilde{m}_1 f\rangle$ *exist.*

PROOF. We introduce f' as in Proposition 5.3, and we evaluate

$$E\langle\tilde{\mathbf{m}}'\cdot\mathbf{b}'f'\rangle' = E\langle\tilde{\mathbf{m}}'\cdot\mathbf{m}'^{p+1}f'\rangle'.$$

The bracket on the right involves $p+1$-replicas and f' depends only on the first p-replicas.

We start with

$$\begin{aligned}E\langle\tilde{\mathbf{m}}'\cdot\mathbf{m}'^{p+1}f'\rangle' &= E\sum_k \eta_k\langle\tilde{\sigma}m_k'^{p+1}f'\rangle'\\ &= \frac{M-1}{N'}E\langle\tilde{\sigma}\sigma_{p+1}f'\rangle'\\ &\quad+\frac{N}{N'}E\sum_{2\le k\le M}\eta_k\langle\tilde{\sigma}m_k'^{p+1}f'\rangle'.\end{aligned}$$

Use of Lemmas 5.4 and 5.5 and integration by parts yield

$$\begin{aligned}E\langle\tilde{\mathbf{m}}'\cdot\mathbf{m}'^{p+1}f'\rangle' \overset{c}{=} {} & \frac{M-1}{N}E\langle\tilde{\sigma}\sigma_{p+1}f\rangle' \\ & +\frac{N}{N'}\beta\sum_{l\le p+1}E\langle\tilde{\sigma}\sigma_l\mathbf{m}^{p+1}\cdot\mathbf{m}^l f\rangle' \\ & -\frac{N}{N'}\beta(p+1)E\langle\tilde{\sigma}\sigma_{p+1}\mathbf{m}^{p+1}\cdot\mathbf{m}^{p+2}f\rangle'.\end{aligned}$$

We now use Corollary 2.7 to see that

$$\begin{aligned}E\langle\tilde{\sigma}\sigma_{p+1}f\rangle' & \overset{c}{=} \beta^2E\left(\frac{1-3\mathrm{th}^2Y}{\mathrm{ch}^2Y}\right)\langle\bar{f}\tilde{\mathbf{m}}\cdot\mathbf{b}\rangle+\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2Y}\langle\bar{f}\tilde{m}_1\rangle \\ & \overset{c}{=} \beta^2E\left(\frac{1-3\mathrm{th}^2Y}{\mathrm{ch}^2Y}\right)E\langle\bar{f}\tilde{\mathbf{m}}\cdot\mathbf{b}\rangle+\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2Y}E\langle\bar{f}\tilde{m}_1\rangle.\end{aligned}$$

As previously, we have

$$\tilde{\sigma}\sigma_1\mathbf{m}^{p+1}\cdot\mathbf{m}^1+\tilde{\sigma}\sigma_2\mathbf{m}^{p+1}\cdot\mathbf{m}^2=\frac{1}{2}(\tilde{\sigma})^2\tilde{\mathbf{m}}\cdot\mathbf{m}^{p+1}$$

and Corollary 2.10 gives

$$E\frac{1}{2}\langle(\tilde{\sigma})^2\tilde{\mathbf{m}}\cdot\mathbf{m}^{p+1}f\rangle \overset{c}{=} E\frac{1}{\mathrm{ch}^2Y}\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^{p+1}f\rangle \overset{c}{=} E\frac{1}{\mathrm{ch}^2Y}E\langle\tilde{\mathbf{m}}\cdot\mathbf{b}f\rangle$$

because all the other terms are of higher order. For $l\ge 3$, Corollary 2.7 gives

$$\begin{aligned}E\langle\tilde{\sigma}\sigma_l\mathbf{m}^{p+1}\cdot\mathbf{m}^l f\rangle' \overset{c}{=} {} & \beta^2E\left(\frac{1-3\mathrm{th}^2Y}{\mathrm{ch}^2Y}\right)\langle\mathbf{m}^{p+1}\cdot\mathbf{m}^l\tilde{\mathbf{m}}\cdot\mathbf{b}f\rangle \\ & +\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2Y}\langle\mathbf{m}^{p+1}\cdot\mathbf{m}^l\tilde{m}_1f\rangle\end{aligned}$$

and also

$$\begin{aligned}E\langle\tilde{\sigma}\sigma_{p+1}\mathbf{m}^{p+1}\cdot\mathbf{m}^{p+2}f\rangle' \overset{c}{=} {} & \beta^2E\left(\frac{1-3\mathrm{th}^2Y}{\mathrm{ch}^2Y}\right)\langle\mathbf{m}^{p+1}\cdot\mathbf{m}^{p+2}\tilde{\mathbf{m}}\cdot\mathbf{b}f\rangle \\ & +\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2Y}\langle\mathbf{m}^{p+1}\cdot\mathbf{m}^{p+2}\tilde{m}_1f\rangle.\end{aligned}$$

In particular, if $l < p+1$, these two expressions coincide modulo converging terms, as is seen by regrouping the terms in their difference. It is simple to show that

$$E\langle\tilde{\mathbf{m}}'\cdot\mathbf{m}'^{p+1}f'\rangle' \overset{c}{=} E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^{p+1}f\rangle$$

so that, if we set

$$A=E\langle\tilde{\mathbf{m}}\cdot\mathbf{b}f\rangle,\qquad B=E\langle\tilde{m}_1f\rangle,$$

we get after regrouping the terms that

$$A \overset{c}{=} UA + VB, \tag{5.41}$$

where

$$U = \beta E \frac{1}{\mathrm{ch}^2 Y} + \beta^2 E\left(\frac{1 - 3\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\right)\Big[\alpha + \beta E\langle \|\mathbf{m}\|^2\rangle - 3E\langle \|\mathbf{b}\|^2\rangle\Big],$$

$$V = \beta E \frac{\mathrm{th} Y}{\mathrm{ch}^2 Y}(E\langle \|\mathbf{m}\|^2\rangle - 3E\langle \|\mathbf{b}\|^2\rangle).$$

A similar, but much easier computation yields

$$B \overset{c}{=} \beta E \frac{1}{\mathrm{ch}^2 Y} B - 2\beta^2 E \frac{\mathrm{th} Y}{\mathrm{ch}^2 Y} A$$

so that

$$B\left(1 - \beta E \frac{1}{\mathrm{ch}^2 Y}\right) \overset{c}{=} -2\beta^2 E \frac{\mathrm{th} Y}{\mathrm{ch}^2 Y} A.$$

Substitution in (5.11), together with the fact that $1 - \beta E(1/\mathrm{ch}^2 Y) + L\rho_1^2 < 1$, then yields $A \overset{c}{=} 0$, and thus $B \overset{c}{=} 0$. □

Our position has strengthened again, because, when f is an expression of order $\tau - 1$, in the application of Proposition 2.6 and its corollaries, we no longer need to consider the terms containing $\tilde{\mathbf{m}} \cdot \mathbf{b} f$, or $m_1 f$.

We now turn to the case of expressions of the type (5.17); without loss of generality, we can consider $\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2$.

PROPOSITION 5.7. *Under $H(\tau - 1)$, if f is an expression of order $\tau - 1$, then*

$$\lim_{N\to\infty} N^{\tau/2} E\langle (\|\tilde{\mathbf{m}}\|^2 - \|\mathbf{m}^*\|^2) f\rangle$$

exists.

PROOF. We introduce f' as in Proposition 5.3, and we evaluate $E\langle (\|\tilde{\mathbf{m}}'\|^2 - \|\mathbf{m}'^*\|^2) f'\rangle'$.

We start with

$$\begin{aligned} E\langle \|\tilde{\mathbf{m}}'\|^2 f'\rangle' &= E \sum_k \eta_k \langle \tilde{\sigma} \tilde{m}'_k f'\rangle' \\ &= \frac{M-1}{N'} E\langle (\tilde{\sigma})^2 f'\rangle' + \frac{N}{N'} E \sum_{2\le k\le M} \eta_k \langle \tilde{\sigma} \tilde{m}_k f'\rangle'. \end{aligned}$$

Using Corollary 2.10,

$$\begin{aligned} E\langle (\tilde{\sigma})^2 f'\rangle' &\overset{c}{=} E\langle (\tilde{\sigma})^2 f\rangle' \\ &\overset{c}{=} E\left(\frac{1}{\mathrm{ch}^2 Y}\langle f\rangle\right) + \frac{\beta^2}{2} E \frac{1}{\mathrm{ch}^2 Y}\left\langle f \sum_{l\ge 1} a_l\right\rangle + \beta E \frac{\mathrm{th} Y}{\mathrm{ch}^2 Y}\left\langle f \sum_{l\ge 3} c_l\right\rangle \end{aligned}$$

because all the other terms from Corollary 2.10 are known to be convergent by Proposition 5.3. Now, integration by parts gives

$$E\sum_{2\le k\le M}\eta_k\langle\tilde\sigma\tilde m_k f'\rangle' \stackrel{c}{=} \beta\sum_{l\le p}E\langle\tilde\sigma\sigma_l\tilde{\mathbf{m}}\cdot\mathbf{m}^l f\rangle'$$
$$-\beta pE\langle\tilde\sigma\sigma_{p+1}\tilde{\mathbf{m}}\cdot\mathbf{m}^{p+1}f\rangle'.$$

