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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS WITH QUADRATIC GROWTH

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We provide existence, comparison and stability results for one-dimensional backward stochastic differential equations (BSDEs) when the coefficient (or generator) $F(t, Y, Z)$ is continuous and has a quadratic growth in Z and the terminal condition is bounded. We also give, in this framework, the links between the solutions of BSDEs set on a diffusion and viscosity or Sobolev solutions of the corresponding semilinear partial differential equations.

1. Introduction. Backward stochastic differential equations (BSDE) are equations of the following type:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \qquad 0 \le t \le T, \tag{1}$$

where $(W_t)_{0\le t\le T}$ is a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$, with $(\mathscr{F}_t)_{0\le t\le T}$ the standard Brownian filtration. The random function $F: [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is generally called a *coefficient*, T the *terminal time*, which may be a stopping time, and the $\mathbb{R}^n$-valued $\mathscr{F}_T$-adapted random variable ξ a *terminal condition*; (F, T, ξ) are the parameters of (1). The integer n is known as the *dimension* of the BSDE.

A *solution* is a couple $(Y_t, Z_t)_{0\le t\le T}$ of processes adapted to the filtration $(\mathscr{F}_t)_{0\le t\le T}$, which have some integrability properties, depending on the framework imposed by the type of assumptions on F. In order to simplify the notations, we sometimes write (Y, Z) for the process $(Y_t, Z_t)_{0\le t\le T}$.

Nonlinear BSDEs were first introduced by Pardoux and Peng [15]. When F is Lipschitz continuous in the variables Y and Z and ξ is square integrable, a solution is a couple $(Y_t, Z_t)_{0\le t\le T}$ of *square integrable* adapted processes. In that framework, Pardoux and Peng gave the first existence and uniqueness results for n-dimensional BSDEs. Since then, BSDEs have been studied with great interest. In particular, many efforts have been made to relax the assumptions on the driver; for instance, Lepeltier and San Martin [14] have proved the existence of a solution for one-dimensional BSDEs when the coefficient is only continuous with linear growth.

The interest in BSDEs comes from their connections with different mathematical fields, such as mathematical finance, stochastic control, and partial

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differential equations (see [10] for an extensive bibliography). We are especially concerned in this paper with the latter connection.

Numerous results (for instance, [16], [1]) show the connections between BSDEs set from a diffusion (or forward–backward system) and solutions of a large class of quasilinear parabolic and elliptic partial differential equations (PDEs). Those results may be seen as a generalization of the celebrated Feynman–Kac formula. Through all these results, a formal dictionary between BSDEs and PDEs can be established, which suggests that existence and uniqueness results which can be obtained on one side should have their counterparts on the other side.

This idea is the starting point of this work, where we consider BSDEs with quadratic growth in Z. Indeed, [4, 5, 6] and [3] have given existence and uniqueness results for quasilinear PDEs set *in a bounded domain* when the nonlinearity has a quadratic growth in the gradient of the solution.

In this paper, we obtain general existence and uniqueness results for one-dimensional BSDEs when the coefficient has a quadratic growth in Z, and connections with both viscosity and Sobolev solutions of PDEs when the nonlinearity has a quadratic growth in the gradient.

Here, because of the quadratic growth of the coefficient, we are looking for solutions such that $(Y_t)_{0\le t\le T} \in \mathscr{H}_T^\infty(\mathbb{R})$, where $\mathscr{H}_T^\infty(\mathbb{R})$ is the set of one-dimensional *progressively measurable* processes which are almost surely bounded, for almost every t (in short, $(Y_t)_{0\le t\le T}$ is a one-dimensional *bounded* process) while $(Z_t)_{0\le t\le T}$ remains in $\mathscr{H}_T^2(\mathbb{R}^d)$ where $\mathscr{H}_T^2(\mathbb{R})$ is the set of *progressively measurable* processes $(Z_t)_{0\le t\le T}$ with values in $\mathbb{R}^d$ such that

$$\mathbb{E}\int_0^T |Z_s|^2\,ds < \infty$$

(in short, $(Z_t)_{0\le t\le T}$ is an integrable adapted process).

Although many ideas of [4, 5, 6] or [3] are used throughout this work, the difference of framework and of the studied subject, require additional arguments. In fact, our results are not the exact counterpart of their results, since they formally correspond to PDEs set on $\mathbb{R}^N$ and not on bounded domains, a rather important difference for PDEs.

In the first section, we are concerned with general BSDEs, whose coefficient is continuous with quadratic growth, whose terminal time is not necessarily deterministic nor bounded and whose terminal condition may be a stopping time. We first give existence results under general assumptions. We next give a uniqueness result: the one-dimensional framework allows us to provide a comparison result which implies uniqueness as a by-product. However, to prove it, we use stronger assumptions on the coefficient F than for the existence result: the quadratic growth is meant, roughly speaking, as linear growth on the partial derivatives of F with respect to Z. We also give a stability result: the solutions (Y^n, Z^n) of BSDEs with parameters (F^n, ξ^n) converge to the unique solution (Y, Z) of the BSDE with parameters (F, ξ) under very general assumptions of the convergence of $(F^n)_n$ to F when F satisfies the assumptions required for the uniqueness result to hold.

The second section provides connections between the solutions of those BSDEs and solutions of related quasilinear PDEs. Of course, when the solution of the PDE is smooth enough, the meaning of this connection is as usual straightforward. Such a regularity may be obtained under strong assumptions on the coefficients of the PDE and of the nonlinearity [15]. Conversely, if one assumes that such assumptions hold on the coefficients of a forward–backward system, the flow thus defined has also a great regularity, and it can define a classical solution of the PDE.

However, when the hypotheses are such that the PDE has to be solved in a "weak" way, difficulties arise. Different approaches can be used relying on different notion of weak solutions for the associated PDEs. A first one consists in using the notion of viscosity solutions. This notion of "weak solutions" was introduced by Crandall and Lions [8] for first-order Hamilton–Jacobi equations, and extended to second order equations by Lions. The connection with BSDEs has been done by Pardoux and Peng [16] for Lipschitz continuous coefficients. In Section 2.1, we give, in our framework, a proof that the BSDEs provide a viscosity solution for the associated PDE. We also prove a uniqueness result for the viscosity solution of this PDE.

Another way of defining weak solutions of PDEs, which is more classical, is the notion of Sobolev solutions. Barles and Lesigne [2] were the first to use this approach in order to connect the solution of the PDE with the associated BSDE. It gives interesting insights, as the solutions of the PDE allow one to obtain the whole solution $(Y_t, Z_t)_{0\le t\le T}$ rather than only $(Y_t)_{0\le t\le T}$ as the viscosity solutions do. It also seems to match better the Hilbertian aspect of stochastic integrals theory. This is the subject of Section 2.2.

2. BSDEs with quadratic growth.

2.1. *Existence.* In order to justify the assumptions we introduce to prove the existence result (more precisely, what we understand by quadratic growth of the coefficient on the one hand, and why we require the boundedness of the terminal condition ξ on the other hand) we give two examples.

These examples are also an occasion to use the techniques of *the exponential change of variable* and of *the application of Itô's formula to a well-chosen function*, which are central tools throughout this chapter.

EXAMPLE 1. We consider the following equation:

$$\forall t \in [0, T], \qquad Y_t = \xi + \int_t^T \tfrac{1}{2}|Z_s|^2\, ds - \int_t^T Z_s\, dW_s. \tag{2}$$

The exponential change of variable $y = \exp(Y)$ transforms formally this equation a.s.:

$$\forall t \in [0, T], \qquad y_t = \exp(\xi) - \int_t^T z_s\, dW_s. \tag{3}$$

The latter equation being linear, we have, when

$$\exp(\xi) \in L^2(\Omega), \tag{4}$$

the existence of a unique solution $(y, z) \in \mathscr{H}_T^2(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ of (3). The process y is given explicitly by

$$\forall t \in [0, T], \qquad y_t = \mathbb{E}[\exp(\xi) | \mathscr{F}_t],$$

and the process z is given by the theorem of representation of continuous martingales (see, e.g., [13]).

Taking $\xi \in L^\infty(\Omega)$ is a sufficient assumption to require on ξ in order to have (4). Indeed, if $\xi \in L^\infty(\Omega)$, $\exp(\xi) \in L^\infty(\Omega) \subset L^2(\Omega)$ and there exists a unique solution of (3). Moreover,

$$\forall t \in [0, T], \qquad y_t \geq \exp(-\|\xi\|_\infty),$$

and one can define

$$\forall t \in [0, T], \qquad Y_t = \ln(y_t), \qquad Z_t = z_t / y_t.$$

It is then easy to check that the pair $(Y, Z) \in \mathscr{H}_T^\infty(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ is a solution of (2).

The uniqueness in $\mathscr{H}_T^\infty(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ comes from the fact that the exponential change of variable is no longer formal and from the uniqueness for equation (3).

EXAMPLE 2 (a priori estimates). Let $a: \mathbb{R}^+ \to \mathbb{R}$ and $b: \mathbb{R}^+ \to \mathbb{R}^+$ be two functions and C be a positive constant. We say that *the coefficient F satisfies condition* (H0) *with a, b, C* if for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$F(t, v, z) = a_0(t, v, z)v + F_0(t, v, z)$$

with

$$\begin{aligned} &a_0(t, v, z) \leq a(t) \quad \text{a.s.,} \\ &|F_0(t, v, z)| \leq b(t) + C|z|^2 \quad \text{a.s.} \end{aligned} \tag{H0}$$

In this example we consider a BSDE with parameters (F, τ, ξ) where the terminal time is a stopping time τ and the terminal condition ξ is bounded.

In this case we call a solution of the BSDE with parameters (F, τ, ξ) a pair of adapted processes,

$$(Y, Z) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d),$$

such that:

(i) $Y_t = \xi$ and $Z_t = 0$ on the set $\{t \geq \tau\}$.
(ii) $\mathbb{E} \int_0^\tau |Z_t|^2 \, dt \leq \infty$.
(iii) For all $0 \leq t \leq T$, $Y_t = Y_T + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s) \, ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \, dW_s$.

PROPOSITION 2.1. *Let* $(Y, Z) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ *be a solution of the BSDE with parameters* (F, τ, ξ), *and suppose that* F *satisfies condition* (H0) *with* a, b, C, *such that*

$$\forall T > 0, \qquad a^+ = \max(a, 0), b \in L^1(0, T) \text{ and } C > 0.$$

Then for all $0 \le t \le T$,

$$Y_t \le \Big[\sup_\Omega (Y_T)\Big]^+ \exp\Big(\int_t^T a_s\,ds\Big) + \int_t^T b_s \exp\Big(\int_t^s a_\lambda\,d\lambda\Big)\,ds \quad a.s.$$

$$\Big(\text{resp. } Y_t \ge \Big[\inf_\Omega (Y_T)\Big]^- \exp\Big(\int_t^T a_s\,ds\Big) - \int_t^T b_s \exp\Big(\int_t^s a_\lambda\,d\lambda\Big)\,ds \quad a.s.\Big).$$

Moreover, there exists a constant K *depending only on* $\|Y\|_\infty$, $\|a^+\|_{L^1_\tau}$ *and* C *such that*

$$\mathbb{E}\int_0^\tau |Z_s|^2\,ds \le K.$$

An immediate consequence of this proposition is the corollary.

COROLLARY 2.2. *Let* $(Y, Z) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ *be a solution of the BSDE with parameters* (F, τ, ξ).

(i) *If* τ *is bounded* ($\tau \le T$ *a.s.*), $\xi \in L^\infty(\Omega)$ *and* F *satisfies condition* (H0) *with* a, b, C *such that* $a^+, b \in L^1(0, T), C > 0$,

$$\|Y\|_\infty \le \big(\|\xi\|_\infty + \|b\|_{L^1(0,T)}\big)\exp\big(\|a^+\|_{L^1(0,T)}\big).$$

(ii) *If* τ *is unbounded*, $\xi \in L^\infty(\Omega)$ *and* F *satisfies condition* (H0) *with* a, b, C *such that there exists a constant* α_0 *such that* $a \le \alpha_0 < 0$, $b \in L^\infty(\mathbb{R}^+)$ *and* $C > 0$,

$$\|Y\|_\infty \le \|\xi\|_\infty + \frac{\|b\|_\infty}{|\alpha_0|}.$$

PROOF OF PROPOSITION 2.1. Let $T \in \mathbb{R}$ be such that $T \le \|\tau\|_\infty$ and consider the solution φ of the ordinary differential equation

$$\varphi_t = \Big[\sup_\Omega (Y_T)\Big]^+ + \int_t^T (a_s\varphi_s + b_s)\,ds.$$

For $0 \le t \le T$,

$$\varphi_t = \Big[\sup_\Omega (Y_T)\Big]^+ \exp\Big(\int_t^T a_s\,ds\Big) + \int_t^T b_s \exp\Big(\int_t^s a_\lambda\,d\lambda\Big)\,ds,$$

our aim is to prove that $Y_t \le \varphi_t$. We apply Itô's formula to the process $Y_t - \varphi_t$

and to an increasing C^2 function Φ yet to be determined:

$$\Phi(Y_t-\varphi_t)=\Phi(Y_T-\varphi_T)+\int_{t\wedge\tau}^{T\wedge\tau}\Phi'(Y_s-\varphi_s)\big[F(s,Y_s,Z_s)-(a_s\varphi_s+b_s)\big]\,ds$$
$$-\int_{t\wedge\tau}^{T\wedge\tau}\tfrac{1}{2}\Phi''(Y_s-\varphi_s)|Z_s|^2\,ds-\int_{t\wedge\tau}^{T\wedge\tau}\Phi'(Y_s-\varphi_s)Z_s\,dW_s.$$

We set, for $0\le s\le T$,

$$\tilde a_s=a_0(s,Y_s,Z_s).$$

The function Φ being increasing, for all $0\le t\le s\le T$, we have

$$\begin{aligned}\Phi'(Y_s-\varphi_s)&\big[F(s,Y_s,Z_s)-(a_s\varphi_s+b_s)\big]\\&\le\Phi'(Y_s-\varphi_s)\big[\tilde a_sY_s+b_s+C|Z_s|^2-(a_s\varphi_s+b_s)\big]\\&\le\Phi'(Y_s-\varphi_s)\big[\tilde a_s(Y_s-\varphi_s)+(\tilde a_s-a_s)\varphi_s+C|Z_s|^2\big]\end{aligned}$$

and since $(\tilde a_s-a_s)\varphi_s\le 0$,

$$\begin{aligned}\Phi(Y_t-\varphi_t)\le\Phi(Y_T-\varphi_T)&+\int_{t\wedge\tau}^{T\wedge\tau}\tilde a_s\Phi'(Y_s-\varphi_s)(Y_s-\varphi_s)ds\\&+\int_{t\wedge\tau}^{T\wedge\tau}[C\Phi'-\tfrac{1}{2}\phi''](Y_s-\varphi_s)|Z_s|^2ds\\&-\int_{t\wedge\tau}^{T\wedge\tau}\Phi'(Y_s-\varphi_s)Z_sdW_s.\end{aligned}\tag{5}$$

We set $M=\|Y\|_\infty+\|\varphi\|_\infty$ and we define on $[-M,M]$ the function Φ by

$$\Phi(u)=\begin{cases}e^{2Cu}-1-2Cu-2C^2u^2, & \text{for } u\in[0,M],\\ 0, & \text{for } u\in[-M,0].\end{cases}$$

For all $u\in[-M,M]$, one can check easily that

$$\begin{aligned}\Phi(u)\ge 0 \text{ and } \Phi(u)=0\quad&\text{if and only if } u\le 0,\\\Phi'(u)&\ge 0,\\0\le u\Phi'(u)&\le 2(M+1)C\Phi(u),\\C\Phi'(u)-\tfrac{1}{2}\Phi''(u)&\le 0.\end{aligned}$$

Hence, setting $k_t=a_t^+2(M+1)C$, the function k is positive and deterministic and, for all $0\le t\le T$,

$$0\le\Phi(Y_t-\varphi_t)\le\int_{t\wedge\tau}^{T\wedge\tau}k_s\Phi(Y_s-\varphi_s)\,ds-\int_{t\wedge\tau}^{T\wedge\tau}\Phi'(Y_s-\varphi_s)Z_s\,dW_s\quad\text{a.s.}$$

and therefore,

$$0\le\Phi(Y_t-\varphi_t)\le\int_t^T k_s\Phi(Y_s-\varphi_s)\,ds-\int_{t\wedge\tau}^{T\wedge\tau}\Phi'(Y_s-\varphi_s)Z_s\,dW_s\quad\text{a.s.}$$

Since $(\Phi'(Y_s-\varphi_s))_{t\le s\le T}$ is bounded, $(\Phi'(Y_s-\varphi_s)Z_s)_{t\le s\le T}\in\mathscr{H}^2_T(\mathbb{R}^d)$, and taking expectations in the above inequality yields

$$0\le \mathbb{E}\Phi(Y_t-\varphi_t)\le\int_t^T k_s\mathbb{E}\Phi(Y_s-\varphi_s)\,ds.$$

Applying Gronwall's lemma,

$$\forall\, t\in[0,T],\qquad \mathbb{E}\Phi(Y_t-\varphi_t)=0;$$

therefore, since $\Phi(u)\ge 0$,

$$\forall\, t\in[0,T],\qquad \Phi(Y_t-\varphi_t)=0\quad\text{a.s.}$$

and since $\Phi(u)=0$ if and only if $u\le 0$ we obtain

$$\forall\, t\in[0,T],\qquad Y_t-\varphi_t\le 0\quad\text{a.s.}$$

The proof of

$$Y_t\ge\Big[\inf_\Omega(Y_T)\Big]^-\exp\bigg(\int_t^T a_s\,ds\bigg)-\int_t^T b_s\exp\bigg(\int_t^s a_\lambda\,d\lambda\bigg)\,ds$$

relies on the same computations. Indeed, if φ is now the solution of the ordinary differential equation

$$\varphi_t=\Big[\inf_\Omega(Y_T)\Big]^-+\int_t^T(a_s\varphi_s-b_s)\,ds,$$

applying Itô's formula to Φ defined as above and to φ_t-Y_t yields to

$$\Phi(\varphi_t-Y_t)\le\Phi(\varphi_T-Y_T)+\int_{t\wedge\tau}^{T\wedge\tau}k_s\Phi(\varphi_s-Y_s)\,ds-\int_{t\wedge\tau}^{T\wedge\tau}k_s\Phi'(\varphi_s-Y_s)Z_s\,dW_s,$$

and the same argument allows us to conclude.

