The Annals of Probability
2000, Vol. 28, No. 2, 537–557

ON THE SMALL TIME ASYMPTOTICS OF DIFFUSION PROCESSES ON HILBERT SPACES

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In this paper, we establish a small time large deviation principle and obtain the following small time asymptotics:

$$\lim_{t\to 0} 2t \log P(X_0 \in B,\ X_t \in C) = -d^2(B, C),$$

for diffusion processes on Hilbert spaces, where $d(B, C)$ is the intrinsic metric between two subsets B and C associated with the diffusions. The case of perturbed Ornstein–Uhlenbeck processes is treated separately at the end of the paper.

1. Introduction. The aim of this paper is to study the small time asymptotics of diffusion processes and heat semigroups on Hilbert spaces, which include solutions of some stochastic evolution equations. Let us start by recalling the basic results in finite dimensions. Let $L = \frac{1}{2}\Delta$ be one-half of the Laplacian operator on R^d. Then we know that the heat kernel is the transition density of the Brownian motion given by

$$P_t(x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{d^2(x, y)}{2t}\right),$$

where $d(x, y)$ stands for the usual distance on R^d. It is clear that the following small time asymptotics holds:

$$\lim_{t\to 0} 2t \log P_t(x, y) = -d^2(x, y). \tag{1.1}$$

Much work has been done to extend the above asymptotics to general situations where the Laplacian is replaced by general elliptic operators, R^d is replaced by some finite-dimensional Riemannian manifolds and $d(x, y)$ is the corresponding Riemannian distance. The results are quite satisfactory; see [8], [24] and references therein.

Formula (1.1) is sometimes called the Varadhan identity. We are here concerned with the above asymptotics in infinite-dimensional cases where L will be the generator of a symmetric diffusion process X_t, $t \geq 0$ on some Hilbert space E. Because of the lack of the transition density, the natural replacement for $P_t(x, y)$ in (1.1) is $P(X_0 \in B,\ X_t \in C)$, where C, B are two Borel subsets. The distance $d(x, y)$ between two points x, y is replaced by the distance of

Received December 1998; revised November 1999.

AMS 1991 *subject classifications*. Primary 60H15, 60F10; secondary 31C25.

Key words and phrases. Dirichlet form, intrinsic metric, large deviation, stochastic evolution equation, Girsanov transform.

the two sets C and B. Specifically, we are interested in getting the following small time asymptotics:

$$\lim_{t\to 0} 2t \log P(X_0 \in B,\ X_t \in C) = -d^2(B, C). \tag{1.2}$$

where d is the appropriate Riemannian distance associated with the diffusion. To this end, we deal with the upper bound and the lower bound separately. The upper bound is proved for any two Borel subsets B, C with positive measures and quite general diffusions with continuous diffusion operators. For the lower bound, we assume that the diffusion is a solution of a stochastic differential equation or a stochastic evolution equation on the Hilbert space. We first establish a small time large deviation principle for solutions of stochastic evolution equations of the type

$$u_t = x - \int_0^t Au_s\,ds + \int_0^t b(u_s)\,ds + \int_0^t \sigma(u_s)\,dW_s. \tag{1.3}$$

Then the lower bound follows from the large deviation principle.

In previous works [10] and [11], small time asymptotics (1.2) was obtained for the standard Ornstein–Uhlenbeck process on classical Wiener space and general Ornstein–Uhlenbeck processes with unbounded linear drifts that include solutions of some simple stochastic partial differential equations. In both cases, the underlying processes are Gaussian. Estimates of the lower bound rely essentially on the special properties of the Gaussian measures, which cannot be found in the present general situation. Instead, we adopted a SDE approach. However, our estimates of the upper bound are similar to those in [10] and [11]. We also notice the recent preprint [1], where a similar problem is addressed. However, our approaches are different and our results cannot cover each other.

Now we discuss the contents of the paper in detail. Section 2 gives the framework. We introduce the Dirichlet forms and the associated diffusions we are going to study. In Section 3, we prove a small time large deviation principle for solutions of stochastic evolution equations. The results are of independent interest and will also be used later. Our idea is to show that the solutions of the stochastic evolution equations have the same asymptotics as the solutions of the corresponding equations without drifts. This is done by several lemmas. Since both operators and drifts are allowed to be unbounded, the proofs are quite involved. Both Itô calculus and the factorization method are used. In Section 4, we prove the asymptotics (1.2) for symmetric diffusions. The Lyons–Zheng decomposition plays an important role in the upper bound estimates. The lower bound estimates follow from the large deviation principle. In Section 5, we study the small time asymptotics of a class of perturbed Ornstein–Uhlenbeck processes which cannot be covered by previous sections. The main tool we use is the Girsanov theorem we proved in [4]. Again because of the unboundness of the drift, careful analysis is carried out. Perturbed Ornstein–Uhlenbeck processes have been extensively studied in the past years in connection with quantum field theory. See [2], [3], [4] and [19]. For example, it is shown in [2] that the Log–Sobolev inequality holds for the Dirichlet forms associated with perturbed Ornstein–Uhlenbeck processes.

2. Framework. Let H be a separable Hilbert space and E be another separable Hilbert space such that H is imbedded in E densely and continuously and the imbedding is Hilbert–Schmidt. Let μ be a mean zero Gaussian measure on $(E, \mathscr{B}(E))$ with the reproducing kernel space H, where $\mathscr{B}(E)$ denotes the Borel σ-field. The (H, E, μ) is an abstract Wiener space in the sense of Gross. More generally, to cover solutions of stochastic evolution equations, let A be a self-adjoint operator on H satisfying $A \geq cI_H$, where $c > 0$ and I_H stands for the identity operator on H. The associated semigroup is denoted by $T_t = e^{-tA}$. Throughout this paper, we impose the following.

ASSUMPTION 2.1. The semigroup $T_t = e^{-tA}$, $t \geq 0$, generated by $-A$, also extends to a strongly continuous semigroup of bounded linear operators on E.

REMARK 2.2. This assumption is not as strong as it looks. According to [5], for a given operator A and a Hilbert space H one can always properly choose E so that the above assumption holds.

Define $H_0 = D(\sqrt{A})$ with inner product $\langle h_1, h_2\rangle_{H_0} = \langle \sqrt{A}h_1, \sqrt{A}h_2\rangle_H$. Then

$$E' \subset H_0 \subset H \subset E \quad \text{densely and continuously.}$$

The inclusion $H_0 \subset E$ is also Hilbert–Schmidt.

Introduce

$$\mathscr{F}C_b^\infty = \Big\{u(x) = f(l_1(x), \dots, l_m(x));\ f \in C_b^\infty(R^m), l_1, \dots, l_m \in E',\ m \geq 1\Big\}.$$

Given $u \in \mathscr{F}C_b^\infty$, denote by $\nabla u(x) \in H_0$ such that for all $k \in H_0$,

$$\langle \nabla u(x), k\rangle_{H_0} = \frac{\partial u(x)}{\partial k} = \lim_{\varepsilon\to 0} \frac{u(x+\varepsilon k) - u(x)}{\varepsilon}. \tag{2.1}$$

Define

$$\mathscr{E}^0(u, v) = \tfrac{1}{2} \int_E \langle \nabla u, \nabla v\rangle_{H_0}\, d\mu, \qquad u, v \in \mathscr{F}C_b^\infty. \tag{2.2}$$

It was shown in [3] (see also [19]) that $(\mathscr{E}^0, \mathscr{F}C_b^\infty)$ is closable on $L^2(E, \mu)$. The closure, denoted by $(\mathscr{E}, D(\mathscr{E}))$, is a Dirichlet form. The diffusion process $M = ((X_t)_{t\geq 0}, \mathscr{F}_t, P_x,\ x \in E)$ associated with $(\mathscr{E}, D(\mathscr{E}))$ is the Ornstein–Uhlenbeck process X_t, $t \geq 0$ on E which solves the following stochastic evolution equation in the weak sense:

$$dX_t = dw_t - \tfrac{1}{2} A X_t\, dt, \tag{2.3}$$

where w_t is an E-valued Brownian motion with covariance space H_0. A simple example is the solution of the following SPDE:

$$\frac{\partial u(t, x)}{\partial t} = \dot{w}_t(x) + \frac{1}{2}\frac{\partial^2 u(t, x)}{\partial x^2} - \frac{1}{2} u(t, x), \tag{2.4}$$

where $H = L^2(R, dx)$, $A = (\partial^2/\partial x^2) - I$, $w_t(x)$ is a cylinder Brownian motion on $H_0 = D(\sqrt{A})$ (the Sobolev space of order 1), E is chosen to be a Hilbert space such that the imbedding $H_0 \subset E$ is Hilbert–Schmidt.

