The Annals of Probability
2000, Vol. 28, No. 2, 909–945

STRONG APPROXIMATION OF QUANTILE PROCESSES BY ITERATED KIEFER PROCESSES

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The notion of a kth iterated Kiefer process $\mathscr{K}(\nu, t; k)$ for $k \in \mathbb{N}$ and $\nu, t \in \mathbb{R}$ is introduced. We show that the uniform quantile process $\beta_n(t)$ may be approximated on [0, 1] by $n^{-1/2}\mathscr{K}(n, t; k)$, at an optimal uniform almost sure rate of $O(n^{-1/2+1/2^{k+1}+o(1)})$ for each $k \in \mathbb{N}$. Our arguments are based in part on a new functional limit law, of independent interest, for the increments of the empirical process. Applications include an extended version of the uniform Bahadur–Kiefer representation, together with strong limit theorems for nonparametric functional estimators.

1. Introduction and statement of main results. The invariance principle approach to asymptotic statistics, initiated by Doob (1949) and Donsker (1952), makes use of Brownian bridge approximations to the sample-based empirical and quantile processes. The original weak laws were followed by a series of *strong invariance principles*, where the original and approximating processes are defined on the same probability space [refer to Billingsley (1968), Csörgő and Révész (1981), Csörgő (1983), Shorack and Wellner (1986), Csörgő and Horváth (1993)]. The main stream of investigations has been concerned with Gaussian approximants, in which case, the results presently available come close to the best achievable rates of convergence.

This paper is motivated by the fact that the optimal rates of Gaussian approximation for quantile processes are, at times, not sharp enough to allow direct applications [see Deheuvels (1997, 1998) and Section 5]. Our purpose is to show that a choice of approximants within the general class of *iterated Gaussian processes* allows us to overcome this difficulty.

There has been much recent interest for kth iterated Gaussian process $Z_1 \circ \cdots \circ Z_k$, where $Z_1, \dots, Z_k$ are Gaussian processes, and, in particular, for *iterated Brownian motions* $W_1 \circ W_2$, where W_1 and W_2 are Wiener processes [see Burdzy (1993), Csáki, Csörgő, Földes and Révész (1989, 1995), Csáki, Földes and Révész (1997), Deheuvels and Mason (1992a), Hu, Pierre-Loti-Viaud and Shi (1995), Khoshnevisan and Lewis (1996) and the references therein]. Here, we set $\{f \circ g\}(t) = f(g(t))$. Below, we introduce a new family of iterated Gaussian processes allowing to derive the appropriate invariance principles. First, we introduce some definitions and notation.

Received October 1998; revised April 1999.

AMS 1991 *subject classifications*. 60F05, 60F15, 60G15, 62G30.

Key words and phrases. Empirical processes, quantile processes, order statistics, law of the iterated logarithm, almost sure convergence, strong laws, strong invariance principles, strong approximation, Kiefer processes, Wiener process, iterated Wiener process, iterated Gaussian processes, Bahadur–Kiefer-type theorems.

Let $\alpha_n(t)$ denote a *uniform empirical process* [see Section 3.1 in Shorack and Wellner (1986) and (2.1) later]. A *Kiefer process* $\mathscr{K}(n,t)$ is a Gaussian process such that [see, e.g., Kiefer (1972), Section 4.2 in Csörgő and Révész (1981), and (1.8) below], for all $m, n \in \mathbb{N}$ and $s, t \in [0, 1]$,

$$\text{(1.1)} \qquad \begin{aligned} \mathbb{E}(\mathscr{K}(n,t)) &= \mathbb{E}\big(n^{1/2}\alpha_n(t)\big) = 0,\\ \mathbb{E}(\mathscr{K}(m,s)\mathscr{K}(n,t)) &= \mathbb{E}\big(m^{1/2}\alpha_m(s)n^{1/2}\alpha_n(t)\big) = (m\wedge n)(s\wedge t - st).\end{aligned}$$

The equalities (1.1) between means and covariances of $n^{1/2}\alpha_n(t)$ and $\mathscr{K}(n,t)$ show that the Kiefer process $\mathscr{K}(n,t)$ is the most natural Gaussian approximant to $n^{1/2}\alpha_n(t)$, considered as a function of (n,t). Kiefer (1972) was first to give an explicit construction of $\alpha_n(t)$ and a Kiefer process $\mathscr{K}_0(n,t)$ on the same probability space, with

$$\text{(1.2)} \qquad \big\|\alpha_n - n^{-1/2}\mathscr{K}_0(n,\mathbf{I})\big\| = O\big(n^{-1/6}(\log n)^{2/3}\big) \to 0 \quad \text{a.s. as } n\to\infty,$$

where we set $\|f\| = \sup_{0\le t\le 1}|f(t)|$, and denote identity by $\mathbf{I}(t) = t$. The best known refinement of (1.2) is due to Komlós, Major and Tusnády (1975) who constructed a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ carrying α_n and a Kiefer process $K_1(n,\mathbf{I})$ with

$$\text{(1.3)} \qquad \big\|\alpha_n - n^{-1/2}K_1(n,\mathbf{I})\big\| = O\big(n^{-1/2}\log^2 n\big) \quad \text{a.s. as } n\to\infty.$$

There is no alternative construction of $K_1(n,\mathbf{I})$ reducing the rate $O(n^{-1/2}\cdot \log^2 n)$ in (1.3) to $o(n^{-1/2}\log n)$ [see Section 4.4 in Csörgő and Révész (1981)], and the optimal a.s. uniform approximation rate of α_n by $n^{-1/2}\mathscr{K}(n,\mathbf{I})$ is (leaving out small order terms) $O(n^{-1/2+o(1)})$.

For $n \ge 0$ and $t \in [0,1]$, denote by $\beta_n(t)$ a *uniform quantile process* [see Section 4.5 in Csörgő and Révész (1981) and (2.1)]. The uniform Bahadur–Kiefer representation [Bahadur (1966), Kiefer (1970) and Deheuvels and Mason (1990)] asserts that

$$\text{(1.4)} \qquad \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\|\alpha_n + \beta_n\| = 2^{-1/4} \quad \text{a.s.,}$$

where we set $\log_2 u = \log_+\log_+ u$ with $\log_+ u = \log(u\vee e)$. By combining (1.3) with (1.4), Csörgő and Révész (1975) showed that the Kiefer process $K_2(n,\mathbf{I}) = -K_1(n,\mathbf{I})$ fulfills

$$\text{(1.5)} \qquad \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\big\|\beta_n - n^{-1/2}K_2(n,\mathbf{I})\big\| = 2^{-1/4} \quad \text{a.s.}$$

The optimality of the rate in (1.5) was discussed by Deheuvels (1997, 1998), who showed that, for any probability space carrying β_n and a Kiefer process $\mathscr{K}(n,\mathbf{I})$,

$$\text{(1.6)} \qquad \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\big\|\beta_n - n^{-1/2}\mathscr{K}(n,\mathbf{I})\big\| > 0 \quad \text{a.s.}$$

As follows from (1.5) and (1.6), the optimal a.s. uniform rate of approximation of β_n by a *normalized Kiefer process* $n^{-1/2}\mathscr{K}(n,\mathbf{I})$ is (leaving out small order terms) $O(n^{-1/4+o(1)})$.

Komlós, Major and Tusnády (1975) and Csörgő and Révész (1978) showed that a replacement of $n^{-1/2}K_r(n,t)$, $r = 1, 2$, by judiciously chosen Brownian bridges allows improving upon (1.3)–(1.5). They proved the existence, on suitable probability spaces, of α_n, β_n, and of sequences of Brownian bridges $\{B'_n\colon n \geq 1\}$ and $\{B''_n\colon n \geq 1\}$, such that

$$\|\alpha_n - B'_n\| = O(n^{-1/2}\log n) \quad \text{and} \quad \|\beta_n - B''_n\| = O(n^{-1/2}\log n) \quad \text{a.s} \quad \text{as } n \to \infty. \tag{1.7}$$

The rates in (1.7) are sharp in that one may not replace $O(n^{-1/2}\log n)$ by $o(n^{-1/2}\log n)$ [see Section 4.4 in Csörgő and Révész (1981)]. However, the dependence with respect to n of B'_n and B''_n in (1.7) being implicit, this approach is not appropriate for the description of the a.s. limiting behavior of most statistics of interest based upon α_n and β_n.

The Kiefer process approximations (1.3) and (1.5) provide the essential tools for deriving such strong laws [see, e.g., Csörgő (1983)]. Unfortunately, the gap between the optimal approximation rates $O(n^{-1/2+o(1)})$ and $O(n^{-1/4+o(1)})$, achieved for α_n and β_n, respectively, does not allow handling a series of interesting statistics based upon β_n. This motivates the need to approximate β_n at uniform almost sure rates in between $O(n^{-1/2+o(1)})$ and $O(n^{-1/4+o(1)})$. Since this may not be achieved by Gaussian processes, new approximants are needed, which should be both explicitly dependent upon (n,t), and closely related to Kiefer processes. This leads us to introduce below the class of *iterated Kiefer processes*.

Given a pair $\{W'(t)\colon t \geq 0\}$ and $\{W''(t)\colon t \geq 0\}$ of independent Wiener processes, a *two-sided Wiener process* $\{W'(t)\colon t \in \mathbb{R}\}$ is defined by $W(t) = W'(t)$ for $t \geq 0$ and $W(t) = W''(-t)$ for $t < 0$, and a *two-sided Brownian bridge* $\{B(t)\colon t \in \mathbb{R}\}$ is defined by $B(t) = W(t) - tW(1)$ for $t \in \mathbb{R}\}$. A *Wiener sheet* is a two-parameter centered Gaussian process $\{W(s,t)\colon s \geq 0, t \geq 0\}$ with [see, e.g., Section 1.11 in Csörgő and Révész (1981)]

$$\mathbb{E}\big(W(s_1,t_1)W(s_2,t_2)\big) = (s_1 \wedge s_2)(t_1 \wedge t_2) \quad \text{for } s_1, s_2, t_1, t_2 \geq 0.$$

If $\{W'(s,t)\colon s \geq 0, t \geq 0\}$ and $\{W''(s,t)\colon s \geq 0, t \geq 0\}$ are independent Wiener sheets, a *two-sided Wiener sheet* $\{W(s,t)\colon s \geq 0, t \in \mathbb{R}\}$ is defined by $W(s,t) = W'(s,t)$ for $s \geq 0$, $t \geq 0$ and $W(s,t) = W''(s,-t)$ for $s \geq 0$, $t < 0$. A *two-sided Kiefer process* is defined by $\mathscr{K}(\nu,t) = W(\nu,t) - tW(\nu,1)$ for $x \geq 0$, $t \in \mathbb{R}$. From now on, the Brownian bridges, Wiener and Kiefer processes we consider will always be assumed to be *two-sided*. For each $\nu > 0$, $\nu^{-1/2}\mathscr{K}(\nu,\mathbf{I})$ is a Brownian bridge, and, for each $n \in \mathbb{N}$, we may write

$$\mathscr{K}(n,t) = \sum_{i=1}^{n} B_i(t) = \sum_{i=1}^{n} \big\{W_i(t) - tW_i(1)\big\} \quad \text{for } t \in \mathbb{R}, \tag{1.8}$$

where $\{B_n\colon n \geq 1\}$ (resp. $\{W_n\colon n \geq 1\}$) are independent and identically distributed (i.i.d.) Brownian bridges (resp. Wiener processes). Here and elsewhere, we set $\sum_{\varnothing}(\cdot) = 0$.

Letting $\{\mathscr{K}(\nu, t): \nu \geq 0, t \in \mathbb{R}\}$ denote a (two-sided) Kiefer process, we define by induction on $k \in \mathbb{N}$ the kth *iterated Kiefer process* pertaining to $\mathscr{K}(\nu, t)$ by setting, for $\nu \geq 0$, $t \in \mathbb{R}$,

$$\begin{aligned} &\mathscr{K}(\nu, t; 0) = 0, \qquad \mathscr{K}(\nu, t; 1) = \mathscr{K}(\nu, t), \\ &\mathscr{K}(\nu, t; k) = \mathscr{K}\big(\nu, t + \nu^{-1}\mathscr{K}(\nu, t; k-1)\big) \quad \text{for } k \geq 1, \end{aligned} \tag{1.9}$$

where, for $\nu = 0$, we use the conventions $0^{-1} = \infty$ and $\infty \times 0 = 0$. In most of the applications considered later, the index $\nu \geq 0$ is integer and replaced by $n \in \mathbb{N}$. Below, we work on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ of (1.3)–(1.5), and consider the kth iterated Kiefer processes $K(n, \mathbf{I}; k)$, $k \in \mathbb{N}$, pertaining to $K(n, \mathbf{I}) := -K_1(n, \mathbf{I})$. We get namely,

$$\begin{aligned} &K(n, \mathbf{I}; 0) = 0, \qquad K(n, \mathbf{I}; 1) = K(n, \mathbf{I}) = K_2(n, \mathbf{I}) = -K_1(n, \mathbf{I}), \\ &K(n, \mathbf{I}; 2) = K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I})), \\ &K(n, \mathbf{I}; 3) = K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I}))), \end{aligned} \tag{1.10}$$

and, in general, for each $k \in \mathbb{N}^* = \mathbb{N} - \{0\}$,

$$K(n, \mathbf{I}; k) = K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I}; k-1)). \tag{1.11}$$

Our first theorem gives the exact rate of uniform approximation of β_n by $n^{-1/2}K(n, \mathbf{I}; k)$.

THEOREM 1.1. *On $(\Omega, \mathscr{A}, \mathbb{P})$, for each $k \in \mathbb{N}$, we have*

$$\begin{aligned} &\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \\ &\qquad \times \big\|\beta_n - n^{-1/2}K(n, \mathbf{I}; k)\big\| = C_k := 2^{-1+(2k+1)2^{-(k+1)}} \quad a.s. \end{aligned} \tag{1.12}$$

REMARK 1.1. (i) Since $C_0 = 2^{-1/2}$ and, by (1.10), $K(n, \mathbf{I}; 0) = 0$, (1.12) for $k = 0$ is in agreement with the Chung (1949) law of the iterated logarithm, which asserts that

$$\limsup_{n\to\infty} (\log_2 n)^{-1/2}\|\beta_n\| = \limsup_{n\to\infty} (\log_2 n)^{-1/2}\|\alpha_n\| = 2^{-1/2} \quad \text{a.s.} \tag{1.13}$$

(ii) Since $C_1 = 2^{-1/4}$ and $K(n, \mathbf{I}; 1) = K(n, \mathbf{I})$, (1.12) reduces for $k = 1$ to (1.5), namely,

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\big\|\beta_n - n^{-1/2}K(n, \mathbf{I})\big\| = 2^{-1/4} \quad \text{a.s.}$$

(iii) Since $C_2 = 2^{-3/8}$ and $K(n, \mathbf{I}; 1) = K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I}))$, (1.12) yields for $k = 2$,

$$\begin{aligned} &\limsup_{n\to\infty} n^{3/8}(\log n)^{-3/4}(\log_2 n)^{-1/8} \\ &\qquad \times \big\|\beta_n - n^{-1/2}K(n, \mathbf{I} + n^{-1}K(n, \mathbf{I}))\big\| = 2^{-3/8} \quad \text{a.s.} \end{aligned} \tag{1.14}$$

(iv) It is readily checked that the constants $\{C_k\colon k \geq 0\}$ in (1.12) fulfill

$$\text{(i) } 1/2 \leq C_k \leq 2^{-1/4}\ \forall\, k \in \mathbb{N}; \qquad \text{(ii) } C_k \to 1/2 \quad \text{as } k \to \infty.$$

Moreover, the sequence C_k is decreasing on $\mathbb{N}^*$ (but not on $\mathbb{N}$).

Our next result establishes an optimal property of the rates achieved via (1.12).

THEOREM 1.2. *Let β_n and a Kiefer process $\mathscr{K}(n, \mathbf{I})$ be defined on the same probability space. For each $k \in \mathbb{N}$, assume that kth iterated Kiefer process $\mathscr{K}(n, \mathbf{I}; k)$ pertaining to $\mathscr{K}(n, \mathbf{I})$ fulfills the following property. There exists an infinite set of indices $S \subseteq \mathbb{N}$ and a positive sequence $\{\theta_k\colon k \in S\}$, such that $\theta_k \to 1/2$ as $k \to \infty$ and, for each $k \in S$,*

$$\limsup_{n\to\infty} n^{\theta_k} \left\| \beta_n - n^{-1/2}\mathscr{K}(n, \mathbf{I}; k) \right\| = 0 \quad a.s. \tag{1.15}$$

Then, we have, for each $k \in \mathbb{N}$,

$$\begin{aligned} &\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \\ &\qquad \times \left\| \beta_n - n^{-1/2}\mathscr{K}(n, \mathbf{I}; k) \right\| = C_k \quad a.s. \end{aligned} \tag{1.16}$$

REMARK 1.2. Theorem 1.2 shows that the rates in (1.12) are optimal when the kth iterated Kiefer processes are all pertaining to the same initial Kiefer process. It leaves open the problem of whether there exists or not an *isolated* value of $k \in \mathbb{N}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \\ &\qquad \times \left\| \beta_n - n^{-1/2}\mathscr{K}(n, \mathbf{I}; k) \right\| = 0 \quad \text{a.s.}, \end{aligned} \tag{1.17}$$

for a suitable kth iterated Kiefer process $\mathscr{K}(n, \mathbf{I}; k)$. We conjecture that (1.17) is impossible for any choice of $\mathscr{K}(n, \mathbf{I})$ and $k \in \mathbb{N}$. For $k = 0$, this result follows obviously from (1.13), whereas for $k = 1$, it is a consequence of (1.6). On the other hand, to settle this conjecture for an arbitrary $k \geq 2$ appears to be a major problem.