Use of Corollary 2.7 shows that only the terms for $l=1,2$ can contribute. They regroup as

$$\tfrac{1}{2}E\langle(\tilde\sigma)^2\|\tilde{\mathbf{m}}\|^2 f\rangle'$$

and use of Corollary 2.10 shows that

$$\frac{1}{2}E\langle(\tilde\sigma)^2\|\tilde{\mathbf{m}}\|^2 f\rangle' \stackrel{c}{=} E\frac{1}{\mathrm{ch}^2 Y}\langle\|\tilde{\mathbf{m}}\|^2 f\rangle+\frac{\beta^2}{2}E\frac{1}{\mathrm{ch}^2 Y}\left\langle\|\tilde{\mathbf{m}}\|^2 f\sum_{l\ge 1}c_l\right\rangle$$
$$+\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}\left\langle\|\tilde{\mathbf{m}}\|^2 f\sum_{l\ge 3}a_l\right\rangle.$$

Regrouping with the contributions from $-E\langle\|\mathbf{m}'^*\|^2 f'\rangle'$, we obtain a total contribution of

$$\frac{N}{N'}\beta E\frac{1}{\mathrm{ch}^2 Y}\langle(\|\tilde{\mathbf{m}}\|^2-\|\mathbf{m}^*\|^2)f\rangle$$
$$\stackrel{c}{=}\frac{N}{N'}\beta E\frac{1}{\mathrm{ch}^2 Y}E\langle(\|\tilde{\mathbf{m}}\|^2-\|\mathbf{m}^*\|^2)f\rangle$$

because $(\|\tilde{\mathbf{m}}\|^2-\|\mathbf{m}^*\|^2)f$ is an extended expression of order τ. [The reader will observe that we do *not* know yet that

$$E\frac{1}{\mathrm{ch}^2 Y}\langle\|\tilde{\mathbf{m}}\|^2 f\rangle \stackrel{c}{=}\left(E\frac{1}{\mathrm{ch}^2 Y}\right)E\langle\|\tilde{\mathbf{m}}\|^2 f\rangle.]$$

In this way we obtain

$$A\stackrel{c}{=}\beta\left(E\frac{1}{\mathrm{ch}^2 Y}\right)A$$

and this completes the proof. □

We have made further progress, because when we apply Theorem 2.5 and its corollaries, we have now controlled the terms containing a_l. Indeed,

$$\langle(\|\dot{\mathbf{m}}^l\|^2-\langle\|\dot{\mathbf{m}}^l\|^2\rangle)f\rangle=\langle(\|\mathbf{m}^l-\mathbf{m}^{p+1}\|^2-\|\mathbf{m}^{p+3}-\mathbf{m}^{p+2}\|^2)f\rangle$$

using three new replicas. So we now know that all terms such as $N^{\tau/2}E(1/\mathrm{ch}^2 Y)\langle a_l f\rangle$ are convergent when f is an expression of order $\tau-1$.

Now, when using Theorem 2.5, we will have to be concerned only about the terms of type I. Still, we are not done.

PROPOSITION 5.8. *Under $H(\tau-1)$, for each expression f of order $\tau-1$, $\lim_{N\to\infty} N^{\tau/2} E\langle \tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle$ exists.*

PROOF. We write

$$\begin{aligned} E\langle \tilde{\mathbf{m}}'\cdot\mathbf{m}'^3 f'\rangle' &= \sum_k E\eta_k\langle\tilde{\sigma} m'^3{}_k f'\rangle' \\ &\overset{c}{=} \frac{M-1}{N'}E\langle\tilde{\sigma}\sigma_3 f\rangle + \frac{N}{N'}\sum_k E\eta_k\langle\tilde{\sigma}m_k^3 f'\rangle'. \end{aligned}$$

By integration by parts,

$$E\sum_k \eta_k\langle\tilde{\sigma}m_k^3 f'\rangle' = \beta\sum_{l\le p} E\langle\tilde{\sigma}\sigma_l\mathbf{m}^3\cdot\mathbf{m}^l f\rangle' - \beta p E(\tilde{\sigma}\sigma_l\mathbf{m}^3\cdot\mathbf{m}^{p+1}\rangle'.$$

As we now care only about the terms of type I of Theorem 2.5, only the terms for $l=1,2$ contribute; they regroup as

$$\beta E\frac{1}{2}\langle(\tilde{\sigma})^2\tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle'$$

and we get

$$E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle \overset{c}{=} \beta E\frac{1}{\mathrm{ch}^2 Y}E\langle\tilde{\mathbf{m}}\cdot\mathbf{m}^3 f\rangle. \qquad \square$$

PROPOSITION 5.9. *Under $H(\tau-1)$, for each expression f of order $\tau-1$, $\lim_{N\to\infty} N^{\tau/2}E\langle(\mathbf{m}^1\cdot\mathbf{m}^2 - E\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle)f\rangle$, $\lim_{N\to\infty}N^{\tau/2}E\langle(m_1^1 - E\langle m_1^1\rangle)f\rangle$ and $\lim_{N\to\infty}N^{\tau/2}E\langle(\|\tilde{\mathbf{m}}\|^2 - E\langle\|\tilde{\mathbf{m}}\|^2\rangle)f\rangle$ exist.*

PROOF. Writing $a = E\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle$, $b = E\|\tilde{\mathbf{m}}\|^2$, $c = E\langle m_1\rangle$, we will first show that

$$\begin{aligned} E\langle(\mathbf{m}^1\cdot\mathbf{m}^2 - a)f\rangle &\overset{c}{=} \alpha(E\mathrm{th}^2 Y\langle f\rangle - (E\mathrm{th}^2Y)E\langle f\rangle) \\ &\quad + \beta\biggl(E\frac{1}{\mathrm{ch}^2Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2 f\rangle \\ &\qquad - E\biggl(\frac{1}{\mathrm{ch}^2Y}\langle\mathbf{m}^1\cdot\mathbf{m}^2\rangle\biggr)E\langle f\rangle\biggr) \\ &\quad + \frac{\beta}{2}(E\mathrm{th}^2Y\langle\|\tilde{\mathbf{m}}\|^2 f\rangle - E\mathrm{th}^2Y\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle f\rangle), \end{aligned} \tag{5.43}$$

$$\begin{aligned} E\langle(\|\tilde{\mathbf{m}}\|^2 - b)f\rangle &\overset{c}{=} 2\alpha\biggl(E\frac{1}{\mathrm{ch}^2Y}\langle f\rangle - \biggl(E\frac{1}{\mathrm{ch}^2Y}\biggr)E\langle f\rangle\biggr) \\ &\quad + \beta E\biggl(\frac{1}{\mathrm{ch}^2Y}\langle\|\tilde{\mathbf{m}}\|^2 f\rangle - E\frac{1}{\mathrm{ch}^2Y}\langle\|\tilde{\mathbf{m}}\|^2\rangle E\langle f\rangle\biggr), \end{aligned} \tag{5.44}$$

$$E\langle(m_1^1 - c)f\rangle = E\mathrm{th}Y\langle f\rangle - (E\mathrm{th}Y)E\langle f\rangle. \tag{5.45}$$

This follows the usual pattern. We write

$$E\langle \mathbf{m}'^1 \cdot \mathbf{m}'^2 f'\rangle' = \frac{M-1}{N} E\langle \sigma_1\sigma_2 f'\rangle' + \frac{N}{N'} \sum_{2\le k\le M} E\eta_k\langle \sigma_1 m_k^2 f'\rangle'$$

and

$$E\langle \sigma_1\sigma_2 f'\rangle \overset{c}{=} E\mathrm{th}^2 Y\langle f\rangle.$$

Also, integration by parts yields

$$(5.46)\qquad \begin{aligned} &\sum_{2\le k\le M} E\eta_k\langle \sigma_1 m_k^2 f'\rangle \\ &\quad \overset{c}{=} \beta \sum_{1\le l\le p} E\langle \sigma_1\sigma_l \mathbf{m}^l\cdot \mathbf{m}^2 f\rangle' - \beta p E\langle \sigma_1\sigma_{p+1}\mathbf{m}^{p+1}\cdot\mathbf{m}^2 f\rangle' \\ &\quad \overset{c}{=} \beta(E\langle \mathbf{m}^1\cdot\mathbf{m}^2 f\rangle + E\mathrm{th}^2 Y\langle \|\mathbf{m}^2\|^2 f\rangle \\ &\qquad + \sum_{3\le l\le p} E\mathrm{th}^2 Y\langle \mathbf{m}^2\cdot\mathbf{m}^l f\rangle - pE\mathrm{th}^2 Y\langle \mathbf{m}^{p+1}\cdot\mathbf{m}^2 f\rangle). \end{aligned}$$

We now observe that, if $l, l' \ge 3$,

$$(5.47)\qquad E\mathrm{th}^2 Y\langle \mathbf{m}^2\cdot\mathbf{m}^l f\rangle \overset{c}{=} E\mathrm{th}^2 Y\langle \mathbf{m}^2\cdot\mathbf{m}^{l'} f\rangle.$$

Indeed, since $\mathbf{m}^2\cdot(\mathbf{m}^l - \mathbf{m}^l)f$ is an extended expression of order τ,

$$E\mathrm{th}^2 Y\langle \mathbf{m}^2\cdot(\mathbf{m}^l-\mathbf{m}^l)f\rangle \overset{c}{=} (E\mathrm{th}^2 Y)E\langle \mathbf{m}^2\cdot(\mathbf{m}^l-\mathbf{m}^{l'})f\rangle \overset{c}{=} 0,$$

where the last equality follows from Proposition 5.8. Thus, (5.46) simplifies to

$$\begin{aligned}\sum_{2\le k\le M} E\eta_k\langle \sigma_1 m_k^2 f'\rangle' \;\overset{c}{=}\; &\beta E\frac{1}{\mathrm{ch}^2 Y}\langle \mathbf{m}^1\cdot\mathbf{m}^2 f\rangle \\ &+\beta E\mathrm{th}^2 Y\langle (\|\mathbf{m}^2\|^2 - \mathbf{m}^1\cdot\mathbf{m}^2)f\rangle.\end{aligned}$$

The replicas of rank 1 and 2 play the same role, so that in the last term we can replace $\|\mathbf{m}^2\|^2 - \mathbf{m}^1\cdot\mathbf{m}^2$ by $\frac{1}{2}(\|\mathbf{m}^1\|^2 + \|\mathbf{m}^2\|^2 - \mathbf{m}^1\cdot\mathbf{m}^2) = \frac{1}{2}\|\tilde{\mathbf{m}}\|^2$. Submitting $E\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle$ to the same computation yields (5.43).