In order to estimate $\mathbb{E}\int_0^\tau|Z_s|^2\,ds$, we use again (5) with $t=0$, $\varphi=0$ and $M=\|Y\|_\infty$ and Φ defined on $[-M,M]$ by

$$\Phi(u)=\frac{1}{2C^2}\big[\exp(2C(u+M))-(1+2C(u+M))\big].$$

It is straightforward to check that, for $u\in[-M,M]$,

$$\begin{gathered}\Phi(u)\ge 0,\\ \Phi'(u)\ge 0,\\ 0\le u\Phi'(u)\le\frac{M}{C}(e^{4CM}-1),\\ \tfrac12\Phi''(u)-C\Phi'(u)=1,\end{gathered}$$

Therefore, (5) gives

$$\begin{aligned}0\le\Phi(Y_0)\le\Phi(Y_T)&+\int_0^{T\wedge\tau}a_s^+\frac{M}{C}(e^{4CM}-1)\,ds\\ &-\int_0^{T\wedge\tau}|Z_s|^2\,ds-\int_0^{T\wedge\tau}\Phi'(Y_s)Z_s\,dW_s,\end{aligned}$$

which leads to

$$\mathbb{E}\int_0^{T\wedge\tau} |Z_s|^2\, ds \le \Phi(M) + \frac{M}{C}(e^{4CM} - 1)\|a^+\|_{L^1}$$

and the proof is completed by letting $T \to \infty$.

The existence and the monotone stability results. Let $\alpha_0, \beta_0, b \in \mathbb{R}$ and c be a continuous increasing function. We say that *the coefficient F satisfies condition* (H1) *with α_0, β_0, b, c* if for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$F(t, v, z) = a_0(t, v, z)v + F_0(t, v, z),$$

with

$$\text{(H1)}\qquad \begin{aligned} &\beta_0 \le a_0(t, v, z) \le \alpha_0 \quad \text{a.s.,} \\ &|F_0(t, v, z)| \le b + c(|v|)|z|^2 \quad \text{a.s.} \end{aligned}$$

The main result of this section is the following theorem.

THEOREM 2.3 (Existence). *Let (F, τ, ξ) be a set of parameters of BSDE* (1) *and suppose that the coefficient F satisfies* (H1) *with $\alpha_0, \beta_0, b \in \mathbb{R}$, and $c\colon \mathbb{R}^+ \to \mathbb{R}^+$ continuous increasing, $\xi \in L^\infty(\Omega)$, and*:

(i) *The terminal time τ is either bounded, ($\tau \le T$ a.s.) or*
(ii) *The terminal time is such that $\tau < \infty$ a.s. and $\alpha_0 < 0$.*

Then the BSDE (1) *has at least one solution (Y, Z) in $\mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ such that the process Y has continuous paths.*

Moreover, there exists a minimal solution solution (Y_*, Z_*) [resp. a maximal solution (Y^*, Z^*)] such that for any set of parameters (G, τ, ζ), if

$$F \le G \quad \textit{and} \quad \xi \le \zeta \ (\textit{resp. } F \ge G \textit{ and } \xi \ge \zeta)$$

and for any solution (Y_G, Z_G) of the BSDE with parameters (G, τ, ζ),

$$Y_* \le Y_G \quad (\textit{resp. } Y^* \ge Y_G).$$

Before giving the proof of Theorem 2.3, we state the following proposition which gives the main argument of the existence. It is presented under general assumptions as it will also be used in the next section.

PROPOSITION 2.4 (Monotone stability). *Let (F, τ, ξ) be a set of parameters and let $(F^n, \tau, \xi^n)_n$ be a sequence of parameters such that*:

(i) *The sequence $(F^n)_n$ converges to F locally uniformly on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, for each $n \in \mathbb{N}$, $\xi^n \in L^\infty(\Omega)$ and $(\xi^n)_n$ converges to ξ in $L^\infty(\Omega)$.*

(ii) *There exists $k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $T > 0$, $k \in L^1(0, T)$ and there exists $C > 0$ such that*

$$\text{(6)}\qquad \forall\, n \in \mathbb{N}, \forall\, (t, u, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d, \qquad |F^n(t, u, z)| \le k_t + C|z|^2.$$

(iii) *For each n, the BSDE with parameters (F^n, τ, ξ^n) has a solution*

$$(Y^n, Z^n) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d),$$

such that the sequence $(Y^n)_n$ is monotonic, and there exists $M > 0$ such that for all $n \in \mathbb{N}$, $\|Y^n\|_\infty \leq M$.

(iv) *The stopping time τ is such that $\tau < \infty$ a.s.*

Then there exists a pair of processes $(Y, Z) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ such that for all $T \in \mathbb{R}^+$,

$$\lim_{n\to\infty} Y^n = Y \text{ uniformly on } [0, T],$$

$$(Z^n)_n \text{ converges to } Z \text{ in } \mathscr{H}_\tau^2(\mathbb{R}^d)$$

and (Y, Z) is a solution of the BSDE with parameters (F, τ, ξ).

In particular, if for each n, Y^n has continuous paths, the process Y has also continuous paths.

REMARK. The limit coefficient F satisfies the assumption of quadratic growth, but not necessarily a comparison principle. Hence the solution we find here might not be unique.

PROOF OF PROPOSITION 2.4. Since for all $t \in \mathbb{R}^+$ the sequence $(Y_t^n)_n$ is monotonic and bounded, it has a limit which we denote Y_t.

In view of Proposition 2.1, there exists a constant $\widetilde{K}$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\int_0^\tau |Z_s|^2\, ds \leq \widetilde{K}.$$

Therefore, there exists a process $Z \in \mathscr{H}_\tau^2(\mathbb{R}^d)$ and a subsequence $(Z^{n_j})_j$ of $(Z^n)_n$ such that

$$Z^{n_j} \rightharpoonup Z \text{ weakly in } \mathscr{H}_\tau^2(\mathbb{R}^d). \tag{7}$$

The point is now to show that in fact *the whole* sequence converges *strongly* to Z in $\mathscr{H}_\tau^2(\mathbb{R}^d)$.

We notice that, by inequality (6), setting $K = 5C$,

$$\left|F^n(t, v, z) - F^p(t, v', z')\right| \leq 2k_t + K\big(|z - z'|^2 + |z' - z''|^2 + |z''|^2\big).$$

STEP 1. The strong convergence of $(Z^n)_n$ in $\mathscr{H}_\tau^2(\mathbb{R}^d)$. The main arguments of this step are adapted from [5]. We have, for all $n, p \in \mathbb{N}$,

$$\left|F^n(Y^n, Z^n) - F^p(Y^p, Z^p)\right| \leq 2k_t + K\big(|Z^n - Z^p|^2 + |Z^n - Z|^2 + |Z|^2\big).$$

Let us apply Itô's formula to the process $(Y_t^n - Y_t^p)_{0\leq t\leq T}$ for $n, p \in \mathbb{N}$, $n \leq p$, and to an increasing function $\psi \in C^2[0, 2M]$, such that $\psi'(0) = 0$ and $\psi(0) = 0$.

The function ψ is yet to be chosen:

$$\begin{aligned}\psi(Y_0^n - Y_0^p) &= \psi(Y_T^n - Y_T^p) + \int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p)\big(F^n(Y_s^n, Z_s^n) - F^p(Y_s^p, Z_s^p)\big)\, ds \\ &\quad - \tfrac{1}{2}\int_0^{T\wedge\tau} \psi''(Y_s^n - Y_s^p)|Z_s^n - Z_s^p|^2\, ds \\ &\quad - \int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p)(Z_s^n - Z_s^p)\, dW_s.\end{aligned}$$

As $\psi'(Y_s^n - Y_s^p) \geq 0$,

$$\begin{aligned}\psi(Y_0^n - Y_0^p) &\leq \psi(Y_T^n - Y_T^p) + \int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p) \\ &\qquad \times \big[2k_s + K(|Z_s^n - Z_s^p|^2 + |Z_s^n - Z_s|^2 + |Z_s|^2)\big]\, ds \\ &\quad - \tfrac{1}{2}\int_0^{T\wedge\tau} \psi''(Y_s^n - Y_s^p)|Z_s^n - Z_s^p|^2\, ds - \int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p)(Z_s^n - Z_s^p)\, dW_s.\end{aligned}$$

We now transfer the terms in $|Z_s^n - Z_s^p|^2$ and $|Z_s^n - Z_s|^2$ to the left-hand side of the inequality, and we take the expectation. As $Y^n - Y^p$ is bounded,

$$\mathbb{E}\int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p)(Z_s^n - Z_s^p)\, dW_s = 0$$

and,

$$\begin{aligned}&\mathbb{E}\psi(Y_0^n - Y_0^p) \\ &\quad + \mathbb{E}\int_0^{T\wedge\tau} [\tfrac{1}{2}\psi'' - K\psi'](Y_s^n - Y_s^p)|Z_s^n \\ &\quad - Z_s^p|^2 - K\psi'(Y_s^n - Y_s^p)|Z_s^n - Z_s|^2\, ds \\ &\qquad \leq \mathbb{E}\psi(Y_T^n - Y_T^p) + \mathbb{E}\int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s^p)(2k_s + K|Z_s|^2)\, ds.\end{aligned}$$

We want to pass to the limit as $p \to \infty$ along the subsequence $(n_j)_j$ defined in (7). The convergence of $Y^p \to Y$ being pointwise, and Y^p being bounded, one has, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}&\mathbb{E}\psi(Y_0^n - Y_0) + \liminf_{p\to\infty,\, p\in(n_j)} \mathbb{E}\int_0^{T\wedge\tau} \big[\tfrac{1}{2}\psi'' - K\psi'\big](Y_s^n - Y_s)|Z_s^n - Z_s^p|^2\, ds \\ &\quad - \mathbb{E}\int_0^{T\wedge\tau} K\psi'(Y_s^n - Y_s)|Z_s^n - Z_s|^2\, ds \\ &\qquad \leq \mathbb{E}\psi(Y_T^n - Y_T) + \mathbb{E}\int_0^{T\wedge\tau} \psi'(Y_s^n - Y_s)(2k_s + K|Z_s|^2)\, ds,\end{aligned}$$

and as

$$\liminf_{p\to\infty,\ p\in(n_j)}\left[-\mathbb{E}\int_0^{T\wedge\tau}\psi'(Y_s^n-Y_s)|Z_s^n-Z_s^p|^2\,ds\right]$$
$$\le -\mathbb{E}\int_0^{T\wedge\tau}\psi'(Y_s^n-Y_s)|Z_s^n-Z_s|^2\,ds,$$

we obtain,

$$\mathbb{E}\psi(Y_0^n-Y_0)+\liminf_{p\to\infty,\ p\in(n_j)}\mathbb{E}\int_0^{T\wedge\tau}\underbrace{([\tfrac{1}{2}\psi''-2K\psi'](Y_s^n-Y_s))}_{=(**)}|Z_s^n-Z_s^p|^2\,ds$$
$$\le \mathbb{E}\psi(Y_T^n-Y_T)+\mathbb{E}\int_0^{T\wedge\tau}\psi'(Y_s^n-Y_s)(2k_s-K|Z_s|^2)\,ds.$$

We now choose ψ such that $(**)=1$, namely,

$$\psi(u)=\frac{1}{4K}(e^{4Ku}-4Ku-1).$$

It is straightforward to check that ψ is a C^∞ function, increasing on $[0,2M]$ and such that $\psi'(0)=\psi(0)=0$.

Noting that by the convexity of the l.s.c. functional,

$$J(Z)=\mathbb{E}\int_0^{T\wedge\tau}|Z_s^n-Z_s|^2\,ds,$$

one has

$$\mathbb{E}\int_0^{T\wedge\tau}|Z_s^n-Z_s|^2\,ds\le\liminf_{p\to\infty,\ p\in(n_j)}\mathbb{E}\int_0^{T\wedge\tau}|Z_s^n-Z_s^p|^2\,ds,$$

we obtain

$$\mathbb{E}\psi(Y_0^n-Y_0)+\mathbb{E}\int_0^{T\wedge\tau}|Z_s^n-Z_s|^2\,ds$$
$$\le \mathbb{E}\psi(Y_T^n-Y_T)+\mathbb{E}\int_0^{T\wedge\tau}\psi'(Y_s^n-Y_s)(2k_s+K|Z_s|^2)\,ds.$$

By Lebesgue's dominated convergence theorem, the right-hand side of this inequality converges to 0 as $n\to\infty$, as well as the first term of the left-hand side. Now, passing to the limit as $n\to\infty$, we find, for all $T>0$,

$$\limsup_{n\to\infty}\mathbb{E}\int_0^{T\wedge\tau}|Z_s^n-Z_s|^2\,ds=0.$$

Consequently the whole sequence $(Z^n)_n$ converges to Z in $\mathscr{H}_\tau^2(\mathbb{R}^d)$.

STEP 2. *The uniform convergence of a subsequence of $(Y^n)_n$ to Y.* At this stage of the proof we know that

$$\text{for all } t\in\mathbb{R}^+,\ \lim_{n\to\infty}Y_t^n=Y_t,\ \text{ the sequence } (Z^n)_n \text{ converges to } Z \text{ in } \mathscr{H}_\tau^2(\mathbb{R}^d).$$

We proceed as in [14], applying the following lemma.

LEMMA 2.5. *There exists a subsequence* $(Z^{n_j})_j$ *of* $(Z^n)_n$ *such that* $(Z^{n_j})_j$ *converges almost surely to* Z *and such that* $\widetilde{Z} = \sup_j |Z^{n_j}| \in \mathscr{H}_\tau^2(\mathbb{R})$.

