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HILBERT SPACE REGULARITY OF THE $(\alpha, d, 1)$-SUPERPROCESS AND ITS OCCUPATION TIME

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The superprocess and its occupation time process are represented as Hilbert space valued solutions of stochastic evolution equations by using the Fourier transform of the process. For appropriate parameter values, the existence of density valued solutions follows. Pathwise regularity of the processes is obtained. As a new tool we develop a maximal inequality. We also extend the Tanaka-like evolution equations developed for local time processes and provide an Ito formula for certain functionals of the superprocess.

1. Introduction. The notation $(\alpha, d, 1)$ superprocess refers to a collection of measure valued stochastic processes arising as scaling limits of the empirical measures of systems of particles. The particles in the approximating systems independently move in $\mathbf{R}^d$ as symmetric stable processes of index $\alpha \in (0, 2]$ and undergo critical binary branching. The special case $\alpha = 2$ is the Dawson-Watanabe (or super-Brownian motion) process.

Each process satisfies a weak stochastic evolution equation [Konno and Shiga (1988), Meleard and Roelly-Coppoleta (1990), Roelly-Coppoleta (1986)]. Many results and background material can be found in Dawson (1993).

In Blount (1996) and Bose and Sundar (1997), Hilbert space regularity results for super Brownian motion on a bounded domain were obtained by using Fourier series methods and a maximal inequality for "Ornstein–Uhlenbeck" like processes. Here, by using Fourier transform methods, we show this approach can be extended to $\mathbf{R}^d$ and to both the $(\alpha, d, 1)$ superprocess and its occupation time process.

In particular, we obtain the following results.

1. The superprocess and its occupation time process are represented as Hilbert space valued solutions of stochastic evolution equations. This is done by using the previously mentioned weak evolution equation to obtain an equation for the Fourier transform of the process. If the initial measure has infinite total mass, a suitable weighting function can be used to first embed the process in the space of finite measures.

For the appropriate parameter values, $\alpha > 1$ if $d = 1$ for the superprocess and $d < 2\alpha$ for the occupation time process, the existence of density valued solutions follows naturally from our approach.

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Pathwise regularity of the processes, as measured by square integrability of their weighted Fourier transforms, is obtained. The regularity results appear to be optimal.

2. We develop a maximal inequality which is used to obtain pathwise regularity results for a process defined by the stochastic convolution of a semigroup with respect to a martingale measure. The semigroup is generated by $-(-\Delta)^{\alpha/2}$, where Δ is the Laplacian. In our problem these operators do not have a discrete spectrum, but we develop a method which reduces the calculations to ones similar to the discrete spectrum case. We also give an example showing the applicability of the maximal inequality for obtaining regularity results for stochastic partial differential equations.

3. In Adler and Lewin (1992), Tanaka-like evolution equations were developed for the local time process. In addition, an Ito formula for certain functionals of the superprocess was obtained. We obtain extensions of these results by expanding the class of objects upon which the superprocess and occupation time process may act.

2. Statement of results. Consider $\mathscr{M}_F(\mathscr{M}_{LF})$, the set of finite (locally finite) Borel measures on $\mathbf{R}^d$. Endow $\mathscr{M}_F$ with the topology of weak convergence. Employ the notation $\nu(\phi)$ to represent the integral of an integrable function ϕ over $\mathbf{R}^d$ with respect to the Borel measure ν. For a measure-valued process, $\nu(t)$, write $\nu(t, \phi)$ instead of $\nu(t)(\phi)$.

Let $\Omega_F = \mathbf{C}([0, \infty)\colon \mathscr{M}_F)$ denote the space of continuous $\mathscr{M}_F$-valued paths with the compact open topology and let $\mathscr{F}$ denote its Borel σ-field. Let $(\mathscr{F}_t)_{t\geq 0}$ denote the canonical right continuous filtration on $(\Omega_F, \mathscr{F})$.

Denote the space of k-times differentiable bounded functions on $\mathbf{R}^d$ with bounded continuous derivatives as $\mathbf{C}_b^k$. For $x \in \mathbf{R}^d$ and $\phi \in \mathbf{C}_b^2$, the fractional Laplacian operator $\mathbf{A}_\alpha = -(-\Delta)^{\alpha/2}$ has the representation

$$\mathbf{A}_\alpha \phi(x) = \begin{cases} \Delta\phi(x), & \text{if } \alpha = 2, \\ \int_{\mathbf{R}^d} \psi(x, y)|y - x|^{-(d+\alpha)}\, dy, & \text{if } 0 < \alpha < 2, \end{cases}$$

where

$$\psi(x, y) = \phi(y) - \phi(x) - \nabla\phi(x)\cdot(y - x)(1 + |x - y|^2)^{-1}$$

($\cdot$ is the scalar product and $|\cdot|$ is the norm in $\mathbf{R}^d$).