To obtain (5.44), we write

$$(5.48)\qquad E\langle \|\tilde{\mathbf{m}}^2\|^2 f'\rangle' \overset{c}{=} \frac{M-1}{N'} E\langle (\tilde\sigma)^2 f\rangle' = \frac{N}{N'} E\sum_{2\le k\le m} \eta_k(\tilde\sigma\tilde m_k f\rangle'.$$

We integrate by parts. Only the terms for $l = 1, 2$ matter, since for the others, due to the factor $\tilde\sigma\sigma_l$ there is no contribution from the terms of type I of Theorem 2.5. These two terms regroup as usual in

$$\beta E\tfrac{1}{2}\langle (\tilde\sigma)^2\|\tilde{\mathbf{m}}\|^2 f\rangle'.$$

This and the same computation for $E\langle\|\tilde{\mathbf{m}}\|^2\rangle$ yield (5.44). As for (5.45), it should be obvious.

To use (5.43) to (5.45), we make a first-order expansion of the terms such as $E\mathrm{th}^2 Y$; we know from Theorem 4.1 that the higher order terms will not contribute. We recall the functions ψ_1, ψ_2 of Section 4, and we write

$$\begin{aligned} A &= E\langle(\mathbf{m}^1 \cdot \mathbf{m}^2 - a)f\rangle, \\ B &= E\langle(\|\tilde{\mathbf{m}}\|^2 - b)f\rangle, \\ C &= E\langle(m_1 - c)f\rangle. \end{aligned}$$

Let us examine what happens on a typical term, say,

$$E\frac{1}{\mathrm{ch}^2 Y}\langle\mathbf{m}^1 \cdot \mathbf{m}^2 f\rangle - E\frac{1}{\mathrm{ch}^2 Y}\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle E\langle f\rangle. \tag{5.49}$$

We write

$$\begin{aligned} \frac{1}{\mathrm{ch}^2 Y} &= \psi_2(a,c) + \langle\mathbf{m}^1 \cdot \mathbf{m}^2 - a\rangle\partial_1\psi_2(a,c) \\ &\quad + \langle m_1 - c\rangle\partial_2\psi_1(a,c) + R, \end{aligned} \tag{5.50}$$

where R is of order N^{-1} [in the sense that $E\langle|R|^n\rangle \le K(n)N^{-n}$].

Substituting in (5.49) gives a contribution

$$\begin{aligned} &\psi_2(a,c)A + \partial_1\psi_2(a,c)(E\langle\mathbf{m}^1 \cdot \mathbf{m}^2 - a\rangle\langle\mathbf{m}^1 \cdot \mathbf{m}^2 f\rangle \\ &\quad -E\langle\mathbf{m}^1 \cdot \mathbf{m}^2 - a\rangle\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle E\langle f\rangle) \\ &\quad +\partial_2\psi_2(a,c)(E\langle m_1 - c\rangle\langle\mathbf{m}^1 \cdot \mathbf{m}^2 f\rangle \\ &\quad -\langle m_1 - c\rangle\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle E\langle f\rangle). \end{aligned} \tag{5.51}$$

Now, in $\langle\mathbf{m}^1 \cdot \mathbf{m}^2 f\rangle$ we can replace $\mathbf{m}^1 \cdot \mathbf{m}^2$ by $E\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle$ because the difference is of higher order. In this manner, (5.59) simplifies to

$$\begin{aligned} &\psi_2(a,c)A + \partial_1\psi_2(a,c)E\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle E\langle\mathbf{m}^1 \cdot \mathbf{m}^2 - a\rangle\langle f\rangle \\ &\quad +\partial_2\psi_2(a,c)E\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle E\langle m_1 - c\rangle\langle f\rangle. \end{aligned} \tag{5.52}$$

Let us first consider the case where f does not depend on replicas of rank 1 and 2, so that $\langle\mathbf{m}^1 \cdot \mathbf{m}^2 f\rangle = \langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle\langle f\rangle$, and (5.52) further simplifies to

$$[\psi_2(a,c) + E\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle\partial_1\psi_2(a,c)]A + \partial_2\psi_2(a,c)E\langle\mathbf{m}^1 \cdot \mathbf{m}^2\rangle C. \tag{5.53}$$

Under the condition that f does not depend on the replicas of order 1 and 2, (5.43) to (5.45) simplify in this manner to

$$\begin{aligned} A \overset{c}{=} \Big[&-\alpha\partial_1\psi_2(a,c)+\beta\psi_2(a,c)+\beta\partial_1\psi_2(a,c)E\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle \\ &-\frac{\beta}{2}\partial_1\psi_2(a,c)E\langle\|\tilde{\mathbf{m}}\|^2\rangle\Big]A+\frac{\beta}{2}(1-\psi_2(a,c))B \\ &+\Big[-\alpha\partial_2\psi_2(a,c)+\beta\partial_2\psi_2(a,c)E\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle \\ &-\frac{\beta}{2}\partial_2\psi_2(a,c)E\langle\|\tilde{\mathbf{m}}\|^2\rangle\Big]C, \end{aligned} \tag{5.54}$$

$$\begin{aligned} B \overset{c}{=} &\big[2\alpha\partial_1\psi_2(a,c)+\beta\partial_1\psi_2(a,c)E\langle\|\tilde{\mathbf{m}}\|^2\rangle\big]A \\ &+\beta\psi_2(a,c)B+\big[2\alpha\partial_2\psi_2(a,c)+\beta\partial_2\psi_2(a,c)E\langle\|\tilde{\mathbf{m}}\|^2\rangle\big]C, \end{aligned} \tag{5.55}$$

$$C \overset{c}{=} \partial_1\psi_1(a,c)A+\partial_2\psi_1(a,c)C. \tag{5.56}$$

To prove that $A\overset{c}{=}0$, $B\overset{c}{=}0$, $C\overset{c}{=}0$, we then have to show that the 3×3 matrix $\mathscr{M}$ given by the right-hand sides of these equations is such that $(\mathrm{Id}-\mathscr{M})^{-1}$ exists and remains bounded in the admissible region. This is a very uninspiring task. We explain the idea only in the case $\beta\le 2$, which is the hardest. Then $\mathrm{Id}-\mathscr{M}=(1-\beta\psi_2(a,c))\mathrm{Id}+\mathscr{M}_1+\mathscr{M}_2$, where the coefficients of $\mathscr{M}_1$ are all bounded by $L\rho_1^2$ and where all the coefficients (m_{ij}) of $\mathscr{M}_2$ are 0, except $m_{1,2}$ and $m_{3,1}$, which remain bounded. This easily implies the result.

This finishes the proof in the case where f does not depend on the replicas of rank 1 and 2. But this special case implies that, for any expression f of order $\tau-1$,

$$E\frac{1}{\mathrm{ch}^2 Y}\langle f\rangle \overset{c}{=} E\frac{1}{\mathrm{ch}^2 Y}E\langle f\rangle. \tag{5.57}$$

This is seen using (5.50) and the fact that we now know that $E\langle \mathbf{m}^1\cdot\mathbf{m}^2-a\rangle\langle f\rangle\overset{c}{=}0$ and $E\langle m_1-c\rangle\langle f\rangle\overset{c}{=}0$ (because there it can be assumed, without loss of generality that f does not depend on replicas of rank 1 and 2). Also, (5.57) remains true if there is a factor $\mathbf{m}^1\cdot\mathbf{m}^2$ or $\|\tilde{\mathbf{m}}\|^2$ in front of f. This is because $E\langle \mathbf{m}^1\cdot\mathbf{m}^2-a\rangle\langle\|\tilde{\mathbf{m}}\|^2 f\rangle\overset{c}{=}0$ (and $E\langle m_1-c\rangle\langle\|\tilde{\mathbf{m}}\|^2 f\rangle\overset{c}{=}0$). Indeed, this would obviously be true if we replace $\|\tilde{\mathbf{m}}\|$ by $E\langle\|\tilde{\mathbf{m}}\|\rangle$; the difference is $\overset{c}{=}0$ as it can be written $E\langle W\rangle$, where W is an expression of order $\tau+1$. Also,

$$E\langle f\rangle E\bigg(\frac{1}{\mathrm{ch}^2 Y}\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle\bigg)\overset{c}{=}E\langle f\rangle E\frac{1}{\mathrm{ch}^2 Y}E\langle \mathbf{m}^1\cdot\mathbf{m}^2\rangle,\ \text{etc.}$$

Thus, (5.43) and (5.44) simplify, respectively, to

$$A \stackrel{c}{=} \beta E \frac{1}{\mathrm{ch}^2 Y} A + \frac{\beta}{2} E \mathrm{th}^2 Y B,$$
$$B \stackrel{c}{=} \beta E \frac{1}{\mathrm{ch}^2 Y} B.$$

The result follows. □

As explained at the end of the previous proof, Proposition 5.9 allows us to dispose of the annoying factors $\mathrm{th}^n Y$ that arise through the use of Theorem 2.5 and that were the last obstacle.