To introduce general diffusions, we let $\mathscr{L}_s(H_0)$ denote the set of all symmetric bounded linear operators on H_0. Let $A(z)$ be a strongly continuous map from E into $\mathscr{L}_s(H_0)$ (i.e., $A(z)h$ is continuous in z for every $h \in H_0$) such that

$$\hat{c}^{-1} I_{H_0} \leq A(z) \leq \hat{c} I_{H_0}, \tag{2.5}$$

where $\hat{c}$ is a positive constant, and I_{H_0} denotes the identity operator on H_0. Introduce the quadratic form

$$\mathscr{Q}^0(u, v) = \tfrac{1}{2} \int_E \Big\langle A(z)\nabla u(z), \nabla v(z)\Big\rangle_{H_0} d\mu, \qquad u, v \in \mathscr{F}\mathscr{C}_b^\infty.$$

This form is also closable on $L^2(E, \mu)$ because of (2.5). Its closure, denoted by $(\mathscr{Q}, D(\mathscr{Q}))$, is a Dirichlet form. Let $M_Q = (X_t,\ t \geq 0,\ Q_x,\ x \in E)$ denote the diffusion associated with $(\mathscr{Q}, D(\mathscr{Q}))$ (see [17] for the existence). The process M_Q is a quite general diffusion with the diffusion operator $\sqrt{A(z)}$ and possibly very singular drift.

REMARK 2.3. The fact that the measure μ is Gaussian is used only in Section 5. In Section 3, the measure μ is not involved. In Section 4, the Gaussian measure μ can be replaced by any Borel measure ν that satisfies:

(a) The integration by parts formula,

$$\int_E \frac{\partial f}{\partial k}\, d\nu = -\int_E \beta_k f\, d\nu \quad \text{for any } k \in E',\ f \in \mathscr{F}C_b^\infty,$$

where $\beta_k \in L^2(E, \nu)$;

(b) The closure $(\mathscr{E}, D(\mathscr{E}))$ is the unique Dirichlet form extending $(\mathscr{E}^o, \mathscr{F}C_b^\infty)$. See [19], [20] for examples.

3. A large deviation principle. In this section, we prove a small time large deviation principle for solutions of a class of stochastic evolution equations. This corresponds not only to small noise but also small drift perturbation where an unbounded operator A and unbounded drifts are involved. The results are of their own interest and will also be used later. Let H, E, A, H_0 be as in Section 2. Let $L_{(2)}(H_0, H)$ denote the set of all Hilbert–Schmidt operators from H_0 into H with the Hilbert–Schmidt norm $\|\cdot\|_{(2)}$. Let b, σ be two measurable mappings. Assume throughout this section:

(I) b: $E \to E$, σ: $E \to L_{(2)}(H_0, H)$ satisfy

$$|b(x) - b(y)|_E \leq c_2|x - y|_E, \qquad \|\sigma(x) - \sigma(y)\|_{(2)} \leq c_1|x - y|_E.$$

$$|b(x)|_E \leq c_2 + c_3|x|_E, \qquad \sup_x \|\sigma(x)\|_{(2)} \leq M, \tag{II}$$

or

(I)′ The imbedding $H \to E$ is a trace class and b: $E \to E$, σ: $E \to \mathscr{L}(H_0)$ satisfy

$$|b(x) - b(y)|_E \le c_2|x - y|_E, \qquad \|\sigma(x) - \sigma(y)\|_{\mathscr{L}(H_0)} \le c_1|x - y|_E.$$

$$\text{(II)}' \qquad |b(x)|_E \le c_2 + c_3|x|_E, \qquad \sup_x \|\sigma(x)\|_{\mathscr{L}(H_0)} \le M,$$

where c_1, c_2, c_3 and M are constants.

REMARK. These assumptions are more than sufficient for the existence and uniqueness of the corresponding stochastic evolution equation, but they are necessary for the proof of Lemma 3.4.

Let $W_t,\ t \ge 0$ be an E-valued Brownian motion with the reproducing Hilbert space H_0 defined on some probability space $(\Omega, \mathscr{F}_t, P)$. Given $x \in E$. Let u_t be the unique solution of the stochastic evolution equation

$$u_t = x - \int_0^t Au_s\,ds + \int_0^t b(u_s)\,ds + \int_0^t \sigma(u_s)\,dW_s \tag{3.1}$$

In general, $u_t,\ t > 0$, will not belong to the domain of A and (3.1) is interpreted in the following sense:

$$u_t = T_t x + \int_0^t T_{(t-s)}(b(u_s))\,ds + \int_0^t T_{(t-s)}\sigma(u_s)\,dW_s, \tag{3.2}$$

where $T_t = e^{-tA}$ is the semigroup generated by $-A$ as in Section 2.

The existence of the solution of the above equation under the assumptions (I), (II) or (I)′, (II)′ is well known (see [17] and [18]). Let $\varepsilon > 0$. It is easy to see that the process $u_{\varepsilon t}$ coincides in law with the solution of the following equation:

$$u_t^\varepsilon = T_{\varepsilon t}x + \varepsilon \int_0^t T_{\varepsilon(t-s)}\big(b(u_s^\varepsilon)\big)\,ds + \varepsilon^{1/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\,dW_s. \tag{3.3}$$

Let μ_ε^x be the law of $u_\cdot^\varepsilon$ on $C([0,1] \to E)$. Define a functional $I(f)$ on $C([0,1] \to E)$ by

$$I(f) = \inf_{h \in \Gamma_f}\left\{\frac{1}{2}\int_0^1 |\dot h(t)|_{H_0}^2\,dt\right\},$$

where

$$\Gamma_f = \Big\{h \in C([0,1] \to H_0);\ h(\cdot) \text{ is absolutely continuous and such that}$$
$$f(t) = x + \int_0^t \sigma\big(f(s)\big)\dot h(s)\,ds,\ 0 \le t \le 1\Big\}.$$

THEOREM 3.1. *μ_ε^x satisfies a large deviation principle with the rate function $I(\cdot)$, that is*;

(i) *For any closed set F*,

$$\limsup_{\varepsilon\to 0,\, x_n\to x} \varepsilon \log \mu_\varepsilon^{x_n}(F) \le -\inf_{f\in F} \big(I(f)\big).$$

(ii) *For any open set G*,

$$\liminf_{\varepsilon\to 0,\, x_n\to x} \varepsilon \log \mu_\varepsilon^{x_n}(G) \ge -\inf_{f\in G} I(f).$$

PROOF. We prove the theorem for $x_n = x$, $n \ge 1$. Slight modifications of the proof led to the general case. Let ν_ε be the law of the solution $v_\cdot^\varepsilon$ of the following stochastic equation:

$$v_t^\varepsilon = x + \varepsilon^{1/2} \int_0^t \sigma(v_s^\varepsilon)\, dW_s, \quad t \ge 0. \tag{3.4}$$