The proofs of Theorems 1.1 and 1.2 are given in the forthcoming Sections 2 and 4. It is relatively easy (see, e.g., Proposition 2.2 in the sequel) to establish that, for each $k \in \mathbb{N}$,

$$\left\| \beta_n - n^{-1/2}K(n, \mathbf{I}; k) \right\| = O\Big(n^{-(1/2)+(1/2)^{k+1}} (\log n)^{1-(1/2)^k} (\log_2 n)^{(1/2)^{k+1}} \Big) \quad \text{a.s.}$$

and the main difficulty is to derive the exact limiting constant C_k in (1.12). This will be achieved in Section 4 by an application of a new functional limit law for empirical process increments. The latter greatly extends Theorem 3.1 of Deheuvels and Mason (1992a, b), and is of interest by and of itself. The corresponding statements and proofs are postponed until Section 3. Some further results, together with examples of applications of our theorems are presented in Section 5.

2. Preliminary results.

2.1. *Some basic notation and useful facts.* We inherit the notation of Section 1 and assume that $(\Omega, \mathscr{A}, \mathbb{P})$ carries the Kiefer process $K(\nu, t) = K_2(\nu, t) = -K_1(\nu, t)$ in (1.3)–(1.5), together with a sequence $\{U_n\colon n \geq 1\}$ of i.i.d. uniform (0,1) random variables (r.v.'s). For each $n \geq 1$, we set $\mathbb{U}_n(t) = n^{-1}\#\{U_i \leq t\colon 1 \leq i \leq n\}$ for $t \in \mathbb{R}$, with $\#A$ denoting cardinality of A, and $\mathbb{V}_n(t) = \inf\{s \geq 0\colon \mathbb{U}_n(s) \geq t\}$ for $t \in [0, 1]$. The uniform empirical and quantile processes in (1.1)–(1.5) are given by

$$\begin{aligned} &\alpha_n(t) = n^{1/2}(\mathbb{U}_n(t) - t) \quad \text{and} \quad \beta_n(t) = n^{1/2}(\mathbb{V}_n(t) - t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n \in \mathbb{N}^* \text{ and } t \in [0, 1], \\ &\alpha_n(t) = \beta_n(t) = 0 \quad \text{for either } n \notin \mathbb{N}^* \text{ or } t \notin [0, 1]. \end{aligned} \tag{2.1}$$

The refinement of (1.3) in Fact 1 below is due to Komlós, Major and Tusnády (1975) [see Laurent-Bonvalot and Castelle (1998) and page 150 in Csörgő and Horváth (1993)]. Let, via (1.8),

$$K_1(n, t) = -K(n, t) = W_1(n, t) - tW_1(n, 1), \tag{2.2}$$

where $W_1(s, t)$ denotes a Brownian sheet.

FACT 1. There exist constants $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3 > 0$ and $n_0 < \infty$ such that, for $n \geq n_0$ and $x \in \mathbb{R}$,

$$\mathbb{P}\Big(\big\|n^{1/2}\alpha_n + K(n, \mathbf{I})\big\| \geq (\mathscr{C}_1 \log n + x)\log n\Big) \leq \mathscr{C}_2 \exp(-\mathscr{C}_3 x). \tag{2.3}$$

Let $\mathscr{C}_4 = \mathscr{C}_1 + 2/\mathscr{C}_3$. By letting $x = (2/\mathscr{C}_3)\log n$ in (2.3), we obtain readily that

$$\sum_{n \geq n_0} \mathbb{P}\Big(\big\|n^{1/2}\alpha_n + K(n, \mathbf{I})\big\| \geq \mathscr{C}_4 \log^2 n\Big) \leq \mathscr{C}_2 \sum_{n \geq n_0} \frac{1}{n^2} < \infty.$$

By the Borel–Cantelli lemma, there exists therefore an $n_1 < \infty$ a.s., such that

$$\big\|\alpha_n + n^{-1/2}K(n, \mathbf{I})\big\| < \mathscr{C}_4 n^{-1/2}\log^2 n \quad \text{for } n \geq n_1. \tag{2.4}$$

The empirical distribution and quantile functions $\mathbb{U}_n$ and $\mathbb{V}_n$are related through the basic identity $\|\mathbb{U}_n(\mathbb{V}_n) - \mathbf{I}\| = n^{-1}$ a.s. A convenient version of this formula in terms of α_n and β_n is stated in Fact 2 below. We refer to (1.6) in Shorack (1982) for details.

FACT 2. We have, almost surely, for each $n \geq 1$,

$$\big\|\beta_n + \alpha_n\big(\mathbf{I} + n^{-1/2}\beta_n\big)\big\| = n^{-1/2}. \tag{2.5}$$

The derivation of (1.4) makes an instrumental use of (2.5) [see, e.g., Kiefer (1970) and Shorack (1982)]. It is natural to combine (1.4) and (2.5) to obtain approximations of β_n based upon the kth *iterated empirical process* $\alpha_{n;k}$, which we introduce below. Set

$$\alpha_{n;0} = 0, \qquad \alpha_{n;1} = \alpha_n,$$

$$\alpha_{n;2} = \alpha_n\big(\mathbf{I} - n^{-1/2}\alpha_n\big), \tag{2.6a}$$

$$\alpha_{n;3} = \alpha_n\big(\mathbf{I} - n^{-1/2}\alpha_n(\mathbf{I} - n^{-1/2}\alpha_n)\big),$$

and, in general, for each $k \in \mathbb{N}^*$,

$$\alpha_{n;k} = \alpha_n\big(\mathbf{I} - n^{-1/2}\alpha_{n;k-1}\big). \tag{2.6b}$$

The uniform Bahadur–Kiefer representation (1.4) draws its usefulness from its ability to replace the relatively complex process $-\beta_n$ by the more easy to handle empirical process $\alpha_n = \alpha_{n;1}$ at the price of an error of $O(n^{-1/4+o(1)})$. It is logical to extend this principle to an arbitrary $k \in \mathbb{N}$, by approximating $-\beta_n$ by the kth iterated empirical process $\alpha_{n;k}$. This problem will be considered in the forthcoming Section 5. Corollary 2.1 below gives a preliminary clue to its solution by showing that, in the applications we have in mind, we may replace $n^{-1/2}K(n, \mathbf{I}; k)$ by $-\alpha_{n;k}$. The corresponding proof will rely on the next fact, due to Stute (1982) [see, e.g., Deheuvels and Mason (1992b)].

FACT 3. Let $\{h_n: n \geq 1\}$ be a sequence of positive constants such that

$$\text{(i) } h_n \downarrow 0; \quad \text{(ii) } nh_n \uparrow \infty; \quad \text{(iii) } nh_n/\log n \to \infty; \tag{2.7}$$
$$\text{(iv) } (\log(1/h_n))/\log_2 n \to \infty.$$

Then, we have, for each $\Lambda > 0$,

$$\limsup_{n\to\infty} (2h_n \log(1/h_n))^{-1/2} \sup_{|u|\leq \Lambda h_n} \|\alpha_n(\mathbf{I}+u) - \alpha_n\| = \Lambda^{1/2} \quad \text{a.s.} \tag{2.8}$$

PROPOSITION 2.1. *Assume that $\mathscr{K}(n, \mathbf{I})$ is a Kiefer process such that, for some specified constants $a > 0$ and $\mathscr{C} > 0$,*

$$\limsup_{n\to\infty} n^{1/2}(\log n)^{-1-a}\big\|\alpha_n + n^{-1/2}\mathscr{K}(n, \mathbf{I})\big\| \leq \mathscr{C} \quad \text{a.s.} \tag{2.9}$$

For $k \in \mathbb{N}$, denote by $\mathscr{K}(n, \mathbf{I}; k)$ the kth iterated Kiefer process based upon $\mathscr{K}(n, \mathbf{I}; 1) = \mathscr{K}(n, \mathbf{I})$. Then, for each $k \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} n^{1/2}(\log n)^{-1-a}\big\|\alpha_{n;k} + n^{-1/2}\mathscr{K}(n, \mathbf{I}; k)\big\| \leq \mathscr{C} \quad \text{a.s.} \tag{2.10}$$

PROOF. Fix any $k \in \mathbb{N}$ and $\varepsilon > 0$, and observe that $h_n = \mathscr{C}(1+\varepsilon)n^{-1}\cdot(\log n)^{1+a}$ fulfills (2.7). Under the assumption that (2.10) holds, there exists with probability 1 an $n_0 < \infty$ such that, for all $n \geq n_0$, $\|n^{-1/2}\alpha_{n;k} + n^{-1}\mathscr{K}(n, \mathbf{I}; k)\| \leq h_n$. Thus, by (2.8),

$$\limsup_{n\to\infty} (2h_n \log(1/h_n))^{-1/2}\big\|\alpha_n\big(\mathbf{I} - n^{-1/2}\alpha_{n;k}\big) - \alpha_n(\mathbf{I} + n^{-1}\mathscr{K}(n, \mathbf{I}; k))\big\| \leq 1 \quad \text{a.s.}$$

Observe that, as $n \to \infty$,

$$(2h_n \log(1/h_n))^{1/2} = (1 + o(1))\{2\mathscr{C}(1+\varepsilon)\}^{1/2} n^{-1/2}(\log n)^{1+(a/2)}$$
$$= o\big(n^{-1/2}(\log n)^{1+a}\big).$$

Since, by (2.6), $\alpha_{n;\,k+1} = \alpha_n(\mathbf{I} - n^{-1/2}\alpha_{n;\,k})$, this entails in turn that, a.s. for all large n,

$$\big\|\alpha_{n;\,k+1} - \alpha_n(\mathbf{I} + n^{-1}\mathscr{K}(n, \mathbf{I}; k))\big\| \le \mathscr{C}(\varepsilon/2) n^{-1/2}(\log n)^{1+a}. \tag{2.11}$$

Next, we infer readily from (1.9) and (2.9), that, a.s. for all large n,

$$\begin{aligned}
&\big\|\alpha_n(\mathbf{I} + n^{-1}\mathscr{K}(n, \mathbf{I}; k)) - n^{-1/2}\mathscr{K}(n, \mathbf{I}; k+1)\big\| \\
&\quad = \big\|\alpha_n(\mathbf{I} + n^{-1}\mathscr{K}(n, \mathbf{I}; k)) - n^{-1/2}\mathscr{K}(n, \mathbf{I} + n^{-1}\mathscr{K}(n, \mathbf{I}; k))\big\| \\
&\quad \le \big\|\alpha_n - n^{-1/2}\mathscr{K}(n, \mathbf{I})\big\| \le \mathscr{C}(1 + (\varepsilon/2)) n^{-1/2}(\log n)^{1+a}.
\end{aligned} \tag{2.12}$$

It follows from (2.11), (2.12) and the triangle inequality that, a.s. for all large n,

$$\big\|\alpha_{n;\,k+1} - n^{-1/2}\mathscr{K}(n, \mathbf{I}; k+1)\big\| \le \mathscr{C}(1+\varepsilon) n^{-1/2}(\log n)^{1+a}.$$

Since $\varepsilon > 0$ is arbitrary in this last inequality, we see that (2.10) holds when k is replaced by $k+1$. Since (2.10) for $k = 0$ reduces to (2.9), the proof follows by induction on k. □

COROLLARY 2.1. *Let $\mathscr{C}_4$ be as in* (2.4). *We have, for each $k \in \mathbb{N}$,*

$$\limsup_{n\to\infty} n^{1/2}(\log n)^{-2}\big\|\alpha_{n;\,k} + n^{-1/2}K(n, \mathbf{I}; k)\big\| \le \mathscr{C}_4 \quad a.s. \tag{2.13}$$

PROOF. By choosing $\mathscr{C} = \mathscr{C}_4$, $a = 1$ and $\mathscr{K}(n, \mathbf{I}) = K(n, \mathbf{I}) = -K_1(n, \mathbf{I})$ in (2.9), we infer readily (2.13) from (2.4) and (2.10). □

2.2. *Rough upper bounds.* In this section, we establish some rough upper bounds, which will be used in Section 4 to prove Theorem 1.1. Namely, we will show that, for each $k \in \mathbb{N}$, the left-hand side of (1.12) is less than or equal to a constant D_k [see (2.17)]. The following facts and lemmas will be needed.

LEMMA 2.1. *Let $\{\gamma_n(t)\colon 0 \le t \le 1\}$, $n = 1, 2, \dots$ be random processes on $(\Omega, \mathscr{A}, \mathbb{P})$ such that there exist constants $a \in (0, 1/2)$, $b, c \in \mathbb{R}^+$ and $d \ge 0$, fulfilling*

$$\limsup_{n\to\infty} n^a(\log n)^{-b}(\log_2 n)^{-c}\|\beta_n - \gamma_n\| \le d \quad a.s. \tag{2.14}$$

Then,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\big\|\beta_n - \alpha_n(\mathbf{I} + n^{-1/2}\gamma_n)\big\| \\
&\quad = \limsup_{n\to\infty} n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2} \\
&\qquad \times \big\|\beta_n - n^{-1/2}K(n, \mathbf{I} + n^{-1/2}\gamma_n)\big\| \le d^{1/2}(2a+1)^{1/2} \quad a.s.
\end{aligned} \tag{2.15}$$

PROOF. Fix $\varepsilon > 0$, and set, for $n \ge 1$, $h_n = (d+\varepsilon)n^{-(2a+1)/2}(\log n)^b(\log_2 n)^c$. By (2.14) there exists an $n_2 < \infty$ a.s. such that $n^{-1/2}\|\beta_n - \gamma_n\| \le h_n$ for all $n \ge n_2$. Since $(2a+1)/2 \in (0,1)$, it is readily checked that h_n fulfills (2.7). Moreover, as $n \to \infty$,

$$
\begin{aligned}
(d+\varepsilon)^{1/2}(2a+1)^{1/2}(2h_n\log(1/h_n))^{-1/2} &\sim n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\\
&= o(n^{1/2}(\log n)^{-2}),
\end{aligned}
$$

where we set $u_n \sim v_n$ whenever $u_n/v_n \to 1$ as $n \to \infty$. Thus, by (2.4) and Facts 2 and 3,

$$
\begin{aligned}
&\limsup_{n\to\infty} n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\|\beta_n - n^{-1/2}K(n, \mathbf{I}+n^{-1/2}\gamma_n)\|\\
&\quad= \limsup_{n\to\infty} n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\\
&\qquad\times \|\alpha_n(\mathbf{I}+n^{-1/2}\beta_n) + n^{-1/2}K(n, \mathbf{I}+n^{-1/2}\gamma_n)\|\\
&\quad= \limsup_{n\to\infty} n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\\
&\qquad\times \|\alpha_n(\mathbf{I}+n^{-1/2}\beta_n) - \alpha_n(\mathbf{I}+n^{-1/2}\gamma_n)\|\\
&\quad\le \limsup_{n\to\infty} (d+\varepsilon)^{1/2}(2a+1)^{1/2}(2h_n\log(1/h_n))^{-1/2}\\
&\qquad\times \sup_{|u|\le h_n} \|\alpha_n(\mathbf{I}+u) - \alpha_n\| = (d+\varepsilon)^{1/2}(2a+1)^{1/2} \quad \text{a.s.}
\end{aligned}
$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, we infer readily (2.15) from this inequality. □

The next proposition provides a crucial step of the forthcoming proof of Theorem 1.1. Introduce the following notation. For each $l \in \mathbb{N}$, set

$$\delta_l = 1 - (1/2)^l, \tag{2.16}$$

and, for each $l \in \mathbb{N}^*$, set

$$D_l = 2^{3\delta_l - 2}\prod_{i=1}^{l-1}\delta_i^{1/2^{l-i}} = 2^{-1/2^l}\prod_{i=1}^{l-1}\left\{2 - \frac{1}{2^{i-1}}\right\}^{1/2^{l-i}}, \tag{2.17}$$

the equality in (2.17) following from (2.16) and the observation that, for each $l \in \mathbb{N}^*$,

$$\delta_{l-1} = \sum_{i=1}^{l-1}\frac{1}{2^{l-i}} = 2\delta_l - 1. \tag{2.18}$$

Note for further use that

$$
\begin{aligned}
&\delta_0 = 0, \quad \delta_1 = 1/2, \quad \delta_2 = 3/4, \quad \delta_3 = 7/8,\\
&D_1 = 2^{-1/2}, \quad D_2 = 2^{-1/4}, \quad D_3 = 2^{-5/8}3^{1/2}.
\end{aligned} \tag{2.19}
$$

PROPOSITION 2.2. *For each* $l \in \mathbb{N}$, *we have*

$$\limsup_{n\to\infty} n^{(1/2)-(1/2)^{l+1}}(\log n)^{-1+(1/2)^l}(\log_2 n)^{-(1/2)^{l+1}} \times \left\|\beta_n - n^{-1/2}K(n,\mathbf{I};l)\right\| \le D_{l+1} \quad a.s. \tag{2.20}$$

PROOF. In view of (1.9) and (1.10), it follows from (1.13), (2.19) and $K(n,\ \mathbf{I};\ 0) = 0$, that (2.20) is an equality for $l = 0$. For $l = 1$, (2.20) is also an equality, since, by (1.9) and (2.19), $K(n,\mathbf{I};\ 1) = K_1(n,\mathbf{I}) = K(n,\mathbf{I})$, $D_2 = 2^{-1/4}$, and the statement reduces to (1.5).