PROPOSITION 5.10. *Under $H(\tau-1)$ if f is an expression of order $\tau-1$, then* $\lim N^{\tau/2} E\langle (1/N)(\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 - E\langle \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 \rangle) f\rangle$ *exists.*

PROOF. Left to the reader.

Now, we can perform the last step in the proof of $H(\tau)$, and hence of Theorem 5.1.

PROPOSITION 5.11. *Under $H(\tau-1)$, if f is an expression of order $\tau-1$, then* $\lim_{N\to\infty} N^{\tau/2} E\langle m_k^1 f\rangle$ *exists for* $k \geq 2$.

PROOF. The remarkable fact is that, despite all our machinery, this is not totally obvious and the same arguments have to be used once more. As a first step, we will prove that

$$\lim_{N\to\infty} N^{\tau/2} E\langle \tilde{m}_k f\rangle \tag{5.58}$$

exists. We compute

$$\begin{aligned} E\langle \tilde{m}'_k f'\rangle' &= E\eta_k \langle \tilde{\sigma} f\rangle' \\ &\stackrel{c}{=} \beta \sum_{l\leq p} E\langle \tilde{\sigma}\sigma_l m_k^l f\rangle - p\beta E\langle \tilde{\sigma}\sigma_{p+1} m_k^{p+1} f\rangle'. \end{aligned}$$

Only the terms for $l = 1, 2$ contribute, because in the others there is no contribution of the terms of type I of Theorem 2.5. These two terms regroup as

$$\frac{\beta}{2} E\langle (\tilde{\sigma})^2 \tilde{m}_k f\rangle \stackrel{c}{=} \beta E \frac{1}{\mathrm{ch}^2 Y} \langle \tilde{m}_k f\rangle$$

so that

$$E\langle \tilde{m} f\rangle \stackrel{c}{=} \beta E \frac{1}{\mathrm{ch}^2 Y} E\langle \tilde{m} f\rangle$$

and this proves (5.58). To prove the proposition, we find as usual

$$\begin{aligned} E\langle m_k^1 f\rangle &\overset{c}{=} \beta \sum_{l\le p} E\langle \sigma_1\sigma_l m_k^l f\rangle' - \beta p E\langle \sigma_1\sigma_{p+1} m_k^{p+1} f\rangle' \\ &\overset{c}{=} \beta E\langle m_k^1 f\rangle + \sum_{1\le l\le p} E\operatorname{th}^2 Y\langle m_k^l f\rangle - \beta p E\operatorname{th}^2 Y\langle m_k^{p+1} f\rangle \\ &\overset{c}{=} \beta E\frac{1}{\operatorname{ch}^2 Y} E\langle m_k^1 f\rangle \end{aligned}$$

because (5.58) implies that

$$E\operatorname{th}^2 Y\langle m_k^l f\rangle \overset{c}{=} E\operatorname{th}^2 Y\langle m_k^{l'} f\rangle$$

for any l, l'. □

6. A priori estimates for the Gibbs measure. In this section we prove (2.13) and (2.14). The proof of (2.13) will start with an estimate of the same nature as (2.14), that is, an estimate on how close $m_1(\boldsymbol{\varepsilon})$ is to m^*. A second separate argument will then deduce (2.13) from this result. It is probably possible to deduce (2.14) from the result of [3]. We feel, however, that seeing first a direct proof of (2.14) would help the reader to understand the much more delicate issues of Corollary 6.13, so we will provide such a proof.

Both (2.13) and (2.14) deal with the image G' of the Gibbs measure under the map $\boldsymbol{\varepsilon} \to (m_k(\boldsymbol{\varepsilon}))_{1\le k\le M}$. Rather than G', it is much easier to use its convolution $\bar{G} = G' * \gamma$ with the Gaussian measure γ on $\mathbb{R}^M$ of density $W\exp(-\beta N \times \|\mathbf{w}\|^2)$, where W is the normalizing coefficient $(\beta N/2\pi)^{M/2}$. This trick, called the Hubbard–Stratonovich transformation, is very useful (and unfortunately very specific to the form of the Hamiltonian). The reason for the success of this method is that $\bar{G}$ has a simple density with respect to Lebesgue's measure, namely,

$$Z^{-1}\exp\psi(\mathbf{z}), \tag{6.1}$$

where

$$\psi(\mathbf{z}) = -\frac{\beta N}{2}\|\mathbf{z}\|^2 + \sum_{i\le N}\log\operatorname{ch}\beta(h + \boldsymbol{\eta}_i\cdot\mathbf{z}), \tag{6.2}$$

where $\boldsymbol{\eta}_i = (\boldsymbol{\eta}_{i,k})_{k\le M}$, the dot product is in $\mathbb{R}^M$ and

$$Z = \int \exp\psi(\mathbf{z})\,d\mathbf{z} \tag{6.3}$$

is the normalization factor. It is well known that γ is sharply concentrated close to the sphere of radius $\sqrt{\alpha/\beta}$ centered at the origin. Thus, to prove that G' is sharply concentrated on a ball of radius R at least as large as $\sqrt{\alpha/\beta}$, a natural method is to prove that $\bar{G} = G' * \gamma$ is concentrated on such a ball; but it seems difficult to obtain results at a finer scale, because convolution with γ "blurs" this information. [In particular, it seems difficult to prove (2.13) by studying $\bar{G}$ only, and this is why we use a separate argument.] The situation is more favorable when we are interested only on the projection G_1 of G on the first copy of $\mathbb{R}$ [i.e., the image of G by the map $\boldsymbol{\varepsilon} \to m_1(\boldsymbol{\varepsilon})$] because the projection $\bar{G}_1$ of $\bar{G}$ on this copy is the convolution of G_1 with a Gaussian measure of variance $1/\beta N$. Thus, to show that G_1 is nearly supported by an interval (the length of which is independent of N), it suffices to show the same for $\bar{G}_1$. Thus, to prove (2.14), it suffices to prove the following result.

PROPOSITION 6.1. *If $\beta \leq 2$, we have*

$$E\bar{G}(\{\mathbf{z}; |z_1 - m^*| \geq \rho_0\}) \leq \exp\left(-\frac{N}{K}\right).$$

Before the proof starts, we collect the probabilistic estimates on which it relies. Throughout the section, we say that an event occurs *with large probability* if its probability is greater than or equal to $1 - \exp(-N/K)$.

LEMMA 6.2. *With large probability the following occur:*

$$\left\| \sum_{i \leq N} \boldsymbol{\eta}_i \right\| \leq LN\sqrt{\alpha}, \tag{6.4}$$

$$\forall \mathbf{w} \in \mathbb{R}^M, \qquad \sum_{i \leq N} (\boldsymbol{\eta}_i \cdot \mathbf{w})^2 \leq LN\|\mathbf{w}\|^2, \tag{6.5}$$

$$\begin{aligned} \forall \mathbf{w} \in \mathbb{R}^M, \qquad \|\mathbf{w}\| \leq a \Rightarrow \sum_{i \leq N} (\boldsymbol{\eta}_i \cdot \mathbf{w})^2 \min(1, |\boldsymbol{\eta}_i \cdot \mathbf{w}|) \\ \leq LN(a^3 + \sqrt{\alpha}a^2). \end{aligned} \tag{6.6}$$

PROOF. Equation (6.4) is obvious. The proof of (6.5) is to be found in the appendix of [T2] and is much easier than the proof of (6.6) that we give now. We replace the function $x^2 \min(1, |x|)$ by $\varphi(x) = \int_0^{|x|} t \min(1, t)\, dt$ that is convex and such that $x^2 \min(1, |x|) \leq L\varphi(x)$, $\varphi(x) \leq Lx^2/2$. It is elementary (and very useful) that there is a subset U of $\mathbb{R}^M$ such that each $\mathbf{u}$ in U satisfies $\|\mathbf{u}\| \leq 2a$, that $\operatorname{card} U \leq 5^M$ and that each $\mathbf{w}$ with $\|\mathbf{w}\| \leq a$ is in the convex hull of U. Thus,

$$\sup_{\|\mathbf{w}\| \leq a} \sum_{i \leq N} \varphi(\boldsymbol{\eta}_i \cdot \mathbf{w}) \leq \sup_{\mathbf{u} \in U} \sum_{i \leq N} \varphi(\boldsymbol{\eta}_i \cdot \mathbf{u}). \tag{6.7}$$

The proof of (6.6) will follow from the following elementary version of Bernstein's inequality: if an r.v. X satisfies $E \exp|X| \leq 2$ and $(X_i)_{i \leq N}$ are i.i.d. copies of X, then, for each $t > 0$,

$$P\left(\sum_{i \leq N} X_i \geq NEX_i + t\right) \leq \exp\left(-\min\left(\frac{t^2}{NL}, \frac{t}{L}\right)\right).$$