If $\mathbf{E}\mathbf{X}(0) = \nu \in \mathscr{M}_F$, then $\mathbf{X}(t)$, the $(\alpha, d, 1)$-super process [in the terminology of Dawson (1993)], is a continuous $\mathscr{M}_F$-valued process which satisfies for $\phi \in \mathbf{C}_b^2$

$$\mathbf{X}(t, \phi) = \mathbf{X}(0, \phi) + \int_0^t \mathbf{X}(s, \mathbf{A}_\alpha\phi)\, ds + \mathbf{M}(t, \phi), \tag{2.1}$$

where $\mathbf{M}(t, \phi)$ is a continuous square integrable $\mathscr{F}_t$ martingale with quadratic variation

$$\big[\mathbf{M}(\cdot, \phi)\big](t) = \int_0^t \mathbf{X}(s, \phi^2)\, ds.$$

Before allowing $\mathbf{X}(0)$ to be an infinite measure, we describe our results for the finite case. We follow Rudin [1973] to introduce the relevant function spaces on $\mathbf{R}^d$. Denote

$$\lambda(dx) = (2\pi)^{-d/2}\,dx, \qquad \mu_\gamma(dx) = (1+|x|^2)^\gamma \lambda(dx) \quad \text{and} \quad e_\theta(x) = \exp(i\theta\cdot x).$$

For $f \in \mathscr{S}$, the usual Schwartz space of smooth and rapidly decreasing functions, let

$$\hat{f}(\theta) := \lambda(e_{-\theta} f) \quad \text{and the norm } \|f\|_\gamma^2 := \mu_\gamma(|\hat{f}|^2).$$

Recall that $\mathbf{H}_\gamma$, Sobolev space of index γ, is the completion of $\mathscr{S}$ in the $\|\cdot\|_\gamma$-norm so that $\mathbf{H}_0 = \mathbf{L}^2(\mathbf{R}^d)$ and $\{\mathbf{H}_\gamma\colon \gamma \in \mathbf{R}\}$ form a scale of Hilbert spaces. Similarly for $\nu \in \mathscr{M}_F$,

$$\hat{\nu}(\theta) := \nu(e_{-\theta}) \quad \text{and} \quad \|\nu\|_\gamma^2 := \mu_\gamma(|\hat{\nu}|^2).$$

For $\gamma < -d/2$, $\mu_\gamma(\mathbf{1}) < \infty$ and we have the following continuous injections

$$\mathscr{M}_F \subset \mathbf{H}_\gamma \quad \text{and} \quad \mathbf{C}\big([0,\infty)\colon \mathscr{M}_F\big) \subset \mathbf{C}\big([0,\infty)\colon \mathbf{H}_\gamma\big).$$

Hence it is natural to write $\hat{\mathbf{X}}(t,\theta)$ for $\mathbf{X}(t, e_{-\theta})$ and similarly for $\mathbf{M}(t, e_{-\theta})$.

Let the Feller semigroup generated by the operator $\mathbf{A}_\alpha$ be $\mathbf{S}_\alpha(t)$. Viewing $\mathbf{M}$ as a martingale measure [Walsh (1986)], one can extend (2.1) [as in Meleard and Roelly-Coppoletta (1990) or as in Prop. 7.1 of Dawson (1993)] for suitable functions ϕ, to obtain in the mild form

$$\mathbf{X}(t,\phi) = \mathbf{X}(0, \mathbf{S}_\alpha(t)\phi) + \mathbf{Y}(t,\phi), \tag{2.2}$$

where

$$\mathbf{Y}(t,\phi) = \int_0^t \int_{\mathbf{R}^d} (\mathbf{S}_\alpha(t-s)\phi)(x)\mathbf{M}(ds, dx).$$

We can extend $\mathbf{A}_\alpha\colon \mathbf{H}_\gamma \to \mathbf{H}_{\gamma-\alpha}$ as a continuous linear mapping by setting, for $g \in \mathscr{S}$,

$$(\widehat{\mathbf{A}_\alpha g})(\theta) = -|\theta|^\alpha \hat{g}(\theta)$$

and similarly

$$\big(\widehat{\mathbf{S}_\alpha(t)g}\big)(\theta) = \exp(-t|\theta|^\alpha)\hat{g}(\theta).$$

Since $\mathbf{A}_\alpha e_{\pm\theta} = -|\theta|^\alpha e_{\pm\theta}$, if also $\mathbf{E}\mathbf{X}(0) \in \mathscr{M}_F$ then (2.1) and (2.2) specialize to

$$\hat{\mathbf{X}}(t,\theta) = \hat{\mathbf{X}}(0,\theta) - |\theta|^\alpha \int_0^t \hat{\mathbf{X}}(s,\theta)\,ds + \hat{\mathbf{M}}(t,\theta) \tag{2.3}$$

and

$$\hat{\mathbf{X}}(t,\theta) = \exp(-t|\theta|^\alpha)\hat{\mathbf{X}}(0,\theta) + \int_0^t \exp(-(t-s)|\theta|^\alpha)\,d\hat{\mathbf{M}}(s,\theta). \tag{2.4}$$

Note (2.4) can also be obtained from (2.3) via variation of constants. If we identify $\mathbf{X}(t)$ with $\hat{\mathbf{X}}(t,\cdot)$ and $\mathbf{M}(t)$ with $\hat{\mathbf{M}}(t,\cdot)$, then the above equations define Hilbert space valued evolution equations whose regularity properties are described in the next theorem.