Then it is known (see, e.g., [7]) that ν_ε satisfies a large deviation principle on $C([0,1] \to E)$ with rate function $I(\cdot)$. Thus by Theorem 4.2.13 in [9] it suffices to show that the two families $\{\mu_\varepsilon\}$, $\{\nu_\varepsilon\}$ of probability measures are so-called exponentially equivalent. That is, the following:

PROPOSITION 3.2. *For any $\delta > 0$*,

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} |u_t^\varepsilon - v_t^\varepsilon|_E > \delta\Big) = -\infty. \tag{3.5}$$

PROOF. Observe that

$$\begin{aligned} u_t^\varepsilon - v_t^\varepsilon &= (T_{\varepsilon t}x - x) + \varepsilon \int_0^t T_{\varepsilon(t-s)}\big(b(u_s^\varepsilon)\big)\, ds \\ &\quad + \varepsilon^{1/2} \int_0^t (T_{\varepsilon(t-s)} - I)\sigma(u_s^\varepsilon)\, dW_s + \varepsilon^{1/2} \int_0^t \big(\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon)\big)\, dW_s \end{aligned} \tag{3.6}$$

Denote the four terms on the right-hand side respectively by I_t^ε, II_t^ε, III_t^ε and IV_t^ε. Since $\lim_{\varepsilon\to 0} I_\varepsilon = 0$, it suffices to establish that for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} |Y|_t > \delta\Big) = -\infty \tag{3.7}$$

for $Y_t = II_t^\varepsilon$, III_t^ε and IV_t^ε. This will be done in the following lemmas.

Let J denote the Hilbert–Schmidt injection mapping from H to E. If $T \in \mathscr{L}(H)$, since $T = JT$ as an operator from H to E, we have $T \in L_{(2)}(H, E)$ and $\|T\|_{(2)} \le \|J\|_{(2)} \|T\|_{\mathscr{L}(H)}$. In the following, c will denote a generic constant which may vary from line to line, but is independent of ε, t, s.

LEMMA 3.3. *Let $\delta > 0$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} |II_t^\varepsilon|_E > \delta\Big) = -\infty. \tag{3.8}$$

PROOF. By assumption (II) and the fact that T_t is also a strongly continuous semigroup on E, we have

$$\begin{aligned}\sup_{0\le s\le t}|\varepsilon u_s^\varepsilon|_E &\le c|\varepsilon x|_E + c\varepsilon^2 + \int_0^t c\varepsilon^2 |u_s^\varepsilon|_E\,ds \\ &\quad + \sup_{0\le t\le 1}\left|\varepsilon^{3/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\,dW_s\right|_E\end{aligned} \tag{3.9}$$

By the Gronwall inequality,

$$\sup_{0\le s\le 1}|\varepsilon u_s^\varepsilon|_E \le \left(c|\varepsilon x|_E + c\varepsilon^2 + \sup_{0\le t\le 1}\left|\varepsilon^{3/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\,dW_s\right|_E\right)e^{\varepsilon c}.$$

Thus,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le s\le 1}|\varepsilon u_s^\varepsilon|_E > \frac{1}{2}\delta\right) \\ &\quad\le \lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}\left|\varepsilon^{3/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)dW_s\right|_E > \frac{1}{4}\delta\right).\end{aligned} \tag{3.10}$$

Since

$$\sup_{0\le t\le 1}|II_t^\varepsilon|_E \le c\varepsilon\left(c_2 + \int_0^1 |u_t^\varepsilon|_E\,dt\right),$$

it follows from (3.10) that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}|II_t^\varepsilon|_E > \delta\right) \\ &\quad\le \lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}\left|\varepsilon^{3/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\,dW_s\right|_E > \tfrac{1}{4}\delta\right)\end{aligned}$$

However, according to Lemma 5.1 in [6],

$$\lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}\left|\varepsilon^{3/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\,dW_s\right|_E > \tfrac{1}{4}\delta\right) = -\infty.$$

This ends the proof. □

LEMMA 3.4. *Let $\delta > 0$. Then*

$$\lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}|III_t^\varepsilon|_E > \delta\right) = -\infty. \tag{3.11}$$

PROOF. We shall use a factorization method which was adopted in [23]. The method is based on the following identity:

$$\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}\,ds = \frac{\pi}{\sin(\pi\alpha)}, \qquad 0\le r\le t,\ 0<\alpha<1/2. \tag{3.12}$$

We write

$$III_t^\varepsilon = \varepsilon^{1/2}\frac{\sin\pi\alpha}{\pi}\biggl[\int_0^t T_{\varepsilon(t-r)}\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}\,ds\,\sigma(u_r^\varepsilon)\,dW_r - \int_0^t\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}\,ds\,\sigma(u_r^\varepsilon)\,dW_r\biggr]$$

$$= \varepsilon^{1/2}\frac{\sin\pi\alpha}{\pi}\biggl[\int_0^t T_{\varepsilon(t-s)}(t-s)^{\alpha-1}Y_1^\varepsilon(s)\,ds - \int_0^t (t-s)^{\alpha-1}Y_2^\varepsilon(s)\,ds\biggr],$$

where

$$Y_1^\varepsilon(s) = \int_0^s T_{\varepsilon(s-r)}(s-r)^{-\alpha}\sigma(u_r^\varepsilon)\,dW_r,$$

$$Y_2^\varepsilon(s) = \int_0^s (s-r)^{-\alpha}\sigma(u_r^\varepsilon)\,dW_r.$$

So

$$III_t^\varepsilon = \varepsilon^{1/2}\frac{\sin\pi\alpha}{\pi}\biggl[\int_0^t T_{\varepsilon(t-s)}(t-s)^{\alpha-1}\bigl(Y_1^\varepsilon(s) - Y_2^\varepsilon(s)\bigr)\,ds + \int_0^t \bigl((T_{\varepsilon(t-s)} - I)(t-s)^{\alpha-1}\bigr)Y_2^\varepsilon(s)\,ds\biggr].$$

Notice that

$$Y_1^\varepsilon(s) - Y_2^\varepsilon(s) = \int_0^s (T_{\varepsilon(s-r)} - I)(s-r)^{-\alpha}\sigma(u_r^\varepsilon)\,dW_r,$$

$$(T_{\varepsilon(t-s)} - I)Y_2^\varepsilon(s) = \int_0^s (T_{\varepsilon(t-s)} - I)(s-r)^{-\alpha}\sigma(u_r^\varepsilon)\,dW_r.$$

Set $\delta_\varepsilon = \sup_{0\le s\le\varepsilon}\|T_s - I\|_{L(H,E)}$. Then $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ since the imbedding $H\to E$ is Hilbert–Schmidt. Applying Theorem 4.1 in Chow and Menaldi [6], we have that for all $s\le 1$,

$$P\bigl(|Y_1^\varepsilon(s) - Y_2^\varepsilon(s)|_E \ge d\bigr) \le 3\exp\biggl(-\frac{d^2}{2c_{\alpha,M}\delta_\varepsilon^2}\biggr),$$

$$P\biggl(\sup_{s\le t\le 1}\Bigl|\Bigl(T_{\varepsilon(t-s)} - I\Bigr)Y_2^\varepsilon(s)\Bigr|_E \ge d\biggr) = P\biggl(|Y_2^\varepsilon(s)|_E \ge \frac{d}{\delta_\varepsilon}\biggr) \le 3\exp\biggl(-\frac{d^2}{2c_{\alpha,M}\delta_\varepsilon^2}\biggr),$$

where $c_{\alpha,M}$ is a constant only depending on α and the constant M appearing in (II). These two inequalities imply that there exists a constant M_1 independent of ε such that

$$\sup_{0\le s\le 1} E\biggl[\exp\biggl(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2}|Y_1^\varepsilon(s) - Y_2^\varepsilon(s)|_E^2\biggr)\biggr] \le M_1, \tag{3.13}$$

$$\sup_{0\le s\le 1} E\biggl[\exp\biggl(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2}\sup_{s\le t\le 1}\bigl|(T_{\varepsilon(t-s)} - I)Y_2^\varepsilon(s)\bigr|_E^2\biggr)\biggr] \le M_1. \tag{3.14}$$