We use this for $t = L\sqrt{\alpha}N$, $X_i = (1/8a^2)\varphi(\boldsymbol{\eta}_i \cdot \mathbf{u})$ so that

$$EX_i \le \frac{1}{8a^2} E|\boldsymbol{\eta}_i \cdot \mathbf{u}|^3 \le La.$$

The condition $E\exp|X_i| \le 2$ follows from the well-known fact that

$$E\exp\frac{1}{4}(\boldsymbol{\eta}_i \cdot \mathbf{x})^2 \le 2$$

if $\|\mathbf{x}\| \le 1$. (See Lemma 6.6.) This proves that, given a, (6.6) occurs with large probability, and it remains to show that, with large probability, it holds for each a. If $a \ge 1$, (6.6) follows from (6.5), and for $a \le \sqrt{\alpha}$, (6.6) is, by homogeneity, a weaker statement than for $a = \sqrt{\alpha}$. Thus, (6.6) holds for each $a > 0$ provided it holds for $a = 2^{-k}$, $\sqrt{\alpha} \le 2^{-k} \le 1$. □

PROOF OF PROPOSITION 6.1. We will use in an essential way that we already know that

$$E\bar{G}(\{\mathbf{z}; \|\mathbf{z} - m^*\mathbf{e}_1\| \ge \rho_1\}) \le \exp\left(-\frac{N}{K}\right), \tag{6.8}$$

where $\mathbf{e}_1$ is the vector $(1, 0, \dots)$ of $\mathbb{R}^M$. Given $\mathbf{w}$ in $\mathbb{R}^M$ with $w_1 = 0$ and t in $\mathbb{R}$, we consider the function

$$\psi_{\mathbf{w}}(t) = \psi(\mathbf{w} + (t + m^*)\mathbf{e}_1) \tag{6.9}$$

and the probability measure $\mu_{\mathbf{w}}$ on $[-\rho_1, \rho_1]$ of density

$$\frac{1}{Z_{\mathbf{w}}} 1_{\{|t| \le \rho_1\}} \exp\psi_{\mathbf{w}}(t) \tag{6.10}$$

with respect to Lebesgue's measure, where

$$Z_{\mathbf{w}} = \int_{-\rho_1}^{\rho_1} \exp\psi_{\mathbf{w}}(t)\, dt \tag{6.11}$$

is the normalizing factor. The reason that we consider only $|t| \le \rho_1$ is that by (6.8) the other values of t do not matter. We will show that, with large probability, we have

$$\|\mathbf{w}\| \le \rho_1 \Rightarrow \mu_{\mathbf{w}}([-\rho_0, \rho_0]) \ge 1 - \exp\left(-\frac{N}{K}\right) \tag{6.12}$$

and this will conclude the proof because the projection of $\bar{G}$ on the first copy of $\mathbb{R}$ is [modulo an exponentially small error due to (6.8)] a mixture of probabilities of the type $\mu_{\mathbf{w}}$.

To prove (6.12), we will simply show that $\psi_{\mathbf{w}}$ is sufficiently concave and has its maximum in $[-\rho_0/2, \rho_0/2]$. We have

$$\psi'_{\mathbf{w}}(t) = -\beta N\left(t + m^* - \frac{1}{N}\sum_{i \le N} \operatorname{th}\beta(h + m^* + t + \boldsymbol{\eta}_i \cdot \mathbf{w})\right), \tag{6.13}$$

$$\psi''_{\mathbf{w}}(t) = -\beta N\left(1 - \frac{\beta}{N}\sum_{i \le N} \frac{1}{\operatorname{ch}^2\beta(h + m^* + t + \boldsymbol{\eta}_i \cdot \mathbf{w})}\right). \tag{6.14}$$

We write, for $\beta \le 2$,

$$|\varphi(x) - \varphi(0) - x\varphi'(0)| \le Lx^2 \tag{6.15}$$

for

$$\varphi(x) = \frac{1}{\operatorname{ch}^2 \beta(h + m^* + x)}$$

and $x = t + \boldsymbol{\eta}_i \cdot \mathbf{w}$. By summation, we get

$$\begin{aligned} &\left| \frac{1}{N} \sum_{i \le N} \frac{1}{\operatorname{ch}^2 \beta(h + m^* + t + \boldsymbol{\eta}_i \cdot \mathbf{w})} - \frac{1}{\operatorname{ch}^2 \beta(h + m^*)} \right| \\ &\qquad \le |\varphi'(0)| \left| t + \frac{1}{N} \right| \left(\sum_{i \le N} \boldsymbol{\eta}_i \right) \cdot \mathbf{w}| + L\left(t^2 + \frac{1}{N} \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \mathbf{w})^2 \right). \end{aligned}$$

We use (6.4), (6.5) and $|t|$, $\|\mathbf{w}\| \le \rho_1$ to see that this is at most

$$L|\varphi'(0)| \left(\frac{\sqrt{\alpha}}{\sqrt{\beta - 1}} + \frac{\alpha}{\sqrt{\beta - 1}} \right) + L \frac{\alpha}{\beta - 1}.$$

Now

$$|\varphi'(0)| \le 2\operatorname{th} \beta(h + m^*) = 2m^* \le L\sqrt{\beta - 1}$$

since we assume h very small; so the above is at most

$$\sqrt{\alpha} + L \frac{\alpha}{\beta - 1}.$$

Since

$$\beta \frac{1}{\operatorname{ch}^2 \beta(h + m^*)} = \beta(1 - m^{*2}) \le 1 - \frac{\beta - 1}{L},$$

we have proved that for α, β in the admissible region we have

$$|t| \le \rho_1 \Rightarrow \psi''_{\mathbf{w}}(t) \le -N(\beta - 1)/L. \tag{6.16}$$

It is more delicate to study (6.13). Considering now

$$\varphi(x) = \operatorname{th} \beta(h + m^* + x),$$

we write

$$\left| \varphi(x) - \varphi(0) - x\varphi'(0) - \frac{x^2}{2} \varphi''(0) \right| \le Lx^2 \min(1, |x|).$$

We use this for $x = \boldsymbol{\eta}_i \cdot \mathbf{w}$, and using (6.4) to (6.6) we get by summation

$$\begin{aligned} &\left| \frac{1}{N} \sum_{i \le N} \operatorname{th} \beta(h + m^* + \boldsymbol{\eta}_i \cdot \mathbf{w}) - \operatorname{th} \beta(m^* + h) \right| \\ &\qquad \le L\sqrt{\alpha} \|\mathbf{w}\| + L\|\mathbf{w}\|^2 |\varphi''(0)| + L(\sqrt{\alpha} \|\mathbf{w}\|^2 + \|\mathbf{w}\|^3). \end{aligned}$$

Now $|\varphi''(0)| \le 2\operatorname{th}\beta(h+m^*) = 2m^* \le L\sqrt{\beta-1}$, and since $\|\mathbf{w}\| \le \rho_1$ this gives a bound

$$L\frac{\alpha}{\sqrt{\beta-1}} + L\frac{\alpha}{\sqrt{\beta-1}} + L\frac{\alpha^{3/2}}{\beta-1} + L\frac{\alpha^{3/2}}{(\beta-1)^{3/2}} \le L\frac{\alpha}{\sqrt{\beta-1}}$$

for $\alpha \le (\beta-1)^2$. Thus, we have shown that

$$|\psi'_{\mathbf{w}}(0)| \le LN\frac{\alpha}{\sqrt{\beta-1}}. \tag{6.17}$$

Comparing with (6.16), we see that the proof is complete: $\psi_{\mathbf{w}}$ has its maximum in $[-L\alpha(\beta-1)^{-3/2}, L\alpha(\beta-1)^{-3/2}]$. □

The previous approach breaks down for large β because for certain values of $\mathbf{w}$ there are too many indexes i for which $h+m^*+\boldsymbol{\eta}_i\cdot\mathbf{w} \simeq 0$. The corresponding terms in (6.14) are then large, and it does not seem possible to bound $\psi''_{\mathbf{w}}(t)$ uniformly from above. The method of proof consists quite naturally of showing that these values of $\mathbf{w}$ are exceptional and irrelevant. This, however, requires a much more detailed analysis, toward which we turn now. Using (6.2), we write

$$\begin{aligned}\psi(m^*\mathbf{e}_1+\mathbf{v}) = &-\frac{\beta N}{2}m^{*2} - \frac{\beta N\|\mathbf{v}\|^2}{2} - \beta N m^*\mathbf{v}\cdot\mathbf{e}_1\\ &+\sum_{i\le N}\log\operatorname{ch}\beta(m^*+h+\boldsymbol{\eta}_i\cdot\mathbf{v})\end{aligned} \tag{6.18}$$

(we do *not* assume that $v_1 = \mathbf{v}\cdot\mathbf{e}_1 = 0$).