THEOREM 1. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_F$.

(a) *For any* $T > 0$, *almost surely*

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}_\alpha \mathbf{X}(s)\,ds + \mathbf{M}(t), \tag{2.5}$$

holds in $\mathbf{C}([0, T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < -d/2$.

(b) *The convolution term* $\int_0^{\cdot} \mathbf{S}_\alpha(\cdot - s)\,d\mathbf{M}(s)$ *is a.s. in* $\mathbf{C}([0, T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < (\alpha - d)/2$.

(c) *For any* $T > 0$, *almost surely*

$$\mathbf{X}(t) = \mathbf{S}_\alpha(t)\mathbf{X}(0) + \int_0^t \mathbf{S}_\alpha(t - s)\,d\mathbf{M}(s), \tag{2.6}$$

holds in $\mathbf{C}((0, T]\colon \mathbf{H}_\gamma) \cap \mathbf{C}([0, T]\colon \mathbf{H}_\beta)$ *provided* $\gamma < (\alpha - d)/2$ *and* $\beta < -d/2$.

Note that Theorem 1 shows that for $d = 1$ and $\alpha > 1$, $\mathbf{X}$ has sample paths in $\mathbf{C}((0, T]\colon H_\gamma)$ for $0 < \gamma < (\alpha - 1)/2$. If $\mathbf{EX}(0) \in \mathscr{M}_{LF}$ with suitable restrictions, then Theorem 3 shows this holds locally. A simple calculation with a nondegenerate, deterministic $\mathbf{X}(0) \in \mathscr{M}_F$ shows $\mathbf{E}(\|\mathbf{X}(t)\|_\gamma^2) < \infty$ for $t > 0$ if and only if $\gamma < (\alpha - d)/2$. Thus our results appear to be essentially optimal. Existence of densities for fixed $t > 0$ and $\mathbf{X}(0) =$ Lebesgue measure was shown in Roelly-Coppoletta (1986), for $d = 1$ and $\alpha > 1$. Konno and Shiga (1988) and Reimars (1989) obtained joint continuity in (x, t) for $d = 1$ and $\alpha > 1$. Their methods are very different.

Define the occupation time process for $\mathbf{X}$ as

$$\mathscr{O}(t) = \int_0^t \mathbf{X}(s)\,ds.$$

The density of $\mathscr{O}(t)$, when it exists, is denoted by $\mathscr{L}(t, x)$ and is also known as the local time process. Results on $\mathscr{O}$ and $\mathscr{L}$ were first obtained in Iscoe (1986a, b) for the case $\alpha = 2$, and he showed existence of $\mathscr{L}$ for $d \leq 3$. More generally, existence of $\mathscr{L}$ is known to hold when $d < 2\alpha$ [Fleischmann (1986), Dynkin (1988)].

Joint continuity results in (x, t) are obtained in Krone (1993), Sugitani (1989) and Adler and Lewin (1992).

Our focus in this paper is regularity as measured by the index of the Sobolev space in which the processes can be embedded. However, pathwise existence of $\mathscr{L}$ when $d < 2\alpha$ follows very easily and naturally using our methods and basic estimates on $\hat{\mathscr{O}}(t, \theta)$ or the analogously defined $\hat{\mathscr{O}}_p(t, \theta)$ (Theorems 2 and 4, respectively). We also show pathwise existence of a distributional derivative for $\mathscr{L}$ when $d = 1$ and $\alpha > 3/2$. In what follows, $\mathbf{D}_i^k f$ will denote the kth derivative of f with respect to the ith variable.

THEOREM 2. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_F$.

(a) *For any* $T > 0$, *almost surely* $\mathscr{O}$ *satisfies the following evolution equations in* $\mathbf{C}([0,T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < (2\alpha - d)/2$:

$$\mathscr{O}(t) = (\mathbf{I} - \mathbf{A}_\alpha)^{-1}\big[\mathbf{X}(0) + \mathscr{O}(t) - \mathbf{X}(t) + \mathbf{M}(t)\big], \tag{2.7}$$

$$\mathscr{O}(t) = \int_0^t \mathbf{S}_\alpha(t-s)[\mathbf{X}(0) + \mathbf{M}(s)]\,ds \tag{2.8}$$

(b) *If* $d < 2\alpha$ *then* $\mathscr{L}(t,\cdot)$ *exists and satisfies* $\mathbf{P}\{\mathscr{L} \in \mathbf{C}([0,T]\colon \mathbf{H}_\gamma)\} = 1$ *for* $\gamma < (2\alpha - d)/2$.