For the other values of l, we proceed by induction, assuming that, for some $\kappa \in \mathbb{N}$, (2.20) holds with $l = \kappa$. We set $\gamma_n = n^{-1/2}K(n,\mathbf{I};\ \kappa)$, $a = a_\kappa := (1/2) - (1/2)^{\kappa+1}$, $b = b_\kappa := 1-(1/2)^\kappa$, $c = c_\kappa := (1/2)^{\kappa+1}$ and $d = d_\kappa := D_{\kappa+1}$ in Lemma 2.2. By (1.9),

$$K(n,\mathbf{I}+n^{-1/2}\gamma_n) = K(n,\mathbf{I}+n^{-1}K(n,\mathbf{I};\ \kappa)) = K(n,\mathbf{I};\kappa+1),$$

and we may write, via (2.14) and (2.15),

$$\limsup_{n\to\infty} n^{(2a_\kappa+1)/4}(\log n)^{-(b_\kappa+1)/2}(\log_2 n)^{-c_\kappa/2} \times \left\|\beta_n - n^{-1/2}K\big(n,\mathbf{I}+n^{-1}K(n,\mathbf{I};\ \kappa)\big)\right\| \le (2a_\kappa+1)^{1/2}d_\kappa^{1/2} \quad \text{a.s.} \tag{2.21}$$

By combining (2.21) with the equalities

$$\begin{aligned}
a_{\kappa+1} &= \frac{1}{2}-\frac{1}{2^{\kappa+2}} = \frac{1}{4}(2a_\kappa+1) = \frac{1}{4}\left(2-\frac{1}{2^\kappa}\right),\\
b_{\kappa+1} &= 1-\frac{1}{2^{\kappa+1}} = \frac{1}{2}(b_\kappa+1) = \frac{1}{2}\left(2+\frac{1}{2^\kappa}\right),\\
c_{\kappa+1} &= \frac{1}{2^{\kappa+2}} = \frac{1}{2}c_\kappa = \frac{1}{2}\left(\frac{1}{2^{\kappa+1}}\right),\\
D_{\kappa+2} &= 2^{-(1/2)^{\kappa+2}}\prod_{i=0}^{\kappa+1}\left\{2-\frac{1}{2^i}\right\}^{(1/2)^{\kappa-i+2}} = (2a_\kappa+1)^{1/2}D_{\kappa+1}^{1/2}\\
&= (2\delta_{\kappa+1}D_{\kappa+1})^{1/2} = \left\{2-\frac{1}{2^\kappa}\right\}^{1/2}\left\{2^{-(1/2)^{\kappa+1}}\prod_{i=0}^{\kappa-1}\left\{2-\frac{1}{2^i}\right\}^{(1/2)^{\kappa-i+1}}\right\}^{1/2},
\end{aligned} \tag{2.22}$$

we see that (2.20) holds for $k = \kappa+1$. The proof is complete by induction. □

LEMMA 2.2. *We have*

$$\text{(i) } 1/2 < D_k < 2 \ \ \forall\ k \in \mathbb{N}^*; \qquad \text{(ii) } D_k \uparrow 2 \ \ \textit{as } k \to \infty. \tag{2.23}$$

PROOF. By (2.19) we have $1/2 < D_1 = 2^{-1/2} < D_2 = 2^{-1/4} < 2$. Assuming that, for some $\kappa \in \mathbb{N}$, $1/2 < D_{\kappa+1} < 2$, we make use of (2.17) to write

$$D_{\kappa+2} = (2a_\kappa+1)^{1/2}D_{\kappa+1}^{1/2} = \left(1-\frac{1}{2^{\kappa+2}}\right)^{1/2}(2D_{\kappa+1})^{1/2} < \left(1-\frac{1}{2^{\kappa+2}}\right)^{1/2}\times 2 < 2,$$

and likewise

$$D_{\kappa+2} > \left(1 - \frac{1}{2^{\kappa+2}}\right)^{1/2} > \left(\frac{3}{4}\right)^{1/2} > \frac{1}{2}.$$

With this implying that $1/2 < D_{\kappa+2} < 2$, the proof of (2.23)(i) follows by induction. The fact that $D_k \uparrow$ follows from the observation that $D_{k+1}/D_k^{1/2} = (2 - 1/2^k)^{1/2} \uparrow$ for $k \geq 1$. Thus, we may write that for $k \geq 1$,

$$\frac{D_{k+1}}{D_k} > \left(\frac{D_k}{D_{k-1}}\right)^{1/2} > \cdots > \left(\frac{D_2}{D_1}\right)^{1/2^k} = \left\{2^{-1/2^2}3\right\}^{1/2^k} > 1,$$

whence $D_{k+1} > D_k$. This, in turn, implies that $D_k \uparrow x$. Since $D_{k+2} = (2a_k + 1)^{1/2} D_{k+2}^{1/2}$, x must fulfill $1/2 \leq x \leq 2$ and $x = (2x)^{1/2}$. The only possibility being $x = 2$, we conclude (2.23) (ii). □

REMARK 2.1. It is easy to check from the expressions (1.12) and (2.19) of the constants C_k and D_{k+1} that $C_k \leq D_{k+1}$ for all $k \in \mathbb{N}$. This inequality holds for $k = 0, 1, 2$, since $C_0 = D_1 = 2^{-1/2}$, $C_1 = D_2 = 2^{-1/4}$ and $C_2 = 2^{-3/8} < D_3 = 3^{-5/8}3^{1/2}$. For higher values of k, we may use an induction argument based upon the fact that, for all $k \geq 2$,

$$\frac{D_{k+1}/\sqrt{D_k}}{C_k/\sqrt{C_{k-1}}} = \frac{\sqrt{2\delta_k}}{2^{(1/2)-\delta_k}} = \left\{\delta_k \times 2^{2\delta_k}\right\}^{1/2} > 1.$$

3. A functional limit law for empirical increments.

3.1. *Statement of the result.* The proof of Theorem 1.1 is based in part upon a functional limit law for the increments of empirical processes which has interest in itself. This result, stated in Theorem 3.1 below, provides a largely extended version of Theorem 3.1 of Deheuvels and Mason (1992b).

First, we introduce some notation. For any $-\infty < a \leq 0 < b < \infty$, denote by $B[a, b]$ (resp. $AC[a, b]$) the set of bounded (resp. absolutely continuous with respect to the Lebesgue measure) functions on $[a, b]$. A set $\mathscr{E}$ endowed with a topology $\mathscr{T}$ will be denoted by $(\mathscr{E}, \mathscr{T})$. The uniform topology, defined by the sup-norm $\|\cdot\|$ on $B[a, b]$ is denoted by $\mathscr{U}$. When $f \in AC[a, b]$, we set $\dot{f} = (d/dt)f$ for the Lebesgue derivative of f. For each $f \in B[a, b]$, set

(3.1)
$$|f|_H = \left\{\int_a^b \dot{f}^2(s)\,ds\right\}^{1/2} \quad \text{if } f \in AC[a, b] \quad \text{and} \quad f(0) = 0,$$
$$|f|_H = \infty \text{ else.}$$

Note for further use that, when $a = -1$, $b = 1$ or $a = 0$, $b = 1$, the inequality $\|f\| \leq |f|_H$ holds for all $f \in B[a, b]$. The Strassen set [Strassen (1964)] is the unit ball, denoted below by $\mathbb{K} = \mathbf{K}_1$ of the reproducing kernel Hilbert space $\mathbb{H} = \{f \in B[a, b]: |f|_H < \infty\}$, of the restriction of a (two-sided) Wiener process on $[a, b]$. An application of the Arzela–Ascoli theorem readily shows that

(3.2)
$$\mathbb{K} = \mathbf{K}_1 = \{f \in AC[a, b]: |f|_H \leq 1\},$$

is compact in $(B[a,b],\mathscr{U})$. In the sequel, $k \geq 1$ will denote an arbitrary, but fixed, integer. We will set $\mathbf{g} = (g_1,\dots,g_k) \in B[a,b]^k$ when $g_1,\dots,g_k \in B[a,b]$ and endow $B[a,b]^k$ with the uniform topology $\mathscr{U}_k$, conveniently defined by the norm

$$\|\mathbf{g}\|_k = \|(g_1,\dots,g_k)\|_k = \max_{1\leq j\leq k} \|g_j\|. \tag{3.3}$$

For each $\varepsilon > 0$ and $A, B \subseteq B[a,b]^k$, $A, B \neq \emptyset$, let

$$A^\varepsilon = \{\mathbf{h} \in B[a,b]^k\colon \exists\, \mathbf{g} \in A, \|\mathbf{h}-\mathbf{g}\|_k < \varepsilon\}, \tag{3.4}$$

and set $\Delta(A,B) = \inf\{\varepsilon > 0\colon A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}$ whenever such an $\varepsilon > 0$ exists, $\Delta(A,B) = \infty$ else, Moreover, for each $\varepsilon > 0$ and $\mathbf{h} \in B[a,b]^k$, set

$$\mathscr{N}_\varepsilon(\mathbf{h}) = \big\{\mathbf{g} \in B[a,b]^k\colon \|\mathbf{g}-\mathbf{h}\|_k < \varepsilon\big\}. \tag{3.5}$$

The notation (3.4) and (3.5) will be used independently of $k \geq 1$, and in particular for $k = 1$. Let $\{h_{n,l}\colon n \geq 1\}$, for $l = 1,\dots,k$, denote k sequences of positive constants fulfilling the assumptions (H1)–(H4).

(H1) For each $l = 1,\dots,k$, $h_{n,l} \downarrow 0$ and $nh_{n,l} \uparrow \infty$.
(H2) $0 < h_{n,k} < \cdots < h_{n,1} < 1$.
(H3) (i) $nh_{n,k}/\log n \to \infty$; (ii) $\{\log(1/h_{n,1})\}/\log_2 n \to \infty$.
(H4) (i) For each $l = 1,\dots,k$, $\{\log(1/h_{n,l})\}/\log(1/h_{n,k}) \to d_l \in (0,1]$.
(ii) $0 < d_1 \leq \cdots \leq d_k = 1$.
(iii) For each $l = 1,\dots,k-1$, $h_{n,l+1}/h_{n,l} \to 0$.

From now on, we will limit overselves to either $[a,b] = [0,1]$ or $[a,b] = [-1,1]$. For $n \geq 1$, $x \in [0,1]$ and $l = 1,\dots,k$, we consider the functions of $t \in [a,b]$, defined by

$$f_{n,l}(x;t) = (2h_{n,l}\log(1/h_{n,l}))^{-1/2}(\alpha_n(x+h_{n,l}t) - \alpha_n(x)). \tag{3.6}$$

For any fixed interval $J = [A,B] \subseteq [0,1]$ with $A < B$, let $\mathscr{F}_{n,k}(J) \subseteq B[a,b]^k$ be defined by

$$\mathscr{F}_{n,k} = \mathscr{F}_{n,k}(J) = \big\{\big(f_{n,1}(x;t_1),\dots,f_{n,k}(x;t_k)\big)\colon x \in J\big\}, t_l,\dots,t_k \in [a,b]. \tag{3.7}$$

In view of (3.2) and (H4)(ii), we set further

$$\mathbf{K}_k = \left\{(f_1,\dots,f_k) \in AC[a,b]^k\colon \forall\, 1 \leq m \leq k, \sum_{l=1}^m d_l|f_l|_H^2 \leq d_m\right\} \subseteq \mathbb{K}^k. \tag{3.8}$$

The main result of this section may now be stated as follows.

THEOREM 3.1. *Fix any $J = [A,B] \subseteq [0,1]$ with $A < B$, and let $[a,b] = [0,1]$ or $[a,b] = [-1,1]$. Then, under* (H1)–(H2)–(H3)–(H4)*, we have*

$$\lim_{n\to\infty} \Delta(\mathscr{F}_{n,k}(J), \mathbf{K}_k) = 0 \quad a.s. \tag{3.9}$$

REMARK 3.1. (i) For $k = 1$, $\mathbf{K}_1 = \mathbb{K}$, and Theorem 2.1 reduces to Theorem 3.1 in Deheuvels and Mason (1992b) [see also Berthet (1997)].

(ii) It is readily verified that, for each $k \in \mathbb{N}^*$, $\mathbf{K}_k$ is a compact subset of $(B[a,b]^k, \mathscr{U}_k)$.

(iii) In Sections 3.2 and 3.3 below we give a detailed proof of Theorem 3.1 under the following restrictions. First, we replace the assumptions (H3) and (H4) by (H3)′ and (H4)′.

(H3)′ (i) $nh_{n,k}/\log^3 n \to \infty$; (ii) $\{\log(1/h_{n,1})\}/\log_2 h \to \infty$.

(H4)′ (i) For each $l = 1, \dots, k$, $\{\log(1/h_{n,l})\}/\log(1/h_{n,k}) \to d_l \in (0,1]$;
(ii) $0 < d_1 < \cdots < d_k = 1$.

The corresponding version of Theorem 3.1 will turn out to be largely sufficient for the sake of proving Theorem 1.1. Second, we will consider only the special cases where $[a,b] = [0,1]$, $J = [0,1]$ and $k = 2$. It will become obvious later on that the proof of the full version of Theorem 3.1 can be completed by a routine but tedious book-keeping-type combination of the present arguments with that of Deheuvels and Mason (1992b) and Deheuvels (1997). We will omit therefore the corresponding lengthy technicalities.

3.2. *Proof of Theorem* 3.1. *Inner bounds.* The main result of this subsection, stated in the forthcoming Proposition 3.1, establishes the inner bound half of Theorem 3.1. In view of Remark 3.1, we will assume from now on that (H1), (H2) and (H3)′, (H4)′ hold. Moreover, we set $[a,b] = [0,1]$ and $k = 2$. We note that, in this case, we have $0 < d_1 < d_2 = 1$, so that (3.8) may be rewritten into

$$\begin{aligned}\mathbf{K}_2 &= \big\{(f_1, f_2) \in AC[0,1]^2\colon d_1|f_1|_H^2 \le d_1 \text{ and } d_1|f_1|_H^2 + d_2|f_2|_H^2 \le d_2\big\}\\ &= \big\{(f_1, f_2) \in AC[0,1]^2\colon |f_1|_H^2 \le 1 \text{ and } |f_2|_H^2 \le 1 - d_1|f_1|_H^2\big\} \subset \mathbb{K}^2,\end{aligned}$$

with $\mathbb{K} = \mathbf{K}_1$ being as in (3.2). Recalling (2.2) and (3.6), for $n \ge 1$, $x \in [0,1]$ and $l = 1, 2$, we consider the functions of $t \in [0,1]$, defined by

$$L_{n,l}(x;t) = (2nh_{n,l}\log(1/h_{n,l}))^{-1/2}\big\{W_1(n, x + h_{n,l}t) - W_1(n,x)\big\}. \tag{3.10}$$

LEMMA 3.1. *Under* (H3)′, *we have*

$$\lim_{n\to\infty} \big\|(f_{n,1}, f_{n,2}) - (L_{n,1}, L_{n,2})\big\|_2 = 0 \quad a.s. \tag{3.11}$$

PROOF. By (2.2)–(2.4), there exists an $n_1 < \infty$ a.s., such that, for all $n \ge n_1$,

$$\big\|\alpha_n - n^{-1/2}K_1(n, \mathbf{I})\big\| < \mathscr{C}_4 n^{-1/2}\log^2 n. \tag{3.12}$$

By (2.2), $K_1(n,t) = W_1(n,t) - tW_1(n,1)$ and $\{W_1(t,1)\colon t \ge 0\}$ is a Wiener process. Thus, by the law of the iterated logarithm, there exists an $n_2 < \infty$ a.s. such that, for all $n \ge n_2$, $|W_1(n,1)| \le 2\sqrt{n\log_2 n}$ a.s. By (H3)′, it follows therefore that, for either $l=1$ or $l=2$,

$$\begin{aligned}\|f_{n,l}-L_{n,l}\| &\le (2h_{n,l}\log(1/h_{n,l}))^{-1/2}\{2\mathscr{C}_4 n^{-1/2}\log^2 n+4h_{n,l}\sqrt{\log_2 n}\}\\ &= O\bigg(\bigg\{\frac{\log^3 n}{nh_{n,l}}\bigg\}^{1/2}\bigg)+O\bigg(\bigg\{\frac{h_{n,l}\log_2 n}{\log(1/h_{n,l})}\bigg\}^{1/2}\bigg)=o(1)\\ &\qquad\text{a.s. as } n\to\infty.\end{aligned}$$

In view of (3.3), this suffices for (3.11). □

For any $A\subseteq B[0,1]$, $A\neq\varnothing$, let

$$J(A)=\inf_{f\in A}|f|_H^2. \tag{3.13}$$

The following fact is due to Schilder (1966) [see, e.g., Deuschel and Stroock (1989), page 12].