PROOF. For the convenience of the reader, we sketch the proof of this lemma. Extracting if necessary a subsequence, we may assume without loss of generality that the sequence $(Z^n)_n$ converges almost surely to Z. Since $(Z^n)_n$ is a Cauchy sequence in $\mathscr{H}_\tau^2(\mathbb{R})$, we can extract a subsequence $(Z^{nj})_j$ such that $\|Z^{n_{j+1}} - Z^{n_j}\|_{\mathscr{H}_\tau^2} \le 1/2^j$, for all $j \in \mathbb{N}$. Then we set

$$g = |Z^{n_0}| + \sum_{j=0}^{\infty} |Z^{n_{j+1}} - Z^{n_j}|.$$

Because of the properties of the sequence $(Z^{n_j})_j$, we have

$$\begin{aligned} \|g\|_{\mathscr{H}_\tau^2} &\le \|Z^{n_0}\|_{\mathscr{H}_\tau^2} + \sum_{j=0}^{\infty} \|Z^{n_{j+1}} - Z^{n_j}\|_{\mathscr{H}_\tau^2} \\ &\le \|Z^{n_0}\|_{\mathscr{H}_\tau^2} + \sum_{j=0}^{\infty} \frac{1}{2^j} \\ &< +\infty. \end{aligned}$$

Moreover, for any $p \in \mathbb{N}$, we also have

$$|Z^{n_p}| \le |Z^{n_0}| + \sum_{j=0}^{p} |Z^{n_{j+1}} - Z^{n_j}| \le g.$$

Therefore, $\widetilde{Z} = \sup_j |Z^{n_j}| \in \mathscr{H}_\tau^2(\mathbb{R})$ and the proof is complete. □

For the sake of simplicity of notations, we still denote by $(Z^n)_n$ the subsequence $(Z^{n_j})_j$ given by Lemma 2.5 [resp. $(Y^n)_n$ and $(F^n)_n$ the sequences $(Y^{n_j})_j$ and $(F^{n_j})_j$] and therefore we have

$$Z^n \to Z \text{ a.s. } dt \otimes d\mathbb{P} \quad \text{and} \quad \widetilde{Z} = \sup_n |Z^n| \in \mathscr{H}_\tau^2(\mathbb{R}).$$

Recalling that the sequence $(F^n)_n$ converges locally uniformly to F, we get, for almost all $\omega \in \Omega$ and $t \in [0, \tau]$,

$$\lim_{n\to\infty} F^n(t, Y_t^n, Z_t^n) = F(t, Y_t, Z_t).$$

Since F^n satisfies condition (6), we have,

$$\left|F^n(t, Y_t^n, Z_t^n)\right| \le k_t + C \sup_n |Z_t^n|^2 = k_t + C\widetilde{Z}^2.$$

Thus, for almost all $\omega \in \Omega$ and, uniformly in $t \in [0, \tau]$, Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_{t\wedge\tau}^{T\wedge\tau} F^n(s, Y_s^n, Z_s^n)\, ds = \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s)\, ds.$$

On the other hand, from the continuity properties of stochastic integral, we get

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| \int_{t\wedge\tau}^{T\wedge\tau} Z_s^n\, dW_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s\, dW_s \right| = 0 \quad \text{in probability.}$$

Extracting a subsequence again if necessary, we may assume that the last convergence is $\mathbb{P}$ a.s.

Finally,

$$|Y_t^n - Y_t^m| \le |Y_T^n - Y_T^m| + \int_{t\wedge\tau}^{T\wedge\tau} \left| F^n(s, Y_s^n, Z_s^n) - F^m(s, Y_s^m, Z_s^m)\right| ds + \left| \int_{t\wedge\tau}^{T\wedge\tau} Z_s^n\, dW_s - \int_t^{\tau} Z_s^m\, dW_s \right|.$$

Therefore taking limits on m and supremum over $t \in [0, \tau]$, we get, for almost all $\omega \in \Omega$,

$$\sup_{0\le t\le T} |Y_t^n - Y_t| \le |Y_T^n - Y_T| + \int_0^{T\wedge\tau} \left| F(s, Y_s^n, Z_s^n) - F^m(s, Y_s, Z_s)\right| ds + \sup_{0\le t\le T} \left| \int_t^{T} Z_s^n\, dW_s - \int_t^{\tau} Z_s\, dW_s \right|,$$

from which we deduce that $(Y^n)_n$ converges to Y uniformly for $t \in [0, T]$ (in particular Y is a continuous process if the Y^n are). We can now pass to the limit in

$$Y_t^p = Y_T^p + \int_{t\wedge\tau}^{T\wedge\tau} F^p(s, Y_s^p, Z_s^p)\, ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s^p\, dW_s,$$

obtaining that (Y, Z) is a solution of the BSDE with parameters (F, ξ).

PROOF OF THEOREM 2.3. The proof consists now in finding a *good* approximation of F, in order to apply the previous theorem. We use a truncation argument in order to control the growth of F in u and an exponential change in order to control its growth in z.

We first suppose that instead of (H1) the coefficient F satisfies the following condition: there exist $\alpha_0, \beta_0 \in \mathbb{R}$, $B, C \in \mathbb{R}^+$, such that, for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$F(t, v, z) = a_0(t, v, z)v + F_0(t, v, z)$$

with

$$\beta_0 \le a_0(t, v, z) \le \alpha_0 \tag{8}$$

and

$$|F_0(t, v, z)| \leq B + C|z|^2 \quad \text{a.s.}$$

and moreover, that either:

(i) The terminal time is bounded $\tau \leq T$ a.s., or
(ii) The terminal time is finite $\tau < \infty$ a.s. and $\alpha_0 < 0$.

Let (G, τ, ζ) be a set of parameters such that

$$G \leq F \quad \text{and} \quad \zeta \leq \xi, \tag{9}$$

and suppose that it has a solution $(Y_G, Z_G) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$. Our aim is to find a solution $(Y, Z) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ of the BSDE with parameters (F, τ, ξ) such that

$$Y_G \leq Y.$$

By Proposition 2.1, setting

$$\widetilde{M} = \begin{cases} (\|\xi\|_\infty + BT)\exp(\alpha_0^+ T), & \text{in case (i)}, \\ \|\xi\|_\infty + B/|\alpha_0|, & \text{in case (ii)}, \end{cases}$$

for any solution (Y, Z) of (F, τ, ξ), one has

$$\|Y\|_\infty \leq \widetilde{M}.$$

We define

$$M = \max(\widetilde{M}, \|Y_G\|_\infty). \tag{10}$$

During the proof we will use several times C^∞ functions $\phi_K: \mathbb{R} \to [0, 1]$ such that

$$\phi_K(u) = \begin{cases} 1, & \text{if } |u| \leq K, \\ 0, & \text{if } |u| \geq K + 1. \end{cases} \tag{11}$$

STEP 1. *The exponential change.* The exponential change $v = e^{2Cu}$ transforms formally a BSDE with parameters (F, τ, ξ) in a BSDE with parameters $(f, \tau, e^{2C\xi})$ where

$$f(t, v, z) = 2CvF\left(t, \frac{\ln(v)}{2C}, \frac{z}{2Cv}\right) - \frac{1}{2}\frac{|z|^2}{v}$$

and

$$g(t, v, z) = 2CvG\left(t, \frac{\ln(v)}{2C}, \frac{z}{2Cv}\right) - \frac{1}{2}\frac{|z|^2}{v}.$$

We consider a function $\psi: \mathbb{R} \to [0, 1]$ such that

$$\psi(u) = \begin{cases} 1, & \text{if } u \in [\exp(-2CM), \exp(2CM)] \\ 0, & \text{if } u \notin [\exp(-2C(M+1)), \exp(2C(M+1))], \end{cases}$$

and we use the convention $0 \times \infty = 0$ for the sake of simplicity of notations; we set, for $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$\tilde{f}(t, v, z) = \psi(v) f(t, v, z) \quad \text{and} \quad \tilde{g}(t, v, z) = \psi(v) g(t, v, z).$$

We have, setting

$$l(v) = \psi(v)(\alpha_0 v \ln(v) + 2CBv)$$

that

$$\psi(v)\left(\beta_0 v \ln(v) - 2CBv - \frac{|z|^2}{v}\right) \le \tilde{f}(t, v, z) \le l(v).$$

We remark that l is a Lipschitz continuous function bounded from above by a constant, say, L.

We also remark that defining

$$y_G = \exp(2CY_G) \quad \text{and} \quad z_G = 2CZ_G \exp(2CY_G),$$

the pair (y_G, z_G) is a solution of the BSDE of parameters $(g, \tau, e^{2C\xi})$.

STEP 2. *The approximation.* We can approximate $\tilde{f}$ by a decreasing sequence of uniformly Lipschitz continuous functions $(f^p)_{p \in \mathbb{N}}$ such that for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$\tilde{g}(t, v, z) \le \tilde{f}(t, v, z) \le f^p(t, v, z) \le l(v) + \frac{1}{2^p}.$$

For instance, if $\tilde{f}^p$ are C^∞ functions such that

$$f + \frac{1}{2^{p+1}} \le \tilde{f}^p \le f + \frac{1}{2^p}$$

(the existence of such functions is given by a standard argument of regularization), and if the functions ϕ_p are defined as in (11) a way of obtaining them is to set, for all $p \in \mathbb{N}$,

$$f^p(t, v, z) = \tilde{f}^p(t, v, z)\phi_p(|v| + |z|) + \left(l(v) + \frac{1}{2^p}\right)[1 - \phi_p(|v| + |z|)].$$

Then classical results of existence and comparison for Lipschitz continuous coefficients give for each p the existence and uniqueness of a solution (y^p, z^p) of the BSDE with parameters (f^p, τ, ξ), and

$$y_G \le y^{p+1} \le y^p \le y^1.$$

Moreover, in case (ii) we remark that the process $(e^{2CM}, 0)_{t \le \tau}$ is the solution of the BSDE with parameters $(0, \tau, e^{2CM})$ and the process $(e^{-2CM}, 0)_{t \le \tau}$ is the solution to the BSDE with parameters $(0, \tau, e^{-2CM})$. Since for p large enough,

$$e^{2CM} \ge e^{2C\xi} \quad \text{and} \quad 0 \ge f^p(e^{2CM}, 0),$$

and for all p,

$$e^{-2CM} \le e^{2C\xi} \quad \text{and} \quad 0 \le f^p(e^{-2CM}, 0),$$

the comparison result for Lipschitz continuous coefficients (cf. [11]) gives, for all p large enough,

$$e^{-2CM} \le y_t^{p+1} \le y_t^p \le e^{2CM} \quad \text{a.s. for all } t \in \mathbb{R}^+.$$

We come back to the first problem by setting

$$F^p(t,u,z) = \frac{f^p(t, e^{2Cu}, 2Ce^{2Cu}z)}{2Ce^{2Cu}} + C|z|^2$$

and

$$\begin{aligned}\widetilde{F}(t,u,z) &= \frac{\tilde{f}(t, e^{2Cu}, 2Ce^{2Cu}z)}{2Ce^{2Cu}} + C|z|^2 \\ &= \psi(e^{2Cu})F(t,u,z) + (1-\psi(e^{2Cu}))C|z|^2.\end{aligned}$$

The pair (Y^p, Z^p) defined by

$$Y_t^p = \frac{\ln(y_t^p)}{2C} \quad \text{and} \quad Z_t^p = \frac{z_t^p}{2Cy_t^p}$$

is a solution of the BSDE with parameters (F^p, τ, ξ). Let us recapitulate what we have obtained.

(i) The sequence $(\widetilde{F}^n)_n$ converges to $\widetilde{F}$ locally uniformly on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, for each $n \in \mathbb{N}$, $\xi^n \in L^\infty(\Omega)$ and $(\xi^n)_n$ converges to ξ in $L^\infty(\Omega)$.

(ii) There exist $K, C > 0$ such that

$$\forall n \in \mathbb{N}, \forall (t,u,z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d, \qquad |F^n(t,u,z)| \le K + C|z|^2.$$

(iii) For each n, the BSDE with parameters $(\widetilde{F}^n, \tau, \xi^n)$ has a solution

$$(Y^n, Z^n) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d),$$

such that the sequence $(Y^n)_n$ is decreasing, and there exists $\overline{M} > 0$ such that for all $n \in \mathbb{N}$, $\|Y^n\|_\infty \le \overline{M}$ [with $\overline{M} = M$ in case (ii)].

(iv) For all $n \in \mathbb{N}$, $Y_G \le Y^n$.

Therefore, applying Theorem 2.4, the process $(Y^p)_p$ converges uniformly to Y and there exists Z, in $\mathscr{H}_T^2(\mathbb{R}^d)$ such that a subsequence of $(Z^p)_p$ converges to Z and (Y, Z) is a solution of

$$Y_t = \xi + \int_t^T \widetilde{F}(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dW_s.$$

Moreover, we prove that

$$\|Y\|_\infty \le M. \tag{12}$$

In view of the remark made above, we only need to give the proof in case (i). Indeed, it follows from Proposition 2.1 as

$$\widetilde{F}(t,v,z) = \tilde{a}_0(t,u,z) + \widetilde{F}_0(t,u,z)$$

with

$$\tilde{a}_0(t, u, z) = \psi(e^{2Cu}) a_0(t, u, z) \le \alpha^+$$

and

$$\widetilde{F}_0(t, u, z) = \psi(e^{2Cu}) F_0(t, u, z) + (1 - \psi(e^{2Cu})) C|z|^2;$$

hence

$$|\widetilde{F}_0(t, u, z)| \le B + 3C|z|^2.$$

By Corollary 2.2 we have,

$$\|Y\|_\infty \le (\|\xi\|_\infty + BT)\exp(\alpha^+ T) = \widetilde{M} \le M.$$

Therefore, (Y, Z) is also a solution of the BSDE with parameters (F, τ, ξ). Moreover,

$$Y_G \le Y \quad \text{and} \quad \|Y\|_\infty \le M.$$

STEP 3. *The truncation.* We now suppose that F satisfies (H1). Let (G, τ, ζ) be a set of parameters such that (9) holds true and let $(Y_G, Z_G) \in \mathscr{H}_\tau^\infty(\mathbb{R}) \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ be a solution of the BSDE with parameters (G, τ, ζ). For M defined by (10) we set for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$\overline{F}(t, v, z) = a_0(t, v, z)v + F_0(t, \phi_M(v)v, z), \qquad \overline{G}(t, v, z) = G(t, \phi_M(v)v, z).$$

The coefficient $\overline{F}$ satisfies (8), and (Y_G, Z_G) is also a solution of the BSDE with parameters $(\overline{G}, \tau, \zeta)$. Applying Steps 1 and 2 we obtain a solution (Y, Z) of the BSDE with parameters $(\overline{F}, \tau, \xi)$ such that $Y_G \le Y$. As $\|Y\|_\infty \le M$, the process (Y, Z) is also a solution with coefficient F. We have proved the existence of a maximal solution.

The proof of the existence of a minimal solution relies on the same proof but with the change of variable $v = e^{-2Cu}$. □

2.2. *Uniqueness and stability.* As we mentioned in the introduction, the one-dimensional frame allows us to prove a *comparison principle* between sub- and supersolutions which implies uniqueness as a by-product. We give it for BSDEs with a bounded terminal condition ξ and with a coefficient F which is locally Lipschitz continuous and has a quadratic growth in Z in a strong sense (i.e., the partial derivatives of F have a linear growth).

We first recall that a *supersolution* (resp. a *subsolution*) of a BSDE with coefficient F and terminal condition ξ is an adapted process $(Y_t, Z_t, C_t)_{0 \le t \le T}$ satisfying

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s + \int_t^T dC_s \biggl(\text{resp.} - \int_t^T dC_s\biggr),$$

where $(C_t)_{0 \le t \le T}$ is a right continuous increasing process ($C \in RCI$) and where, in the classical framework (i.e., when F is Lipschitz continuous in Y and Z, and when ξ is square integrable), the process $(Y_t, Z_t)_{0 \le t \le T}$ is assumed to be square integrable (this notion is introduced in [11]). Because of the quadratic growth of the coefficient, we will assume here that $(Y_t)_{0\le t\le T}$ is a one-dimensional *bounded* process and $(Z_t)_{0\le t\le T}$ is a square integrable process $(Y_t \in \mathscr{H}_T^\infty(\mathbb{R}), Z_t \in \mathscr{H}_T^2(\mathbb{R}^d))$.

We say that *the coefficient F satisfies condition* (H2) *on* $[-M, M]$ with l, k and C if for all $t \in \mathbb{R}^+$, $u \in [-M, M]$, $z \in \mathbb{R}^d$,

$$\text{(H2)}\qquad \begin{aligned} |F(t,u,z)| &\le l(t) + C|z|^2 \quad \text{a.s.},\\ \left|\frac{\partial F}{\partial z}(t,u,z)\right| &\le k(t) + C|z| \quad \text{a.s.} \end{aligned}$$

and *the coefficient F satisfies condition* (H3) *with* c_ε and ε if for all $t \in \mathbb{R}^+$, $v \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$\text{(H3)}\qquad \frac{\partial F}{\partial u}(t,u,z) \le l_\varepsilon(t) + \varepsilon|z|^2 \quad \text{a.s.}$$

Our main result is the following theorem.