(c) *If* $d = 1$ *and* $1 < \alpha \le 2$, *then almost surely a version of* $\mathscr{L}(t,x)$, *which is jointly continuous in* x *and* t, *is given by* $\mathscr{L}(t,x) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty}\exp(i\theta\cdot x)\,\hat{\mathscr{O}}(t,\theta)\,d\theta$.

(d) *If* $d = 1$ *and* $\frac{3}{2} < \alpha \le 2$, $\mathbf{P}\{\mathbf{D}_1\mathscr{L} \in \mathbf{C}([0,T]\colon \mathbf{H}_\gamma)\} = 1$ *for* $\gamma < (2\alpha - 3)/2$.

To start initially from a σ-finite measure, we need to consider a subset of $\mathscr{M}_{LF}$. Towards this end, consider $\mathbf{C}$ the space of continuous functions on $\mathbf{R}^d$ and let $\phi_p(x) = 1/(1+|x|^2)^p$. Then define $\mathbf{C}_p = \{f \in \mathbf{C} : \|f\|_p < \infty\}$ where $\|f\|_p := \sup_{x\in\mathbf{R}^d}|f(x)/\phi_p(x)|$. Let us introduce the notation $\mathbf{C}_p^k$ to denote functions in $\mathbf{C}_p$ which are k-times differentiable with the derivatives also in $\mathbf{C}_p$.

Now define $\mathscr{M}_p = \{\mu \in \mathscr{M}_{LF}\colon \mu(\phi_p) < \infty\}$. The topology on $\mathscr{M}_p$ is defined by the convergence $\mu_n \to \mu$ in $\mathscr{M}_p$ if and only if $\mu_n(g) \to \mu(g)$ for $g \in \mathbf{C}_p$. Introduce the path space Ω_p as in the finite case. In discussing $\mathscr{M}_p$, we assume $p > d/2$ if $\alpha = 2$ and $d/2 < p < (d+\alpha)/2$ if $0 < \alpha < 2$.

For $\mathbf{E}\mathbf{X}(0) \in \mathscr{M}_p$, $\mathbf{X}(t)$, the $(\alpha, d, 1)$-super process, is a continuous $\mathscr{M}_p$-valued process that satisfies for $f \in \mathbf{C}_b^2$

$$\mathbf{X}(t, f\phi_p) = \mathbf{X}(0, f\phi_p) + \int_0^t \mathbf{X}(s, \mathbf{A}_\alpha(f\phi_p))\,ds + \mathbf{M}(t, f\phi_p), \tag{2.9}$$

where $\mathbf{M}(t, f\phi_p)$ is a continuous martingale with quadratic variation process

$$[\mathbf{M}(\cdot, f\phi_p)](t) = \int_0^t \mathbf{X}(s, (f\phi_p)^2)\,ds.$$

Define $\mathbf{X}_p(t,f) = \mathbf{X}(t, \phi_p f)$ and $\mathbf{M}_p(t,f) = \mathbf{M}(t,\phi_p f)$. Note that a.s. for all t we have the Radon–Nikodym derivatives

$$\frac{d\mathbf{X}_p(t)}{d\mathbf{X}(t)}(x) = \phi_p(x), \qquad \frac{d\mathbf{X}(t)}{d\mathbf{X}_p(t)}(x) = \frac{1}{\phi_p(x)}.$$

Equation (2.9) is sometimes stated in the literature with more restrictive conditions on f. But it holds for $f \in \mathbf{C}_c^\infty$, the space of infinitely differentiable functions of compact supports; and using subsequent Lemmas 4 and 5, if $f \in \mathbf{C}_b^2$, then one can choose $f_n \in \mathbf{C}_c^\infty$ with $f_n \to f$ and $\phi_p^{-1}\mathbf{A}_\alpha(f_n\phi_p) \to \phi_p^{-1}\mathbf{A}_\alpha(f\phi_p)$ boundedly and pointwise. Now (2.9) follows using standard approximation arguments. Note also that $\mathbf{X}_p$ has sample paths in $\mathbf{C}([0,\infty)\colon \mathscr{M}_F)$.

In order to derive an evolution equation for $\hat{\mathbf{X}}_p(t,\theta)$, we need the Leibnitz formula for $\mathbf{A}_\alpha$ which we now develop.