FACT 4. For each $\lambda>0$, set $W_{\{\lambda\}}(s)=(2\lambda)^{-1/2}W(s)$ for $s\in[0,1]$, where $\{W(t)\colon t\ge 0\}$ is a Wiener process. Then, for each closed (resp. open) subset F (resp. G) of $(B[0,1],\mathscr{U})$,

$$\limsup_{\lambda\to\infty}\lambda^{-1}\log\mathbb{P}(W_{\{\lambda\}}\in F)\le -J(F), \tag{3.14}$$

$$\liminf_{\lambda\to\infty}\lambda^{-1}\log\mathbb{P}(W_{\{\lambda\}}\in G)\ge -J(G). \tag{3.15}$$

Set $\overline{A}=B[0,1]-A$ and $\rho A=\{\rho f\colon f\in A\}$ for $\rho\in\mathbb{R}$ when $A\subseteq B[0,1]$. The next fact is a version of Lemma 2.5 in Deheuvels (1997), given in view of an application of (3.14), (3.15) to $F=\overline{\{\rho\mathbb{K}^\varepsilon\}}=\overline{\{(\rho\mathbb{K})^{\rho\varepsilon}\}}$ and $G=\mathscr{N}_\varepsilon(f)$, as defined in (3.4), (3.5), for $\rho>0$ and $\varepsilon>0$.

FACT 5. For each $\rho>0$, $\varepsilon\in(0,1)$ and $f\in\mathbb{K}\subseteq B[0,1]$ such that $0<\varepsilon<|f|_H\le 1$,

$$\begin{aligned}&\text{(i)}\quad J\,(\overline{\{\rho\mathbb{K}^\varepsilon\}})\ge\rho^2(1+\varepsilon)^2;\\ &\text{(ii)}\quad J\,(\mathscr{N}_\varepsilon(f))\le(|f|_H-\varepsilon)^2\le|f|_H^2(1-\varepsilon)^2.\end{aligned} \tag{3.16}$$

Denote by $\lfloor u\rfloor\le u<\lfloor u\rfloor+1$ (resp. $\lceil u\rceil-1<u\le\lceil u\rceil$) the lower (resp. upper) integer part of u. Let $0<\varepsilon<1/2$ and $\delta>0$ denote two constants to be specified later on, and set $\theta=1-(1-\varepsilon)^3$. For each $n\ge 1$, set $x_i=(i+\delta)h_{n,1}$ for $i\le m_n:=-1+\lfloor(1/h_{n,1})-\delta\rfloor$. Next, for each $i\le m_n$ and $j\le M_n:=\lfloor\delta h_{n,1}/h_{n,2}\rfloor$, set $x_{i,j}=(i+\delta)h_{n,1}-jh_{n,2}$. Note for further use that, for all large n and $1\le i\le m_n$, $1\le j\le M_n$, we have

$$0\le x_i-\delta h_{n,1}\le x_{i,j}<x_{i,j}+h_{n,2}\le x_i<x_i+h_{n,1}\le 1. \tag{3.17}$$

It follows from (H4)$'$(i) and these definitions that, ultimately as $n\to\infty$,

$$\begin{aligned}h_{n,1}&=h_{n,2}^{d_1+o(1)},\qquad m_n=h_{n,2}^{-d_1+o(1)}\ge h_{n,2}^{-d_1+\theta/2},\\ M_n&=h_{n,2}^{d_1-1+o(1)}\ge h_{n,2}^{-d_1-1+\theta/2}.\end{aligned} \tag{3.18}$$

For each event E, we set $\overline{E} = \Omega - E$ for the complement of E. For any $(f_1, f_2) \in AC[0,1]^2$, $n \geq 1$ and $i = 0, \dots, m_n$, consider the events

$$
\begin{aligned}
E_{i,n} &= E_{i,n}(f_1, f_2; \varepsilon, \delta) \\
&= \{L_{n,1}(x_i; \mathbf{I}) \in \mathscr{N}_\varepsilon(f_1)\} \cap \left\{ \bigcup_{j=1}^{M_n} \{L_{n,2}(x_{i,j}, \mathbf{I}) \in \mathscr{N}_\varepsilon(f_2)\} \right\}, \\
E_n &= E_n(f_1, f_2; \varepsilon, \delta) = \bigcup_{i=1}^{m_n} E_{i,n}(f_1, f_2; \varepsilon, \delta).
\end{aligned} \tag{3.19}
$$

LEMMA 3.2. *Let $(f_1, f_2) \in \mathbf{K}_2$ be such that $|f_1|_H^2 < 1$ and $d_1|f_1|_H^2 + d_2|f_2|_H^2 < 1$. Then, for each $\delta > 0$ and $0 < \varepsilon < (1/32)\min\{1 - d_1|f_1|_H^2 - d_2|f_2|_H^2, d_1(1 - |f_1|_H^2)\}$, we have*

$$
\mathbb{P}\big(\overline{E}_n(f_1, f_2; \varepsilon, \delta) \text{ i.o. }\big) = 0 \tag{3.20}
$$

PROOF. Fix any $(f_1, f_2) \in \mathbf{K}_2$ with $|f_1|_H^2 < 1$ and $d_1|f_1|_H^2 + d_2|f_2|_H^2 < 1$. Select an arbitrary $\varepsilon \in [0, (1/32)\min\{1 - d_1|f_1|_H^2 - d_2|f_2|_H^2, d_1(1 - |f_1|_H^2)\}]$, and set $\theta = 1 - (1-\varepsilon)^3$. It follows form (3.17) that, for all large n, $1 \leq i \leq m_n$ and $1 \leq j \leq M_n$,

$$
\begin{aligned}
&(x_i, x_i + h_{n,1}) \cap (x_{i,j}, x_{i,j} + h_{n,2}) \\
&\quad = \big((i+\delta)h_{n,1}, (i+1+\delta)h_{n,1}\big) \\
&\qquad \cap \big((i+\delta)h_{n,1} - jh_{n,2}, (i+\delta)h_{n,1} - (j-1)h_{n,2}\big) = \varnothing.
\end{aligned}
$$

In view of (3.10) and (3.19), it follows from the scale invariance properties, and independence of Wiener process increments for nonoverlapping intervals, that

$$
\begin{aligned}
\mathbb{P}(\overline{E}_n) &= \mathbb{P}\left(\bigcap_{i=1}^{m_n} \overline{E}_{i,n}\right) = \{1 - \mathbb{P}(E_{1,n})\}^{m_n} \leq \exp(-m_n \mathbb{P}(E_{1,n})), \\
m_n \mathbb{P}(E_{1,n}) &= m_n \mathbb{P}\big(L_{n,1}(0, \mathbf{I}) \in \mathscr{N}_\varepsilon(f_1)\big) \\
&\quad \times \Big(1 - \big\{1 - \mathbb{P}(L_{n,2}(0, \mathbf{I}) \in \mathscr{N}_\varepsilon(f_2))\big\}^{M_n}\Big) =: m_n P_{1,n} \times P_{2,n}.
\end{aligned} \tag{3.21}
$$

Now, letting $G = \mathscr{N}_\varepsilon(f_1)$ in (3.15), we readily infer from (3.10), (3.16), (3.18) and (3.21) that, for all large n,

$$
\begin{aligned}
m_n P_{1,n} &= m_n \mathbb{P}\big(L_{n,1}(0, \mathbf{I}) \in \mathscr{N}_\varepsilon(f_1)\big) = m_n \mathbb{P}\big(W_{\{\log(1/h_{n,1})\}} \in \mathscr{N}_\varepsilon(f_1)\big) \\
&\geq m_n \exp\big(-(1-\varepsilon) J(\mathscr{N}_\varepsilon(f_1)) \log(1/h_{n,1})\big) \\
&\geq m_n h_{n,1}^{(1-\varepsilon)^3 |f_1|_H^2} = h_{n,2}^{(1-\varepsilon)^3 d_1 |f_1|_H^2 - d_1 + o(1)} \geq h_{n,2}^{(1-\varepsilon)^3 d_1 |f_1|_H^2 - d_1 + \theta/2}.
\end{aligned} \tag{3.22}
$$

Likewise, by combining (3.15) with the inequalities $1 - (1-u)^M \geq 1 - e^{-Mu} \geq \frac{1}{2}\min\{Mu, 1\}$ for $0 < u < 1$ and $M \geq 0$, we obtain readily from (3.18) and (3.21) that, for all large n,

$$
\begin{aligned}
P_{2,n} &\geq 1-\exp\Bigl(-M_n\mathbb{P}\bigl(W_{\{\log(1/h_{n,2})\}}\in\mathscr{N}_\varepsilon(f_2)\bigr)\Bigr)\\
&\geq 1-\exp\Bigl(-M_n h_{n,2}^{(1-\varepsilon)^3|f_2|_H^2}\Bigr)\geq \tfrac{1}{2}\min\Bigl\{M_n h_{n,2}^{(1-\varepsilon)^3 d_2|f_2|_H^2},1\Bigr\}\\
&\geq \tfrac{1}{2}\min\Bigl\{h_{n,2}^{(1-\varepsilon)^3 d_2|f_2|_H^2+d_1-1+\theta/2},1\Bigr\},
\end{aligned}
\tag{3.23}
$$

where we have used the fact that $d_2 = 1$. By combining (3.21), (3.22) and (3.23), and recalling that $\theta = 1-(1-\varepsilon)^3$, we obtain that, for all large n,

$$
\begin{aligned}
m_n\mathbb{P}(E_{1,n}) &\geq \frac{1}{2}\min\Bigl\{h_{n,2}^{(1-\varepsilon)^3\{d_1|f_1|_H^2+d_2|f_2|_H^2\}-1+\theta},h_{n,2}^{(1-\varepsilon)^3 d_1|f_1|_H^2-d_1+\theta}\Bigr\}\\
&\geq \frac{h_{n,2}^{\theta}}{2}\min\Bigl\{h_{n,2}^{(1-\varepsilon)^3\{d_1|f_1|_H^2+d_2|f_2|_H^2-1\}},h_{n,2}^{(1-\varepsilon)^3 d_1\{|f_1|_H^2-1\}}\Bigr\}\\
&\geq \frac{h_{n,2}^{\theta-2\rho}}{2},
\end{aligned}
\tag{3.24}
$$

where we set $\rho = (1/4)\min\{1-d_1|f_1|_H^2-d_2|f_2|_H^2, d_1(1-|f_1|_H^2)\}\in(0,1/4)$, and make use of the rough inequality $(1-\varepsilon)^3\geq 1/4$, implied by the lemma's assumption that $0<\varepsilon<\rho/8<1/32$. Since $\theta = 1-(1-\varepsilon)^3\leq 3\varepsilon$, the fact that $0<\varepsilon<\rho/8$ entails further that $\rho>8\varepsilon>3\varepsilon>\theta$. This, when combined with (2.24), implies that, for all large n,

$$
m_n\mathbb{P}(E_{1,n})\geq \frac{h_{n,2}^{\theta-\rho}}{2}\times h_{n,2}^{-\rho}\geq h_{n,2}^{-\rho}.
$$

Since (H3)$'$(ii) implies that, ultimately in $n\to\infty$, $h_{n,2}^{-\rho}\geq 2\log n$, we infer from (3.21) and the above inequalities that, for all n sufficiently large,

$$
\mathbb{P}(\overline{E}_n)\leq\exp\bigl(-m_n\mathbb{P}(E_{1,n})\bigr)\leq\exp\bigl(-h_{n,2}^{-\rho}\bigr)\leq n^{-2},
$$

which is summable, so that the proof of (3.20) is completed by the Borel–Cantelli lemma. □

LEMMA 3.3. *For each $\delta>0$, we have*

$$
\limsup_{n\to\infty}\biggl\{\max_{1\leq i\leq m_n}\max_{1\leq j\leq M_n}\bigl\|L_{n,1}(x_{i,j};\mathbf{I})-L_{n,1}(x_i;\mathbf{I})\bigr\|\biggr\}\leq 4\delta^{1/2}\quad a.s.
\tag{3.25}
$$

PROOF. By (H4)$'$ [or (H4)(iii)], $h_{n,2}/h_{n,1}\to 0$ as $n\to\infty$. We have, therefore, for all large n and uniformly over all $1\leq i\leq m_n$ and $1\leq j\leq M_n = \lfloor\delta h_{n,1}/h_{n,2}\rfloor$,

$$
|x_i - x_{i,j}| = jh_{n,2}\leq M_n h_{n,2}\leq(\delta h_{n,1}/h_{n,2})h_{n,2}\leq\delta h_{n,1}.
\tag{3.26}
$$

Recall the definition (3.6) of $f_{n,1}(x;t)$. By combining (3.26) with (3.11) and the triangle inequality, we next observe, via (2.7) and (2.8), that

$$\limsup_{n\to\infty}\left\{\max_{1\le i\le m_n}\max_{1\le j\le M_n}\left\|L_{n,1}(x_{i,j};\mathbf{I})-L_{n,1}(x_i;\mathbf{I})\right\|\right\}$$
$$=\limsup_{n\to\infty}\left\{\max_{1\le i\le m_n}\max_{1\le j\le M_n}\left\|f_{n,1}(x_{i,j};\mathbf{I})-f_{n,1}(x_i;\mathbf{I})\right\|\right\}$$
$$\le 4\limsup_{n\to\infty}(2h_{n,1}\log(1/h_{n,1}))^{-1/2}\sup_{|u|\le\delta h_{n,1}}\|\alpha_n(\mathbf{I}+u)-\alpha_n\|=4\delta^{1/2}\quad\text{a.s.,}$$

which is (3.25). □

PROPOSITION 3.1. *For any $\epsilon>0$, we have almost surely for all n sufficiently large,*

$$\mathbf{K}_2\subset\mathscr{F}^{\epsilon}_{n,2}. \tag{3.27}$$

PROOF. $\mathbf{K}_2$ being a compact subset of $(B[0,1]^2,\mathscr{U})$, for each $\epsilon>0$, there exists a finite sequence $\{\mathbf{g}_\mu=(g_1^{(\mu)},g_2^{(\mu)})\colon 1\le\mu\le\mathscr{M}\}\subset\mathbf{K}_2$ such that

$$\mathbf{K}_2\subseteq\bigcup_{\mu=1}^{\mathscr{M}}\mathscr{N}_{\epsilon/4}(\mathbf{g}_\mu). \tag{3.28}$$

Since, for each $1\le\mu\le\mathscr{M}$, there exists an $\mathbf{f}_\mu=(f_1^{(\mu)},f_2^{(\mu)})\in\mathscr{N}_{\epsilon/4}(\mathbf{g}_\mu)$ with $|f_1^{(\mu)}|_H^2<1$ and $d_1|f_1^{(\mu)}|_H^2+d_2|f_2^{(\mu)}|_H^2<1$, it follows from (3.28) and the triangle inequality that

$$\mathbf{K}_2\subseteq\bigcup_{\mu=1}^{\mathscr{M}}\mathscr{N}_{\epsilon/2}(\mathbf{f}_\mu). \tag{3.29}$$

By (3.29) and the triangle inequality, the proof of (3.27) boils down to showing that, for each specified $\mu=1,\dots,\mathscr{M}$ and $(f_1,f_2):=\mathbf{f}_\mu=(f_1^{(\mu)},f_2^{(\mu)})$, the following property holds. With probability 1 for all n sufficiently large, there exists an $x(n)\in[0,1]$ such that

$$\left\|(f_{n,1}(x(n),\mathbf{I}),f_{n,2}(x(n),\mathbf{I}))-(f_1,f_2)\right\|_2<\varepsilon/2. \tag{3.30}$$

Where we set

$$\varepsilon=\min\left\{\epsilon,\frac{1}{32}\min\left\{1-d_1|f_1|_H^2-d_2|f_2|_H^2,d_1(1-|f_1|_H^2)\right\}\right\}.$$

By (3.11), (3.30) will hold if our choice of $x(n)\in[0,1]$ is such that, a.s. for all large n,

$$\left\|\left(L_{n,1}(x(n),\mathbf{I}),L_{n,2}(x(n),\mathbf{I})\right)-(f_1,f_2)\right\|_2<\varepsilon/4. \tag{3.31}$$

To establish (3.31), we choose $\delta = (1/4096)\varepsilon^2$, so that $4\delta^{1/2} = \varepsilon/16 < \varepsilon/8$. In view of (3.25), this implies that, with probability 1 for all large n, $1 \le i \le m_n$ and $1 \le j \le M_n$,

$$\big\| L_{n,1}(x_{i,j}; \mathbf{I}) - L_{n,1}(x_i; \mathbf{I}) \big\| \le \varepsilon/8. \tag{3.32}$$