Let $Z_t^\varepsilon = \varepsilon^{-1/2}(\pi/\sin(\pi\alpha))III_t^\varepsilon$. By Hölder inequality, for $m \geq [1/2\alpha]+1$,

$$
\begin{aligned}
&\sup_{0\le t\le 1} |Z_t^\varepsilon|_E^{2m} \\
&\quad \le \sup_{0\le t\le 1}\left(\int_0^t (t-s)^{(\alpha-1)(2m/2m-1)}\,ds\right)^{2m-1} 2^{2m-1} \\
&\qquad \times \sup_{0\le t\le 1}\left(\int_0^t \left|Y_1^\varepsilon(s)-Y_2^\varepsilon(s)\right|_E^{2m} + \left|(T_{\varepsilon(t-s)}-I)Y_2^\varepsilon(s)\right|_E^{2m}\right)ds\Big) \\
&\quad \le \rho_\alpha^{2m}\int_0^1 \left(|Y_1^\varepsilon(s)-Y_2^\varepsilon(s)|_E^{2m} + \sup_{s\le t\le 1}|(T_{\varepsilon(t-s)}-I)Y_2^\varepsilon(s)|_E^{2m}\right)ds,
\end{aligned} \tag{3.15}
$$

where

$$
\rho_\alpha = \frac{2}{1+\alpha-2([1/\alpha]+1)/2[1/\alpha]+1}.
$$

Thus we can find a constant c_α^1 such that

$$
\begin{aligned}
&E\left[\exp\left(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\sup_{0\le t\le 1}|Z_t^\varepsilon|_E^2\right)\right] \\
&\quad = \sum_{m=0}^\infty \frac{1}{m!}E\left(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\sup_{0\le t\le 1}|Z_t^\varepsilon|_E^2\right)^m \\
&\quad \le c_\alpha^1\sum_{m=0}^\infty \frac{1}{m!}E\left[\int_0^1\left\{\left(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2}|Y_1^\varepsilon(s)-Y_2^\varepsilon(s)|_E^2\right)^m\right.\right. \\
&\qquad\qquad \left.\left. + \left(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2}\sup_{s\le t\le 1}\left|(T_{\varepsilon(t-s)}-I)Y_2^\varepsilon(s)\right|_E^2\right)^m\right\}\text{as}\right] \\
&\quad \le 2c_\alpha^1 M_1,
\end{aligned} \tag{3.16}
$$

where the last inequality follows from (3.13) and (3.14). For $\delta > 0$, applying Chebyshev's inequality and (3.16),

$$
\begin{aligned}
&P\left(\sup_{0\le t\le 1}|III_t^\varepsilon|_E > \delta\right) \\
&\quad = P\left(\frac{1}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\sup_{0\le t\le 1}|Z_t^\varepsilon|_E^2 > \frac{\delta^2}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\left(\frac{\pi}{\sin(\pi\alpha)}\right)^2\frac{1}{\varepsilon}\right) \\
&\quad \le 2c_\alpha^1 M_1\exp\left(-\frac{\delta^2}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\left(\frac{\pi}{\sin(\pi\alpha)}\right)^2\frac{1}{\varepsilon}\right).
\end{aligned} \tag{3.17}
$$

This gives

$$
\lim_{\varepsilon\to 0}\varepsilon\log P\left(\sup_{0\le t\le 1}|III_t^\varepsilon|_E > \delta\right) \le -\lim_{\varepsilon\to 0}\left(\frac{\delta^2}{3c_{\alpha,M}\delta_\varepsilon^2\rho_\alpha^2}\left(\frac{\pi}{\sin(\pi\alpha)}\right)^2\right) = -\infty,
$$

which proves the lemma. □

LEMMA 3.5. *Let $\delta > 0$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} |IV_t^\varepsilon|_E > \delta\Big) = -\infty.$$

PROOF. Set $X_t^\varepsilon = I_t^\varepsilon + II_t^\varepsilon + III_t^\varepsilon$. For notional convenience, in the following we use y_t^ε to denote IV_t^ε. For any $\rho > 0$, introduce stopping times

$$\tau_1^\varepsilon = \inf\{t > 0;\ |X_t^\varepsilon| > \rho\}, \qquad \tau_2^\varepsilon = \inf\{t > 0;\ |y_t^\varepsilon| > \delta\},$$

and define $\phi_\lambda(h) = (\rho^2 + |h|_E^2)^\lambda$, $h \in E$, where $\lambda > 2$ will be chosen later. Put $\tau = \tau_1^\varepsilon \wedge \tau_2^\varepsilon$. By Itô's formula,

$$\begin{aligned} \phi_\lambda(y_{t\wedge\tau}^\varepsilon) &= \rho^{2\lambda} + \varepsilon^{1/2}\int_0^{t\wedge\tau} \langle \phi_\lambda'(y_s^\varepsilon), (\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))\, dW_s\rangle \\ &\quad + \tfrac{1}{2}\varepsilon \textstyle\int_0^t \operatorname{tr}\big[\phi_\lambda''(y_{s\wedge\tau}^\varepsilon)(\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon))(\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon))^*\big]\, ds \\ &= \rho^{2\lambda} + M_t^\varepsilon + \int_0^t g_s\, ds, \end{aligned} \tag{3.18}$$

where M_t^ε denotes the martingale part and

$$\begin{aligned} g_s &= \tfrac{1}{2}\varepsilon\Big[2\lambda\big(\rho^2 + |y_{s\wedge\tau}^\varepsilon|_E^2\big)^{\lambda-1} \operatorname{tr}\big(\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon)\big)\big(\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon)\big)^* \\ &\qquad + 4\lambda(\lambda-1)\big(\rho^2 + |y_{s\wedge\tau}^\varepsilon|_E^2\big)^{\lambda-2} \operatorname{tr}((y_{s\wedge\tau}^\varepsilon \otimes y_{s\wedge\tau}^\varepsilon)\big(\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon)\big) \\ &\qquad\qquad \times (\sigma(u_{s\wedge\tau}^\varepsilon) - \sigma(v_{s\wedge\tau}^\varepsilon))^*\Big] \\ &\le \tfrac{1}{2}\varepsilon c(4\lambda^2 - 2\lambda)(\rho^2 + |y_{s\wedge\tau}^\varepsilon|_E^2)^{\lambda-1} |u_{s\wedge\tau}^\varepsilon - v_{s\wedge\tau}^\varepsilon|_E^2. \end{aligned}$$

By the choice of τ,

$$|u_{s\wedge\tau}^\varepsilon - v_{s\wedge\tau}^\varepsilon|_E^2 \le 2\big(|X_{s\wedge\tau}^\varepsilon|_E^2 + |y_{s\wedge\tau}^\varepsilon|_E^2\big) \le 2(\rho^2 + |y_{s\wedge\tau}^\varepsilon|_E^2). \tag{3.19}$$

Thus,

$$g_s \le \varepsilon c 4\lambda^2 \phi(y_{s\wedge\tau}^\varepsilon). \tag{3.20}$$

Taking expectation in (3.18), we get from (3.20) that

$$E[\phi(y_{t\wedge\tau}^\varepsilon)] \le \rho^{2\lambda} + \varepsilon c 4\lambda^2 \int_0^t E(\phi_\lambda(y_{s\wedge\tau}^\varepsilon))\, ds.$$