For $f, g \in \mathbf{C}_b^2$,

$$\mathbf{A}_\alpha(fg) = \mathbf{A}_\alpha(f)\,g + f\mathbf{A}_\alpha(g) + \mathbf{B}_\alpha(f,g), \tag{2.10}$$

where

$$\begin{aligned}
&\mathbf{B}_\alpha(f,g)(x)\\
&\quad = \begin{cases} 2\nabla f(x)\cdot\nabla g(x), & \alpha = 2,\\ \int_{\mathbf{R}^d}[f(x+y)-f(x)][g(x+y)-g(x)]|y|^{-(d+\alpha)}\,dy, & 0<\alpha<2; \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_\alpha(e_{-\theta}\phi_p)(x) &= -|\theta|^\alpha e_{-\theta}(x)\phi_p(x) + e_{-\theta}(x)\mathbf{A}_\alpha\phi_p(x)\\
&\quad + e_{-\theta}(x)e_\theta(x)\mathbf{B}_\alpha(e_{-\theta},\phi_p)(x)\\
&= -|\theta|^\alpha e_{-\theta}(x)\phi_p(x) + e_{-\theta}(x)\mathbf{R}_{\alpha,\theta}\phi_p(x),
\end{aligned} \tag{2.11}$$

where

$$\mathbf{R}_{\alpha,\theta}f = \mathbf{A}_\alpha f + e_\theta\mathbf{B}_\alpha(e_{-\theta},f).$$

For $f = e_{-\theta}$, (2.9) becomes

$$\begin{aligned}
\hat{\mathbf{X}}_p(t,\theta) &= \hat{\mathbf{X}}_p(0,\theta) - |\theta|^\alpha\int_0^t \hat{\mathbf{X}}_p(s,\theta)\,ds\\
&\quad + \int_0^t \mathbf{X}_p(s, e_{-\theta}\phi_p^{-1}\mathbf{R}_{\alpha,\theta}\phi_p)\,ds + \hat{\mathbf{M}}_p(t,\theta).
\end{aligned} \tag{2.12}$$

Now define $\mathbf{G}_{\alpha,p}$ by setting, for $\nu \in \mathscr{M}_F$,

$$(\widehat{\mathbf{G}_{\alpha,p}\nu})(\theta) := \nu\big(e_{-\theta}\phi_p^{-1}\mathbf{R}_{\alpha,\theta}\phi_p\big).$$

THEOREM 3. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_p$ *and* $T > 0$.

(a) *Almost surely the following holds in* $\mathbf{C}([0,T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < -d/2$:

$$\mathbf{X}_p(t) = \mathbf{X}_p(0) + \int_0^t (\mathbf{A}_\alpha + \mathbf{G}_{\alpha,p})\mathbf{X}_p(s)\,ds + \mathbf{M}_p(t), \tag{2.13}$$

(b) *Almost surely, the convolution terms* $\int_0^\cdot \mathbf{S}_\alpha(\cdot - s)\,d\mathbf{M}_p(s)$ *and* $\int_0^\cdot \mathbf{S}_\alpha(\cdot - s)\mathbf{G}_{\alpha,p}\,\mathbf{X}_p(s)\,ds$ *are in* $\mathbf{C}([0,T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < (\alpha-d)/2$.

(c) *Almost surely, the following equation holds in* $\mathbf{C}((0,T]\colon \mathbf{H}_\gamma)\cap\mathbf{C}([0,T]\colon \mathbf{H}_\beta)$ *provided* $\gamma < (\alpha-d)/2$ *and* $\beta < -d/2$:

$$\mathbf{X}_p(t) = \mathbf{S}_\alpha(t)\mathbf{X}_p(0) + \int_0^t \mathbf{S}_\alpha(t-s)\,d\mathbf{M}_p(s) + \int_0^t \mathbf{S}_\alpha(t-s)\mathbf{G}_{\alpha,p}\mathbf{X}_p(s)\,ds. \tag{2.14}$$

THEOREM 4. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_p$ *and* $T > 0$.

(a) *The following evolution equation holds a.s. in* $\mathbf{C}([0,T]\colon \mathbf{H}_\gamma)$ *for* $\gamma < (2\alpha - d)/2$:

$$\mathscr{O}_p(t) = \int_0^t \mathbf{S}_\alpha(t-s)[\mathbf{X}_p(0) + \mathbf{M}_p(s)]\,ds + \int_0^t \mathbf{S}_\alpha(t-s)\int_0^s \mathbf{G}_{\alpha,p}\mathbf{X}_p(u)\,du\,ds. \tag{2.15}$$

(b) *If* $d < 2\alpha$ *then* $\mathbf{P}\{\mathscr{L}_p \in \mathbf{C}([0,T]\colon \mathbf{H}_\gamma)\} = 1$ *for* $\gamma < (2\alpha - d)/2$.

(c) *If* $d = 1$ *and* $1 < \alpha \le 2$, *then almost surely a version of* $\mathscr{L}_p$, *which is jointly continuous in* x *and* t, *is given by* $\mathscr{L}_p(t,x) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} \exp(i\theta\cdot x)\hat{\mathscr{O}}_p(t,\theta)\,d\theta$.