Next, in view of (3.3), we observe that (3.19) and (3.20) imply the existence, a.s. for all large n, of indices $1 \le i \le m_n$ and $1 \le j \le M_n$ (depending upon n) such that

$$\big\| \big(L_{n,1}(x_i, \mathbf{I}), L_{n,2}(x_{i,j}, \mathbf{I})\big) - (f_1, f_2) \big\|_2 < \varepsilon/8. \tag{3.33}$$

The conclusion (3.31) is achieved by setting $x(n) = x_{i,j}$ as above and by combining (3.32), (3.33) with the triangle inequality. □

3.3. *Proof of Theorem* 3.1. *Outer bounds.* We now turn to the second half of the proof of Theorem 3.1. The following additional notation will be used. Let $\gamma > 0$ and $\delta > 0$ be constants which will be precised later on. Consider the sequence of indices $\nu_m = \lfloor (1+\gamma)^m \rfloor$, for $m \in \mathbb{N}$, and denote by m_0 the smallest index such that $1 \le \nu_{m-1} < \nu_m$ for all $m \ge m_0$. It will be convenient to set, for $m \ge 1$, $i \le T_m := 1 + \lceil 1/(2\delta h_{\nu_m,1}) \rceil$ and $j \le Q_m := \lceil h_{\nu_m,1}/h_{\nu_m,2} \rceil$,

$$y_i = (i+1)\delta h_{\nu_m,1} \quad \text{and} \quad y_{i,j} = (i-1)\delta h_{\nu_m,1} + j\delta h_{\nu_m,2}. \tag{3.34}$$

It is noteworthy that, for all large m and $1 \le i \le T_m$, $1 \le j \le Q_m$,

$$\begin{aligned} 0 \le y_i - 2\delta h_{\nu_m,1} \le y_{i,j} \le y_{i,j} + h_{\nu_m,2} &\le y_i - \delta h_{\nu_m,1} + (\delta+1) h_{\nu_m,2} \\ < y_i < y_i + h_{\nu_m,1} &\le (1/2) + (3\delta+1) h_{\nu_m,1} < 1 \end{aligned} \tag{3.35}$$

and

$$1/2 < y_{T_m,Q_m} < y_{T_m} < (1/2) + 3\delta h_{\nu_m,1} < 1. \tag{3.36}$$

Moreover, it follows readily from (H4)$'$ and these definitions that, ultimately as $m \to \infty$,

$$\begin{aligned} T_m &\le 2 + \{1/(2\delta h_{\nu_m,1})\} \le 1/(\delta h_{\nu_m,1}), \\ Q_m &\le 1 + \{h_{\nu_m,1}/h_{\nu_m,2}\} \le 2h_{\nu_m,1}/h_{\nu_m,2}. \end{aligned} \tag{3.37}$$

Recall from (H1) that, for $l = 1, 2$ and $\nu_{m-1} < n \le \nu_m$, we have $h_{\nu_m,l} \le h_{n,l}$. For each $x \in [0, 1/2]$, $m \ge m_0$ and $\nu_{m-1} < n \le \nu_m$, set $z_{n,1}(x) = y_I$ and $z_{n,2}(x) = y_{I,J}$ where $I = I(x,n)$ and $J = J(x,n)$ are such that $1 \le I \le T_m$, $1 \le J \le Q_m$ and

$$\begin{aligned} |x - z_{n,2}(x)| = |x - y_{I,J}| &= \min_{1 \le i \le T_m} \min_{1 \le j \le Q_m} |x - y_{i,j}| \\ &\le \delta h_{\nu_m,2} \le 2\delta h_{n,2}. \end{aligned} \tag{3.38}$$

Note for further use that

$$|x - z_{n,1}(x)| = |x - x_I| \le 2\delta h_{\nu_m,1} \le 2\delta h_{n,1}. \tag{3.39}$$

LEMMA 3.4. *For each* $0 < \gamma < 1$ *and* $l = 1, 2$, *we have*

$$\limsup_{n\to\infty}\left\{\sup_{0\le x\le 1/2}\ \sup_{1-\gamma\le\theta\le 1}\|L_{n,l}(x,\mathbf{I}) - L_{n,l}(z_{n,l}(x),\theta\mathbf{I})\|\right\} < 4(\delta^{1/2}+\gamma^{1/2}) \quad a.s. \tag{3.40}$$

PROOF. Recall (3.6). By combining (2.8) with (3.38) and (3.39) and the triangle inequality, we see that, uniformly over $x \in [0, 1/2]$, for $l = 1, 2$,

$$\begin{aligned}\sup_{1-\gamma\le\theta\le 1}\|f_{n,l}(x,\theta\mathbf{I}) - f_{n,l}(z_{n,l}(x),\theta\mathbf{I})\| &\le \|f_{n,l}(x,\mathbf{I}) - f_{n,l}(z_{n,l}(x),\mathbf{I})\|\\ &\le 2\sup_{|u|\le 2\delta h_{n,l}}(2h_{n,l}\log(1/h_{n,l}))^{-1/2}\|\alpha_n(\mathbf{I}+u)-\alpha_n\|\\ &\to 2^{3/2}\delta^{1/2} \qquad \text{a.s. as } n\to\infty\end{aligned}$$

and

$$\begin{aligned}&\sup_{1-\gamma\le\theta\le 1}\|f_{n,l}(x,\mathbf{I}) - f_{n,l}(x,\theta\mathbf{I})\|\\ &\quad\le \sup_{|u|\le\gamma h_{n,l}}(2h_{n,l}\log(1/h_{n,l}))^{-1/2}\|\alpha_n(\mathbf{I}+u)-\alpha_n\| \to \gamma^{1/2} \quad \text{a.s. as } n\to\infty.\end{aligned}$$

By combining these statements with (3.3) and (3.11), we obtain readily that, for $l = 1, 2$,

$$\begin{aligned}&\limsup_{n\to\infty}\left\{\sup_{0\le x\le 1/2}\ \sup_{1-\gamma\le\theta\le 1}\left\|L_{n,l}(x,\mathbf{I}) - L_{n,l}(z_{n,l}(x),\theta\mathbf{I})\right\|\right\}\\ &\quad= \limsup_{n\to\infty}\left\{\sup_{0\le x\le 1/2}\ \sup_{1-\gamma\le\theta\le 1}\left\|f_{n,l}(x,\mathbf{I}) - f_{n,l}(z_{n,l}(x),\theta\mathbf{I})\right\|\right\}\\ &\quad\le \limsup_{n\to\infty}\left\{\sup_{0\le x\le 1/2}\ \sup_{1-\gamma\le\theta\le 1}\left\|f_{n,l}(x,\mathbf{I}) - f_{n,l}(x,\theta\mathbf{I})\right\|\right\}\\ &\qquad+ \limsup_{n\to\infty}\left\{\sup_{0\le x\le 1/2}\ \sup_{1-\gamma\le\theta\le 1}\left\|f_{n,l}(x,\theta\mathbf{I}) - f_{n,l}(z_{n,l}(x),\theta\mathbf{I})\right\|\right\}\\ &\quad\le \gamma^{1/2} + 2^{3/2}\delta^{1/2} \quad \text{a.s.,}\end{aligned}$$

which readily implies (3.40). □

In view of the definition (3.10), for each $m \ge m_0$, $\nu_{m-1} < n \le \nu_m$, $x \in [0, 1]$ and $l = 1, 2$, introduce the functions of $t \in [0, 1]$ defined by

$$\begin{aligned}H_{n,l}(x;t) &= (2\nu_m h_{\nu_m,l}\log(1/h_{\nu_m,l}))^{-1/2}\{W_1(n, x+h_{\nu_m,l}t) - W_1(n,x)\}\\ &= \left\{\frac{nh_{n,l}\log(1/h_{n,l})}{\nu_m h_{\nu_m,l}\log(1/h_{\nu_m,l})}\right\}^{1/2} L_{n,l}(x;th_{\nu_m,l}/h_{n,l}).\end{aligned} \tag{3.41}$$

Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be constants which will be precised later. Recalling the notation $\rho\mathscr{F} = \{\rho f: f \in \mathscr{F}\}$ for $\mathscr{F} \in B[0,1]$ and $\rho \in \mathbb{R}$, introduce the events, for $\varepsilon > 0$, $m \geq m_0$, $\nu_{m-1} < n \leq \nu_m$, $1 \leq i \leq T_m$ and $1 \leq j \leq Q_m$,

$$\mathscr{E}'_{n,i}(\varepsilon) = \{H_{n,1}(y_i; \mathbf{I}) \notin \lambda_1 \mathbb{K}^{\varepsilon}\}, \quad \mathscr{E}''_{n,i,j}(\varepsilon) = \{H_{n,2}(y_{i,j}; \mathbf{I}) \notin \lambda_2 \mathbb{K}^{\varepsilon}\},$$

$$\mathscr{E}'_{m,i}(\varepsilon) = \left\{ \bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}'_{n,i}(\varepsilon) \right\} \cap \left\{ \bigcup_{j=1}^{Q_m} \left\{ \bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}''_{n,i,j}(\varepsilon) \right\} \right\}, \tag{3.42}$$

$$\mathscr{E}(\varepsilon) = \bigcup_{i=1}^{T_m} \mathscr{E}_{m,i}(\varepsilon).$$

We will make use of the following general inequality, in the spirit of Lemma 3.4 in Deheuvels and Mason (1992a, b). Let $\eta_r(t)$, for $1 \leq r \leq R$ and $t \in [0,1]$, denote $(B[0,1], \mathscr{U})$-valued random functions such that the following condition holds:

(C) For each $1 \leq r \leq R$, $\eta_R - \eta_r$ and $\{\eta_i: 1 \leq i \leq r\}$ are independent.

LEMMA 3.5. *Let $A \neq \emptyset$ be a Borel subset of $(B[0,1], \mathscr{U})$. Then under* (C), *for each $\epsilon > 0$ such that $\mathbb{P}(\|\eta_R - \eta_r\| \leq \epsilon) \geq 1/2$ for $1 \leq r < R$, we have*

$$\mathbb{P}\left(\bigcup_{r=1}^{R} \{\eta_r \notin A^{2\epsilon}\} \right) \leq 2\mathbb{P}(\eta_r \notin A^{\epsilon}). \tag{3.43}$$

PROOF. Set, for each $r = 1, \ldots, R$ and $\epsilon > 0$, $E_r(\epsilon) = \{\eta_r \notin A^{\epsilon}\}$ and $E_0(\epsilon) = \emptyset$. Observe that, for each $r = 1, \ldots, R$, $E_r(2\epsilon) \cap \{\|\eta_R - \eta_r\| \leq \epsilon\} \subseteq E_R(\epsilon)$. Since, for each $r = 1, \ldots, R$, $\mathbb{P}(\|\eta_R - \eta_r\| \leq \epsilon) \geq 1/2$, we may use (C) to write

$$\begin{aligned}
\tfrac{1}{2}\mathbb{P}\left(\bigcup_{r=1}^{R} E_r(2\epsilon) \right) &= \tfrac{1}{2} \sum_{r=1}^{R} \mathbb{P}\left(E_r(2\epsilon) \cap \bigcap_{i=0}^{r-1} \overline{E}_i(2\epsilon) \right) \\
&\leq \sum_{r=1}^{R} \mathbb{P}\left(E_r(2\epsilon) \cap \bigcap_{i=0}^{r-1} \overline{E}_i(2\epsilon) \right) \mathbb{P}(\|\eta_R - \eta_r\| \leq \epsilon) \\
&= \sum_{r=1}^{R} \mathbb{P}\left(\{E_r(2\epsilon) \cap \{\|\eta_R - \eta_r\| \leq \epsilon\}\} \cap \bigcap_{i=0}^{r-1} \overline{E}_i(2\epsilon) \right) \\
&\leq \mathbb{P}\left(\bigcup_{r=1}^{R} \left\{ E_R(\epsilon) \cap E_r(2\epsilon) \cap \bigcap_{i=0}^{r-1} \overline{E}_i(2\epsilon) \right\} \right) \leq \mathbb{P}(E_R(\epsilon)),
\end{aligned}$$

which is (3.43). □

LEMMA 3.6. *For all m sufficiently large, we have, for all $1 \leq i \leq T_m$ and $1 \leq j \leq Q_m$,*

$$\mathbb{P}\left(\bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}'_{n,i}(\varepsilon) \right) \leq \mathbb{P}\left(\mathscr{E}'_{\nu_m,i}(\varepsilon/2)\right) \tag{3.44}$$

and

$$\mathbb{P}\biggl(\bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}''_{n,i,j}(\varepsilon)\biggr) \le \mathbb{P}\bigl(\mathscr{E}''_{\nu_m,i,j}(\varepsilon/2)\bigr). \tag{3.45}$$

PROOF. Fix any $1 \le i \le T_m$ and $1 \le j \le Q_m$. We make use of Lemma 3.5 with the following choices of $\epsilon > 0$, $\{\eta_r\colon 1 \le r \le R\}$ and A. For either $l = 1$ or $l = 2$, and for $1 \le r \le R := R_m = \nu_m - \nu_{m-1}$, we set $\eta_r = H_{\nu_{m-1}+r,l}(y_i, \mathbf{I})$, which, by (3.41), obviously fulfills (C). Moreover, for each $\epsilon > 0$, we have, uniformly over $1 \le r < R$,

$$\mathbb{P}\bigl(\|\eta_R - \eta_r\| > \varepsilon\bigr) = \mathbb{P}\biggl(\|W\| > \varepsilon\biggl\{\frac{2\log(1/h_{\nu_m,l})}{R-r}\biggr\}^{1/2}\biggr)$$
$$\le 4\exp\bigl(-\varepsilon^2 \log(1/h_{\nu_m,l})\bigr) \to 0,$$

where we have used the inequality $\mathbb{P}(\|W\| \ge u) \le 4\exp(-u^2/2)$ for $u \ge 0$ [combine (1.1.1) and (1.5.1) in Csörgő and Révész (1981)]. We may therefore apply (3.43), to obtain (3.44), (3.45), after setting $A = \lambda_l \mathbb{K}$ and $\epsilon = (\lambda_l \varepsilon)/2$ for $l = 1, 2$. □

LEMMA 3.7. *For each $\varepsilon > 0$, we have, for all m sufficiently large,*

$$\mathbb{P}(\mathscr{E}_m(\varepsilon)) \le \min\bigl\{h_{\nu_m,2}^{(1+\varepsilon)d_1\lambda_1^2 - d_1}, h_{\nu_m,2}^{(1+\varepsilon)\{d_1\lambda_1^2 + d_2\lambda_2^2\} - d_2}\bigr\}. \tag{3.46}$$

PROOF. We infer from (3.42) and the Bonferroni inequalities that

$$\begin{aligned}\mathbb{P}(\mathscr{E}_m(\varepsilon)) &\le T_m \mathbb{P}(\mathscr{E}_{m,1}(\varepsilon)) = \mathscr{Q}_{m,1} \times \mathscr{Q}_{m,2}\\ &:= \biggl\{T_m \mathbb{P}\biggl(\bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}'_{n,1}(\varepsilon)\biggr)\biggr\} \times \mathbb{P}\biggl(\bigcup_{j=1}^{Q_m}\biggl\{\bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}''_{n,1,j}(\varepsilon)\biggr\}\biggr),\end{aligned} \tag{3.47}$$

where we make use of the independence of Wiener process increments on nonoverlapping intervals. Further, we may write, via (3.35), (3.41), (3.42) and (3.44),

$$\begin{aligned}\mathscr{Q}_{m,1} &\le T_m \mathbb{P}\bigl(\mathscr{E}'_{\nu_m,1}(\varepsilon/2)\bigr) = T_m \mathbb{P}\bigl(H_{\nu_m,1}(0;\mathbf{I}) \notin \lambda_1 \mathbb{K}^{\varepsilon/2}\bigr)\\ &= T_m \mathbb{P}\bigl(W_{\{\log(1/h_{\nu_m,1})\}} \in \overline{\{\lambda_1 \mathbb{K}^{\varepsilon/2}\}}\bigr).\end{aligned} \tag{3.48}$$

By (3.45) and another application of the Bonferroni inequalities, we get, likewise,

$$\begin{aligned}\mathscr{Q}_{m,2} &\le \min\biggl\{1, Q_m \mathbb{P}\biggl(\bigcup_{n=\nu_{m-1}+1}^{\nu_m} \mathscr{E}''_{n,1,j}(\varepsilon)\biggr)\biggr\}\\ &\le \min\bigl\{1, Q_m \mathbb{P}\bigl(\mathscr{E}''_{\nu_m,1,1}(\varepsilon/2)\bigr)\bigr\}\\ &= \min\bigl\{1, Q_m \mathbb{P}\bigl(H_{h_{\nu_m,2}}(0;\mathbf{I}) \notin \lambda_2 \mathbb{K}^{\varepsilon/2}\bigr)\bigr\}\\ &= \min\Bigl\{1, Q_m \mathbb{P}\Bigl(W_{\{\log(1/h_{\nu_m,2})\}} \in \overline{\{\lambda_2 \mathbb{K}^{\varepsilon/2}\}}\Bigr)\Bigr\}.\end{aligned} \tag{3.49}$$