From now on, we assume (without loss of generality since in Proposition 6.1 the condition $\beta \le 2$ can be replaced by $\beta \le L$) that β is large enough that $m^* \ge 3/4$, and we consider the function

$$\xi(x) = \frac{\beta^2x^2}{\operatorname{ch}^2(\beta/4)} + 2\beta\left(|x|-\frac{1}{2}\right)^+. \tag{6.19}$$

We set

$$\psi_0(m^*\mathbf{e}_1+\mathbf{v}) = Nb^* - \frac{\beta N}{2}\|\mathbf{v}\|^2 + \beta m^*\left(\sum_{i\le N}\boldsymbol{\eta}'_i\right)\cdot\mathbf{v}, \tag{6.20}$$

where $\boldsymbol{\eta}'_i = \boldsymbol{\eta}_i - \mathbf{e}_1$ and where

$$b^* = \log\operatorname{ch}\beta(m^*+h) - \frac{\beta m^{*2}}{2}.$$

LEMMA 6.3. *We have*

$$\psi_0(m^*\mathbf{e}_1+\mathbf{v}) \le \psi(m^*\mathbf{e}_1+\mathbf{v}) \le \psi_0(m^*\mathbf{e}_1+\mathbf{v}) + \sum_{i\le N}\xi(\boldsymbol{\eta}_i\cdot\mathbf{v}). \tag{6.21}$$

PROOF. The function

$$\xi_1(x) = \log \operatorname{ch} \beta(m^* + h + x) - \log \operatorname{ch} \beta(m^* + h) - \beta x m^*$$

is convex positive [since $m^* = \operatorname{th} \beta(m^* + h)$ and hence $\xi'(0) = 0$]. For $|x| \le 1/2$, we have $m^* + x + h \ge 1/4$, so that

$$\xi''(x) = \frac{\beta^2}{\operatorname{ch}^2 \beta(m^* + h + x)} \le \frac{\beta^2}{\operatorname{ch}^2(\beta/4)}. \tag{6.22}$$

On the other hand, for all x,

$$\xi'(x) = \beta \operatorname{th} \beta(m^* + h + x) - \beta m^*$$

so that

$$|\xi'(x)| \le 2\beta. \tag{6.23}$$

It is elementary to deduce from (6.22), (6.23) that $\xi_1(x) \le \xi(x)$, so that

$$\begin{aligned} \log \beta(m^* + h) + \beta x m^* &\le \log \operatorname{ch} \beta(m^* + h + x) \\ &\le \log \beta(m^* + h) + \beta x m^* + \xi(x). \end{aligned}$$

Using this for $x = \boldsymbol{\eta}_i \cdot \mathbf{v}$ and summation yield the result. □

We now consider the vector $\boldsymbol{\theta} = (1/N) m^* \sum_{i \le N} \boldsymbol{\eta}'_i$, whose importance is revealed by (6.20). Thus,

$$\psi_0(m^* \mathbf{e}_1 + \mathbf{v}) = N b^* - \frac{\beta N}{2} \|\mathbf{v} - \boldsymbol{\theta}\|^2 + \frac{\beta N}{2} \|\boldsymbol{\theta}\|^2. \tag{6.24}$$

LEMMA 6.4. *For a subset A of $\mathbb{R}^M$, we have*

$$\bar{G}(A + m^* \mathbf{e}_1) \le W \int_A \exp(-\beta \|\mathbf{z} - \boldsymbol{\theta}\|^2 + \sum_{i \le N} \xi(\boldsymbol{\eta}_i \cdot \mathbf{z}))\, d\mathbf{z}. \tag{6.25}$$

PROOF. By the lower bound of Lemma 6.3, we have

$$\begin{aligned} Z = \int \psi(\mathbf{z})\, d\mathbf{z} &\ge \exp N b^* \int \exp\biggl(-\frac{\beta N}{2} \|\mathbf{v}\|^2 + \beta N \boldsymbol{\theta} \cdot \mathbf{v}\biggr) d\mathbf{v} \\ &= W^{-1} \exp\biggl(N b^* + \frac{\beta N}{2} \|\boldsymbol{\theta}\|^2\biggr) \end{aligned}$$

and the result follows from (6.1), the upper bound of (6.21) and (6.24). □

It is natural in (6.25) to make the change of variable $\mathbf{z} = \boldsymbol{\theta} + \mathbf{w}$. Since, by convexity,

$$\xi(x + y) \le \tfrac{1}{2}(\xi(2x) + \xi(2y)),$$

we then have the following result.

LEMMA 6.5. *For a subset A of $\mathbb{R}^M$, we have*

$$\begin{aligned} &\bar{G}(A + m^*\mathbf{e}_1 + \boldsymbol{\theta}) \\ &\quad \le \exp\biggl(\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\boldsymbol{\theta})\biggr)\int_A \exp\biggl(\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\mathbf{w})\biggr)\,d\gamma(\mathbf{w}). \end{aligned} \tag{6.26}$$

What we would like now is to show that $\bar{G}$ is essentially concentrated on a ball of center $\boldsymbol{\theta}$ and radius $L\sqrt{\alpha/\beta}$ (the best accuracy we can hope for) and that, moreover, only the part of this ball where the exponent in the integral (6.26) is not too large matters. But, of course, before we can even start this program we need to be able to control this exponent. The part

$$\frac{\beta^2}{\operatorname{ch}^2\beta/4}x^2$$

of ξ is not dangerous because the coefficient of x^2 is very small for large β. We will use the following elementary lemma that reformulates (2.25).

LEMMA 6.6. *If $t\|\mathbf{w}\|^2 < 1/2$, we have*

$$E\exp t\sum_{i\le N}(\boldsymbol{\eta}_i\cdot\mathbf{w})^2 \le \biggl(\frac{1}{1-2t\|\mathbf{w}\|^2}\biggr)^{N/2}.$$

To handle the dangerous part of ξ, that is, $\beta(|x|-1/2)^+$, we will use the following result.

LEMMA 6.7. *If*

$$16t\|\mathbf{w}\|^2 \le 1, \tag{6.27}$$

then

$$E\exp t\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{w}| - \frac{1}{4}\biggr)^+ \le \exp\biggl(2N\exp\biggl(-\frac{1}{128\|\mathbf{w}\|^2}\biggr)\biggr). \tag{6.28}$$

PROOF. We recall the "sub-Gaussian inequality," valid for all u,

$$E\exp u\boldsymbol{\eta}_i\cdot\mathbf{w} \le \exp\frac{u^2}{2}\|\mathbf{w}\|^2 \tag{6.29}$$

so that

$$E\exp u|\boldsymbol{\eta}_i\cdot\mathbf{w}| \le 2\exp\frac{u^2}{2}\|\mathbf{w}\|^2 \tag{6.30}$$

and by Chebyshev's inequality,

$$P(|\boldsymbol{\eta}_i\cdot\mathbf{w}| \ge 1/4) \le 2\exp-\frac{1}{32\|\mathbf{w}\|^2}. \tag{6.31}$$

We write

$$\begin{aligned}
&E\exp t\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{w}|-\frac{1}{4}\biggr)^+\\
&\qquad\le 1+E(1_{\{|\boldsymbol{\eta}_i\cdot\mathbf{w}|\ge 1/4\}}\exp t|\boldsymbol{\eta}_i\cdot\mathbf{w}|)\\
&\qquad\le 1+\exp\biggl(-\frac{1}{64\|\mathbf{w}\|^2}\biggr)\exp 2t^2\|\mathbf{w}\|^2
\end{aligned}$$

using (6.30), (6.31) and the Cauchy–Schwarz inequality. The result follows. □

LEMMA 6.8. *Given $a>0$, with large probability, we have*

$$\sup_{\|\mathbf{w}\|\le a}\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{w}|-\frac{1}{4}\biggr)^+\le La^2\biggl(\alpha+\exp-\frac{1}{La^2}\biggr). \tag{6.32}$$

PROOF. As in Lemma 6.1, we use a set U of $\mathbb{R}^M$ of cardinality less than or equal to 5^M, consisting of vectors of length less than or equal to $2a$, and whose convex hull contains the ball of center 0 of radius a. Thus,

$$\sup_{\|\mathbf{w}\|\le a}\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{w}|-\frac{1}{4}\biggr)^+\le\sup_{\mathbf{u}\in U}\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{u}|-\frac{1}{4}\biggr)^+. \tag{6.33}$$

Now we use (6.28) with $t=1/64a^2$ to get

$$P\biggl(\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\mathbf{u}|-\frac{1}{4}\biggr)^+\ge y\biggr)\le\exp N\biggl(-\frac{y}{64a^2}+\exp-\frac{1}{La^2}\biggr).$$

The result follows easily. □

LEMMA 6.9. *With large probability, we have*

$$\sum_{i\le N}\biggl(|\boldsymbol{\eta}_i\cdot\boldsymbol{\theta}|-\frac{1}{4}\biggr)^+\le LN\exp-\frac{1}{L\alpha}. \tag{6.34}$$

PROOF. It is pretty obvious that

$$E\biggl(|\boldsymbol{\eta}_i\cdot\boldsymbol{\theta}|-\frac{1}{4}\biggr)^+\le L\exp-\frac{1}{L\alpha}.$$

The problem, however, is that, as i varies, the variables $|\boldsymbol{\eta}_i\cdot\boldsymbol{\theta}|$ are not independent. (The reader is advised to skip the rest of this purely technical argument.) All that we need is (e.g.) an inequality like

$$\exp\biggl(\frac{1}{16}\sum_{i\le N}\biggl(\biggl|\frac{1}{N}\boldsymbol{\eta}_i\cdot\sum_{j\ne i}\boldsymbol{\eta}_i\biggr|-\frac{1}{8}\biggr)^+\biggr)\le\exp\biggl(LN\exp-\frac{1}{L\alpha}\biggr). \tag{6.35}$$

To prove this, we will show that for each subset I of $\{1, \dots, N\}$,

$$E \exp \frac{1}{4} \sum_{i \in I} \left(\left| \frac{1}{N} \boldsymbol{\eta}_i \cdot \left(\sum_{j \notin I} \boldsymbol{\eta}_i \right) \right| - \frac{1}{16} \right)^+ \leq \exp\left(LN \exp - \frac{1}{L\alpha} \right). \tag{6.36}$$

This will imply (6.35) by averaging over I since

$$\operatorname*{Av}_I \sum_{i \in I} \left(\left| \frac{1}{N} \boldsymbol{\eta}_i \cdot \left(\sum_{j \notin I} \boldsymbol{\eta}_j \right) \right| - \frac{1}{16} \right)^+ \geq \frac{1}{2} \sum_{i \leq N} \left(\left| \frac{1}{2} \frac{1}{N} \boldsymbol{\eta}_i \cdot \sum_{j \neq i} \boldsymbol{\eta}_j \right| - \frac{1}{16} \right)^+.$$