(d) *If* $d = 1$ *and* $\alpha > \frac{3}{2}$, $P\{\mathbf{D}_1\mathscr{L}_p \in \mathbf{C}([0,T]\colon H_\alpha)\} = 1$ *for* $\gamma < (2\alpha - 3)/2$.

Define

$$\mathbf{B}_{\alpha,2}(f,g)(x) := \begin{cases} \displaystyle\int_{\mathbf{R}^d} \mathbf{B}_\alpha(f(\cdot+z) - f(\cdot), g(\cdot+z) \\ \qquad -g(\cdot))(x)|z|^{-(d+\alpha)}\,dz, & \text{if } 0 < \alpha < 2, \\ \displaystyle\sum_{i,j}(\mathbf{D}_i\mathbf{D}_j f)(x)(\mathbf{D}_i\mathbf{D}_j g)(x), & \text{if } \alpha = 2. \end{cases}$$

For $f, g \in \mathbf{C}_b^2$,

$$\mathbf{A}_\alpha\mathbf{B}_\alpha(f,g) = \mathbf{B}_\alpha(\mathbf{A}_\alpha f, g) + \mathbf{B}_\alpha(f, \mathbf{A}_\alpha g) + \mathbf{B}_{\alpha,2}(f,g). \tag{2.16}$$

Let us introduce $\mathbf{B}_{\alpha,0}(f,g) := fg$ and $\mathbf{B}_{\alpha,1}(f,g) := \mathbf{B}_\alpha(f,g)$. Then for $i = 1, 2$ one defines (formally),

$$\begin{aligned}\mathbf{F}_{\alpha,i}(\mathbf{V},\mathbf{W},f,g)(t) = {}& \mathbf{V}\big(0, \mathbf{B}_{\alpha,i-1}(f,g)\big) + \int_0^t \mathbf{V}\big(s, \mathbf{B}_{\alpha,i-1}(f,g)\big)\,ds \\ & + \mathbf{W}\big(t, \mathbf{B}_{\alpha,i-1}(f,g)\big) - \mathbf{V}\big(t, \mathbf{B}_{\alpha,i-1}(f,g)\big) \\ & + \int_0^t \mathbf{V}\big(s, \mathbf{B}_{\alpha,i-1}(\mathbf{A}_\alpha f, g)\big)\,ds + \int_0^t \mathbf{V}\big(s, \mathbf{B}_{\alpha,i}(f,g)\big)\,ds.\end{aligned}$$

Recall $\mathbf{Y}$ as defined in (2.2).

THEOREM 5. *Assume* $\mathbf{EX}(0)$ *has a bounded Lebesgue density and* $f \in \mathbf{C}_b^3$.

(a) *For each* t *and* $g \in \mathbf{H}_{-\alpha}$, *a.s.*,

$$\int_0^t \mathbf{Y}(s, fg)\,ds = \mathbf{F}_{\alpha,1}(\mathbf{Y}, \mathbf{M}, f, (\mathbf{I} - \mathbf{A}_\alpha)^{-1}g)(t), \tag{2.17}$$

where the last expression of (2.17) *in the expansion of* $\mathbf{F}_{\alpha,1}$ *satisfies*

$$\int_0^t \mathbf{Y}\big(s, \mathbf{B}_\alpha(f,g)\big)\,ds = \mathbf{F}_{\alpha,2}(\mathbf{Y}, \mathbf{M}, f, (\mathbf{I} - \mathbf{A}_\alpha)^{-2}g)(t) \tag{2.18}$$

(b)

$$\sup_{t\le T}\mathbf{E}\left[\left(\int_0^t \mathbf{Y}(s, fg)ds\right)^2\right] \le C(T, f)\|g\|_{-\alpha}^2.$$

(c) *If* (2.18) *is substituted in* (2.17), *then all time integrals and martingale terms on the right-hand side of the resulting single equation are a.s. continuous in* t.

(d) *If* $d < 2\alpha$, *then the previous results hold with* $g = \delta_a$, *the probability measure with unit mass at* $a \in \mathbf{R}^d$.

THEOREM 6. *Assume* $\mathbf{E}[\|\mathbf{X}(0)\|_0^2] < \infty$ *in addition to the assumptions of Theorem* 5. *Then the conclusion of Theorem* 5 *holds with* $\mathbf{X}$ *in place of* $\mathbf{Y}$.