Recalling (3.18), (3.37) and $d_2 = 1$, we set $G = \overline{\{\lambda_1 \mathbb{K}^{\varepsilon/2}\}}$ in (3.15) and use the inequalities $\lambda_1^2(1+\varepsilon/2)^2 > \lambda_1^2(1+\varepsilon+\varepsilon^2/8) > \lambda_1^2(1+\varepsilon)$ to obtain via (3.48) that, for all large m,

$$(3.50)\qquad \begin{aligned} &T_m \mathbb{P}\big(W_{\{\log(1/h_{\nu_m,1})\}} \in \overline{\{\lambda_1 \mathbb{K}^{\varepsilon/2}\}}\big) \\ &\quad \le (1/\delta) \times h_{\nu_m,1}^{-1} \times h_{\nu_m,1}^{\lambda_1^2(1+\varepsilon+\varepsilon^2/8)} \le h_{\nu_m,2}^{(1+\varepsilon)d_1\lambda_1^2 - d_1}. \end{aligned}$$

Likewise, by setting $G = \overline{\{\lambda_2 \mathbb{K}^{\varepsilon/2}\}}$ in (3.15), we obtain via (3.46) that, for all large m,

$$(3.51)\qquad \begin{aligned} &T_m \mathbb{P}\big(W_{\{\log(1/h_{\nu_m,1})\}} \in \overline{\{\lambda_1 \mathbb{K}^{\varepsilon/2}\}}\big) Q_m \mathbb{P}\big(W_{\{\log(1/h_{\nu_m,2})\}} \in \overline{\{\lambda_2 \mathbb{K}^{\varepsilon/2}\}}\big) \\ &\quad \le (1/\delta) \times h_{\nu_m,2}^{-1} \times h_{\nu_m,1}^{\lambda_1^2(1+\varepsilon+\varepsilon^2/8)} \times h_{\nu_m,2}^{\lambda_2^2(1+\varepsilon+\varepsilon^2/8)} = h_{\nu_m,2}^{(1+\varepsilon)\{d_1\lambda_1^2+d_2\lambda_2^2\}-d_2}. \end{aligned}$$

The conclusion (3.46) is a direct consequence of (3.47) and (3.48)–(3.51). □

LEMMA 3.8. *Let $\Lambda_1 > 0$ and $\Lambda_2 > 0$ be any two constants such that either $d_1\Lambda_1^2 \ge d_1$ or $d_1\Lambda_1^2 + d_2\Lambda_2^2 \ge d_2$. Fix any $0 < \varepsilon < 1$, and select $\delta > 0$ and $\gamma > 0$ such that $0 < \max\{\delta, \gamma\} < (\varepsilon^2/64)\min\{1, \Lambda_1^2, \Lambda_2^2\})$. Then, we have almost surely for all large n,*

$$(3.52)\qquad f_{n,l}(x, \mathbf{I}) \in \Lambda_l \mathbb{K}^{\varepsilon} \quad \textit{for } l = 1, 2 \textit{ and all } x \in [0,1].$$

PROOF. Note for further use that our assumptions imply that $4(\delta^{1/2} + \gamma^{1/2}) \le (\Lambda_l \varepsilon)/2$ for $l = 1, 2$. Let for convenience $\epsilon = \varepsilon/2$. Set $\lambda_l = (1+\epsilon)^{-1/4}\Lambda_l$ for $l = 1, 2$ in Lemma 3.7. We infer readily from (3.46) that, for some $m_1 < \infty$,

$$\sum_{m \ge m_1} \mathbb{P}(\mathscr{E}_m(\epsilon)) \le \sum_{m \ge m_1} \min\Big\{ h_{\nu_m,2}^{(1+\epsilon)d_1\lambda_1^2 - d_1}, h_{\nu_m,2}^{(1+\epsilon)\{d_1\lambda_1^2+d_2\lambda_2^2\}-d_2} \Big\}.$$

Since, for each $\eta > 0$, (H3)′(ii) [or equivalently (H3)(ii)] implies that, ultimately for all large n, $h_{n,2} \le (\log n)^{-2/\eta}$, the fact that $\nu_m = \lfloor (1+\gamma)^m \rfloor$ readily implies that

$$\sum_m h_{\nu_m,2}^{\eta} = O\bigg(\sum_m \frac{1}{m^2}\bigg) < \infty.$$

In view of (3.41) and (3.42), the Borel–Cantelli lemma implies therefore that, a.s. for all m sufficiently large, we have uniformly over $x \in [0, 1/2]$, $l = 1, 2$ and $\nu_{m-1} < n \le \nu_m$,

$$(3.53)\qquad \begin{aligned} &\bigg\{ \frac{n h_{n,l} \log(1/h_{n,l})}{\nu_m h_{\nu_m,l} \log(1/h_{\nu_m,l})} \bigg\}^{1/2} L_{n,l}(z_{n,l}(x); t h_{\nu_m,l}/h_{n,l}) \in \lambda_l \mathbb{K}^{\epsilon} \\ &\quad = (1+\epsilon)^{-1/4} \Lambda_l \mathbb{K}^{\epsilon}. \end{aligned}$$

Now, it follows from (H1) that, for $l = 1, 2$ and $\nu_{m-1} < n \le \nu_m$,

$$h_{\nu_m,l} \le h_{n,l} \le \bigg\{ \frac{\nu_m}{n} \bigg\} h_{\nu_m,l} \le \bigg\{ \frac{\nu_m}{\nu_{m-1}} \bigg\} h_{\nu_m,l} \sim (1+\gamma) h_{\nu_m,l} \quad \text{as } m \to \infty.$$

This, in turn, implies that, for all large m, $l = 1, 2$ and $\nu_{m-1} < n \le \nu_m$,

$$1 \le \left\{ \frac{\nu_m h_{\nu_m, l} \log(1/h_{\nu_m, l})}{n h_{n, l} \log(1/h_{n, l})} \right\}^{1/2} \le 1 + \gamma < (1 + \epsilon)^{1/4} \tag{3.54}$$

and

$$1 \ge \frac{h_{\nu_m, l}}{h_{n, l}} = \frac{1 + o(1)}{1 + \gamma} > 1 - \gamma. \tag{3.55}$$

Here, we have used the fact, following from our assumptions, that $0 < \gamma < \epsilon/8$ and $0 < \epsilon < 1$, and hence, $\gamma < \epsilon/8 < (1+\epsilon)^{1/4} - 1$. By (3.53), (3.54), we see that, a.s. for all m sufficiently large, we have, uniformly over $x \in [0, 1/2]$, $l = 1, 2$ and $\nu_{m-1} < n \le \nu_m$,

$$L_{n, l}(z_{n, l}(x); \mathbf{I} h_{\nu_m, l}/h_{n, l}) \in \Lambda_l \mathbb{K}^{\varepsilon} = \{\Lambda_l \mathbb{K}\}^{\Lambda_l \epsilon}.$$

This, when combined with (3.40) and (3.55), shows in turn that, a.s. for all large n, we have, uniformly over all $x \in [0, 1/2]$,

$$L_{n, l}(x; \mathbf{I}) \in \{\Lambda_l \mathbb{K}\}^{\Lambda_l \epsilon + 4(\delta^{1/2} + \gamma^{1/2})} \subseteq \{\Lambda_l \mathbb{K}\}^{\Lambda_l \varepsilon} = \Lambda_l \mathbb{K}^{\varepsilon}.$$

This, together with a similar argument on [1/2, 1], where we repeat the previous steps via the mapping $x \to 1 - x$, completes the proof of (3.52). □

PROPOSITION 3.2. *For any $\epsilon > 0$, we have almost surely for all n sufficiently large,*

$$\mathscr{F}_{n, 2} \subseteq \mathbf{K}_2^{\epsilon}. \tag{3.56}$$

PROOF. It follows from (3.52) in combination with the analytic fact that, for each $\epsilon > 0$, there exists a finite sequence $\{(\Lambda_{1, r}, \Lambda_{2, r})\colon 1 \le r \le R\}$ together with an $\varepsilon > 0$, such that;

$$\begin{aligned} &\text{(i)} \quad \Lambda_l > 0 \text{ for } l = 1, 2; \quad d_1 \Lambda_{1, r}^2 \le d_1, \quad d_1 \Lambda_{1, r}^2 + d_2 \Lambda_{2, r}^2 \le d_2; \\ &\text{(ii)} \quad \min\{d_1 - d_1 \Lambda_{1, r}^2, d_2 - d_1 \Lambda_{1, r}^2 + d_2 \Lambda_{2, r}^2\} = 0; \\ &\text{(iii)} \quad \bigcup_{r=1}^{R} (\Lambda_{1, r} \mathbb{K}^{\varepsilon}, \Lambda_{2, r} \mathbb{K}^{\varepsilon}) \subseteq \mathbf{K}_2^{\epsilon}. \end{aligned} \tag{3.57}$$

We omit the details. □

PROOF OF THEOREM 3.1. The proof of (3.9) is obtained by combining (3.27) and (3.57). □

4. Proofs of Theorems 2.1 and 2.2.

4.1. *Proof of Theorem* 2.1. This subsection, devoted to the proof of Theorem 2.1, will make use of the notation and results of Sections 1–3. From now on, $k \ge 1$ will be arbitrary but fixed.

Select an $\varepsilon > 0$ and, recalling (2.16)–(2.19), set for each $l \in \mathbb{N}^*$, $n \geq 2$ and $t \in [-1, 1]$,

$$a_{n;l} = (1+\varepsilon)^{1/2^l} D_l n^{-1+(1/2)^l} (\log n)^{1-(1/2)^{l-1}} (\log_2 n)^{(1/2)^l}, \tag{4.1}$$

$$\begin{aligned} \phi_{n;l}(x,t) = &-\big(2a_{n,l}\log(1/a_{n;l})\big)^{-1/2} \\ &\times\big\{\alpha_n(x+n^{-1/2}\beta_n(x)+ta_{n,l}) - \alpha_n(x+n^{-1/2}\beta_n(x))\big\} \end{aligned} \tag{4.2}$$

and, for each $l \in \mathbb{N}$,

$$\delta_l = 1-(1/2)^l \quad \text{and} \quad \Delta_{l,k} := \frac{\delta_l}{\delta_k}. \tag{4.3}$$

Set further, for $n \geq 2$ and $l \in \mathbb{N}$,

$$t_{n;l} = t_{n;l}(x) = -\left\{\frac{n^{-1/2}(\alpha_{n;l-1}(x)+\beta_n(x))}{a_{n;l}}\right\}, \tag{4.4}$$

so that, via (2.6), (4.1) and (4.2),

$$\begin{aligned} &\phi_{n;l}(x, t_{n;l}(x)) \\ &\quad = -(2a_{n;l}\log(1/a_{n;l}))^{-1/2}\big\{\alpha_{n;l}(x) - \alpha_n(x+n^{-1/2}\beta_n(x))\big\}. \end{aligned} \tag{4.5}$$

LEMMA 4.1. *There exists an $n_0 < \infty$ a.s., such that, for all $n \geq n_0$ and $1 \leq l \leq k$,*

$$\|t_{n;l}\| \leq (1+\varepsilon/2)^{-1/2^l} < 1. \tag{4.6}$$

Moreover, we have, for each $l \in \mathbb{N}$,

$$\lim_{n\to\infty} \|t_{n;l+1} - \phi_{n;l}(t_{n;l})\| = 0 \quad \textit{a.s.} \tag{4.7}$$

PROOF. By combining (2.13) and (2.20) with (4.1)–(4.4), we obtain that, for each $l \in \mathbb{N}^*$,

$$\limsup_{n\to\infty} \|t_{n;l}\| = (1+\varepsilon)^{-1/2^l} < (1+\varepsilon/2)^{-1/2^l} < 1 \quad \text{a.s.,} \tag{4.8}$$

which yields readily (4.6). Next, we use the triangle inequality to write

$$\begin{aligned} \|t_{n;l+1} - \phi_{n;l}(t_{n;l})\| \leq \|t_{n;l+1}\| \times &\left|1 - \left\{\frac{a_{n;l+1}}{n^{-1/2}} \times (2a_{n,l}\log(1/a_{n,l}))^{-1/2}\right\}\right| \\ &+ \left\|\left\{\frac{a_{n;l+1}}{n^{-1/2}} \times (2a_{n;l}\log(1/a_{n,l}))^{-1/2}\right\} t_{n;l+1} - \phi_{n;l}(t_{n;l})\right\| \\ = \mathscr{R}'_n + \mathscr{R}''_n. \end{aligned}$$

An application of (4.8) shows that $\|t_{n;l+1}\| = O(1)$ a.s. Since, by (2.22) and (4.1),

$$\frac{a_{n;l+1}}{n^{-1/2}} \times (2a_{n,l}\log(1/a_{n;l}))^{-1/2} = \left\{\frac{D_{l+1}}{\sqrt{2\delta_l D_l}}\right\} \times \left\{\frac{2\delta_l \log n}{2\log(1/a_{n;l})}\right\}^{1/2}$$
$$= \left\{\frac{2\delta_l \log n}{2\log(1/a_{n;l})}\right\}^{1/2} \to 1 \quad \text{as } n \to \infty,$$

we have therefore $\mathscr{R}'_n \to 0$ a.s. Next, we infer from (2.5) that, almost surely,

$$\mathscr{R}''_n = (2a_{n;l}\log(1/a_{n;l}))^{-1/2}\|\beta_n + \alpha_n(\mathbf{I} + n^{-1/2}\beta_n)\|$$
$$= O\big((2na_{n;l}\log(1/a_{n;l}))^{-1/2}\big) \to 0,$$

which completes the proof of (4.7). □

In view of (2.16) and (4.3), observe that, for $l = 1, \dots, k$,

$$\frac{1}{2} \le \Delta_{l,k} = \frac{\delta_l}{\delta_k} = \frac{1-(1/2)^l}{1-(1/2)^k} = \lim_{n\to\infty} \frac{\log(1/a_{n;l})}{\log(1/a_{n;k})} \le 1. \tag{4.9}$$

Set

$$\mathbb{L}_k = \left\{(\phi_1, \dots, \phi_k) \in AC[-1,1]^k \colon \forall\ 1 \le m \le k, \sum_{l=1}^m \delta_l |\phi_l|_H^2 \le \delta_m\right\}. \tag{4.10}$$

LEMMA 4.2. *We have*

$$\limsup_{n\to\infty} \|t_{n;k+1}\| \le \sup_{(\phi_1,\dots,\phi_k)\in\mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| \quad \textit{a.s.} \tag{4.11}$$

PROOF. For each $J = [A, B] \subseteq [0, 1]$ with $A < B$, set

$$\mathscr{G}_{n,k}(J) = \big\{(\phi_{n;1}(x, \mathbf{I}), \dots, \phi_{n;k}(x, \mathbf{I})) \colon x \in J\big\}. \tag{4.12}$$

It follows from (4.1), (4.3) and (4.9) that the sequences $h_{n,l} = a_{n;l}$ fulfill the assumptions (H1), (H2) and (H3)′, (H4)′. We may therefore apply Theorem 3.1 in this case. By so doing, we obtain readily from (3.9) and (4.10) that, for each specified J as in the theorem,

$$\lim_{n\to\infty} \Delta(\mathscr{G}_{n,k}(J), \mathbb{L}_k) = 0 \quad \text{a.s.} \tag{4.13}$$

It is easy to check from (4.10) that the functions of $\mathbb{L}_k \subseteq \mathbb{K}^k$ are uniformly equicontinuous. This, when combined with (4.6), (4.7) and (4.13), readily implies that

$$\begin{aligned} \limsup_{n\to\infty} \|t_{n;k+1}\| &= \limsup_{n\to\infty} \left\{\sup_{0\le x\le 1} \big|\{\phi_{n;k}(x,\mathbf{I}) \circ \cdots \circ \phi_{n;1}(x,\mathbf{I})\}(t_{n;1}(x))\big|\right\} \\ &\le \limsup_{n\to\infty} \left\{\sup_{0\le x\le 1}\left(\sup_{(\phi_1,\dots,\phi_k)\in\mathbb{L}_k} \big|\{\phi_k \circ \cdots \circ \phi_1\}(t_{n;1}(x))\big|\right)\right\} \\ &\le \sup_{(\phi_1,\dots,\phi_k)\in\mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| \quad \text{a.s.,} \end{aligned} \tag{4.14}$$

which, in turn, yields (4.11). □

PROPOSITION 4.1. *For each $k \in \mathbb{N}^*$, we have*

$$\sup_{(\phi_1,\dots,\phi_k)\in\mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| = \sup_{(\phi_1,\dots,\phi_k)\in\mathbb{L}_k} \{\phi_k \circ \cdots \circ \phi_1\}(1)$$

$$(4.15) \qquad = \mathscr{S}_k := \left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{D_k\delta_k}\right\}^{1/2}$$

$$= \left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{\delta_k}\right\}^{1/2} \times \left\{2^{3\delta_k-2}\prod_{l=1}^{k-1}\delta_l^{1/2^{k-l}}\right\}^{-1/2},$$

the supremum $\mathscr{S}_k$ being reached for functions $\phi_l = \tilde{\phi}_l$, $l = 1,\dots,k$, of the form

$$(4.16) \qquad \tilde{\phi}_l(t) = \begin{cases} \lambda_l t, & \text{for } 0 \le t \le \prod_{i=1}^{l-1}\lambda_i, \\ 0, & \text{else,} \end{cases}$$

with $\lambda_1 = 1$ and $0 < \lambda_l \le 1$ for $1 \le l \le k$.