To prove (6.36), we denote by E_0 integration in $(\boldsymbol{\eta}_i)_{i \in I}$ alone, so that, as in (6.28), denoting $\boldsymbol{\eta}_I = \sum_{i \notin I} \boldsymbol{\eta}_i$,

$$E_0 \exp \frac{1}{4} \sum_{i \in I} \left(\left| \frac{1}{N} \boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_I \right| - \frac{1}{16} \right)^+ \leq \exp\left(LN \exp - \frac{N^2}{L \|\boldsymbol{\eta}_I\|^2} \right), \tag{6.37}$$

provided $\|\boldsymbol{\eta}_I\|^2 \leq N^2/L$. Also,

$$E_0 \exp \frac{1}{4} \sum_{i \in I} \left| \frac{1}{N} \boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_I \right| \leq 2^N \exp L \|\boldsymbol{\eta}_I\|^2 / N \tag{6.38}$$

by (6.30). Using that, from Lemma 6.6,

$$E \exp \frac{1}{4M} \|\boldsymbol{\eta}_I\|^2 = \prod_{i \in I} \exp \frac{1}{4M} \left(\sum_{k \leq M} \eta_{i,k} \right)^2 \leq L^N,$$

we see that if $t \geq 1$, $P(\|\boldsymbol{\eta}_{\mathbf{I}}\|^2 \geq Lt\alpha N^2) \leq \exp(-N)$. The result follows easily from (6.67), (6.68). □

PROPOSITION 6.10. *If $\beta \geq \beta_0$, $\alpha \leq \alpha_0$, then*

$$E\bar{G}\left(\left\{ \mathbf{w}; \|\mathbf{w} - m^* \mathbf{e}_1 - \boldsymbol{\theta}\| \geq 2\sqrt{\frac{\alpha}{\beta}} \right\} \right) \leq \exp - \frac{N}{K}. \tag{6.39}$$

PROOF. Given $a > 0$, consider the set

$$C_a = \left\{ \mathbf{w} \in \mathbb{R}^M;\ \frac{a}{2} \leq \|\mathbf{w}\| \leq a \right\}.$$

Combining the estimates (6.4), (6.5), (6.32), (6.34), we have shown that, with large probability, we have

$$\begin{aligned} \forall \|\mathbf{w}\| \in C_a, \qquad & \frac{1}{2} \sum_{i \leq N} \big(\xi(2\boldsymbol{\eta}_i \cdot \boldsymbol{\theta}) + \xi(2\boldsymbol{\eta}_i \cdot \mathbf{w}) \big) \\ & \leq L\left((a^2 + \alpha) \frac{\beta^2}{\operatorname{ch}^2(\beta/4)} + \beta a^2 \left(\alpha + \exp - \frac{1}{La^2} \right) \right) =: \delta(a) \end{aligned}$$

so that, by (6.26),

$$\bar{G}(C_a + m^* \mathbf{e}_1 + \boldsymbol{\theta}) \leq \gamma(C_a) \exp N\delta(a).$$

Now we use the elementary fact that, for $a \geq 4\sqrt{\alpha/\beta}$, we have $\gamma(C_a) \leq \exp -N\beta a^2/L$. Thus, we have

$$\bar{G}(C_a + m^*\mathbf{e}_1 + \boldsymbol{\theta}) \leq \exp -N/K$$

as soon as a satisfies

$$a^2\left(\alpha + \exp -\frac{1}{La^2}\right) \leq \frac{a^2}{L}$$

and

$$(a^2 + \alpha)\frac{\beta^2}{\mathrm{ch}^2(\beta/4)} \leq \frac{a^2}{L}.$$

For $\alpha \leq \alpha_0$, $\beta \geq \beta_0$, this is satisfied for

$$a \leq a_0, \qquad a^2 \geq L\alpha\beta^2/\mathrm{ch}^2(\beta/4).$$

Since we must in any case have $a^2 \geq 16\alpha/\beta$, for β_0 large enough this is satisfied. Combining this with Proposition 9.2 of [7], we see that the case $a \geq a_0$ is irrelevant. This completes the proof. $\square$

What Proposition 6.10 shows is that in (6.26) we have to be concerned only with those $\mathbf{w}$ for which $\|\mathbf{w}\| \leq 2\sqrt{\alpha/\beta}$.

LEMMA 6.11. *If* $\|\mathbf{w}\| \leq 2\sqrt{\alpha/\beta}$ *we have*

$$E\exp\frac{1}{2}\sum_{i\leq N}\xi(2\boldsymbol{\eta}_i\cdot\mathbf{w}) \leq \exp\left(LN\alpha\exp\left(-\frac{\beta}{L}\right)\right). \tag{6.40}$$

PROOF. Using the Cauchy–Schwarz inequality, the left-hand side is bounded by

$$\left(E\exp\sum_{i\leq N}(\boldsymbol{\eta}_i\cdot\mathbf{w})^2\frac{8\beta^2}{\mathrm{ch}^2(\beta/4)}\right)^{1/2}\left(E\exp\sum_{i\leq N}2\beta\left(|2\boldsymbol{\eta}_i\cdot\mathbf{w}| - \frac{1}{2}\right)^+\right)^{1/2}.$$

We then use Lemmas 6.6 and 6.7 to get a bound

$$\exp\left(LN\left(\exp -\frac{\beta}{L\alpha} + \frac{\alpha\beta}{\mathrm{ch}^2(\beta/4)}\right)\right) \leq \exp\left(LN\alpha\exp\left(-\frac{\beta}{L}\right)\right)$$

for $\alpha \leq \alpha_0$, $\beta \geq \beta_0$. $\square$

PROPOSITION 6.12. *We have*

$$EG'\left(\left\{m_1 \leq m^* - L\left(\frac{1}{\beta}\exp -\frac{1}{L\alpha} + \frac{\alpha}{\beta}\exp -\frac{\beta}{L}\right)^{1/2}\right\}\right) \leq \exp\left(-\frac{N}{K}\right).$$

PROOF. Consider $t > 0$. Using (6.8), (6.26), we have with large probability that

$$\bar{G}(\{\mathbf{w}; w_1 \le m^* - t\}) \le \exp\left(-\frac{N}{K}\right) + \exp\left(\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\boldsymbol{\theta})\right)\int_{A_0}\exp\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\mathbf{w})\,d\gamma(\mathbf{w}),$$

where

$$A_0 = \{\mathbf{w}\in\mathbb{R}^M; \|\mathbf{w}\| \le 2\sqrt{\alpha/\beta},\ w_1 \le t\},$$

and where we have used the fact that the first component of $\boldsymbol{\theta}$ is 0.

By (6.34) with large probability we have

$$\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\boldsymbol{\theta}) \le LN\left(\exp-\frac{1}{L\alpha} + \alpha\frac{\beta^2}{\mathrm{ch}^2(\beta/4)}\right).$$

Using (6.40), we see that

$$\begin{aligned}E\int_{A_0}\exp\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\mathbf{w})\,d\gamma(\mathbf{w}) &\le \gamma(A_0)\exp 2N\left(\alpha\exp\left(-\frac{\beta}{L}\right)\right)\\ &\le \exp N\left(-\frac{\beta t^2}{2} + 2\alpha\exp\left(-\frac{\beta}{L}\right)\right)\end{aligned}$$

so that

$$P\left(\int_{A_0}\exp\frac{1}{2}\sum_{i\le N}\xi(2\boldsymbol{\eta}_i\cdot\mathbf{w})\,d\gamma(\mathbf{w}) \ge \exp N\left(-\frac{\beta t^2}{2} + 3\alpha\exp\left(-\frac{\beta}{L}\right)\right)\right) \le \exp\left(-\frac{N}{K}\right).$$

It follows that with probability greater than or equal to $1-\exp(-N/K)$ one has

$$\begin{aligned}&\bar{G}(\{\mathbf{w}; w_1 \le m^* - t\})\\ &\quad\le \exp\left(-\frac{N}{K}\right) + \exp N\left(-\frac{\beta t^2}{2} + L\left(\exp-\frac{1}{L\alpha} + \alpha\exp-\frac{\beta}{L}\right)\right).\end{aligned}$$

The result should now be obvious. □

COROLLARY 6.13. *If $\alpha \le 1/(L\log\beta)$, then (for β large enough)*

$$EG\left(\left\{\boldsymbol{\varepsilon}; m_1(\boldsymbol{\varepsilon}) \le m^* - \frac{1}{\beta^4}\right\}\right) \le \exp\left(-\frac{N}{K}\right).$$

COMMENT. There is nothing specific about the power β^{-4} that could be replaced by a larger power.

Corollary 6.13 completes the proof of (2.14).