By the Gronwall inequality, we have that

$$E[\phi(y_{1\wedge\tau}^\varepsilon)] \le \rho^{2\lambda} \exp(4\varepsilon c\lambda^2).$$

Consequently,

$$\begin{aligned} &P\Big(\sup_{0\le s\le 1}|y_s^\varepsilon|_E>\delta,\ \sup_{0\le t\le 1}|X_t^\varepsilon|\le\rho\Big)(\rho^2+\delta^2)^\lambda \\ &\quad\le P(\tau_2^\varepsilon\le 1,\ \tau_1^\varepsilon>1)(\rho^2+\delta^2)^\lambda \\ &\quad\le E[\phi(y^\varepsilon_{1\wedge\tau})]\le\rho^{2\lambda}\exp(4\varepsilon c\lambda^2). \end{aligned} \tag{3.21}$$

Let $\lambda=1/\varepsilon$. It follows that

$$P\Big(\sup_{0\le s\le 1}|y_s^\varepsilon|_E>\delta,\ \sup_{0\le t\le 1}|X_t^\varepsilon|\le\rho\Big)\le\left(\frac{\rho^2}{\rho^2+\delta^2}\right)^{1/\varepsilon}\exp\left(4c\frac{1}{\varepsilon}\right).$$

Hence,

$$\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le s\le 1}|y_s^\varepsilon|_E>\delta,\ \sup_{0\le t\le 1}|X_t^\varepsilon|\le\rho\Big)\le\log\left(\frac{\rho^2}{\rho^2+\delta^2}\right)+4c. \tag{3.22}$$

On the other hand, Lemmas 3.3, 3.4 imply that for any $\rho>0$,

$$\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le t\le 1}|X_t^\varepsilon|>\rho\Big)=-\infty. \tag{3.23}$$

Combination of (3.22) and (3.23) yields

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le s\le 1}|IV_s^\varepsilon|_E>\delta\Big) \\ &\quad\le\Big(\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le s\le 1}|y_s^\varepsilon|_E>\delta,\ \sup_{0\le t\le 1}|X_t^\varepsilon|\le\rho\Big)\Big) \\ &\qquad\vee\Big(\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le t\le 1}|X_t^\varepsilon|>\rho\Big)\Big)\le\log\left(\frac{\rho^2}{\rho^2+\delta^2}\right)+4c. \end{aligned}$$

Sending ρ to 0, we get that

$$\limsup_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le s\le 1}|IV_s^\varepsilon|_E>\delta\Big)=-\infty.$$

Combination of Lemmas 3.3, 3.4, 3.5 gives Proposition 3.2, hence Theorem 3.1. □

As an illustration of the result, we look at the following simple example. For more sophisticated examples, please see [7].

EXAMPLE 3.6. Let D be a smooth bounded domain in R^d. Set $H=L^2(D)$. Let $A=-\Delta$ be the Laplacian operator on H with Dirichlet boundary. Then $H_0=H_0^1(D)$, defined in Section 2, is the Sobolev space of order 1. Take E to be a Hilbert space such that the imbedding $H_0^1(D)\to E$ is a trace class.

Theorem 3.1 applies to the solution of the following stochastic partial differential equation:

$$\frac{\partial u_t(x)}{\partial t} = \Delta u_t(x) + \sigma(u_t(x))\dot{W}_t(x),$$

where $W_t(x)$ is a cylindrical Brownian motion on $H_0^1(D)$.

4. The small time asymptotics. Let $M_Q = (X_t,\ t \geq 0,\ Q_x,\ x \in E)$ be the diffusion as in Section 2. Set, for $u, v \in \mathscr{F}\mathscr{C}_b^\infty$,

$$\Gamma(u, v)(z) = \langle A(z)\nabla u(z), \nabla v(z)\rangle_{H_0}. \tag{4.1}$$

The intrinsic metric $d(x, y)$ on $E \times E$ determined by the Dirichlet form $(\mathscr{Q}, D(\mathscr{Q}))$ is defined as

$$d(x, y) = \sup_{u \in \mathscr{F}C_b^\infty,\ \Gamma(u, u) \leq 1} (u(x) - u(y)). \tag{4.2}$$

From the definition, it is easy to see that $d(x, y) < \infty$ if and only if $x - y \in H_0$. Let B be a Borel subset of E. We define $d(x, B) = \inf\{d(x, y);\ y \in B\}$. For two Borel subsets B, C with $\mu(B) > 0$ and $\mu(C) > 0$, let

$$d(B, C) = \sup\Big(\operatorname*{ess\,inf}_{z \in B} d(z, C),\ \operatorname*{ess\,inf}_{z \in C} d(z, B)\Big). \tag{4.3}$$

Set $u(x) = d(x, B)$. Since $d(\cdot, \cdot)$ is lower semicontinuous on $E \times E$, we see that $u(x)$ is measurable if B is a union of a sequence of compact sets.

PROPOSITION 4.1. *Assume $B = \bigcup_n K_n$, where K_n is compact. Then for any constant $c > 0$, $u_c(x) = u(x) \wedge c \in D(\mathscr{Q})$ and $\Gamma(u_c, u_c)(z) \leq 1$.*

PROOF. Let $h \in H_0$. Then we see that

$$\begin{aligned} &|u_c(x+h) - u_c(x)| \\ &\quad \leq d(x+h, x) \\ &\quad = \sup_{v \in \mathscr{F}C_b^\infty,\ \Gamma(v, v) \leq 1} (v(x+h) - v(x)) \\ &\quad = \sup_{v \in \mathscr{F}C_b^\infty,\ \Gamma(v, v) \leq 1} \left| \int_0^1 \langle \nabla v(x+sh), h\rangle_{H_0}\, ds \right| \\ &\quad \leq \sup_{v \in \mathscr{F}C_b^\infty,\ \Gamma(v, v) \leq 1} \left| \int_0^1 |\nabla v(x+sh)|_{H_0} |h|_{H_0}\, ds \right| \\ &\quad = \sup_{v \in \mathscr{F}C_b^\infty,\ \Gamma(v, v) \leq 1} \left| \int_0^1 \big|A^{-1/2}A^{1/2}(x+sh)\nabla v(x+sh)\big|_{H_0} |h|_{H_0}\, ds \right| \\ &\quad \leq c \sup_{v \in \mathscr{F}C_b^\infty,\ \Gamma(v, v) \leq 1} \int_0^1 (\Gamma(v, v)(x+sh))^{1/2} |h|_{H_0}\, ds \leq c|h|_{H_0}. \end{aligned} \tag{4.4}$$

By Theorem 2.3 in [19], this implies that $u_c \in D(\mathscr{Q})$. Next, we are going to prove $\Gamma(u_c, u_c)(x) \le 1$. Since $\Gamma(u_c, u_c)(x) = |A^{1/2}(x)\nabla u_c(x)|^2_{H_0}$, it is enough to establish

$$\left|\left\langle A^{1/2}(x)\nabla u_c(x),\ h\right\rangle_{H_0}\right| \le |h|_{H_0} \quad \text{for all } h \in H_0$$