THEOREM 2.6 (Comparison principle). *Let (F^1, τ, ξ^1) and (F^2, τ, ξ^2) be two sets of parameters for BSDEs and suppose that*:

(i) *$\xi^1 \le \xi^2$ a.s. and $F^1 \le F^2$.*

(ii) *For all ε, $M > 0$ there exists $l, l_\varepsilon \in L^1_\tau$, $k \in L^2_\tau$, $C \in \mathbb{R}$ such that either F^1 or F^2 satisfies both condition* (H2) *on $[-M, M]$ with l, k, C and satisfies both condition* (H3) *on $[-M, M]$ with l_ε and ε.*

Then if $(Y^1_t, Z^1_t, C^1_t)_{0\le t\le T}$ [resp. $(Y^2_t, Z^2_t, C^2_t)_{0\le t\le T}$] $\in \mathscr{H}_T^\infty(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d) \times RCI(\mathbb{R})$ is a subsolution (resp. a supersolution) of the BSDE *with parameters (F^1, τ, ξ^1) [resp. (F^2, τ, ξ^2)], one has*

$$\forall\, t \in \mathbb{R}^+, \qquad Y^1_t \le Y^2_t \quad a.s.$$

REMARK 2.7. It holds true if either $F^1(t, Y^1_t, Z^1_t) \le F^2(t, Y^1_t, Z^1_t)$ a.s. for all t and F^2 satisfy (H2) and (H3), or if $F^1(t, Y^2_t, Z^2_t) \le F^2(t, Y^2_t, Z^2_t)$ a.s. for all t and F^1 satisfy (H2) and (H3).

We postpone the proof to give an important application.

THEOREM 2.8 (Stability of BSDEs). *Let $(F^n, \tau, \xi^n)_n$ be a sequence of parameters of BSDEs such that*:

(i) *There exists $\alpha_0, \beta_0, b \in \mathbb{R}$ and an increasing function c such that for all $n \in \mathbb{N}$ the coefficient F^n satisfies condition* (H1) *with $\alpha_0, \beta_0, b \in \mathbb{R}$ and c.*

(ii) *For all n there exists a solution (Y^n, Z^n) to the BSDE with parameters (F^n, τ, ξ^n).*

Let (F, τ, ξ) be a set of parameters of BSDEs such that the coefficient F satisfies the assumptions of Theorem 2.6.

Then, if the sequence $(F^n)_n$ *converges to* F *locally uniformly on* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, *and if the sequence* $(\xi^n)_n$ *converges to* ξ *in* $L^\infty(\Omega)$, *there exists a pair of adapted processes* $(Y, Z) \in \mathscr{H}_\tau^\infty \times \mathscr{H}_\tau^2(\mathbb{R}^d)$ *such that sequence* $(Y^n)_n$ *converges to* Y *uniformly on* $[0, T]$ *for all* T, $(Z^n)_n$ *converges to* Z *in* $\mathscr{H}_\tau^2(\mathbb{R}^d)$ *and* (Y, Z) *is the solution of the BSDE with parameters* (F, ξ).

PROOF. We define

$$G^n = \sup_{p \ge n} F^p, \qquad H^n = \inf_{p \ge n} F^p$$

and

$$\xi^{n*} = \sup_{p \ge n} \xi^p, \qquad \xi_*^n = \inf_{p \ge n} \xi^p$$

and we consider the maximal solutions (Y^{n*}, Z^{n*}) of the BSDE with parameters (H^n, ξ^{n*}) and the minimal solutions $(Y_*^n Z_*^n)$ of the BSDE with parameters (G^n, ξ_*^n), as both:

(i) The sequence $(\xi^{n*})_n$ is decreasing and the sequence $(G^n)_n$ is decreasing and converges locally uniformly to F.

(ii) The sequence $(\xi_*^n)_n$ is increasing and the sequence $(H^n)_n$ is increasing and converges locally uniformly to F.

Then we have:

(i) The sequence $(Y^{n*})_n$ is bounded and decreasing, and for all $n \in \mathbb{N}$

$$Y^{n*} \ge Y^n;$$

therefore by Theorem 2.4, there exists (Y^*, Z^*) such that $(Y^{n*})_n$ converges uniformly to Y^*, and (Y^*, Z^*) is a solution of the BSDE with parameters (F, τ, ξ).

(ii) The sequence $(Y_*^n)_n$ is bounded and decreasing, and for all $n \in \mathbb{N}$,

$$Y_*^n \le Y^n;$$

therefore by Theorem 2.4, there exists (Y_*, Z_*) such that $(Y_*^n)_n$ converges uniformly to Y_*, and (Y_*, Z_*) is a solution of the BSDE with parameters (F, τ, ξ).

(iii) By Theorem 2.6, we have both

$$\forall n \qquad Y_*^n \le Y^n \le Y^{n*} \quad \text{and} \quad Y_* = Y^* = Y;$$

therefore the sequence $(Y^n)_n$ converges uniformly to Y. □

We now turn to the following proof.

PROOF OF THEOREM 2.6. Following the method used by [3] for PDEs, we first show the comparison principle under a *structure condition* on the coefficient. We next complete the proof by giving a change of variable that transforms a BSDE with a coefficient satisfying hypotheses (H2) and (H3) into a BSDE with a coefficient satisfying this structure condition.

STEP 1. *The comparison result under structure condition.* Let $a \in \mathbb{R}$ and b: $\mathbb{R}^+ \to \mathbb{R}$ be a function. We say that *the coefficient f satisfies condition* (STR) *with a* and b if for all $(t, v, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$\text{(STR)} \qquad \frac{\partial f}{\partial u}(t, u, z) + a\left|\frac{\partial f}{\partial z}\right|^2(t, u, z) \le b(t) \quad \text{a.s.}$$

PROPOSITION 2.9. *Let (f^1, ξ^1), and (f^2, ξ^2) be two parameters of BSDE, and let $(Y_t^1, Z_t^1, C_t^1)_t$, and $(Y_t^2, Z_t^2, C_t^2)_t$ be associated supersolution and subsolution. Suppose*

$$\text{(13)} \qquad \begin{gathered} \xi^1 \le \xi^2 \quad a.s., \\ f^1(t, u, z) \le f^2(t, u, z) \quad a.s. \text{ for all } t, u, z, \end{gathered}$$

and suppose that there exist $a > 0$ and $b \in L_\tau^1$ such that either f^1 or f^2 satisfy condition (*STR*) *with a, b, then,*

$$\forall\, t \in [0, T], \qquad Y_t^1 \le Y_t^2 \quad a.s.$$

REMARK 2.10.

(i) It holds true in $\mathscr{H}_T^p(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ instead of $\mathscr{H}_T^\infty(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ when b is bounded instead of being integrable and when $a > 1/2(p-1)$ instead of $a > 0$.

(ii) It still holds true if either $f^1(t, Y_t^1, Z_t^1) \le f^2(t, Y_t^1, Z_t^1)$ a.s. for all t and f^2 satisfies the structure condition (STR), or if $f^1(t, Y_t^2, Z_t^2) \le f^2(t, Y_t^2, Z_t^2)$ a.s. for all t and f^1 satisfies the structure condition (STR).

PROOF OF PROPOSITION 2.9. We set $Y_t = Y_t^1 - Y_t^2$, and $Z_t = Z_t^1 - Z_t^2$. Tanaka's formula applied to the process $Y^+ = \max(0, Y)$ yields

$$-dY_t^+ = \mathbf{1}_{\{Y^+ \ge 0\}} \delta f_t \, dt - \mathbf{1}_{\{Y^+ \ge 0\}} Z_t \, dW_t + dK_t^+ + \underbrace{dC_t^1 - dC_t^2}_{= dC_t},$$

where K_t^+ is a nondecreasing process and grows only at those points t for which $Y_t = 0$.

Now Itô's formula gives for $p \in \mathbb{N}$, $p \ge 2$,

$$\begin{aligned} (Y_t^+)^p &+ \frac{p(p-1)}{2} \int_t^T \mathbf{1}_{\{Y^+ \ge 0\}} (Y_s^+)^{p-2} Z_s^2 \, ds \\ &= p \int_t^T \delta f_s (Y_s^+)^{p-1} \, ds - p \int_t^T (Y_s^+)^{p-1} Z_s \, dW_s \\ &\quad + p \underbrace{\int_t^T (Y_s^+)^{p-1} \, dK_s^+}_{=0} + p \underbrace{\int_t^T (Y_s^+)^{p-1} \, dC_s}_{\le 0}, \end{aligned}$$

where

$$\delta f_s = f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)$$
$$= \underbrace{f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^1, Z_s^1)}_{\le 0} + f^2(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2),$$

$$\delta f_s \le \left(\int_0^1 \frac{\partial f^2}{\partial u}(*)\,d\lambda\right)(Y_s^1 - Y_s^2) + \left(\int_0^1 \frac{\partial f^2}{\partial z}(*)\,d\lambda\right)(Z_s^1 - Z_s^2)$$

with

$$(*) = (s, \lambda Y_s^1 + (1-\lambda)Y_s^2, \lambda Z_s^1 + (1-\lambda)Z_s^2).$$

We then use the well-known inequality $2(\alpha, \beta) \le |\alpha^2| + |\beta^2|$ with

$$\alpha = \sqrt{2a}\frac{\partial f^2}{\partial z}(*)(Y_s^+)^{p/2},$$
$$\beta = \frac{1}{\sqrt{2a}}(Y_s^+)^{(p-2)/2} Z_s \mathbf{1}_{\{Y^+ \ge 0\}}.$$

Hence, with $M = \max(\|Y^1\|_\infty, \|Y^2\|_\infty)$,

$$\delta f_s (Y_s^+)^{p-1} \le \underbrace{\int_0^1 \frac{\partial f^2}{\partial u} + a\left|\frac{\partial f^2}{\partial z}\right|^2 (*)\,d\lambda}_{\le b(s)}(Y_s^+)^p + \frac{1}{4a}(Y_s^+)^{p-2}|Z_s|^2 \mathbf{1}_{\{Y^+ \ge 0\}}.$$

Coming back to Itô's formula,

$$(14)\qquad \begin{aligned} &(Y_t^+)^p + \frac{p}{2}\left((p-1) - \frac{1}{2a}\right)\int_t^T \mathbf{1}_{\{Y^+ \ge 0\}}(Y_s^+)^{p-2}|Z_s|^2\,ds \\ &\quad \le \int_t^T b(s)(Y_s^+)^p\,ds - p\int_t^T (Y_s^+)^{p-1} Z_s\,dW_s. \end{aligned}$$

As Y is bounded, $(Y^+)^{p-1}Z \in \mathscr{H}_T^2(\mathbb{R}^d)$, and taking the expectation,

$$\begin{aligned} &\mathbb{E}(Y_t^+)^p + \frac{p}{2}\left((p-1) - \frac{1}{2a}\right)\mathbb{E}\int_t^T \mathbf{1}_{\{Y^+ \ge 0\}}(Y_s^+)^{p-2}|Z_s|^2\,ds \\ &\quad \le p\int_t^T b(s)\mathbb{E}(Y_s^+)^p\,ds. \end{aligned}$$

For p large enough, $(p-1) - (1/2a) \ge 0$. It now follows that Gronwall's inequality that for all $t \in [0, T]$, $\mathbb{E}(Y_t^+)^p \le 0$; therefore for all $t \in [0, T]$,

$$Y_t^1 \le Y_t^2 \quad \text{a.s.}$$

STEP 2. *The change of variable.* We now look for a change of variable that transforms parameters of BSDE satisfying conditions (H2) and (H3) into parameters satisfying the structure condition (STR).

Let $(Y_t, Z_t)_{0 \le t \le T}$ be a solution in $\mathscr{H}_T^\infty(\mathbb{R}) \times \mathscr{H}_T^2(\mathbb{R}^d)$ of a BSDE with coefficient F and bounded terminal condition ξ.

Take $M \in \mathbb{R}$ such that $\|Y\|_\infty < M$, and consider the change of variable $\tilde{u} = \phi^{-1}(u)$ where ϕ is a regular increasing function yet to be chosen. Setting $w(u) = \phi'(\tilde{u})$, $\widetilde{Y}_t = \phi^{-1}(Y_t)$, $\widetilde{Z}_t = Z_t/w(Y_t)$, the couple $(\widetilde{Y}_t, \widetilde{Z}_t)_{0\le t\le T}$ is the solution to the BSDE with coefficient f and terminal condition $\phi^{-1}(\xi)$, where

$$f(t,u,z) = \frac{1}{\phi'(u)}\bigg(F(t,\phi(u),\phi'(u)z) + \frac{1}{2}\phi''(u)z^2\bigg).$$

Writing u for $\phi(\tilde{u})$, z for $\phi'(\tilde{u})\tilde{z}$ and w for $w(Y)$, a straightforward computation gives

$$\begin{aligned}\frac{\partial f}{\partial \tilde{u}}(t,\tilde{u},\tilde{z}) &= -\frac{w'}{w}F(t,u,z) + \frac{\partial F}{\partial u}(t,u,z) + \frac{1}{2}\frac{w''}{w}|z^2| + \frac{w'}{w}\frac{\partial F}{\partial z}(t,u,z)z \\ &= \frac{1}{w}\bigg(\frac{1}{2}w''|z|^2 + w'\bigg(\frac{\partial F}{\partial z}z - F\bigg) + w\frac{\partial F}{\partial u}\bigg),\\ \frac{\partial f}{\partial \tilde{z}}(t,\tilde{u},\tilde{z}) &= \frac{\partial F}{\partial z}(t,u,z) + z\frac{w'}{w}.\end{aligned}$$

We now show that a good choice of ϕ allows f to satisfy the structure condition (STR). Indeed, if ϕ is such that $w > 0$ and $w' > 0$, then

$$\begin{aligned}&\bigg(\frac{\partial f}{\partial \tilde{u}} + a\bigg|\frac{\partial f}{\partial \tilde{z}}\bigg|^2\bigg)(t,\tilde{u},\tilde{z})\\ &\quad= \frac{1}{w}\bigg(\frac{1}{2}w''|z|^2 + w'\bigg(\frac{\partial F}{\partial z}z - F\bigg) + w\frac{\partial F}{\partial u}\bigg) + a\bigg|\frac{\partial F}{\partial z}(t,u,z) + z\frac{w'}{w}\bigg|^2\\ &\quad\le \frac{1}{w}\bigg[\frac{1}{2}w''|z|^2 + w'\big(k(t)|z| + l(t) + 2C|z|^2\big)\bigg]\\ &\qquad + l_\varepsilon(t) + \varepsilon|z|^2 + a\bigg(k(t) + \bigg(C + \frac{w'}{w}\bigg)|z|\bigg)^2\\ &\quad\le |z|^2\bigg[\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + \varepsilon + a\bigg(C + \frac{w'}{w}\bigg)^2\bigg]\\ &\qquad + |z|\bigg[\frac{w'}{w}k(t) + 2ak(t)\bigg(C + \frac{w'}{w}\bigg)\bigg] + \frac{w'}{w}l(t) + l_\varepsilon(t) + a(k(t))^2\\ &\quad\le |z|^2\bigg[\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + \bigg(\frac{w'}{w}\bigg)^2 + \varepsilon + 2a\bigg(C + \frac{w'}{w}\bigg)^2\bigg]\\ &\qquad + \frac{w'}{w}l(t) + l_\varepsilon(t) + (1+2a)(k(t))^2.\end{aligned}$$

Thus, if we find ϕ satisfying all the required assumptions and such that on $[-M, M]$,

$$\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + \bigg(\frac{w'}{w}\bigg)^2 < -\delta < 0, \tag{15}$$

then choosing a and ε small enough, the coefficient before $|z|^2$ is nonpositive for all u. Therefore (STR) is satisfied.

Setting

$$\phi(v) = \frac{1}{\lambda}\ln\left(\frac{e^{\lambda A\tilde{v}}+1}{A}\right) - M,$$

a straightforward yet tedious computation gives $w(u) = A - \exp(-\lambda(u+M))$; when $A > 1$ and $\lambda > 0$, we have $w > 0$, $w' > 0$, on $[-M, M]$ and moreover as

$$\begin{aligned}
&\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + \left(\frac{w'}{w}\right)^2 \\
&\quad = \frac{\exp(-\lambda(u+M))}{(A-\exp(-\lambda(u+M)))^2} \\
&\quad \times \left[\lambda^2\left(-\frac{A}{2} + \frac{3}{2}\exp(-\lambda(u+M))\right) + \lambda 2C\big(A - \exp(-\lambda(u+M))\big)\right]
\end{aligned}$$

is nonpositive on $[-M, M]$ for a proper choice of A and λ.