PROOF. Set first $k = 1$. We first recall from (2.19) that $\delta_1 = 1/2$ and $D_1 = 2^{-1/2}$. It follows that

$$\mathscr{S}_1 = \left\{\frac{2^{-3/2}}{2^{-1/2}\times(1/2)}\right\} = 1.$$

Since, via (3.2) and (4.10), $\mathbb{L}_1 = \mathbb{K}$, it is straightforward that the supremum

$$\sup_{\phi\in\mathbb{K}}\|\phi\| = 1 = \mathscr{S}_1$$

is reached for $\phi(t) = t$ for $0 \le t \le 1$, $\phi(t) = 0$ else. Thus, (4.15) and (4.16) hold in this case.

Having proved the proposition for $k = 1$, we assume from now on that $k \ge 2$. We first note that for $(\phi_1,\dots,\phi_k) \in \mathbb{L}_k$ we have $\|\phi_l\| \le |\phi_l|_H \le 1$ for each $l = 1,\dots,k$. An easy induction shows therefore that $\{\phi_k \circ \cdots \circ \phi_1\}(t)$ is defined for each $t \in [-1,1]$. Next, we use the fact that if $f \in \mathbb{H}$ and $t_1 \in (0,1]$ are such that $\|f\| = |f(t_1)|$, the function defined by

$$\tilde{f}(t) = \begin{cases} (t/t_1)f(t_1), & \text{for } 0 \le t \le t_1, \\ 0, & \text{else,} \end{cases}$$

fulfills $\|\tilde{f}\| = \|f\| = \tilde{f}(t_1) = f(t_1)$ and $|\tilde{f}|_H \le |f|_H$. Namely, by the Schwarz inequality,

$$|\tilde{f}|_H^2 = \frac{f(t_1)^2}{t_1} = \frac{1}{t_1}\left\{\int_0^{t_1}\dot{f}(t)\,dt\right\}^2 \le \left\{\int_0^{t_1}\dot{f}(t)^2\,dt\right\} \le |f|_H^2.$$

A repeated application of this property shows that, given any $(\phi_1,\dots,\phi_k)$ and $t_1 \in (0,1]$ such that $\|\phi_1 \circ \cdots \circ \phi_k\| = |\{\phi_1 \circ \cdots \circ \phi_k\}(t_1)|$, there exist functions of the form, for $l = 1,\dots,k$,

$$(4.17) \qquad \tilde{\phi}_l(t) = \begin{cases} \lambda_l t, & \text{for } 0 \le t \le t_l := t_1\Lambda_l, \\ 0, & \text{else,} \end{cases} \qquad \text{with } \Lambda_l = \prod_{j=1}^{l-1}\lambda_j,$$

fulfilling $|\tilde{\phi}_l|_H \le |\phi_l|_H$ for $l = 1, \dots, k$ and $\|\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\| = \|\phi_k \circ \cdots \circ \phi_1\|$. Moreover, if

$$\hat{\phi}_1(t) = \begin{cases} (\lambda_1 t_1)t, & \text{for } 0 \le t \le 1, \\ 0, & \text{else,} \end{cases} \quad \text{and} \quad \hat{\phi}_l = \tilde{\phi}_l \quad \text{for } 2 \le l \le k,$$

then it is easily cheched that $|\hat{\phi}_l|_H \le |\tilde{\phi}_l|_H$ for $l = 1, \dots, k$ and $\|\hat{\phi}_k \circ \cdots \circ \hat{\phi}_1\| = \|\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\|$. In view of the definition (4.10) of $\mathbb{L}_k$, we may therefore restrict ourselves to the functions $\hat{\phi}_l$ or $\tilde{\phi}_l$ with $t_1 = 1$. Since the supremum in (4.15) is reached for a suitable choice of these functions, (4.16) is established. We set, from now on, $t_1 = 1$ in (4.17) and observe that

$$|\tilde{\phi}_l|_K^2 = \lambda_l \prod_{j=1}^{l} \lambda_j \quad \text{for } l = 1, \dots, k.$$

By all this, we are led to evaluate the numerical value of the supremum of

$$\|\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\| = \prod_{l=1}^{k} \lambda_l,$$

given that

$$(\mathscr{I}_m) \qquad \sum_{l=1}^{m} \delta_l |\tilde{\phi}_l|_H^2 = \sum_{l=1}^{m} \left\{ \delta_l \lambda_l \prod_{j=1}^{l} \lambda_j \right\} \le \delta_m$$

holds for $m = 1, \dots, k$. By the change of variables $x_l = \lambda_1 \cdots \lambda_l$ for $l = 1, \dots, k$, our problem reduces to finding the supremum of $x_k > 0$ when $x_1 > 0$, $\dots, x_{k-1} > 0$ vary in such a way that

$$(\mathscr{J}_m) \qquad \sum_{l=1}^{m} \delta_l \left\{ \frac{x_l^2}{x_{l-1}} \right\} \le \delta_m$$

holds for $m = 1, \dots, k$. Here, we set for convenience $x_0 = 1$. Let us first limit ourselves to $(\mathscr{J}_k)$, written as an equality. We consider the function of $x_1, \dots, x_{k-1}$,

$$\delta_k x_k^2 = \Psi(x_1, \dots, x_{k-1}) := x_{k-1} \left(\delta_k - \sum_{l=1}^{k-1} \delta_l \left\{ \frac{x_l^2}{x_{l-1}} \right\} \right). \tag{4.18}$$

By setting $\hat{x}_0 = 1$ and letting $\hat{x}_1, \dots, \hat{x}_{k-1}$ denote the solutions of the set of equations

$$\frac{\partial}{\partial x_l} \Psi(x_1, \dots, x_{k-1}) = 0 \quad \text{for } l = 1, \dots, k-2,$$

we obtain the equalities

$$\delta_l \left\{ \frac{\hat{x}_l^2}{\hat{x}_{l-1}} \right\} = 2\delta_{l-1} \left\{ \frac{\hat{x}_{l-1}^2}{\hat{x}_{l-2}} \right\} \quad \text{for } 1 \le l \le k-1. \tag{4.19}$$

A recursive application of (4.19) shows that, for each $1 \le l \le k-1$,

$$\delta_l\left\{\frac{\hat{x}_l^2}{\hat{x}_{l-1}}\right\} = 2^{l-1}\delta_1\left\{\frac{\hat{x}_l^2}{x_0}\right\} = 2^{l-2}\hat{x}_1^2 \quad \text{for } l = 1,\dots,k-1, \tag{4.20}$$

where we have used, via (2.19), the fact that $\delta_1 = 1/2$. This, in turn, implies that

$$\hat{x}_{k-1} = \prod_{l=1}^{k-1}\left\{\frac{\hat{x}_l^2}{\hat{x}_{l-1}}\right\}^{1/2^{k-l}} = \prod_{l=1}^{k-1}\left\{\frac{2^{l-2}\hat{x}_1^2}{\delta_l}\right\}^{1/2^{k-l}} = \hat{x}_1^{2\delta_{k-1}}V_k, \tag{4.21}$$

where $\prod_{\varnothing}(\cdot) = 1$, and we set for convenience, for each $\kappa \in \mathbb{N}^*$,

$$V_\kappa = \prod_{l=1}^{\kappa-1}\left\{\frac{2^{l-2}}{\delta_l}\right\}^{1/2^{\kappa-l}} = 2^{\kappa-1-3\delta_{\kappa-1}}\prod_{l=1}^{\kappa-1}\delta_l^{-1/2^{\kappa-l}}. \tag{4.22}$$

In (4.21) and (4.22), we have used the routine equalities (recall that $k \ge 2$)

$$\sum_{l=1}^{k-1}\frac{l-2}{2^{k-l}} = k-4+\frac{1}{2^{k-1}}+\frac{1}{2^{k-2}} = k-1-3\delta_{k-1}, \tag{4.23}$$

$$\sum_{l=1}^{k-1}\frac{1}{2^{k-l}} = 1-\frac{1}{2^{k-1}} = \delta_{k-1}. \tag{4.24}$$

Since $\delta_k - \delta_{k-1} = 2^{1-k} - 2^{-k} = 2^{-k} = 1 - \delta_k$, it follows from (2.17) and (4.22) that

$$D_kV_k = \left\{2^{3\delta_k-2}\prod_{l=1}^{k-1}\delta_l^{1/2^{k-l}}\right\}\left\{2^{k-1-3\delta_{k-1}}\prod_{l=1}^{k-1}\delta_l^{-1/2^{k-l}}\right\} = 2^{k-3\delta_k}. \tag{4.25}$$

By replacing in (4.18) $x_0 = 1, x_1,\dots,x_{k-1}$ by their expressions $\hat{x}_0 = 1$, $x_1,\dots,\hat{x}_{k-1}$ in terms of $\hat{x}_1$ following from (4.20), (4.21), we obtain, via (4.3), the equation

$$\begin{aligned}\delta_k x_k^2 = \psi(\hat{x}_1) &:= \Psi(\hat{x}_1,\dots,\hat{x}_{k-1}) = \hat{x}_{k-1}\left(\delta_k - \hat{x}_1^2\sum_{l=1}^{k-1}2^{l-2}\right)\\ &= V_k\hat{x}_1^{2\delta_{k-1}}(\delta_k - 2^{k-2}\delta_{k-1}\hat{x}_1^2).\end{aligned} \tag{4.26}$$

The supremum $\hat{x}_k$ of $\psi(\hat{x}_1)$ in (4.26) is obviously reached when $\hat{x}_1$ fulfills, via (2.6),

$$2^{k-2}\hat{x}_1^2 = \frac{\delta_k}{1+\delta_{k-1}} = \frac{1}{2} \quad \Leftrightarrow \quad \hat{x}_1 = 2^{-(k-1)/2}. \tag{4.27}$$

This, in turn, yields $\delta_k - 2^{k-2}\delta_{k-1}\hat{x}_1^2 = (2\delta_k - \delta_{k-1})/2 = 1/2$, whence, by (4.25)–(4.27),

$$\begin{aligned}\hat{x}_k &= \left\{\frac{\psi(\hat{x}_1)}{\delta_k}\right\}^{1/2} = \left\{\frac{V_k\hat{x}_1^{2\delta_{k-1}}}{2\delta_k}\right\}^{1/2} = \left\{\frac{V_kD_k2^{-(k-1)\delta_{k-1}}}{2D_k\delta_k}\right\}^{1/2}\\ &= \left\{\frac{2^{-1+k-3\delta_k-(k-1)(2\delta_k-1)}}{D_k\delta_k}\right\}^{1/2} = \left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{D_k\delta_k}\right\}^{1/2} = \mathscr{S}_k,\end{aligned}$$

as given in (4.15). The proof of the equalities in (4.15) is completed by checking that $\hat{x}_1, \ldots, \hat{x}_k$ fulfill $(\mathscr{J}_m)$ for $m = 1, \ldots, k-1$. By combining (4.3), (4.20) and (4.27), we get

$$\sum_{l=1}^{m} \delta_l \left\{ \frac{\hat{x}_l^2}{\hat{x}_{l-1}} \right\} = \sum_{l=1}^{m} 2^{l-2} \hat{x}_1^2 = \sum_{l=1}^{m} \frac{2^{l-2}}{2^{k-1}} = 2^{m-k} \delta_m \leq \delta_m \quad \text{for } m = 1, \ldots, k.$$

which is sufficient for our needs. □

LEMMA 4.3. *We have*

$$\limsup_{n \to \infty} \|t_{n;\,k+1}\| \geq \left\{ \frac{1-\varepsilon}{1+\varepsilon} \right\} \sup_{(\phi_1, \ldots, \phi_k) \in \mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| \quad \text{a.s.} \tag{4.28}$$

PROOF. The proof is inspired by arguments of Shorack (1982). Let $\tilde{\phi}_l$, $l = 1, \ldots, k$ be functions, of the form given in (4.16), and such that

$$\sup_{(\phi_1, \ldots, \phi_k) \in \mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| = \{\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\}(1). \tag{4.29}$$

Obviously, if 1_A denotes the indicator function of A, for each $t \in [-1, 1]$,

$$\{\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\}(t) = t 1_{[0,1]} \sup_{(\phi_1, \ldots, \phi_k) \in \mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\|. \tag{4.30}$$

We obtain readily from (4.29) and (4.30), in combination with (3.9) and (4.13), (4.14), that for each specified interval $J = [A, B] \subseteq [0, 1]$ with $A < B$,

$$\begin{aligned} &\limsup_{n \to \infty} \|t_{n;\,k+1}\| \\ &\quad \geq \limsup_{n \to \infty} \left\{ \sup_{x \in J} \left| \{\phi_{n;\,k}(x, \mathbf{I}) \circ \cdots \circ \phi_{n;\,1}(x, \mathbf{I})\}(t_{n;\,1}(x)) \right| \right\} \\ &\quad \geq \limsup_{n \to \infty} \left\{ \inf_{x \in J} \left(\{\tilde{\phi}_k \circ \cdots \circ \tilde{\phi}_1\}(t_{n;\,1}(x)) \right) \right\} \\ &\quad \geq \left(\limsup_{n \to \infty} \left\{ \inf_{x \in J} 1_{[0,1]}(t_{n;\,1}(x)) \right\} \right) \sup_{(\phi_1, \ldots, \phi_k) \in \mathbb{L}_k} \|\phi_k \circ \cdots \circ \phi_1\| \quad \text{a.s.} \end{aligned} \tag{4.31}$$

Thus, by (4.31), we need only show the existence of $0 \leq A_\varepsilon < B_\varepsilon \leq 1$ such that the event

$$\left\{ \frac{1-\varepsilon}{1+\varepsilon} \leq t_{n;\,1}(x) \colon \forall\, x \in [A_\varepsilon, B_\varepsilon] \right\} \tag{4.32}$$

holds a.s. i.o. in n. Now, the Finkelstein (1971) law of the iterated logarithm, in combination with (1.4), shows that the sequence $\{(2 \log_2 n)^{-1/2} \beta_n \colon n \geq 1\}$ is a.s. compact in $(B[0, 1], \mathscr{U})$ with limit set equal to

$$\mathscr{F} := \{f \in AC[0, 1] \colon f(0) = f(1) = 0 \text{ and } |f|_H \leq 1\}.$$

Set $f(t) = -\min\{t, 1-t\} 1_{[0,1]}$ for $t \in [0, 1]$. Since $f \in \mathscr{F}$, we have therefore

$$\limsup_{n \to \infty} \|(2 \log_2 n)^{-1/2} \beta_n - f\| = 0 \quad \text{a.s.},$$

so that the choices of $A_\varepsilon = (1-\frac{1}{2}\varepsilon)/2$ and $B_\varepsilon = (1+\frac{1}{2}\varepsilon)/2$ ensure that the event

$$\Big\{(1-\varepsilon)/2 \le -(2\log_2 n)^{-1/2}\beta_n(x) \le (1+\varepsilon)/2, \forall\, x \in [A_\varepsilon, B_\varepsilon]\Big\} \tag{4.33}$$

holds a.s. i.o. in n. By the definition (4.4) of $t_{n;1}$, we infer from (2.19) and (4.1) that

$$t_{n;1}(x) = -\left\{\frac{2}{1+\varepsilon}\right\}(2\log_2 n)^{-1/2}\beta_n(x).$$

Thus, by (4.33), the event (4.32) holds a.s. i.o. in n, as sought. □

PROOF OF THEOREM 1.1. By (2.13), (2.22), and the definition (4.4) of $t_{n,\,k+1}$,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}}(\log n)^{-1+(1/2)^k}(\log_2 n)^{-(1/2)^{k+1}}\|\beta_n - n^{-1/2}K(n,\mathbf{I};k)\|\\
&\quad= \limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}}(\log n)^{-1+(1/2)^k}(\log_2 n)^{-(1/2)^{k+1}}\|\beta_n + \alpha_{n;k}\|\\
&\quad= \big\{D_{k+1}(1+\varepsilon)^{1/2^{k+1}}\big\}\limsup_{n\to\infty}\|t_{n;k+1}\|\\
&\quad= \big\{(2\delta_k D_k)^{1/2}(1+\varepsilon)^{1/2^{k+1}}\big\}\limsup_{n\to\infty}\|t_{n;k+1}\| \quad \text{a.s.}
\end{aligned} \tag{4.34}$$

On the other hand, we get, by combining (4.11), (4.15) and (4.28),

$$\begin{aligned}
\left\{\frac{1-\varepsilon}{1+\varepsilon}\right\}\left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{D_k\delta_k}\right\}^{1/2} &\le \limsup_{n\to\infty}\|t_{n;k+1}\|\\
&\le \left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{D_k\delta_k}\right\}^{1/2} \quad \text{a.s.}
\end{aligned} \tag{4.35}$$

Recalling that $\delta_k = 1-2^{-k}$, it is readily checked that

$$(2\delta_k D_k)^{1/2}\left\{\frac{2^{2(k-1)(1-\delta_k)-3\delta_k}}{D_k\delta_k}\right\}^{1/2} = 2^{\{1+2(k-1)(1-\delta_k)-3\delta_k\}/2} = 2^{-1+(2k+1)2^{-(k+1)}},$$

which, when combined with (4.34) and (4.35), and the fact that $\varepsilon > 0$ may be rendered arbitrarily small, yields (1.12). □

4.2. *Proof of Theorem* 1.2. Let β_n and a Kiefer process $\mathscr{K}(n,\mathbf{I})$ be defined on the same probability space. An application of Lemma A1 of Berkes and Philipp (1979) enables enlarging this space to carry also $K(n,\mathbf{I}) = -K_1(n,\mathbf{I})$ as in (1.3)–(1.5). Therefore, from now on, we may and do assume that β_n, $\mathscr{K}(n,\mathbf{I})$ and $K(n,\mathbf{I})$ are jointly defined on the probability space $(\Omega,\mathscr{A},\mathbb{P})$ of Sections 1–3. The following Proposition gives a key argument for the proof of Theorem 1.2.