Observe that

$$\begin{aligned}
&\left|u_c(x + sA^{1/2}(x)h) - u_c(x)\right| \le d(x + sA^{1/2}(x)h, x)\\
&\quad\le \sup_{v\in\mathscr{F}C_b^\infty,\ \Gamma(v,v)\le 1.} \left|(v(x+sA^{1/2}(x)h) - v(x))\right|\\
&\quad= \sup_{v\in\mathscr{F}C_b^\infty,\ \Gamma(v,v)\le 1.} \left|\int_0^s \left\langle \nabla v(x+tA^{1/2}(x)h), A^{1/2}(x)h\right\rangle_{H_0} dt\right|\\
&\quad\le \sup_{v\in\mathscr{F}C_b^\infty,\ \Gamma(v,v)\le 1.} \left|\int_0^s \left\langle \nabla v(x+tA^{1/2}(x)h),\right.\right.\\
&\qquad\qquad \left.\left. A^{1/2}(x+tA^{1/2}(x)h)h\right\rangle_{H_0} dt\right| \qquad (4.5)\\
&\qquad + \sup_{v\in\mathscr{F}C_b^\infty,\ \Gamma(v,v)\le 1.} \left|\int_0^s \left\langle \nabla v(x+tA^{1/2}(x)h), (A^{1/2}(x)\right.\right.\\
&\qquad\qquad \left.\left. - A^{1/2}(x+tA^{1/2}(x)h))h\right\rangle_{H_0} dt\right|\\
&\quad\le \sup_{v\in\mathscr{F}C_b^\infty,\ \Gamma(v,v)\le 1.} \int_0^s \Gamma(v,v)^{1/2}(x+tA^{1/2}(x)h)|h|_{H_0}\, dt\\
&\qquad + c\int_0^s \left|(A^{1/2}(x) - A^{1/2}(x+tA^{1/2}(x)h))h\right|_{H_0} dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\left\langle A^{1/2}(x)\nabla u_c(x),\ h\right\rangle_{H_0}\right|\\
&\quad= \left|\left\langle \nabla u_c(x),\ A^{1/2}(x)h\right\rangle_{H_0} = \left|\frac{\partial u_c(x)}{\partial A^{1/2}(x)h}\right|\right.\\
&\quad= \left|\lim_{s\to 0}\left(\frac{u_c(x+sA^{1/2}(x)h) - u_c(x)}{s}\right)\right|\\
&\quad\le \limsup_{s\to 0}\left|\frac{u_c(x+sA^{1/2}(x)h) - u_c(x)}{s}\right|\\
&\quad\le \limsup_{s\to 0}\left(\frac{1}{s}\int_0^s |h|_{H_0}\, dt\right)\\
&\qquad + c\limsup_{s\to 0}\frac{1}{s}\int_0^s \left|(A^{1/2}(x) - A^{1/2}(x+tA^{1/2}(x)h))h\right|_{H_0} dt\\
&\quad= |h|_{H_0},
\end{aligned}$$

which proves the proposition. □

Set

$$Q(\cdot) = \int_E Q_x(\cdot) d\mu.$$

THEOREM 4.2. *Let B, C be two Borel subsets with $\mu(B) > 0$, $\mu(C) > 0$. Then*

$$\limsup_{t\to 0} 2t \log Q(X_0 \in B,\ X_t \in C) \le -d^2(B, C).$$

PROOF. We proceed as in [10] and [11]. First, note that we can assume B and C are a union of compact sets. Otherwise, we replace B and C by subsets that also have the same measures as B and C. Furthermore, we can assume $d(B, C) > 0$. Let λ be any positive number such that $\lambda < d(B, C)$. Then, $\operatorname{ess\,inf}_{x\in B} d(x, C) > \lambda$ or $\operatorname{ess\,inf}_{x\in C} d(x, B) > \lambda$. We assume, for example, the latter holds. This implies that there exists a Borel set $K \subset C$ with $\mu(K) = \mu(C)$ such that

$$d(x, B) > \lambda \quad \text{for all } x \in K. \tag{4.6}$$

Now fix an integer $n > \lambda$. Define $u(x) = d(x, B) \wedge n$. By Proposition 4.1, we know $u \in D(\mathscr{E})$. Then, by the Lyons–Zheng's decomposition (see [15], [16]), under Q the following holds:

$$u(X_s) - u(X_0) = \tfrac{1}{2} M_s^u - \tfrac{1}{2}\big(M_t^u(\gamma_t(\omega)) - M_{t-s}^u(\gamma_t(\omega))\big) \quad \text{for } 0 \le s \le t, \tag{4.7}$$

where M_u is an $\mathscr{F}_t = \sigma(X_s,\ s \le t)$-square integrable martingale with

$$\langle M^u \rangle_t = \int_0^t \Gamma(u, u)(X_s)\, ds \tag{4.8}$$

and γ_t is the reverse operator such that $X_s(\gamma_t(\omega)) = w(t - s),\ 0 \le s \le t$.

Remarking that μ is an invariant measure of the diffusion process, we have

$$\begin{aligned} Q(X_0 \in B,\ X_t \in C) &= Q(X_0 \in B,\ X_t \in K) \\ &\le Q\big(u(X_t) > \lambda,\ u(X_0) = 0\big) \le Q\big(u(X_t) - u(X_0) > \lambda\big) \\ &= Q\bigg(\frac{1}{2}(M_t^u - M_t^u(\gamma_t(\omega))) > \lambda\bigg) \\ &\le Q(M_t^u > \lambda) + Q(-M_t^u > \lambda) \\ &\le 4\int_{\lambda/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\bigg(-\frac{s^2}{2}\bigg)\, ds, \end{aligned} \tag{4.9}$$

where we have used the reversibility of the diffusion and $\langle M^u \rangle_t \le t$ which follows from (4.8) and $\Gamma(u, u)(z) \le 1$. Thus we get from (4.9) that

$$\limsup_{t\to 0} 2t \log P(X_0 \in B,\ X_t \in C) \le -\lambda^2. \tag{4.10}$$

Letting $\lambda \to d(B, C)$, the assertion follows. □

For the lower bound, more assumptions on $A(z)$ are needed. Basically, the assumption can be formulated as follows.

ASSUMPTION 4.3. The operator function $A(z)$ is smooth enough so that the diffusion M_Q is a solution of a stochastic differential equation or stochastic evolution equation on the Hilbert space E with coefficients satisfying (I), (II) or (I)′, (II)′ in Section 3. For example, $A(z) \in C_b^2(E \to \mathscr{L}_s(H_0))$, $A = I_{H_0}$.

THEOREM 4.4. *Assume assumption* 4.3 *holds*. *Let* B, C *be two Borel subsets with* $\mu(B) > 0$, $\mu(C) > 0$. *If* B *or* C *is open, then*

$$\liminf_{t\to 0} 2t \log Q(X_0 \in B,\ X_t \in C) \geq -d^2(B, C).$$

PROOF. Assume, for example, C is open. Let $\lambda = d(B, C)$ and $\varepsilon > 0$. Since $\operatorname{ess\,inf}_{x\in B} d(x, C) < \lambda+\varepsilon$, there exists a compact subset $K \subset B$ with $\mu(K) > 0$ such that $d(x, C) < \lambda + \varepsilon$ on K. For any $x \in K$, applying the large deviation principle in Theorem 3.1 we have

$$\liminf_{t\to 0,\, x_n \to x.} 2t \log Q_{x_n}(X_t \in C) \geq -d(x, C)^2 \geq -(\lambda+\varepsilon)^2. \tag{4.11}$$

Since C is open, $2t \log P(X_t^x \in C)$ is lower semicontinuous on $[0, 1] \times K$. Hence it is lower bounded by (4.11). Applying Fatou's lemma and Jensen's inequality,

$$\begin{aligned}
&\liminf_{t\to 0} 2t \log Q(X_0 \in B,\ X_t \in C)\\
&\quad\geq \liminf_{t\to 0} 2t \log Q(X_0 \in K,\ X_t \in C)\\
&\quad= \liminf_{t\to 0} 2t \log \left(\int_K Q_x(X_t \in C)\mu(dx) \right) \geq \liminf_{t\to 0} 2t \log(\mu(K))\\
&\qquad+ \liminf_{t\to 0} 2t \log \left(\frac{1}{\mu(K)} \int_K Q_x(X_t \in C)\mu(dx) \right)\\
&\quad\geq \liminf_{t\to 0} \frac{1}{\mu(K)} \int_K 2t \log Q_x(X_t \in C)\mu(dx)\\
&\quad\geq \frac{1}{\mu(K)} \int_K \liminf_{t\to 0} 2t \log Q_x(X_t \in C)\mu(dx)\\
&\quad\geq -(\lambda+\varepsilon)^2 \frac{1}{\mu(K)} \int_K \mu(dx) = -(\lambda+\varepsilon)^2,
\end{aligned}$$

where we have used (4.11) for the last inequality. Since ε is arbitrary, the theorem is proved. □

5. The perturbed Ornstein–Uhlenbeck processes. In this section, we study the small time asymptotics of perturbed Ornstein–Uhlenbeck processes,

$$dX_t = dw_t - \tfrac{1}{2}AX_t\,dt + b(X_t)\,dt, \tag{5.1}$$

where b is a drift of gradient type, but can be very singular and unbounded.