We now turn to the proof of (2.13). We write, setting $\rho_i = \rho_i(\boldsymbol{\varepsilon}) = \epsilon_i - m_1(\boldsymbol{\varepsilon})$,

$$\begin{aligned} -H(\boldsymbol{\varepsilon}) &= \frac{N}{2}\sum_{k\le M} m_k^2(\boldsymbol{\varepsilon}) + hNm_1(\boldsymbol{\varepsilon}) \\ &= \sum_{1\le k\le M}\frac{1}{2N}\Big(\sum_{i\le N}\eta_{k,i}\Big)^2 m_1(\boldsymbol{\varepsilon})^2 \\ &\quad + \sum_{2\le k\le M}\frac{1}{N}\Big(\sum_{i\le N}\eta_{k,i}\rho_i\Big)\Big(\sum_{j\le N}\eta_{k,j}\Big)m_1(\boldsymbol{\varepsilon}) \\ &\quad + hNm_1(\boldsymbol{\varepsilon}) \\ &\quad + \frac{N}{2}\sum_{2\le k\le M}\Big(\frac{1}{N}\sum_{i\le N}\eta_{k,i}\rho_i\Big)^2. \end{aligned} \tag{6.41}$$

The basic idea is that, since

$$\sum_{i\le N}\rho_i^2 = N(1-m_1(\boldsymbol{\varepsilon})^2), \tag{6.42}$$

this sum is small for $m_1(\boldsymbol{\varepsilon})$ close to 1 (the only case we have to consider by Corollary 6.13). Thus, the last two terms of (6.48) can be seen as small perturbation terms. We will show that the only configurations that really contribute to the Gibbs measure are such that $\sum_{2\le k\le M}((1/N)\sum_{i\le N}\eta_{k,i}\rho_i)^2$ is small. If we denote by $\mathbf{c}$ the (random) point of $\mathbb{R}^M$ given by

$$\mathbf{c} = \Big(\frac{1}{N}\sum_{i\le N}\eta_{k,i}\Big)_{k\le M},$$

then

$$\|\mathbf{m}(\boldsymbol{\varepsilon}) - \mathbf{c}m_1(\boldsymbol{\varepsilon})\|^2 = \sum_{2\le k\le M}\Big(\frac{1}{N}\sum_{i\le N}\eta_{k,i}\rho_i\Big)^2 \tag{6.43}$$

so that

$$\|\mathbf{m}(\boldsymbol{\varepsilon}) - \mathbf{c}\|^2 \le 2|1-m_1(\boldsymbol{\varepsilon})|^2\|\mathbf{c}\|^2 + 2\sum_{2\le k\le M}\Big(\frac{1}{N}\sum_{i\le N}\eta_{k,i}\rho_i\Big)^2 \tag{6.44}$$

will be small and we will have proved (2.13).

LEMMA 6.14. *If $\boldsymbol{\varepsilon}$ is such that $L\beta^2 \sum_{i\le N} \rho_i^2 \le 1$, $u \ge L\alpha \sum_{i\le N} \rho_i^2$, then we have*

$$
\begin{aligned}
&E\exp\biggl[\beta \sum_{2\le k\le M} \frac{1}{N}\biggl(\sum_{i\le N} \eta_{k,\,i}\rho_i\biggr)\biggl(\sum_{j\le N} \eta_{k,\,j}\biggr) m_1(\boldsymbol{\varepsilon}) \\
&\qquad + \beta N \sum_{2\le k\le M} \biggl(\frac{1}{N}\sum_{i\le N} \eta_{k,\,i}\rho_i\biggr)^2\biggr] 1_{\{\sum_{2\le k\le M}((1/N)\sum_{i\le N}\eta_{k,\,i}\rho_i)^2 \ge u\}} \qquad (6.45)\\
&\qquad \le \exp\biggl(-\frac{Nu}{L\sum_{i\le N}\rho_i^2}\biggr).
\end{aligned}
$$

PROOF. The proof relies on Lemma 6.6. Using the inequality $ab \le a^2/12\beta + 6\beta b^2$ and Hölder's inequality, we bound the left-hand side of (6.45) by $U^{1/3} \times V^{1/3}\, W^{1/3}$, where

$$
\begin{aligned}
U &= E\exp \sum_{2\le k\le M} \frac{1}{4N}\biggl(\sum_{i\le N} \eta_{k,\,i}\biggr)^2,\\
V &= E\exp(3\beta + 18\beta^2) \sum_{2\le k\le M} \frac{1}{N}\biggl(\sum_{i\le N}\eta_{k,\,i}\rho_i\biggr)^2,\\
W &= E 1_{\{\sum_{2\le k\le M}((1/N)\sum_{i\le N}\eta_{k,\,i}\rho_i)^2\ge u\}}.
\end{aligned}
$$

It should be obvious that by Lemma 6.6 one has

$$
U \le 2^M, \qquad V \le \exp LM\beta^2\biggl(\sum_{i\le N}\rho_i^2\biggr).
$$

By Lemma 6.6 again,

$$
E\exp\frac{1}{4\sum_{i\le N}\rho_i^2}\sum_{2\le k\le M}\frac{1}{N}\biggl(\sum_{i\le N}\eta_{k,\,i}\rho_i\biggr)^2 \le 2^M
$$

and, thus, by Chebyshev's inequality, we have

$$
W \le 2^M \exp\biggl(-\frac{Nu}{4\sum_{i\le N}\rho_i^2}\biggr).
$$

The result follows. □

Let us now recall the Chernov large deviation function,

$$
\varphi(t) = \tfrac{1}{2}((1+t)\log(1+t) + (1-t)\log(1-t)),
$$

so that, as well known, if $tN \in \mathbb{N}$, we have

$$
\frac{2^N}{L\sqrt{N}}\exp(-N\varphi(t)) \le \operatorname{card}\biggl\{\boldsymbol{\varepsilon};\ \sum_{i\le N}\epsilon_i = tN\biggr\} \le 2^N\exp(-N\varphi(t)). \tag{6.46}
$$

COROLLARY 6.15. *With large probability, for each t with $Nt \in \mathbb{N}$, $L\beta^2 \times (1-t^2) \le 1$, we have*

$$\sum \exp -\beta H(\boldsymbol{\varepsilon}) \le 2^N \exp\left(-N\varphi(t) - \frac{N\alpha}{L} + \frac{t^2\beta S}{2} + \beta h t N\right),$$

where the summation is taken over all the configurations $\boldsymbol{\varepsilon}$ with

$$m_1(\boldsymbol{\varepsilon}) = t, \qquad \sum_{2\le k\le M}\left(\frac{1}{N}\sum_{i\le N}\eta_{k,i}\rho_i\right)^2 \ge L\alpha(1-t^2)$$

and where

$$S = \frac{1}{N}\sum_{k\le M}\left(\sum_{i\le N}\eta_{k,i}\right)^2. \tag{6.47}$$

PROOF. This should be obvious from (6.41) and Lemma 6.14, taking $u = L\alpha(1-m_1(\boldsymbol{\varepsilon})^2) = L\alpha(1-t^2)$ and using (6.42). □

LEMMA 6.16. *If S is as in* (6.47), *we have*

$$\sum_{\boldsymbol{\varepsilon}} \exp -\beta H(\boldsymbol{\varepsilon}) \ge 2^N \sup_{0\le t\le 1} \frac{1}{L\sqrt{N}} \exp\left(-N\varphi(t) + \frac{t^2\beta S}{2} + \beta h t N\right).$$

PROOF. This follows from Jensen's inequality, since the average of $-\beta H(\boldsymbol{\varepsilon})$ over $m_1(\boldsymbol{\varepsilon}) = t$ fixed is at least $\beta t^2 S/2 + \beta h t N$ and using (6.46). □

The following proves (2.13).

PROPOSITION 6.17. *If $\alpha \le 1/(L\log\beta)$, then for β large enough we have*

$$EG\left(\left\{\boldsymbol{\varepsilon}; \sum_{k\le M}\left(m_k(\boldsymbol{\varepsilon}) - \left(\frac{1}{N}\sum_{i\le N}\eta_{i,k}\right)\right)^2 \ge \frac{L}{\beta^4}\right\}\right) \le \exp\left(-\frac{N}{K}\right).$$

PROOF. Combine Corollary 6.13, Corollary 6.15, Lemma 6.16 and (6.44). □

REFERENCES

[1] AMIT, D. J., GUTFREUND, H. and SOMPOLINSKY, H. (1987). Statistical mechanics of neural networks near saturation. *Ann. Phys.* **173** 30–67.

[2] BOVIER, A. and GAYRARD, V. (1997). The retrival phase of the Hopfield model. *Probab. Theory Related Fields* **107** 61–98.

[3] BOVIER, A. and GAYRARD, V. (1997). Hopfield models as generalized random mean field models. *Progr. Probab.* **41** 3–89.

[4] BOVIER, A. and GAYRARD, V. (1997). An almost sure central limit theorem for the Hopfield model. *Markov Proc. Related Fields* **3** 151–173.

[5] GUERRA, F. (1996). About the overlap distribution in mean field spin glass models. *Internat. J. Modern Phys. B* 1675–1684.

[6] TALAGRAND, M. (1998). The Sherrington–Kirkpatrick model: a challenge to mathematicians. *Probab. Theory Related Fields* **110** 109–176.

[7] TALAGRAND, M. (1998). Rigorous results for the Hopfield model with many patterns. *Probab. Theory Related Fields* **110** 177–276.
[8] TALAGRAND, M. (2000). The high temperature case of the K-sat problem. *Probab. Theory Related Fields*. To appear.
[9] TALAGRAND, M. (2000). Intersecting random half-spaces: toward the Gardner–Derrida formula. *Ann. Probab.* **28** 725–758.
[10] TALAGRAND, M. (2000). Replica symmetry breaking and exponential inequalities for the Sherrington–Kirkpatrick model. *Ann. Probab.* **28** 1018–1062.
[11] TALAGRAND, M. (1998). Huge random structures and mean field models for spin glasses. *Proc. Internat. Congress of Mathematicians. Documenta Math.* Extra Volume I. (electronic)

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