THEOREM 7. *Assume* $\mathbf{X}(0)$ *is Lebesgue measure and* $f \in \mathbf{C}_b^3$. *Then the following hold.*

(a) *For each* t *and* $g \in \mathbf{H}_{-\alpha}$ *with compact support, a.s.,*

$$\int_0^t \mathbf{X}(s, fg)\,ds = t\langle g, f\rangle + \int_0^t \mathbf{Y}(s, fg)\,ds.$$

Here, the second term on the right-hand side has the properties given in Theorem 5 *and in addition* $\langle g, f\rangle$ *is given by the duality between functions locally in* $\mathbf{H}_\alpha$ *and distributions in* $\mathbf{H}_{-\alpha}$ *of compact support.*

(b) *If* $d < 2\alpha$, *then* (a) *holds with* $g = \delta_a$ *where* δ_a *is the probability measure with unit mass at* $a \in \mathbf{R}^d$.

In Adler and Lewin (1992) an analogue of Theorem 7 is proved assuming that either of the following conditions hold:

1. $d < 2\alpha$, $g = \delta_0$, and $f \equiv 1$.
2. $d \le 3$, $\alpha = 2$, $g = \delta_0$, and $f \in C_b^2$ with compact support and $f(0) = 1$.

Adler and Lewin do not use a version of (2.18), and their result is stated in terms of $\mathbf{X}$ rather than $\mathbf{Y}$. Some computations show their formula agrees with ours if $g \in C_c^\infty$, and taking limits gives agreement with $g = \delta_0$. By using integrability properties of $(I-\Delta)^{-1}\delta_0$, continuity in t is also obtained in their paper. To obtain continuity in t in Theorem 5 would require continuity in t for $\mathbf{Y}(t, g)$ with $g \in H_0$ being arbitrary. We don't know if this is true. However, Theorem 5 holds for $g \in H_{-\alpha}$ rather than the particular case $g = \delta_a$ with $d < 2\alpha$, and we have eliminated the compact support assumption on f. Adler and Lewin did not use the Leibnitz formula for $\alpha \neq 2$, which restricted them to the case $f \equiv c$ or $\alpha = 2$ if f is not constant.

THEOREM 8 (Itô Formula). *Let* $\mathbf{X}(0)$ *be Lebesgue measure.*

(a) *If* $g \in \mathbf{H}_{\alpha/2} \cap \mathbf{L}^1(\mathbf{R}^d)$, *then a.s.* $\mathbf{X}(\cdot, g)$ *is continuous and satisfies*

$$\mathbf{X}(\cdot, g) = \mathbf{X}(0, g) + \int_0^{\cdot} \mathbf{Y}(s, \mathbf{A}_\alpha g)\,ds + \mathbf{M}(\cdot, g).$$

(b) *If* $g_i \in \mathbf{H}_{\alpha/2} \cap \mathbf{L}^1(\mathbf{R}^d)$, $1 \le i \le n$ *and* $F \in \mathbf{C}_b^2(\mathbf{R}^n)$, *then a.s. for* $t \ge 0$,

$$\begin{aligned}
&F(\mathbf{X}(t, g_1), \dots, \mathbf{X}(t, g_n)) \\
&\quad = F(\mathbf{X}(0, g_1), \dots, \mathbf{X}(0, g_n)) \\
&\qquad + \int_0^t \nabla F(\mathbf{X}(s, g_1), \dots, \mathbf{X}(s, g_n)) \cdot (\mathbf{Y}(s, \mathbf{A}_\alpha g_1), \dots, \mathbf{Y}(s, \mathbf{A}_\alpha g_n))\, ds \\
&\qquad + \int_0^t \nabla F(\mathbf{X}(s, g_1), \dots, \mathbf{X}(s, g_n)) \cdot (d\mathbf{M}(s, g_1), \dots, d\mathbf{M}(s, g_n)) \\
&\qquad + \tfrac{1}{2} \int_0^t \sum_{i,j} \mathbf{D}_i \mathbf{D}_j F(\mathbf{X}(s, g_1), \dots, \mathbf{X}(s, g_n)) \mathbf{X}(s, g_i g_j)\, ds.
\end{aligned}$$

In Adler and Lewin (1992) an analogue of Theorem 8 is proved in which only $\mathbf{X}$, rather than $\mathbf{Y}$, appears. They assume each $g \in \{g_1, \dots, g_n\}$ satisfies $g \in \mathbf{H}_2$ and

$$\int_{\mathbf{R}^d} |\mathbf{D}_1^{m_1} \cdots \mathbf{D}_d^{m_d} g(x)|\, dx < \infty$$

for $\sum_{i=1}^d m_i \le 2$, which is more restrictive than our conditions. As an example, let $g(x) = I_{[-1,1]}(x_1)\phi(x_2, \dots, x_d)$ where $I_{[-1,1]}(x_1)$ is the indicator function of $[-1, 1]$ and $\phi \in C_c^\infty(\mathbf{R}^{d-1})$. Then $\hat{g}(\theta) = (2\sin(\theta_1)/\theta_1)h(\theta_2, \dots, \theta_d)$ where h is rapidly decreasing in $\theta_2, \dots, \theta_d$. Thus $g \in \mathbf{H}_\gamma$ for $\gamma < 1/2$, and so $g \in \mathbf{H}_{\alpha/2}$ for $\alpha < 1$. Theorem 8(a) provides a semi-martingale representation for $\mathbf{X}(\cdot, g)$ if $\alpha < 1$, but g does not satisfy the regularity conditions of Adler and Lewin.