The proof is now complete. □

3. BSDEs and PDEs. We consider the following *forward–backward* system

$$\text{(16a)}\qquad \begin{aligned} dX_s^{t,x} &= b(s, X_s^{t,x})\,ds + \sigma(s, X_s^{t,x})\,dW_s \quad \text{for } t \le s \le T, \\ X_t^{t,x} &= x \in \mathbb{R}^n, \end{aligned}$$

$$\text{(16b)}\qquad \begin{aligned} -dY_s^{t,x} &= F(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\,ds - Z_s^{t,x}\,dW_s \quad \text{for } t \le s \le T, \\ Y_T^{t,x} &= g(X_T^{t,x}), \end{aligned}$$

where b and σ are Lipschitz continuous functions on $(0, T)\times\mathbb{R}^n$ taking values, respectively, in $\mathbb{R}^n$ and in the space of $n \times d$ matrices such that there exists a constant K such that for $t \in [0, T]$ and $x, y \in R^n$,

$$\text{(H4)}\qquad \begin{aligned} &|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y|, \\ &\qquad\qquad |b(t,x)|^2 + |\sigma(t,x)|^2 \le K^2(1+|x|^2). \end{aligned}$$

F is a *real*-valued continuous function defined on $(0, T)\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, and g is a real-valued bounded continuous function defined on $\mathbb{R}^n$.

The diffusion (16a) is associated with the second-order elliptic operator L defined by

$$Lu = -\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i,j=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i}, \tag{17}$$

where a is the symmetric positive matrix defined by $a = \sigma\sigma^T$ where σ^T denotes the transposed of σ.

The forward–backward system (16) [or BSDE (16b) set on the diffusion (16a)] is connected to the following PDE:

$$(18)\qquad -\frac{\partial u}{\partial t} + Lu - F(t, x, u, \sigma(t, x)Du) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \qquad u(T, x) = g(x) \quad \text{in } \mathbb{R}^n.$$

Indeed, suppose that u is a solution of (18) of class C^2; applying Itô's formula to $u(s, X_s^{t,x})$ gives that the process $(Y_s^{t,x}, Z_s^{t,x})_{t\le s\le T}$ defined by

$$(19a)\qquad Y_s^{t,x} = u(s, X_s^{t,x})$$

$$(19b)\qquad Z_s^{t,x} = (\sigma^T Du)(s, X_s^{t,x}),$$

is a solution of the BSDE (16b).

In this section we generalize this connection between PDEs and BSDEs when the solution u of (18) does not have such a regularity. We first use the notion of viscosity solutions of PDEs. This method allows one only to justify equation (19a). We then use the notion of Sobolev solutions of PDEs. It allows giving a meaning to both (19a) and (19b), and it is better suited for expressing the Hilbertian aspect of stochastic integrals.

REMARK 3.1. (i) The assumptions (H4) on b and σ give the existence and uniqueness of the diffusion process X, as well as the continuity of the flow $(t, x) \mapsto (X_s^{t,x})_{s\ge t}$.

(ii) We will precise later the assumptions taken on the coefficient F of (16). They will assure the existence and uniqueness of the solution and the continuity of the process $(t, x, s) \mapsto Y_s^{t,x}$ when needed.

(iii) We want to emphasize that the representations of solutions of semilinear by (18) hold only for those PDEs whose nonlinearity f has the peculiar form given by

$$f(t, x, u, p) = F\big(t, x, u, \sigma(t, x)p\big).$$

3.1. *Viscosity solutions.* The notion of viscosity solution was introduced by Crandall and Lions [8] in order to solve first-order Hamilton–Jacobi equations, and then extended to second-order partial differential equations by Lions.

Consider an equation of the form

$$(20)\qquad -\frac{\partial u}{\partial t} + H(t, x, u, Du, D^2u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n.$$

This equation is said to be parabolic if H satisfies the following *ellipticity condition*:

$$H(t, x, u, p, M) \le H(t, x, u, p, N) \quad \text{if } M \ge N,$$

for any $t \in [0, T]$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $M, N \in S^n$ where S^n is the space of $n \times n$ symmetric matrices. In our case H is given by

$$H(t, x, u, p, M) = -\mathrm{Tr}(aM) - (bp) - F\big(t, x, u, \sigma^T(t, x)p\big).$$

We recall that a lower semicontinuous (resp. upper semicontinuous) function u is a *viscosity subsolution* (resp. *viscosity supersolution*) of (20) if for any $\phi \in C^2([0,T] \times \mathbb{R}^n)$ and $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t,x) \geq u(t,x)$ [resp. $\phi(t,x) \leq u(t,x)$] on $[0,T] \times \mathbb{R}^n$, one has

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \phi(t_0, x_0), D\phi(t_0, x_0), D^2\phi(t_0, x_0)) \leq 0$$

$$\left[\text{resp. } -\frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \phi(t_0, x_0), D\phi(t_0, x_0), D^2\phi(t_0, x_0)) > 0\right].$$

The function u is a *viscosity solution* if it is both a super and a subsolution. For further details, we refer to [7].

In this section we first provide a proof of a uniqueness result for viscosity solutions of a PDE with quadratic growth with respect to the gradient. Then we show that the forward–backward system (16) provides a viscosity solution of (18).

3.1.1. *Uniqueness for viscosity solutions.* We suppose that F satisfies the following assumptions: *there exist a positive constant C and, for any $\varepsilon > 0$, there exists a constant c_ε, such that for all $t \in [0,T]$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $q \in \mathbb{R}^d$,*

(H5)

$$\text{(a)} \quad \left|F(t,x,u,\sigma^T(t,x)q)\right| \leq C(1 + |\sigma^T(t,x)q|^2,$$

$$\text{(b)} \quad \left|\frac{\partial F}{\partial z}(t,x,u,\sigma^T(t,x)q)\right| \leq C(1 + |\sigma^T(t,x)q|),$$

$$\text{(c)} \quad \frac{\partial F}{\partial \mu}(t,x,u,\sigma^T(t,x)q) \leq c_\varepsilon + \varepsilon|\sigma^T(t,x)q|^2,$$

$$\text{(d)} \quad \left|\frac{\partial F}{\partial x}(t,x,u,\sigma^T(t,x)q)\right| \leq C(1 + |\sigma^T(t,x)q|^2).$$

THEOREM 3.2 (Uniqueness for viscosity solutions). *Under assumptions* (H4) *and* (H5), *there is a comparison result for the viscosity solutions of* (18). *More precisely, if u is a bounded upper semicontinuous viscosity subsolution of* (18) *and v a lower bounded semicontinuous viscosity supersolution of* (18), *such that*

$$u(T,x) \leq v(T,x) \quad \textit{in } \mathbb{R}^n,$$

then

$$u \leq v \quad \textit{on } [0,T) \times \mathbb{R}^n.$$

REMARK 3.3. Assumptions (H5)(a)(b) are very close to condition (H2), and (H5)(c) to condition (H3). (H5)(d) corresponds to the assumption we need on the coefficient of the BSDE (16) in order to prove the continuity of the flow $(t,x) \mapsto (Y_s^{t,x})_{s \geq 0}$.

PROOF OF THEOREM 3.2. For the same reasons as for the proof of the comparison result for BSDE, we first make a change of variable, which preserves viscosity sub- and supersolutions and which transforms the equation into an equation easier to use. In fact, we want to have a nonlinearity H which is increasing with respect to u, hence F to be decreasing in that variable.

STEP 1. *The change of variable.* We set $M = \max(\| u \|, \| v \|) + 1$ and we consider the real-valued function

$$\phi(v) = \frac{1}{\lambda}\ln\left(\frac{e^{\lambda A v} + 1}{A}\right).$$

ϕ is one-to-one from $\mathbb{R}$ onto $(-(\ln A)/\lambda, +\infty)$.

For A and λ such that $-(\ln A)/\lambda \leq M$ we consider the change of variable $\overline{u} = \phi^{-1}(e^{Kt}(u - M))$, where K is a positive parameter and ϕ a positive increasing function, both yet to be chosen.

We set $w(u) = e^{-Kt}\phi'(\overline{u}) = \partial u/\partial\overline{u}$ and $p = w(u)\overline{p}$. Equation (18) gives way to the following equaiton:

$$-\frac{\partial \overline{u}}{\partial t} + \overline{H}(t, x, \overline{u}, D\overline{u}, D^2\overline{u}) = 0, \tag{21}$$

where

$$\overline{H}(t, x, \overline{u}, \overline{p}, M) = -\mathrm{Tr}\big(\sigma\sigma^T(t, x)M\big) - b(x)\overline{p} - \overline{F}\big(t, x, \overline{u}, \sigma^T(t, x)\overline{p}\big)$$

with $\overline{F}$ defined by

$$\begin{aligned}\overline{F}\big(t, x, \overline{u}, \sigma^T(t, x)\overline{p}\big) &= \frac{\phi''(\overline{u})}{\phi'(\overline{u})}\big|\sigma^T(t, x)\overline{p}\big|^2 - K\frac{\phi(\overline{u})}{\phi'(\overline{u})}\\ &\quad + \frac{e^{Kt}}{\phi'(\overline{u})}F(t, x, e^{-Kt}\phi(\overline{u}) + M, e^{-Kt}\phi'(\overline{u})\sigma^T(t, x)\overline{p}).\end{aligned}$$

Then

$$\begin{aligned}\overline{F}\big(t, x, \overline{u}, \sigma^T(t, x)\overline{p}\big) &= w'(u)\big|\sigma^T(t, x)\overline{p}\big|^2\\ &\quad - K\frac{u - M}{w(u)} + \frac{1}{w(u)}F(t, x, u, w(u)\sigma^T(t, x)\overline{p})\end{aligned} \tag{22}$$

One has, after some computation, using (H5) and supposing that $w'(u) > 0$,

$$\begin{aligned}&\frac{\partial\overline{F}}{\partial\overline{u}}\big(t, x, \overline{u}, \sigma^T(t, x)\overline{p}\big)\\ &\quad\leq \frac{|\sigma^T(t, x)p|^2}{w(u)}\big(w''(u) + 2Cw'(u) + \varepsilon w(u)\big) + \frac{|\sigma^T(t, x)p|}{w(u)}c_\varepsilon w'(u)\\ &\quad\quad - K\left(1 + (M - u)\frac{w'(u)}{w(u)}\right) + c_\varepsilon + C\frac{w'(u)}{w(u)}.\end{aligned}$$

Noting that

$$|\sigma^T(t,x)p|\frac{w'(u)}{w(u)}c_\varepsilon \le \frac{|\sigma^T(t,x)p|^2}{w(u)}w'(u)+\frac{w'(u)}{w(u)}c_\varepsilon^2,$$

we obtain

$$\begin{aligned}\frac{\partial \overline{F}}{\partial \overline{u}}\big(t,x,\overline{u},\sigma^T(t,x)\overline{p}\big) &\le \frac{|\sigma^T(t,x)p|^2}{w(u)}\big(w''(u)+(2C+1)w'(u)+\varepsilon w(u)\big)\\ &\quad - K + c_\varepsilon + \frac{w'(u)}{w(u)}\big(-K(M-u)+C+c_\varepsilon^2\big).\end{aligned}$$

In order to have

$$\frac{\partial \overline{F}}{\partial \overline{u}} \le -\widetilde{K}(1+|\sigma^T(t,x)p|^2) \tag{23}$$

(which corresponds to a *proper* equation in [7]), we now choose λ and ε such that $w''(u)+(2C+1)w'(u)+\varepsilon w(u) \le -\delta < 0$ on $[-M,M]\times[0,T]$. Then choosing K great enough, the second term can also be made smaller than $-\delta$. Choosing then A great enough and the change of variable is valid, one has for all $t\in[0,T]$, $x\in\mathbb{R}^n$, $\overline{u}\in I$, $\overline{p}\in\mathbb{R}^n$, $\overline{F}$ satisfies the following conditions:

$$(24)\quad \begin{aligned}&\text{(a)}\quad \frac{\partial \overline{F}}{\partial \overline{u}}\big(t,x,\overline{u},\sigma^T(t,x)\overline{p}\big) \le -\widetilde{K}(1+|\sigma^T(t,x)p|^2),\\ &\text{(b)}\quad \frac{\partial \overline{F}}{\partial x}\big(t,x,\overline{u},\sigma^T(t,x)\overline{p}\big) \le \overline{C}(1+|\sigma^T(t,x)p|^2),\\ &\text{(c)}\quad \frac{\partial \overline{F}}{\partial z}\big(t,x,\overline{u},\sigma^T(t,x)\overline{p}\big) \le \overline{C}(1+|\sigma^T(t,x)p|)\end{aligned}$$

for some positive constants $\widetilde{K}$ an $\overline{C}$.

In order to simplify the statement of the next result, we set, for all x, y in $\mathbb{R}$, p,q in $\mathbb{R}^n$,

$$\mathscr{K}(t,x,y,p,q) = 1+\frac{|\sigma^T(t,x)p|^2}{2}+\frac{|\sigma^T(t,y)q|^2}{2}.$$

LEMMA 3.4. *Suppose that $\overline{F}$ satisfies assumption* (24), *then for all $t\in(0,T)$, $x,y\in\mathbb{R}^n$, $u,v\in\mathbb{R}$ and $p,q\in\mathbb{R}^n$, we have, if $u-v>0$,*

$$\begin{aligned}&\overline{F}\big(t,x,u,\sigma^T(t,x)p\big)-\overline{F}\big(t,y,v,\sigma^T(t,y)q\big)\\ &\quad \le \mathscr{K}(t,x,y,p,q)\Big(-\widetilde{K}(u-v)+\overline{C}|x-y|+\overline{C}\big|\sigma^T(t,x)p-\sigma^T(t,y)q\big|\Big).\end{aligned}$$

PROOF. For all $x,y\in\mathbb{R}^n$, $u,v\in\mathbb{R}$ and $p,q\in\mathbb{R}^n$ we have

$$\begin{aligned}&\overline{F}\big(t,x,u,\sigma^T(t,x)p\big)-\overline{F}\big(t,y,v,\sigma^T(t,y)q\big)\\ &\quad = \overline{F}\big(t,x,u,\sigma^T(t,x)p\big)-\overline{F}\Big(t,x,\frac{u+v}{2},\sigma^T(t,x)p\Big)\end{aligned}$$

$$+\overline{F}\left(t, x, \frac{u+v}{2}, \sigma^T(t,x)p\right) - \overline{F}\left(t, y, \frac{u+v}{2}, \sigma^T(t,y)q\right)$$
$$+\overline{F}\left(t, y, \frac{u+v}{2}, \sigma^T(t,y)q\right) - \overline{F}\big(t, y, v, \sigma^T(t,y)q\big),$$

hence

$$\begin{aligned}
&\overline{F}\big(t, x, u, \sigma^T(t,x)p\big) - \overline{F}\big(t, y, v, \sigma^T(t,y)q\big)\\
&\quad= \int_0^1 \frac{\partial \overline{F}}{\partial u}(*_1)\,d\lambda\left(\frac{u-v}{2}\right) + \int_0^1 \frac{\partial \overline{F}}{\partial x}(*_2)(x-y)\\
&\qquad+ \frac{\partial \overline{F}}{\partial z}(*_2)\big(\sigma^T(t,x)p - \sigma^T(t,y)q\big)\,d\lambda\\
&\qquad+ \int_0^1 \frac{\partial \overline{F}}{\partial u}(*_3)\,d\lambda\left(\frac{u-v}{2}\right),
\end{aligned}$$

with

$$\begin{aligned}
*_1 &= \big(x, \lambda u + (1-\lambda)(u+v)/2, \sigma^T(t,x)p\big),\\
*_2 &= \big(\lambda x + (1-\lambda)y, (u+v)/2, \lambda\sigma^T(t,x)p + (1-\lambda)\sigma^T(t,y)q\big),\\
*_3 &= \big(y, \lambda(u+v)/2 + (1-\lambda)v, \sigma^T(t,y)q\big).
\end{aligned}$$

The proof is now straightforward.

We now prove the uniqueness result with $\overline{F}$ as nonlinearity.

STEP 2. *Uniqueness under the structure conditions* (24). Let u be a subsolution and v a supersolution. As u and v are bounded $u - v$ has a supremum M. The proof consists in showing that $M \le 0$.

2a. *The regular case.* Suppose that u and v are regular $C^2([0,T] \times \mathbb{R}^n)$ functions and that there exists $(t,x) \in [0,T) \times \mathbb{R}^n$ such that M is reached for (t,x). If $t = 0$, then $M \le 0$.