This case cannot be covered by Section 4. We have to adopt a different approach. The main tool is the Girsanov theorem. First we give the construction of the perturbed process. Let $\varphi > 0$, $\varphi \in D(\mathscr{E})$, Define $m(dx) = \varphi^2(x)\mu(dx)$. Consider the quadratic form

$$\mathscr{E}^0_\varphi(u, v) = \tfrac{1}{2}\int_E \langle \nabla u, \nabla v\rangle_{H_0}\varphi^2(x)d\mu, \quad u, v \in \mathscr{F}C^\infty_b; \tag{5.2}$$

It is known (see [19] and [20]) that the form $(\mathscr{E}^0_\varphi(u, v), \mathscr{F}C^\infty_b)$ is closable on $L^2(E, m)$. Its closure $(\mathscr{E}_\varphi, D(\mathscr{E}_\varphi))$ is a Dirichlet form. The associated diffusion $((X_t)_{t\geq 0}, \mathscr{F}_t, \widehat{Q}_x,\ x \in E)$ solves (5.1) with $b = \nabla\phi/\phi$ in the weak sense.

The intrinsic metric $d(x, y)$ associated with the Dirichlet form $(\mathscr{E}_\varphi, D(\mathscr{E}_\varphi))$ can be exactly defined as in Section 3 with Γ there replaced by $\Gamma(u, v)(z) = \langle \nabla u(z), \nabla v(z)\rangle_{H_0}$ here. Let B be a Borel subset of E; we define $d(x, B) = \inf\{d(x, y);\ y \in B\}$. For two Borel subsets B, C with $\mu(B) > 0$ and $\mu(C) > 0$, let

$$d(B, C) = \sup\Big(\operatorname*{ess\,inf}_{z\in B} d(z, C),\ \operatorname*{ess\,inf}_{z\in C} d(z, B)\Big). \tag{5.3}$$

Define

$$\hat{Q}(\cdot) = \int_E \hat{Q}_x(\cdot)Q^2(x)\,d\mu.$$

We first recall some known results.

THEOREM 5.1 [11]. *Let B, C be two Borel subsets with $\mu(B) > 0$, $\mu(C) > 0$. If B or C is open, then*

$$\liminf_{t\to 0} 2t\log P(X_0 \in B,\ X_t \in C) \geq -d^2(B, C), \tag{5.4}$$

where P is the probability measure corresponding to the Ornstein–Uhlenbeck process defined in Section 2.

REMARK 5.2. The above theorem is now also contained in Theorem 4.4.

THEOREM 5.3 [4]. *The diffusion $((X_t)_{t\geq 0}, \mathscr{F}_t,\ \hat{Q}_x,\ x \in E)$ is given by*

$$d\hat{Q}_x|_{\mathscr{F}_t} = \exp\big\{M^{\ln\varphi}_t - \tfrac{1}{2}\langle M^{\ln\varphi}\rangle_t\big\}\,dP_x|_{\mathscr{F}_t}, \tag{5.5}$$

where $M^{\ln\varphi}$ stands for the martingale part of Fukushima's decomposition of the additive functional $\ln\varphi(X_t) - \ln\varphi(X_0)$ (*see* [12]) *and*

$$\langle M^{\ln\varphi}\rangle_t = \int_0^t \left|\frac{\nabla\varphi}{\varphi}\right|^2_{H_0}(X_s)\,ds \tag{5.6}$$

is the bracket.

We now prepare some preliminary results.

LEMMA 5.4. *Let $b(x)$ be a nonnegative measurable function on E. Assume that there exists a $\delta > 0$ such that $\int_E \exp(\delta b(x))\,du < \infty$. Then for any $M > 0$,*

$$\lim_{t\to 0} E\left[\exp\left(M\int_0^t b(X_s)\,ds\right)\right] = 1, \tag{5.7}$$

where E stands for the expectation with respect to P.

PROOF. By Jensen's inequality,

$$\begin{aligned}1 \le E\left[\exp\left(M\int_0^t b(X_s)\,ds\right)\right] &\le E\left[\frac{1}{t}\int_0^t \exp(Mtb(X_s))\,ds\right]\\ &= \frac{1}{t}\int_0^t E\big[\exp(Mtb(X_s))\big]\,ds = \int_E \exp(Mtb(x))\,d\mu,\end{aligned}$$

where we have used the fact that μ is an invariant measure of the Ornstein–Uhlenbeck process.

The assertion now follows by the dominated convergence theorem.

LEMMA 5.5. *Let B, C be two Borel subsets with $\mu(B) > 0$, $\mu(C) > 0$. Let B_n, $n \ge 1$ be a sequence of subsets such that $B_n \uparrow B$. Then*

$$\lim_{n\to\infty} d(B_n, C) = d(B, C). \tag{5.8}$$

PROOF. Let $L = \lim_{n\to\infty} d(B_n, C)$. Clearly, $L \ge d(B, C)$. Suppose $L > d(B, C)$. Then $\operatorname{ess\,inf}_{z\in B} d(z, C) < L$ and $\operatorname{ess\,inf}_{z\in C} d(z, B) < L$. This implies that there exists a subset $K \subset B$ such that $\mu(K) > 0$ and $d(x, C) < L$ for all $x \in K$. Since $\mu(K \cap B_n) > 0$ for big enough n, it follows that $\operatorname{ess\,inf}_{z\in B_n} d(z, C) < L$ for big enough n. To show $d(B_n, C) < L$, we need also to prove $\operatorname{ess\,inf}_{z\in C} d(z, B_n) < L$. Since $\operatorname{ess\,inf}_{z\in C} d(z, B) < L$, it holds that $\mu(C \cap \{x; d(x, B) < L\}) > 0$.

On the other hand,

$$C \cap \{x; d(x, B) < L\} = \bigcup_n C \cap \{x; d(x, B_n) < L\}.$$

Hence, if n is big enough, $\mu(C \cap \{x; d(x, B_n) < L\}) > 0$ which gives $\operatorname{ess\,inf}_{z\in C} d(z, B_n) < L$. Thus we have proved that $d(B_n, C) < L$ for big enough n, which is a contradiction. The proof is complete. □

Let B be a Borel subset. Define the function $u(x) = d(x, B)$ and set $u_n(x) = u(x) \wedge n$.

LEMMA 5.6. *For any $n \ge 1$, we have $u_n \in D(\mathscr{E})$ and $\Gamma(u_n, u_n)(z) \le 1$.*

PROOF. Fix any $x \in E$ and $h \in H_0$; we have

$$|u_n(x+h) - u_n(x)| \le d(x+h, x) = |th|_{H_0}. \tag{5.9}$$

This implies that as a function on the real line, $u_n(x+h)$ is absolutely continuous and $|du_n(x+th)/dt| \le |h|_{H_0}$ which already gives us the desired result according to the uniqueness result for $D(\mathscr{E})$ given in [19].