PROPOSITION 4.2. *Assume that* $k \in \mathbb{N}$, $a \in (0,1/2)$, $b, c \in \mathbb{R}^+$ *and* $d \ge 0$ *are such that*

$$\limsup_{n\to\infty} n^a(\log n)^{-b}(\log_2 n)^{-c}\|\beta_n - n^{-1/2}\mathscr{K}(n,\mathbf{I};k)\| \le d \quad \textit{a.s.} \tag{4.36}$$

Then, for each $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} n^{-(\varepsilon/2)-(1-\delta_{k+1}\vee(1-2a)/2)}\|K(n,\mathbf{I}) - \mathscr{K}(n,\mathbf{I})\| = 0 \quad a.s. \tag{4.37}$$

PROOF. We set $\gamma_n = n^{-1/2}\mathscr{K}(n,\mathbf{I};k)$ in Lemma 2.1, to obtain, via (2.15), that

$$\begin{aligned}\limsup_{n\to\infty} & n^{(2a+1)/4}(\log n)^{-(b+1)/2}(\log_2 n)^{-c/2}\\ &\times \|\beta_n - n^{-1/2}K(n,\mathbf{I}+n^{-1/2}\mathscr{K}(n,\mathbf{I};k))\|\\ &\le d^{1/2}(2a+1)^{1/2} \quad \text{a.s.}\end{aligned} \tag{4.38}$$

On the other hand, it follows from (1.12) that

$$\begin{aligned}\limsup_{n\to\infty} & n^{-(1/2)+\delta_{k+2}}(\log n)^{-\delta_{k+1}}(\log_2 n)^{1-\delta_{k+2}}\\ &\times\big\|\beta_n - n^{-1/2}K(n,\mathbf{I}+n^{-1/2}K(n,\mathbf{I};k))\big\| = C_{k+1} \quad \text{a.s.}\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}\limsup_{n\to\infty} & n^{-(1/2)+\delta_{k+1}}(\log n)^{-\delta_k}(\log_2 n)^{1-\delta_{k+1}}\\ &\times\big\|\beta_n - n^{-1/2}K(n,\mathbf{I};k)\big\| = C_k \quad \text{a.s.}\end{aligned} \tag{4.40}$$

Since $K(n,\mathbf{I})$ and $\mathscr{K}(n,\mathbf{I})$ are identically distributed, we have, by (4.39), (4.40),

$$\begin{aligned}\limsup_{n\to\infty} & n^{-(1/2)+\delta_{k+1}}(\log n)^{-\delta_k}(\log_2 n)^{1-\delta_{k+1}}\\ &\times\big\|n^{-1/2}\mathscr{K}(n,\mathbf{I};k) - n^{-1/2}\mathscr{K}(n,\mathbf{I}+n^{-1/2}\mathscr{K}(n,\mathbf{I};k))\big\|\\ &= \limsup_{n\to\infty} n^{-(1/2)+\delta_{k+1}}(\log n)^{-\delta_k}(\log_2 n)^{1-\delta_{k+1}}\\ &\quad\times\big\|n^{-1/2}K(n,\mathbf{I};k) - n^{-1/2}K(n,\mathbf{I}+n^{-1/2}K(n,\mathbf{I};k))\big\| = C_k \quad \text{a.s.}\end{aligned} \tag{4.41}$$

The triangle inequality in combination with (4.36), (4.38) and (4.41) shows readily that

$$\begin{aligned}\limsup_{n\to\infty} & n^{-A}(\log n)^{-B}(\log_2 n)^{-C}\|K(n,\mathbf{I}) - \mathscr{K}(n,\mathbf{I})\|\\ &= \limsup_{n\to\infty} n^{-A}(\log n)^{-B}(\log_2 n)^{-C}\\ &\quad\times\|K(n,\mathbf{I}+n^{-1/2}\mathscr{K}(n,\mathbf{I};k)) - \mathscr{K}(n,\mathbf{I}+n^{-1/2}\mathscr{K}(n,\mathbf{I};k))\| \le D \ \text{a.s.},\end{aligned} \tag{4.42}$$

where $A = \max\{1-\delta_{k+1}, (1-2a)/2\}$; $B = \max\{\delta_k, b\}$, $C = \max\{1-\delta_{k+1}, c\}$ and $D = d + C_k$. This obviously implies (4.37). □

PROOF OF THEOREM 1.2. Select an arbitrary $\varepsilon > 0$. By setting $a = \theta_k$ and $b = c = d = 0$ in Proposition 4.1, we may rewrite (4.37) into

$$\limsup_{n\to\infty} n^{-(\varepsilon/2)-(1-\delta_{k+1}\vee(1-2\theta_k)/2)}\|K(n,\mathbf{I}) - \mathscr{K}(n,\mathbf{I})\| = 0 \quad \text{a.s.} \tag{4.43}$$

Since our assumptions imply that $1 - \delta_{k+1} \vee (1 - 2\theta_k)/2 \to 0$ as $k \to \infty$, with $k \in S$, we infer from (4.43) that, for any specified $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} n^{-\varepsilon} \| K(n, \mathbf{I}) - \mathscr{K}(n, \mathbf{I}) \| = 0 \quad \text{a.s.} \tag{4.44}$$

By applying (4.40) for each value of $k \in \mathbb{N}$, (1.12) and (4.44) readily imply (1.16). □

5. Applications and examples.

5.1. *Extended Bahadur–Kiefer-type representations.* Let $\alpha_{n;k}$ be the kth iterated empirical process in (2.6). The following theorem gives an extended form of the uniform Bahadur–Kiefer representation (1.4), obtained for $k = 1$.

THEOREM 5.1. *For each $k, l \in \mathbb{N}$ with $l \geq k+1$, we have*

$$\begin{aligned} &\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \| \beta_n + \alpha_{n;k} \| \\ &\quad = \limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \| \alpha_{n;k} - \alpha_{n;l} \| \\ &\quad = C_k = 2^{-1+(2k+1)2^{-(k+1)}} \quad \textit{a.s.} \end{aligned} \tag{5.1}$$

Moreover (5.1) *holds with the formal replacement of $\alpha_{n;k}$ by $-n^{-1/2} K(n, \mathbf{I}; k)$.*

For the proof, combine (1.12) and (2.13).

EXAMPLE 5.1. By setting $k = 2$ in (5.1), we obtain the following version of (1.14) [see (3.3), page 100 in Stute (1982)]:

$$\limsup_{n\to\infty} n^{3/8} (\log n)^{-3/4} (\log_2 n)^{-1/8} \| \beta_n + \alpha_n (\mathbf{I} - n^{-1/2} \alpha_n) \| = 2^{-3/8} \text{ a.s.} \tag{5.2}$$

5.2. *Strong limit theorems for quantile density estimators.* Let $\{X_n : n \geq 1\}$ be i.i.d. r.v.'s with distribution function $F(x) = \mathbb{P}(X_1 \leq x)$, and endpoints $x_0 = \inf\{x : F(x) > 0\} < x_1 = \sup\{x : F(x) < 1\}$. Let $Q(t) = \inf\{x : F(x) \geq t\}$ for $0 < t < 1$ be the corresponding quantile function, and make the following assumptions.

(Q1) F is twice continuously differentiable on (x_0, x_1), with derivatives f and f'.

(Q2) $f(x) > 0$ on (x_0, x_1).

(Q3) For some $\gamma < \infty$, we have

$$\sup_{0<t<1} t(1-t) |f'(Q(t))| / f^2(Q(t)) \leq \gamma. \tag{5.3}$$

(Q4) For some $\Gamma < \infty$, we have

$$\sup_{0<t<1} \{t(1-t)\}^2 | / f^2(Q(t)) \leq \Gamma. \tag{5.4}$$

For each $n \geq 1$, denote by $X_{1,n} < \cdots < X_{n,n}$ the order statistics of $X_1, \ldots, X_n$ [a.s. distinct by (Q1)]. Set $Q_n(t) = X_{\lceil nt \rceil, n}$ for $0 < t \leq 1$ and define the *quantile process* by

$$\rho_n(t) = n^{1/2} f(Q(t))(Q_n(t) - Q(t)) \quad \text{for } n \geq 1 \text{ and } 0 < t < 1. \tag{5.5}$$

Fact 6 below follows from Theorem 3.1 of Csörgő, Csörgő, Horváth and Révész (1984). Let $U_n = F(X_n)$ for $n \geq 1$ and assume, without loss of generality, that $\{X_n \colon n \geq 1\}$ is defined on $(\Omega, \mathscr{A}, \mathbb{P})$.

FACT 6. Under (Q1)–(Q3), for each $\varepsilon > 0$, we have

$$\sup_{1/(n+1) \leq t \leq n/(n+1)} |\rho_n(t) - \beta_n(t)| = O\big(n^{-1/2} (\log n)^{(1+\varepsilon)(\gamma-1)}\big) \quad \text{a.s.} \tag{5.6}$$

An easy corollary of Theorem 1.1 and Fact 6 is as follows.

COROLLARY 5.1. *On $(\Omega, \mathscr{A}, \mathbb{P})$, for each $k \in \mathbb{N}$, we have*

$$\begin{aligned} &\limsup_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \\ &\quad \times \sup_{1/(n+1) \leq t \leq n/(n+1)} \big|\rho_n(t) - n^{-1/2} K(n, \mathbf{I}; k)\big| = C_k \quad \textit{a.s.} \end{aligned} \tag{5.7}$$

PROOF. In view of (5.5) and (5.6), we need only show that, for $J_n = [0, 1/(n+1)]$ or $J_n = [n/(n+1), 1]$ we have

$$\begin{aligned} &\lim_{n\to\infty} n^{(1/2)-(1/2)^{k+1}} (\log n)^{-1+(1/2)^k} (\log_2 n)^{-(1/2)^{k+1}} \\ &\quad \times \sup_{t \in J_n} |\beta_n(t) - n^{-1/2} K(n, \mathbf{I}; k)| = 0. \end{aligned}$$

This can be achieved by routine arguments which we omit. □

For constants $0 < a < 1$ and $0 < b < 1$, set $h_n = n^{-a}$, $\epsilon_n = n^{-b}$, and consider the naive estimator of the *quantile density function* $q(t) = 1/f(Q(t))$ defined by

$$q_n(t) = \frac{Q_n(t + h_n) - Q_n(t - h_n)}{2h_n} \quad \text{for } \epsilon_n < t < 1 - \epsilon_n.$$

The next fact is due to Csörgő and Révész (1984).

FACT 7. Under (Q1)–(Q4), whenever $a, b, d > 0$ fulfill the inequalities $3b + d < a < 1/2$ and $2d + 4b + a < 1$, we have

$$\lim_{n\to\infty} n^d \sup_{\epsilon_n \leq t \leq 1-\epsilon_n} |q_n(t) - q(t)| = 0 \quad \text{a.s.} \tag{5.8}$$

A crucial step in the proof of (5.8) is a Kiefer process approximation (1.6) of β_n at an a.s. uniform rate of $O(n^{-1/4+o(1)})$. This leads to the condition $a < 1/2$, imposed in the statement of Fact 7. The replacement of (1.6) by the kth iterated Kiefer process approximation (1.12) allows an a.s. uniform rate of $O(n^{-1/2+\varepsilon})$, where $\varepsilon > 0$ can be chosen as small as desired for a suitably large k. Similar methods as that used by Csörgő and Révész (1984) then allow treating the case of $0 < a < 1$. Because this example is being given only to illustrate the applications of our theorems, the details will be given elsewhere.

5.3. *Limit theorems for kernel density estimators.* Our last example is an application of Theorem 3.1 in the setting of *kernel estimation* of the density f [refer to Devroye and Györfi (1985), Bosq and Lecoutre (1987), Scott (1992) and the references therein]. When two different derivatives of f are estimated, the optimal bandwidths and kernels are typically different [see, e.g., Section 6.2 in Scott (1992)]. Theorem 5.1 below gives a description of the joint limiting behavior of the corresponding estimators.

Let $X_1, X_2, \dots$ be i.i.d. r.v.'s with distribution function $F(x) = \mathbb{P}(X_1 \le x)$ and density $f(x) = f^{(0)}(x) = F'(x)$ continuous and positive on $[A, B]$ ($\infty < A < B < \infty$). For $l = 1, 2$ and $p_l \in \mathbb{N}$, denote by $H_l(\cdot)$ a p_l times differentiable function such that, for $l = 1, 2$:

(K1) $H_l^{(p_l)}(\cdot)$ is of bounded variation on $\mathbb{R}$.
(K2) For some $0 < M_l < \infty$, $H_l^{(p_l)}(u) = 0$ for all $|u| \ge M_l$.
(K3) $\int_{-\infty}^{\infty} H_l(u)\,du = 1$.

For $l = 1, 2$, let $\{\lambda_{n;l}\colon n \ge 1\}$ be positive constants fulfilling the conditions:

(L1) (i) $\lambda_{n;l} \downarrow 0$; (ii) $n\lambda_{n;l} \uparrow$.
(L2) (i) $n\lambda_{n;l}/\log n \to \infty$; (ii) $\{\log(1/\lambda_{n;l})\}/\log_2 n \to \infty$.
(L3) (i) $\{\log(1/\lambda_{n;1})\}/\log(1/\lambda_{n;2}) \to d \in (0, 1]$; (ii) $\lambda_{n;2}/\lambda_{n;1} \to 0$.

For $l = 1, 2$, introduce the estimator of the p_lth derivative $f^{(p_l)}$ of f given by

$$f_{n;l}^{(p_l)}(x) = \frac{1}{n\lambda_{n;l}^{p_l+1}} \sum_{i=1}^{n} K_l^{(p_l)}\left(\frac{x - X_i}{\lambda_{n;l}}\right) \quad \text{for } x \in \mathbb{R}. \tag{5.9}$$

Consider the functions of $x \in [A, B]$ defined by

$$\begin{aligned} Y_{n,l}(x) = {} & \left\{\frac{f_{n;l}^{(p_l)}(x) - \mathbb{E}(f_{n;l}^{(p_l)}(x))}{\sqrt{f(x)}}\right\} \\ & \times \left\{\frac{2\log_+(1/\lambda_{n;l})}{n\lambda_{n;l}^{2p_l+1}} \int_{-\infty}^{\infty} H^{(p_l)}(u)^2\,du\right\}^{-1/2} \end{aligned} \tag{5.10}$$

and introduce the random subset of $\mathbb{R}$ defined (for all large n) by

$$\mathscr{D}_n = \{(Y_{n;1}(x), Y_{n;2}(x))\colon A \le x \le B\}. \tag{5.11}$$

COROLLARY 5.2. *We have, almost surely,*

$$\mathscr{D}_n \to \mathscr{D}_\infty(d) := \{(y_1, y_2) \in \mathbb{R}^2: y_1^2 \le 1, dy_1^2 + y_2^2 \le 1\}, \tag{5.12}$$

with convergence under the Hausdorff set-metric generated by the usual distance in $\mathbb{R}^2$.

The result follows by repeating the arguments of Section 4.2. in Deheuvels and Mason (1992b) in the setting of Theorem 3.1. We omit details.

REMARK 5.1. (i) The arguments of Section 4.2 in Deheuvels and Mason (1992b) show that, for $l = 1, 2$,

$$\{Y_{n,l}(x): A \le x \le B\} \to [-1, 1] \quad \text{a.s.},$$

with convergence under the Hausdorff set-metric generated by the usual distance in $\mathbb{R}$.

(ii) For an arbitrary $0 \le d \le 1$, we have, with the notation given in (5.12),

$$\mathscr{D}_\infty(0) = \{(y_1, y_2): y_1^2 + y_2^2 \le 1\} \subseteq \mathscr{D}_\infty(d) \subseteq \mathscr{D}_\infty(1) = [-1, 1]^2.$$

(iii) (5.12) holds independently of the existence of the derivatives $f^{(p_l)}$ of f.

Acknowledgment. We thank the referee for a careful reading of our manuscript and for insightful comments leading, in particular, to a simplified proof of Lemma 2.2.

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