Suppose now $t \in (0,T)$ and $M > 0$. We have

$$\begin{aligned}
u(t,x) &= v(t,x) + M, &\quad \frac{\partial u}{\partial t}(t,x) &= \frac{\partial v}{\partial t}(t,x),\\
Du(t,x) &= Dv(t,x), &\quad D^2u(t,x) &\le D^2v(t,x).
\end{aligned}$$

Then, as u is a subsolution,

$$-\frac{\partial u}{\partial t} + \overline{H}(t, x, u(t,x), Du(t,x), D^2u(t,x)) \le 0, \tag{25}$$

and, as v is a supersolution,

$$-\frac{\partial v}{\partial t}\overline{H}\big(t, x, v(t,x), Dv(t,x), D^2v(t,x)\big) \ge 0. \tag{26}$$

Subtracting (25) from (26), and using (24)(a), we have, as $M > 0$,

$$0 \le \overline{F}\big(t, x, u(t, x), \sigma^T(t, x)Du(t, x)\big) - \overline{F}\big(t, x, v(t, x), \sigma^T(t, x)Du(t, x)\big)$$
$$= M \int_0^1 \frac{\partial \overline{F}}{\partial \overline{u}}\big(t, x, \lambda u(t, x) + (1-\lambda)v(t, x), \sigma^T(t, x)Du(t, x)\big)\, d\lambda < 0$$

which is a contradiction.

2b. *The general case.* The difficulty is double. First, the functions u and v are only supposed to be semicontinuous and second, the maximum of $u - v$ is not necessarily reached on $[0, T) \times \mathbb{R}^n$. The method consists in *penalizing*.

We define

$$M = \sup_{x \in \mathbb{R}^n,\, t \in [0, T]} [u(t, x) - v(t, x)],$$

the supremum of the bounded function $u - v$, and also

$$M(h) = \sup_{|x-y| \le h} [u(t, x) - v(t, y)] \quad \text{and} \quad M' = \lim_{h \to 0} M(h).$$

One has, of course, $M \le M'$. One purpose is to prove that $M' \le 0$.

Let us consider

$$\psi_{\varepsilon, \eta}(t, x, y) = u(t, x) - v(t, y) - \frac{|x-y|^2}{\varepsilon^2} - \eta(|x|^2 + |y|^2).$$

Let $M_{\varepsilon, \eta}$ be a maximum of $\psi_{\varepsilon, \eta}$ and $(t_{\varepsilon, \eta}, x_{\varepsilon, \eta}, y_{\varepsilon, \eta})$ the point at which this maximum is reached.

As we seek to prove that $u \le v$, we assume to the contrary that $u(s, z) > v(s, z)$ for some s and z; it follows that

$$M_{\varepsilon, \eta} \ge u(s, z) - v(s, z) = \delta > 0 \quad \text{for all } \varepsilon, \eta.$$

We need the equivalent of Lemma 3.1 of [7].

For the sake of simplicity of notations, we write $\tilde{c}$ instead of $t_{\varepsilon, \eta}$ (resp. $\hat{x}$, $\hat{y}$), and we introduce the following notation: if $(a_{\varepsilon, \eta})$ is a sequence we write

$$\limsup_{\varepsilon \ll \eta \to 0} [a_{\varepsilon, \eta}] = \limsup_{\eta \to 0} \Big[\limsup_{\varepsilon \to 0} a_{\varepsilon, \eta}\Big]$$

and

$$\liminf_{\varepsilon \ll \eta \to 0} [a_{\varepsilon, \eta}] = \liminf_{\eta \to 0} \Big[\liminf_{\varepsilon \to 0} a_{\varepsilon, \eta}\Big].$$

If

$$\limsup_{\varepsilon \ll \eta \to 0} [a_{\varepsilon, \eta}] = \liminf_{\varepsilon \ll \eta \to 0} [a_{\varepsilon, \eta}] = a,$$

we write

$$a = \lim_{\varepsilon \ll \eta \to 0} a_{\varepsilon, \eta}.$$

LEMMA 3.5.

$$\text{(i)}\quad \lim_{\varepsilon \ll \eta \to 0} M_{\varepsilon,\eta} = M, \qquad \lim_{\varepsilon \ll \eta \to 0} u(\tilde{c}, \hat{x}) - v(\tilde{c}, \hat{y}) = M,$$

$$\text{(ii)(a)}\quad \lim_{\eta \ll \varepsilon \to 0} M_{\varepsilon,\eta} = M', \qquad \text{(b)}\quad \lim_{\eta \ll \varepsilon \to 0} u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) = M',$$

$$\text{(c)}\quad \lim_{\eta \ll \varepsilon \to 0} \frac{|\hat{x} - \hat{y}|}{\varepsilon} = 0, \qquad \text{(d)}\quad \lim_{\eta \ll \varepsilon \to 0} \eta(|\hat{x}|^2 + |\hat{y}|^2) = 0.$$

We postpone the proof of this lemma and continue the main stream of our proof.

Theorem 8.3 of [7] allows us to state the following.

LEMMA 3.6. *We set $p = 2(\hat{x} - \hat{y})/\varepsilon^2 + 2\eta\hat{x}$ and $q = 2(\hat{x} - \hat{y})/\varepsilon^2 - 2\eta\hat{y}$. There exists $X, Y \in \mathscr{S}^N$ such that*

$$\overline{H}(\tilde{c}, \hat{x}, u(\tilde{c}, \hat{x}), p, X) \le 0 \le \overline{H}(\tilde{c}, \hat{y}, v(\tilde{c}, \hat{y}), q, Y) \tag{27}$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{28}$$

Multiplying (28) by $\binom{\sigma(\tilde{c},\hat{x})}{\sigma(\tilde{c},\hat{y})}$ from the right side and by ${}^t\binom{\sigma(\tilde{c},\hat{x})}{\sigma(\tilde{c},\hat{y})}$ from the left side and taking the Trace, we find

$$\begin{aligned} &\operatorname{Tr}\big(\sigma\sigma^T(\tilde{c}, \hat{x})X\big) - \operatorname{Tr}(\sigma\sigma^T(\tilde{c}, \hat{y})Y) \\ &\quad \ge \frac{2\|\sigma(\tilde{c}, \hat{x}) - \sigma\big(\tilde{c}, \hat{y}\big)\|_2^2}{\varepsilon^2} + 2\eta(\|\sigma(\tilde{c}, \hat{x})\|_2^2 + \|\sigma(\tilde{c}, \hat{y})\|_2^2) \\ &\quad \ge -\|\sigma\|_{\mathrm{Lip}}^2\bigg(\frac{2|\hat{x} - \hat{y}|^2}{\varepsilon^2} + 2\eta(|\hat{x}|^2 + |\hat{y}|^2)\bigg). \end{aligned}$$

In order to complete our proof we only need to show that (27) is in contradiction with $M' > 0$. Indeed, (27) gives

$$\begin{aligned} &-\operatorname{Tr}\big(\sigma\sigma^T(\hat{t}, \hat{x})X\big) + \operatorname{Tr}\big(\sigma\sigma^T(\hat{t}, \hat{y})Y\big) - b\big(\hat{t}, \hat{x}\big)p + b\big(\hat{t}, \hat{y}\big)q \\ &\quad \le \overline{F}\big(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), p\big) - \overline{F}\big(\hat{t}, \hat{y}, v(\hat{t}, \hat{y}), q\big). \end{aligned}$$

We use the majoration provided by Lemma 3.4 for the right-hand side, hence

$$\begin{aligned} &-\Big(\|\sigma\|_{\mathrm{Lip}}^2 + \|b\|_{\mathrm{Lip}}\Big)\bigg(\frac{2|\hat{x} - \hat{y}|^2}{\varepsilon^2} + 2\eta\big(|\hat{x}|^2 + |\hat{y}|^2\big)\bigg) \\ &\quad \le \mathscr{K}(\hat{t}, \hat{x}, \hat{y}, p, q)\big(-\widetilde{K}(u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}))\big). \end{aligned}$$

Therefore, by Lemma 3.5,

$$o(1) \le \mathscr{K}(\hat{t}, \hat{x}, \hat{y}, p, q)\big(-\widetilde{K}(M' + o(1))\big).$$

As $\mathscr{K}(\hat{t}, \hat{x}, \hat{y}, p, q) \geq 1$, we can divide by $\mathscr{K}(\hat{t}, \hat{x}, \hat{y}, p, q)$, and pass to the limit. We get

$$0 \leq -\widetilde{K} M',$$

which is the expected contradiction. □

PROOF OF LEMMA 3.5. For all x, y, t,

$$\begin{aligned} &u(t, x) - v(t, y) - \frac{|x-y|^2}{\varepsilon^2} - \eta(|x|^2 + |y|^2) \\ &\leq M_{\varepsilon, \eta} = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - \frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2} - \eta(|\hat{x}|^2 + |\hat{y}|^2) \\ &\leq u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}). \end{aligned} \tag{29}$$

Using the first inequality of (29) with $x = y = 0$ one has

$$\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2} + \eta(|\hat{x}|^2 + |\hat{y}|^2) \leq 2(\|u\|_\infty + \|v\|_\infty);$$

hence, with $C = \sqrt{2(\|u\|_\infty + \|v\|_\infty)}$, we have the following first estimate:

$$|\hat{x}-\hat{y}| \leq C\varepsilon, \qquad |\hat{x}|, |\hat{y}| \leq \frac{C}{\eta}. \tag{30}$$

STEP 1. We have, using (30),

$$u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq \underbrace{u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x})}_{\leq M} + \underbrace{v(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y})}_{\substack{\to 0 \text{ when} \\ \eta \text{ is fixed and } \varepsilon \to 0}}.$$

Let (t_h, x_h) be a sequence such that

$$\lim_{h\to 0} u(t_h, x_h) - v(t_h, x_h) = M.$$

Inequality (29) gives, when η is fixed,

$$u(t_h, x_h) - v(t_h, x_h) - 2\eta|x_h|^2 \leq M_{\varepsilon, \eta} \leq u(\hat{t}, \hat{x}) - v(\overline{s}, \hat{y}) \leq M + o(\varepsilon).$$

Letting successively $\varepsilon \to 0$, $\eta \to 0$ and $h \to 0$ in the inequality above gives both

$$M \leq \liminf_{\varepsilon \ll \eta \to 0} M_{\varepsilon, \eta} \leq \limsup_{\varepsilon \ll \eta \to 0} M_{\varepsilon, \eta} \leq M$$

and

$$M \leq \liminf_{\varepsilon \ll \eta \to 0} u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq \limsup_{\varepsilon \ll \eta \to 0} u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq M.$$

STEP 2. Using (30) we have

$$u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq M(C\varepsilon).$$

Let (t_h, x_h, y_h) be a sequence such that

$$|x_h - y_h| \leq h \quad \text{and} \quad \lim_{h\to 0} u(t_h, x_h) - v(t_h, y_h) = M'.$$

It gives in (29),

$$u(t_h, x_h) - v(t_h, y_h) - \frac{h}{\varepsilon^2} - \eta(|x_h|^2 + |y_h|^2) \leq M_{\varepsilon,\eta} \leq u(\hat{t}, \hat{x}) - v(\bar{s}, \hat{y}) \leq M(C\varepsilon).$$

Letting first $\eta \to 0$, $h \to 0$, then $\varepsilon \to 0$,

$$M' \leq \liminf_{\eta \ll \varepsilon \to 0} M_{\varepsilon,\eta} \leq \limsup_{\eta \ll \varepsilon \to 0} M_{\varepsilon,\eta} \leq M',$$

$$M' \leq \liminf_{\eta \ll \varepsilon \to 0} u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq \limsup_{\eta \ll \varepsilon \to 0} u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \leq M'.$$

Now passing to the limit in the previous order in (29) gives the last three results. This ends the proof of the lemma. □

3.1.2. *Existence of viscosity solutions given by BSDE.* Let $(X_s^{t,x})_{t\leq s\leq T}$ be the diffusion defined by (16) under assumption (H4), and suppose that F satisfies (H5). In view of Remark 3.1(i) and of (H5)(d), the function $(s, u, z) \mapsto F(s, X_s^{t',x'}, u, z)$ converges locally uniformly to $(s, u, z) \mapsto F(s, X_s^{x,t}, u, z)$ as $(t', x') \to (t, x)$. The immediate application of Theorem 2.8 allows us to state the following theorem.

THEOREM 3.7 (Continuity). *If σ and b satisfy assumptions* (H4) *and F satisfies assumptions* (H5), *the flow $(t, x) \mapsto (Y_s^{t,x})_{s\geq t}$ is a.s. continuous. In particular, the deterministic function $(x, t) \mapsto Y_t^{t,x}$ is continuous.*

The main result of our section is this theorem.

THEOREM 3.8 (BSDE and viscosity solutions). *Under assumptions* (H4) *on L and* (H5) *on F, the function defined on $[0, T] \times \mathbb{R}^d$ by $u(t, x) = Y_t^{t,x}$ is a viscosity solution of* (18).

The proof of this result does not depend on a comparison result. It is a local proof and therefore closer to the spirit of viscosity solutions. The main argument is given by the following theorem.

THEOREM 3.9 (Touching). *Let $(Y_t)_{0\leq t\leq T}$ be a continuous adapted process such that*

$$dY_t = b(t)\,dt + \sigma(t)\,dW_t,$$

where b and σ are continuous adapted processes such that b, $|\sigma|^2$ are integrable. If $Y_t \geq 0$ a.s. for all t, then for all t,

$$\mathbf{1}_{\{Y_t=0\}}\sigma(t) = 0 \quad a.s.,$$
$$\mathbf{1}_{\{Y_t=0\}}b(t) \geq 0 \quad a.s.$$

PROOF. Let $\phi: \mathbb{R}^+ \to \mathbb{R}$ be a positive C^2 bounded increasing function such that $\phi(0) = 0$, $\phi'(0) = \alpha > 0$, $\phi''(0) = \beta < 0$.

Set $\widetilde{Y}_t = \phi(Y_t)$. Itô's formula gives

$$d\widetilde{Y}_t = \tilde{b}(t)\,dt + \tilde{\sigma}(t)\,dW_t,$$

where $\tilde{\sigma}(t) = \phi'(Y_t)\sigma(t)$ and $\tilde{b}(t) = \phi'(Y_t)b(t) + \frac{1}{2}\phi''(Y_t)|\sigma(t)|^2$.

Let us apply now Itô's formula to $\psi_\varepsilon(\widetilde{Y}_t)$ where $(\psi_\varepsilon)_\varepsilon$ is a sequence of $C^2(\mathbb{R})$ convex functions that converges to t^- (with $\psi'_\varepsilon = -1$ on $\mathbb{R}^-$):

$$\psi_\varepsilon(\widetilde{Y}_t) - \psi_\varepsilon(\widetilde{Y}_s) = \int_s^t \psi'_\varepsilon(\widetilde{Y}_\tau)\tilde{b}(\tau) + \tfrac{1}{2}\psi''_\varepsilon(\widetilde{Y}_\tau)|\tilde{\sigma}(\tau)|^2\,d\tau + \int_s^t \psi'_\varepsilon(\widetilde{Y}_\tau)\tilde{\sigma}(\tau)\,dW_\tau$$

for all $0 \leq s \leq t$. Hence

$$0 \geq \int_s^t -\mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\tilde{b}(\tau)\,d\tau - \int_s^t \mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\tilde{\sigma}(\tau)\,dW_\tau. \tag{31}$$

Taking the expectation we get

$$\int_s^t \mathbb{E}\Big(\mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\tilde{b}(\tau)\Big)\,d\tau \geq 0,$$

dividing by $t-s$ and letting $t \to s$ and as $s \mapsto \mathbb{E}(\mathbf{1}_{\{\widetilde{Y}_s=0\}}\tilde{b}(s))$ is lower semicontinuous, we get, for all $s \in (0, T)$,

$$\mathbb{E}\Big(\mathbf{1}_{\{Y_s=0\}}\tilde{b}(s)\Big) \geq 0.$$

This implies that, for all $\alpha > 0$ and $\beta < 0$,

$$\mathbb{E}\Big(\mathbf{1}_{\{Y_s=0\}}(\alpha b(s) + \tfrac{1}{2}\beta|\sigma(s)|^2)\Big) \geq 0.$$

Letting β to $-\infty$, one gets

$$\mathbf{1}_{\{Y_s=0\}}|\sigma(s)|^2 = 0 \quad \text{a.s. for all } 0 \leq s \leq t.$$

This implies that for all $0 \leq \tau \leq t$, one has both

$$\mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\tilde{\sigma}(\tau) = 0,$$
$$\mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\tilde{b}(\tau) = \mathbf{1}_{\{\widetilde{Y}_\tau=0\}}\alpha b(\tau).$$

Back in (31), it gives the second inequality and completes the proof. □

PROOF OF THEOREM 3.8. For the sake of simplicity of notations let us write (X_t) for the diffusion process $(X_t^{t_0,x_0})$ starting from x_0 at t_0 and (Y_t, Z_t) for the solution $(Y_t^{t_0,x_0}, Z_t^{t_0,x_0})$ of the BSDE driven by this diffusion.