THEOREM 5.7. *Let B, C be two Borel subsets with $\mu(B) > 0$, $\mu(C) > 0$. Then*

$$\limsup_{t\to 0} 2t \log \hat{Q}(X_0 \in B,\ X_t \in C) \le -d^2(B, C). \tag{5.10}$$

The proof is the same as that of Theorem 4.2; therefore, we omit it.

THEOREM 5.8. *Assume $\int_E \exp(\delta|\nabla\varphi/\varphi|^2_{H_0}(x))\, d\mu < \infty$ for some small $\delta > 0$. Let B, C be two Borel subsets with $\mu(B) > 0$, $\mu(C) > 0$. If B or C is open, then*

$$\liminf_{t\to 0} 2t \log \hat{Q}(X_0 \in B,\ X_t \in C) \ge -d^2(B, C). \tag{5.11}$$

PROOF. Let us say, for example, that C is open. Set

$$Z_t = \exp\left\{ M_t^{\ln\varphi} - \tfrac{1}{2}\langle M^{\ln\varphi}\rangle_t \right\} \tag{5.12}$$

Then, according to Theorem 5.3, $d\hat{Q}_x|_{\mathscr{F}_t} = Z_t\, dP_x|_{\mathscr{F}_t}$. Choose a sequence $\{\delta_n,\ n \ge 1\}$ of positive numbers satisfying $\delta_n \downarrow 0$. Define $B_n = B \cap \{x; \varphi^2(x) > \delta_n\}$.

Now, for any $n \ge 1$,

$$\begin{aligned} \hat{Q}(X_0 \in B,\ X_t \in C) &= \int_E \hat{Q}_x(X_0 \in B, X_t \in C)\varphi^2(x)\, d\mu \\ &= \int_E \hat{Q}_x[X_0 \in B,\ X_t \in C,\ \varphi^2(X_0)]\, d\mu \\ &= \int_E P_x[Z_t \varphi^2(X_0);\ X_0 \in B,\ X_t \in C]\, d\mu \\ &\ge \delta_n \int_E P_x[Z_t;\ \varphi^2(X_0) > \delta_n;\ X_0 \in B,\ X_t \in C]\, d\mu \\ &= \delta_n \int_E P_x[Z_t;\ X_0 \in B_n,\ X_t \in C]\, d\mu. \end{aligned} \tag{5.13}$$

Hence,

$$\begin{aligned} &\liminf_{t\to 0} 2t \log \hat{Q}(X_0 \in B,\ X_t \in C) \\ &\quad \ge \liminf_{t\to 0} 2t\big[\log \delta_n + \log P[Z_t;\ X_0 \in B_n,\ X_t \in C]\big] \\ &\quad = \liminf_{t\to 0} 2t \log P[Z_t;\ X_0 \in B_n,\ X_t \in C]. \end{aligned} \tag{5.14}$$

Since $B_n \uparrow B$, applying Lemma 5.5 we see that the theorem follows if we have proved

$$\liminf_{t\to 0} 2t \log P[Z_t;\ X_0 \in B_n,\ X_t \in C] \geq -d^2(B_n, C) \quad \text{for any } n \geq 1. \tag{5.15}$$

Let us now prove (5.15). Drop the index n for simplicity. We may assume $\lambda = d^2(B, C) < \infty$. For any $\varepsilon_1 > 0$, it follows that

$$\begin{aligned} &P[Z_t;\ X_0 \in B,\ X_t \in C] \\ &\quad \geq \exp\Big(-\frac{\varepsilon_1}{2t}\Big) P\Big(Z_t > \exp\Big(-\frac{\varepsilon_1}{2t}\Big),\ X_0 \in B,\ X_t \in C\Big) \qquad (5.16) \\ &\quad \geq \exp\Big(-\frac{\varepsilon_1}{2t}\Big)\Big[P(X_0 \in B,\ X_t \in C) - P\Big(Z_t < \exp\Big(-\frac{\varepsilon_1}{2t}\Big)\Big)\Big], \end{aligned}$$

where $P(Z_t < \exp(-\varepsilon_1/2t))$ can be further estimated as follows: let $a > 0$,

$$\begin{aligned} (5.17)\ &P\Big(Z_t < \exp\Big(-\frac{\varepsilon_1}{2t}\Big)\Big) \\ &\quad = P\Big(Z_t^{-1} > \exp\Big(\frac{\varepsilon_1}{2t}\Big)\Big) \\ &\quad = P\Big(\exp\Big(-aM_t^{\ln\varphi} + \frac{a}{2}\int_0^t \Big|\frac{\nabla\varphi}{\varphi}\Big|_{H_0}^2 (X_s)\,ds\Big) > \exp\Big(\frac{a\varepsilon_1}{2t}\Big)\Big) \\ (5.18)\ &\quad \leq \exp\Big(-\frac{a\varepsilon_1}{2t}\Big) E\Big[\exp\Big(-2aM_t^{\ln\varphi} - 2a^2\int_0^t \Big|\frac{\nabla\varphi}{\varphi}\Big|_{H_0}^2 (X_s)\,ds\Big)\Big]^{1/2} \\ &\qquad \times E\Big[\exp\Big((2a^2 + a)\int_0^t \Big|\frac{\nabla\varphi}{\varphi}\Big|_{H_0}^2 (X_s)\,ds\Big)\Big]^{1/2} \\ (5.19)\ &\quad \leq \exp\Big(-\frac{a\varepsilon_1}{2t}\Big) D_a(t), \end{aligned}$$

where

$$D_a(t) = E\Big[\exp\Big((2a^2 + a)\int_0^t \Big|\frac{\nabla\varphi}{\varphi}\Big|_{H_0}^2 (X_s)\,ds\Big)\Big]^{1/2} \tag{5.20}$$

and from (5.18) to (5.19) we used that $\exp(-2aM_t^{\ln\varphi} - 2a^2\int_0^t |\nabla\varphi/\varphi|_{H_0}^2 (X_s)\,ds)$ is a super-martingale.

On the other hand, for any $\varepsilon_2 > 0$, by Theorem 5.1 there is $t_1 > 0$ such that if $t \leq t_1$,

$$P(X_0 \in B,\ X_t \in C) \geq \exp\Big(-\frac{\lambda + \varepsilon_2}{2t}\Big). \tag{5.21}$$

Take $a_0 = (\lambda + 2\varepsilon_2)/\varepsilon_1$. It follows from (5.16), (5.19) and (5.21) that if $t \le t_1$,

$$P[Z_t; X_0 \in B,\ X_t \in C]$$

$$\ge \exp\left(-\frac{\varepsilon_1}{2t}\right)\left[\exp\left(-\frac{\lambda+\varepsilon_2}{2t}\right) - D_{a_0}(t)\exp\left(-\frac{\lambda+2\varepsilon_2}{2t}\right)\right] \tag{5.22}$$

$$= \exp\left(-\frac{\varepsilon_1}{2t}\right)\exp\left(-\frac{\lambda+\varepsilon_2}{2t}\right)\left[1 - \exp\left(-\frac{\varepsilon_2}{2t}\right)D_{a_0}(t)\right].$$

By Lemma 5.4, we have that

$$\lim_{t\to 0}\left[1 - \exp\left(-\frac{\varepsilon_2}{2t}\right)D_{a_0}(t)\right] = 1. \tag{5.23}$$

Hence, (5.22) and (5.23) imply that

$$\liminf_{t\to 0} 2t \log P[Z_t;\ X_0 \in B_n,\ X_t \in C] \ge -\varepsilon_1 - \lambda - \varepsilon_2.$$

Since ε_1, ε_2 are arbitrary, (5.15) follows; hence the theorem. □

Acknowledgments. I thank the anonymous referees for their careful reading and helpful suggestions. The paper was revised during my stay at Cornell University. I thank Professor L. Gross and the Mathematics Department for their hospitality.

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