First notice that $u(t, X_t) = Y_t$. This is readily seen from the Markovian property of the diffusion process X_t and from the uniqueness of the BSDE.

Let ϕ be a C^2 function such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t, x) \geq u(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$. Hence $\phi(t, X_t) \geq Y_t$.

We now show that u is a viscosity subsolution of (18). One has

$$-dY_t = F(Y_t, Z_t)\,dt - Z_t\,dW_t; Y_\tau,$$
$$-d\phi(t, X_t) = -\left(\frac{\partial \phi}{\partial t} + L\phi\right)(t, X_t)\,dt - \sigma^T D\phi(t, X_t)\,dW_t; \phi(\tau, X_\tau).$$

As $\phi(t, X_t) \geq Y_t$, Theorem 3.9 gives, for all t,

$$1_{\{\phi(t, X_t)=Y_t\}}\left(-\left(\frac{\partial \phi}{\partial t} + L\phi\right)(t, X_t) - F(Y_t, Z_t)\right) \leq 0 \quad \text{a.s.},$$
$$1_{\{\phi(t, X_t)=Y_t\}}|-Z_t + \sigma^T D\phi(t, X_t)|^2 = 0 \quad \text{a.s.}$$

As $\phi(t_0, X_{t_0}) = Y_{t_0}$ for $t = t_0$, the second equation gives $Z_{t_0} = \sigma^T D\phi(t_0, X_{t_0})$, and the first inequality gives the expected result.

3.2. *Sobolev solutions and BSDEs.*

3.2.1. *Sobolev solutions for PDEs with quadratic growth.* We first give existence and uniqueness results of Sobolev solutions for a quasilinear elliptic PDE, and then some regularity results for its linear part together with properties of its first eigenvalue.

The homogeneous Dirichlet problem. In this paragraph we study the PDE

$$\begin{aligned} &Lu - f(x, u, Du) = 0 \quad \text{in } \mathscr{O}, \\ &u \in H_0^1(\mathscr{O}), \end{aligned} \tag{32}$$

where $\mathscr{O}$ is an open bounded subset of $\mathbb{R}^n$ and the operator L is, throughout this section, considered in the divergence form

$$Lu = -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right), \tag{33}$$

where we assume without loss of generality that $a_{i,j} = a_{j,i}$, and we first recall existence and uniqueness results of a solution of (32).

For the *existence result* we suppose that

$$a_{ij} \in L^\infty(\mathscr{O}), \qquad 0 \leq i, j \leq n,$$

and there exists $\alpha > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{i,j}\xi^i\xi^j \geq \alpha|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^n \textit{ and a.e. } x \in \mathscr{O} \tag{34}$$

and that the function f *is a Caratheodory function defined on* $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$ (i.e., $f(x, ., .)$ is continuous for a.e. $x \in \mathcal{O}$ and $f(\cdot, u, p)$ is measurable for all $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$) *such that for almost all* $x \in \mathcal{O}$ *and for all* $u \in \mathbb{R}$, $p \in \mathbb{R}^n$,

$$f(x, u, p) = -a_0(x)u - f_0(x, u, p),$$

with

$$0 < \alpha_0 \le a_0(x) \le \beta_0 \quad \text{a.e. in } \mathcal{O} \tag{35}$$

and

$$|f_0(x, u, p)| \le C_0 + b(|u|)|p|^2$$

for a positive constant C_0 *and an increasing function* b.

THEOREM 3.10 [5]. *Under assumptions* (34) *and* (35), *there exists at least a solution in* $H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ *of* (32).

For the *uniqueness result* we suppose that *the function* f *defined on* $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$ *is a Caratheodory function, such that for a.e.* $x \in \mathcal{O}$ *the function* $(u, p) \mapsto f(x, u, p)$ *is locally Lipschitz continuous with*

$$\begin{aligned} &\left|\frac{\partial f}{\partial p}(x, u, p)\right| \le C_0(|u|)(1 + |p|) \quad \text{for a.e. } x \in \mathcal{O}, u \in \mathbb{R}, p \in \mathbb{R}^n, \\ &|f(x, u, 0)| \le C_1(|u|) \quad \text{for a.e. } x \in \mathcal{O}, u \in \mathbb{R}, p \in \mathbb{R}^n, \\ &\frac{\partial f}{\partial u}(x, u, p) \le -\alpha_0 < u \quad \text{for a.e. } x \in \mathcal{O}, u \in \mathbb{R}, p \in \mathbb{R}^n, \end{aligned} \tag{H6}$$

for a constant α_0 *and for increasing functions* C_0 *and* C_1.

THEOREM 3.11 [3]. *Assume that* (34) *and* (H6) *hold. If* u_1 *and* u_2 *belong to* $H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ *and are, respectively, a subsolution and a supersolution of* (32) *such that* $(u_1 - u_2)^+ \in H_0^1(\mathcal{O})$, *then*

$$u_1 \le u_2 \quad \text{in } \mathcal{O}.$$

In particular (32) *has at most one solution in* $H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$.

An immediate consequence of this comparison principle is the corollary.

COROLLARY 3.12. *Assume that* (34) *and* (H6) *hold. If* $u \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ *is a solution of* (32) *then*

$$\|u\|_\infty \le \frac{C_1(0)}{\alpha_0}.$$

PROOF. Indeed, the function defined by

$$u_1(x) = C_1(0)/\alpha_0 = C$$

is a supersolution of (32), since

$$\begin{aligned} Lu_1(x) - f(x, u_1(x), Du_1(x)) &= -f(x, C, 0) \\ &= -f(x, 0, 0) - \int_0^C \frac{\partial f}{\partial u}(x, u, 0) \\ &\geq -C_1(0) + \alpha_0 C = 0. \end{aligned}$$

In a similar way, we can show that $-C$ is a subsolution.

PROPOSITION 3.13. *Assume that* (35) *and* (H6) *hold, then equation* (32) *has a unique solution in* $H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$.

PROOF. The proof relies on a truncation argument. Indeed, for all $K > 0$ we consider a C^2 function, ϕ_K, such that $\phi(u) = u$ for $-K \leq u \leq K$, ϕ is constant when $|u| > K + 2$ and $0 \leq \phi'_K(u) \leq 1$ for all $u \in \mathbb{R}$, and we set

$$f_K(x, u, p) = -\alpha_0 u + f(x, \phi_K(u), p) + \alpha_0 \phi_K(u).$$

The function f_K obviously satisfies assumptions (H6) with the same constants as f and satisfies also assumptions (35) with $\alpha_{0,K}(x) = \alpha_0$, $C_{0,K} = \alpha_0(K+1) + C_0(K+1) + C_1(K+1)$ and $b_K(u) = 2C_0(K+1)$, hence applying Theorems 3.10 and 3.11, equation $Lu - f_K(x, u, Du) = 0$ has a unique solution u_K in $H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$, and by Corollary 3.12, $\|u_K\|_\infty \leq C_1(0)/\alpha_0$. As $f_K(c, u, p) = f(x, u, p)$ for all $x \in \mathscr{O}$, $|u| \leq K$ and $p \in \mathbb{R}^n$, u_K is also the unique solution of (32) as soon as $K > C_1(0)/\alpha_0$.

The nonhomogenous Dirichlet problem. We now consider the following PDE:

$$\begin{aligned} Lu - f(x, u, Du) = 0 \quad \text{in } \mathscr{O} \\ u - g \in H_0^1(\mathscr{O}), \end{aligned} \tag{36}$$

where L is defined by (33), and

(H7)
- (a) The boundary of $\mathscr{O}$ has a $C^{1,1}$ regularity.
- (b) $g \in W^{2,p}(\mathscr{O})$ with $p > N$.
- (c) $a_{ij}(x) \in W^{2,\infty}(\mathscr{O})$, $1 \leq i, j \leq N$, and there exists $\alpha > 0$ such that $\sum_{i,j=1}^N a_{ij}(x)\xi^i\xi^j \geq \alpha|\xi|^2$ for a.e. $x \in \mathscr{O}$ and $\xi \in \mathbb{R}^n$.

THEOREM 3.14 (Nonhomogenous Dirichlet problem). *Assume that* (H6) *and* (H7) *hold. Then there exists a unique Sobolev solution u in* $H^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$ *of* (36).

PROOF. Under assumption (H7), the Dirichlet problem

$$L\hat{g} = 0 \quad \text{in } \mathscr{O},$$
$$\hat{g} - g \in H_0^1(\mathscr{O})$$

has a unique solution $\hat{g} \in W^{2,p}(\mathscr{O})$ (cf. Theorem 9.15, page 241 of [12]). As $p > N$, Sobolev imbeddings show that $\hat{g} \in C^1(\overline{\mathscr{O}}) \cap C^2(\mathscr{O})$.

Consider now

$$(37) \qquad \begin{aligned} Lw - \hat{f}(x, w, Dw) = 0 \quad \text{in } \mathscr{O}, \\ w \in H_0^1(\mathscr{O}), \end{aligned}$$

where $\hat{f}$ is defined by $\hat{f}(x, u, p) = f(x, u - \tilde{g}(x), p - D\tilde{g}(x))$.

As $\hat{g} \in C^1(\overline{\mathscr{O}})$, it is obvious that $u \in H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$ is the solution of (36) if and only if $w = u - \hat{g}$ is the solution of (37), but as $\hat{f}$ satisfies the assumptions of Proposition 3.13, (37) has a unique solution in $H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$, which completes the proof. □

The linear operator. We recall some classical results on the linear equation

$$(38) \qquad \begin{aligned} Lu + \alpha_0 u = \tilde{f}_0 \quad \text{in } \mathscr{O}, \\ u = g \quad \text{on } \partial\mathscr{O}, \end{aligned}$$

where L is defined by (34) and α_0 is a nonnegative constant.

PROPOSITION 3.15. (i) *If $\tilde{f}_0 \in L^2(\mathscr{O})$ and $g = 0$, there exists a constant C depending only on $\mathscr{O}$ and α_0 such that*

$$\|u\|_{H_0^1(\mathscr{O})} \le C\|\tilde{f}_0\|_{L^2(\mathscr{O})}.$$

(ii) *Assume $a_{ij} \in C^{1,\alpha}(\mathscr{O})$ for all $1 \le i, j \le N$, $\tilde{f}_0 \in C^{0,\alpha}(\mathscr{O})$; $\partial\mathscr{O} \in C^{1,1}$, $g \in C^{1,\alpha}(\partial\mathscr{O})$; then the unique solution u of* (38) *belongs to $C^{2,\alpha}(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$ and*

$$\|u\|_\infty \le \max\left(\frac{\|\tilde{f}_0\|_\infty}{\alpha_0}, \|g\|_\infty\right);$$

in particular, this result holds true under assumptions (H7).

(iii) *Assume* (34); *then the operator L is self-adjoint strictly elliptic, its first eigenvalue $\lambda_1 > 0$ and the associated eigenfunction e_1 is unique up to a multiplicative constant $e_1 > 0$ in $\mathscr{O}$. Moreover, if we suppose that 2 holds true, $e_1 \in C^{2,\alpha}(\mathscr{O})$.*

For the proof, see, for instance, [12].

3.3. *Connections with BSDEs.* We first define the system of forward–backward equations associated with (36).

The diffusion process—the exit time problem. From now on, we suppose that (H7) holds true, and we define

$$b_i = -\frac{1}{2}\sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}$$

and $\sigma\colon \mathscr{O} \to \mathbb{R}^{n\times n}$ is such that $\sigma\sigma^T = a$ and, for all $i, j \le n$, $\sigma_{i,j}, b_i \in W^{1,\infty}(\mathscr{O})$.

L is the infinitesimal generator of the *diffusion process*

$$dX_t^x = b(X_t^x)\,dt + \sigma(X_t^x)\,dW_t \quad \text{for } 0 \le t \le +\infty,$$
$$X_0^x = x \ (x \in \mathscr{O}).$$

We note τ to be *the first exit time* of the process X from $\mathscr{O}$,

$$\tau(x, \omega) = \inf\{t > 0,\ X_t^x(\omega) \notin \mathscr{O}\}.$$

As a is not degenerated, τ is almost surely finite. In fact, the exit time and the first eigenvalue of L are connected in the following way:

$$\mathbb{E}\big(e^{\lambda\tau^x}\big) < \infty \quad \text{for all } \lambda < \lambda_1, \tag{39}$$

for all $x \in \mathscr{O}$ [9].

The flow of BSDEs. For all $x \in \mathscr{O}$, we define the coefficient set on the diffusion starting from x by

$$F_x(t, v, z)(\omega) = f(X_t^x(\omega), v, (\sigma^T(X_t^x(\omega)))^{-1}z)$$

for all $\omega \in \Omega, t \le \tau^x(\omega), v \in \mathbb{R}, z \in \mathbb{R}^n$, and we consider the BSDE with parameters $(F_x, \tau^x, g(X_{\tau^x}^x))$,

$$Y_t^x = g(X_{\tau^x}^x) + \int_{t\wedge\tau^x}^{\tau^x} F_x(s, Y_s^x, Z_s^x)\,ds - \int_{t\wedge\tau^x}^{\tau^x} Z_s^x\,dW_s. \tag{40}$$

THEOREM 3.16 (BSDE). *Assume that f satisfies* (H6) *and that* (H7) *holds true; then for all $x \in \mathscr{O}$, there exists a unique solution $(Y_t^x, Z_t^x)_{0\le t\le\tau^x}$ of* (40).

The proof is a direct consequence of the results of Section 1, since all the required assumptions are obviously satisfied.

We now give the Feynman–Kac formula corresponding to equation (38) expressed in terms of BSDEs.

LEMMA 3.17 (Feynman–Kac formula). *Assume that* (H7) *holds true and $\tilde{f}_0 \in C^{0,\alpha}$, and let u be the solution of* (38). *Setting*

$$Y_t^x = u(X_t^x),$$
$$Z_t^x = (\sigma^T Du)(X_t^x),$$

for every $x \in \mathscr{O}$, the process $(Y_t^x, Z_t^x)_{t\le\tau^x}$ is in $\mathscr{H}_{\tau^x}^{\infty}(\mathbb{R}) \times \mathscr{H}_{\tau^x}^{2}(\mathbb{R}^n)$ and is the solution of

$$-dY_t^x = -\alpha_0 Y_t^x + \tilde{f}_0(X_t^x)\,dt - Z_t^x\,dW_t,$$
$$Y_{\tau^x}^x = g(X_{\tau^x}^x).$$

PROOF. It is a simple application of Itô's formula to the C^2 solution u of (38) and to the process $(X_t^x)_{0\le t\le \tau^x}$.

Our main result is the following theorem.

THEOREM 3.18. *Assume that* (H6) *and* (H7) *hold true and let u be the solution of* (36). *Setting*

$$Y_t^x = u(X_t^x),$$
$$Z_t^x = (\sigma^T Du)(X_t^x),$$

then:

(i) *For almost every $x \in \mathscr{O}$,*

$$(Y_t^x)_{0\le t\le\tau^x} \in \mathscr{H}_{\tau^x}^{\infty}(\mathbb{R}),$$
$$(Z_t^x)_{0\le t\le\tau^x} \in \mathscr{H}_{\tau^x}^{2}(\mathbb{R}^n).$$

(ii) *For almost all $x \in \mathscr{O}$, the process $(Y_t^x, Z_t^x)_{0\le t\le\tau^x}$ is the solution of*

$$-dY_t^x = f(X_t^x, Y_t^x, (\sigma^T(X_t^x))^{-1}Z_t^x)\,dt - Z_t^x\,dW_t,$$
$$Y_{\tau^x}^x = g(X_{\tau^x}^x).$$

PROOF OF THEOREM 3.18 *Part* 1. We want to show that if ϕ is in $L^\infty(\mathscr{O})$ or in $L^2(\mathscr{O})$, then for almost every $x \in \mathscr{O}$, $\phi(X_t^x) \in \mathscr{H}_{\tau^x}^{\infty}(\mathbb{R})$ or $\mathscr{H}_{\tau^x}^{2}(\mathbb{R})$. This has been done in [2] when the domain is the entire set $\mathbb{R}^n$. Indeed, these authors have shown that, under assumptions (H7)(c), there exist two nonnegative constants K_1 and K_2 such that for all $\phi \in L^2(\mathbb{R}^n)$,

$$K_1\|\phi\|_{L^2(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \|\phi(X_t^x)\|_{\mathscr{H}_T^2(\mathbb{R}^d)}\,dx \le K_2\|\phi\|_{L^2(\mathbb{R}^n)},$$

where

$$\|\phi(X_t^x)\|_{\mathscr{H}_T^2(\mathbb{R}^d)} = \mathbb{E}\int_0^T |\phi(X_t^x)|^2\,dt.$$

This equivalence of norm is linked in our case with the first eigenvalue problem as can be seen in the following result.

We introduce the following measure on the space $\mathscr{O} \times \mathbb{R}^+ \times \Omega$;

$$d\mu(x, s, \omega) = e_1(x)\mathbf{1}_{s\le\tau^x}\,dx\,ds\,d\mathbb{P}.$$

LEMMA 3.19. *For all $\phi \in L^1(\mathscr{O})$, $\phi(X_t^x) \in L^1(\mathscr{O} \times \mathbb{R}^+ \times \Omega, \mu)$ and*

$$\int_{\mathscr{O}} \phi(x)e_1(x)\,dx = \lambda_1\mathbb{E}\int_{\mathscr{O}}\int_0^{\tau^x} \phi(X_s^x)\,d\mu. \tag{41}$$

In particular:

(i) *If* $\phi \in [L^\infty(\mathscr{O})]^d$ *then, for almost all* $x \in \mathscr{O}$, $\phi(X_t^x) \in \mathscr{H}^\infty_{\tau^x}(\mathbb{R}^d)$, *and*

$$\|\phi(X_t^x)\|_{\mathscr{H}^\infty_{\tau^x}(\mathbb{R}^d)} \le \|\phi\|_{L^\infty(\mathscr{O})}.$$

(ii) *If* $\phi \in [L^1(\mathscr{O})]^d$ *then, for almost all* $x \in \mathscr{O}$, $\phi(X_t^x) \in \mathscr{H}^1_{\tau^x}(\mathbb{R}^d)$ *and*

$$\int_{\mathscr{O}} \|\phi(X_t^x)\|_{\mathscr{H}^1_{\tau^x}(\mathbb{R}^d)} e_1(x)\,dx \le \|\phi\|_{L^1(\mathscr{O})}.$$

(iii) *If* $\phi^\varepsilon \to \phi$ *in* $[L^1(\mathscr{O})]^d$ *then one can extract a subsequence such that, for almost all* $x \in \mathscr{O}$, $\phi^{\varepsilon'}(X_t^x) \to \phi(X_t^x)$ *in* $\mathscr{H}^1_{\tau^x}(\mathbb{R}^d)$.

PROOF.

STEP 1. Take $\phi \in C^{0,\alpha}(\mathscr{O})$; consider the solution $u \in C^2(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$ of the equation

$$\begin{aligned} Lu &= \phi \quad \text{in } \mathscr{O}, \\ u &= 0 \quad \text{on } \partial\mathscr{O}. \end{aligned}$$

By the Feynman–Kac formula $u(x) = \mathbb{E}\int_0^\tau \phi(X_t^x)\,dt$ and in view of Theorem 3.15, the functions u, e_1 belong to $C^{2,\alpha}(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$. This allows justifying the following computations based on Green's formula:

$$\begin{aligned} &\int_{\mathscr{O}} \phi(x) e_1(x)\,dx \\ &\quad = \int_{\mathscr{O}} Lu(x) e_1(x)\,dx \\ &\quad = -\frac{1}{2}\sum_{i,j=1}^N \int_{\mathscr{O}} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) e_1(x)\,dx \\ &\quad = -\frac{1}{2}\sum_{i,j=1}^N \mu \int_{\mathscr{O}} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial e_1}{\partial x_j}\right) dx + \int_{\partial\mathscr{O}} \left(\frac{\partial u}{\partial n}\underbrace{e_1}_{=0} + \underbrace{u}_{=0}\frac{\partial e_1}{\partial n}\right)(x)\,d\sigma(x) \\ &\quad = \int_{\mathscr{O}} u(x) L e_1(x)\,dx \\ &\quad = \lambda_1 \mathbb{E}\int_{\mathscr{O}} \int_0^{\tau^x} \phi(X_s^x) e_1(x)\,dx\,ds. \end{aligned}$$

A first consequence of this regular case is that for any negligible subset A of $\mathscr{O}$,

$$\{(x, s, \omega); X_s^x(\omega) \in A\} \in \mathscr{O} \times \mathbb{R}^+ \times \Omega$$

is also negligible for the measure $d\mu$. This implies (i).

STEP 2. Take $\phi \in L^1(\mathscr{O})$, and let ϕ^ε be a sequence of $C^{0,\alpha}(\mathscr{O})$ functions converging to ϕ in $L^1(\mathscr{O})$. As Step 1 holds for $(\phi^\varepsilon)_\varepsilon$,

$$\int_{\mathscr{O}} |\phi^\varepsilon(x) - \phi^{\varepsilon'}(x)| e_1(x)\,dx = \lambda_1 \mathbb{E} \int_{\mathscr{O}} \int_0^{\tau^x} |\phi^\varepsilon(X_s^x) - \phi^{\varepsilon'}(X_s^x)| e_1(x)\,dx\,ds,$$

hence $\phi^\varepsilon(X_t^x)$ is a Cauchy sequence in $L^1(\Omega \times \mathscr{O} \times \mathbb{R}^+, \mathbf{1}_{s \le \tau^x} e_1(x)\,dx\,ds\,\mathbb{P})$ and it has a limit Y when $\varepsilon \to 0$. Extracting a subsequence if necessary, one can suppose that ϕ^ε converges to ϕ for almost all x and $\phi^\varepsilon(X_s^x)$ converges to Y for almost all x, t, ω. Hence there exists a negligible subset A of $\mathscr{O}$ such that $\phi^\varepsilon(x) \to \phi(x)$ when $\varepsilon \to 0$ for all $x \notin A$; consequently $\phi^\varepsilon(X_t^x) \to \phi(X_t^x)$ when $\varepsilon \to 0$ for all (ω, x, t) such that $X_t^x \notin A$.

As $\{(\omega, x, t);\ X_t^x(\omega) \in A\}$ is negligible, one can identify the limit Y with $\phi(X_s^x)$. Hence passing to the limit in

$$\int_{\mathscr{O}} \phi^\varepsilon(x) e_1(x)\,dx = \lambda_1 \mathbb{E} \int_{\mathscr{O}} \int_0^{\tau^x} \phi^\varepsilon(X_s^x)\,d\mu,$$

we have proved (41).

Let us now prove (ii) and (iii).

Let $\phi \in [L^1(\mathscr{O})]^d$. Then, as

$$\int_{\mathscr{O}} \mathbb{E} \int_0^{\tau^x} |\phi(X_t^x)| e_1(x)\,dt\,dx = \frac{1}{\lambda_1} \|\phi\|_{L^1(\mathscr{O}, e_1(x)\,dx)} < \infty,$$

we have

$$\mathbb{E} \int_0^{\tau^x} |\phi(X_t^x)|\,dt\,dx < \infty \quad \text{for almost all } x \in \mathscr{O}.$$

Let $\phi \in [L^1(\mathscr{O})]^d$, and suppose $\phi^\varepsilon \in [L^1(\mathscr{O})]^d$ is a sequence converging to ϕ. We have

$$\int_{\mathscr{O}} |\phi^\varepsilon(x) - \phi(x)| e_1(x)\,dx = \lambda_1 \mathbb{E} \int_{\mathscr{O}} \int_0^{\tau^x} |\phi^\varepsilon(X_s^x) - \phi(X_s^x)| e_1(x)\,dx\,ds;$$

hence

$$\int_{\mathscr{O}} \mathbb{E} \int_0^{\tau^x} |\phi^\varepsilon(X_s^x) - \phi(X_s^x)| e_1(x)\,dx\,ds \to 0 \quad \text{as } \varepsilon \to 0,$$

and therefore one can extract a subsequence ε' such that, for almost all $x \in \mathscr{O}$,

$$\mathbb{E} \int_0^{\tau^x} |\phi^{\varepsilon'}(X_s^x) - \phi(X_s^x)| e_1(x)\,dx\,ds \to 0 \quad \text{as } \varepsilon \to 0.$$

This completes the proof of Lemma 3.19. □

Let $u \in H^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$ be the solution of (36). Lemma 3.19 gives that, for almost all $x \in \mathscr{O}$, one has both

$$(u(X_t^x))_{0 \le t \le \tau^x} \in \mathscr{H}_{\tau^x}^\infty(\mathscr{O}),$$
$$((\sigma^T u)(X_t^x))_{0 \le t \le \tau^x} \in \mathscr{H}_{\tau^x}^2(\mathscr{O}).$$

PROOF OF THEOREM 3.18 *Part* 2. The idea is to argue by approximation. Finding a good approximation is not easy. We first show our result when f is bounded, and then we use a result of $[x]$.

The bounded case. Suppose that

$$f(x, u, p) = -\alpha_0 u + f_0(x, u, p)$$

on $\mathscr{O} \times \mathbb{R} \times \mathbb{R}^n$ where $\alpha_0 > 0$ and f_0 is a *bounded continuous* function and suppose that there exists a solution u in $H^1(\mathscr{O}) \cap L^\infty(\mathscr{O})$ of (36). Define for all $x \in \mathscr{O}$,

$$F_x(t, Y, Z) = f\big(X_t^x, Y, (\sigma^T(X_t^x))^{-1} Z\big).$$

LEMMA 3.20. *Under the above assumptions on f, the process*

$$\big(u(X_t^x), (\sigma^T Du)(X_t^x)\big)_{t \le \tau^x}$$

is a solution of the BSDE with parameters $(F_x, \tau^x, g(X_{\tau^x}^x))$ for almost all $x \in \mathscr{O}$. Moreover,

$$\|u(X_\bullet^x)\|_\infty \le \max\left(\frac{\|f_0\|_\infty}{\alpha_0}, \|g\|_\infty\right).$$

PROOF. We define

$$\tilde{f}_0(x) = f_0(x, u(x), Du(x)) \quad \text{for } x \in \mathscr{O}.$$

The function $\tilde{f}_0 \in L^\infty(\mathscr{O})$, let $\tilde{f}_0^\varepsilon$ be a sequence of $C^{0,\eta}(\mathscr{O})$ functions uniformly bounded such that $\tilde{f}_0^\varepsilon \to \tilde{f}_0$ as $\varepsilon \to 0$ in $L^2(\mathscr{O})$. In view of Theorem (3.15), the equation

$$Lu^\varepsilon + \alpha_0 u^\varepsilon = \tilde{f}^\varepsilon(x) \quad \text{in } \mathscr{O},$$
$$u^\varepsilon = g \quad \text{on } \delta\mathscr{O},$$

has a unique solution $u^\varepsilon \in C^2(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$,

$$\|u^\varepsilon\|_\infty \le \max\left(\frac{\|\tilde{f}_0^\varepsilon\|_\infty}{\alpha_0}, \|g\|_\infty\right).$$

In particular, u^ε is uniformly bounded. Furthermore,

$$\|u^\varepsilon - u\|_{H^1(\mathscr{O})} \le C\|\tilde{f}_0^\varepsilon - \tilde{f}_0\|_{L^2(\mathscr{O})}.$$

Hence according to Lemma 3.17, the process $(Y_t^{x,\varepsilon}, Z_t^{x,\varepsilon})_{0 \le t \le \tau^x}$ defined by

$$Y_t^{x,\varepsilon} = u^\varepsilon(X_t^x),$$
$$Z_t^{x,\varepsilon} = (\sigma^T Du^\varepsilon)(X_t^x)$$

is the solution of

$$Y_t^{x,\varepsilon} = g(X_{\tau^x}^x) + \int_t^{\tau^x} \tilde{f}^\varepsilon(X_s^x)\, dt - Z_s^{x,\varepsilon}\, dW_s. \tag{42}$$

Applying Lemma 3.19 and taking a subsequence if necessary, one may assume that, for almost all $x \in \mathscr{O}$:

(i) $u^\varepsilon(X_t^x)$ is uniformly bounded and $u^\varepsilon(X_t^x) \to u(X_t^x)$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R})$ when $\varepsilon \to 0$;
(ii) $\sigma^T Du^\varepsilon(X_t^x) \to \sigma^T Du(X_t^x)$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R}^n)$ when $\varepsilon \to 0$;
(iii) $\tilde{f}^\varepsilon(X_t^x) \to \tilde{f}(X_t^x)$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R})$ when $\varepsilon \to 0$.

Hence, passing to the limit as $\varepsilon \to 0$ in (42), we obtain

$$u(X_t^x) = g(X_{\tau^x}) + \int_t^{\tau^x} \underbrace{\alpha_0 Y_s^x + \tilde{f}_0(X_s^x)}_{F_x(s,\, u(X_s^x),\, (\sigma^T Du)(X_s^x))} \, ds - \int_t^{\tau^x} (\sigma^T Du)(X_s^x)\, dW_s.$$

We have proved that the process $(u(X_t^x), \sigma^T Du(X_t^x))_{0 \le t \le \tau^x}$ is the solution of the BSDE with $(F_x, \tau^x, g(X_{\tau^x}))$ as parameters.

The general case. We now suppose that f satisfies assumption (H6).

LEMMA 3.21 (Approximation [5]). *There exists an approximation of f by bounded functions f^ε, such that the associated solutions u^ε satisfy*

$$\|u^\varepsilon\|_\infty \le \max\left(\frac{C_0(0)}{\alpha_0}, \|g\|_\infty\right), \tag{43}$$

$$\lim_{\varepsilon \to 0}(u^\varepsilon - u) = 0 \quad \text{in } H_0^1(\mathscr{O}) \text{ strong} \tag{44}$$

and

$$\lim_{\varepsilon \to 0} f^\varepsilon(x, u^\varepsilon(x), Du^\varepsilon(x)) \to f(x, u(x), Du(x)) \quad \text{in } L^1(\mathscr{O}) \text{ strong}. \tag{45}$$

The proof of (43) is nothing but the application of the maximum principle property; the proof of (44) is difficult; the proof of (45) relies on the Vitalli theorem. See [5] for details. The approximation of f given in this paper is

$$f^\varepsilon(x, u, p) = \frac{f(x, u, p)}{1 + \varepsilon |f(x, u, p)|}.$$

Consider f^ε and u^ε given by Lemma 3.21. According to Lemma 3.20, the process $(Y_t^{x,\varepsilon}, Z_t^{x,\varepsilon})_{0 \le t \le \tau^x}$ defined for almost all x by

$$Y_t^{x,\varepsilon} = u^\varepsilon(X_t^x),$$
$$Z_t^{x,\varepsilon} = (\sigma^T Du^\varepsilon)(X_t^x),$$

is the solution of the BSDE with parameters $(F_x^\varepsilon, \tau^x, g(X_{\tau^x}^x))$.

Applying Lemma 3.19 and taking a subsequence if necessary, we have, for almost all $x \in \mathscr{O}$:

(i) $u^\varepsilon(X_t^x)$ is uniformly bounded and $\lim_{\varepsilon \to 0} u^\varepsilon(X_t^x) = u(X_t^x)$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R})$;
(ii) $\lim_{\varepsilon \to 0} \sigma^T Du^\varepsilon(X_t^x) = \sigma^T Du(X_t^x)$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R}^n)$;

(iii) $\lim_{\varepsilon\to 0} f^\varepsilon(X_t^x, u^\varepsilon(X_t^x), (\sigma^T Du^\varepsilon)(X_t^x)) = f(X_t^x, u(X_t^x), (\sigma^T Du)(X_t^x))$ in $\mathscr{H}^2_{\tau^x}(\mathbb{R})$.

Therefore, passing to the limit when $\varepsilon \to 0$ in

$$Y_t^x = g(X_{\tau^x}^x) + \int_t^{\tau^x} f^\varepsilon(X_s^x, u^\varepsilon(X_s^x), (\sigma^T Du^\varepsilon)(X_s^x))\, ds,$$

we obtain the expected result.

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