We now state a maximal inequality for real valued nonnegative processes which is a key ingredient in our proof of Theorems 1 and 3.

THEOREM 9 (Maximal inequality). *Suppose nonnegative processes* $x(\cdot)$, $g(\cdot)$ *with* $g(\cdot) \ge ax(\cdot)$ *for* $a > 0$ *and* $f(\cdot)$ *are adapted to the filtration* $\mathcal{G}_t$ *and satisfy the following S.D.E.*:

$$x(t) = x(0) - \int_0^t g(s)\, ds + \int_0^t f(s)\, ds + m(t),$$

where $m(t)$ *is a continuous* $\mathcal{G}_t$*-local martingale. Assume that* $\mathbf{E}x(0) < \infty$ *and the quadratic variation of m satisfies* $[m](t) = \int_0^t k(s)\, ds$ *where* $k(t) \le h(t)x(t)$. *Assume, in addition, that for a* $\mathcal{G}_t$*-stopping time* τ, *we have*

$$\sup_{t \le T} f(t \wedge \tau) \le b \quad \text{and} \quad \sup_{t \le T} h(t \wedge \tau) \le 2c, \quad \text{for strictly positive constants } b, c.$$

Then

$$\mathbf{P}\Big[\sup_{t \le T} x(t \wedge \tau) \ge q^2\Big] \le \frac{T + \mathbf{E}F(x(0))}{F(q^2)},$$

where $F(v)$ *is the function given by*

$$F(v) = a^{-1} \sum_{n=1}^\infty \frac{(av)^n}{n \prod_{j=0}^{n-1}(b + cj)}.$$

COROLLARY 1 OF THE MAXIMAL INEQUALITY. *If* $\Gamma = \max(b, c)$ *and* $x(0) = 0$ *then*

$$\mathbf{P}\Big[\sup_{t\le T} x(t\wedge\tau) \ge q^2\Big] \le \frac{aT}{(\exp(aq^2/2\Gamma) - 1)}.$$

COROLLARY 2 OF THE MAXIMAL INEQUALITY. *If* $\Gamma = \max(b, c)$ *and* $x(0) = 0$ *then for* $q^2 > 0$, *one has*

$$\mathbf{E}\Big[\sup_{t\le T} x(t\wedge\tau)\Big] \le q^2 + 2T\Gamma\bigg(\exp\bigg(\frac{aq^2}{2\Gamma}\bigg) - 1\bigg)^{-1}.$$

COROLLARY 3 OF THE MAXIMAL INEQUALITY. *Let* $y(t) = \int_0^t \exp(-\beta(t-s))\sqrt{\alpha(s)}\,dW(s)$ *where* $\beta > 0$, $W(t)$ *is a* $\mathscr{G}_t$ *Brownian motion*, α *is nonnegative* $\mathscr{G}_t$ *adapted, and* α *is bounded by a constant* $\Gamma/2$. *Then the following hold*:

(a) $P(\sup_{t\le T} y^2(t) \ge q^2) \le 2\beta T(\exp(\beta q^2/\Gamma) - 1)^{-1}$.
(b) *For* $q^2 > 0$,

$$E\Big[\sup_{t\le T} y^2(t)\Big] \le q^2 + 2T\Gamma(\exp(\beta q^2/\Gamma) - 1)^{-1}.$$

EXAMPLE. We give an example which also contains the proof of Corollary 3 of Theorem 9. The calculations are similar to some of the results of Dawson (1972) which developed a now standard technique for Hilbert space regularity of stochastic partial differential equations driven by space–time white noise. A version of (a) of Corollary 3 is used in Blount (1996) and Bose and Sundar (1997).

Let W be a $\mathscr{G}_t$ Brownian motion and α be nonnegative $\mathscr{G}_t$ adapted and bounded by $\Gamma/2$. Consider for $\beta > 0$,

$$\begin{aligned} y(t) &= \int_0^t \exp(-\beta(t-s))\sqrt{\alpha(s)}\,dW(s) \\ &= -\beta\int_0^t y(s)\,ds + \int_0^t \sqrt{\alpha(s)}\,dW(s), \\ y^2(t) &= -2\beta\int_0^t y^2(s)\,ds + \int_0^t 2y(s)\sqrt{\alpha(s)}\,dW(s) + \int_0^t \alpha(s)\,ds. \end{aligned}$$

Now let $x(t) = y^2(t)$, so that

$$x(t) = -a\int_0^t x(s)\,ds + \int_0^t f(s)\,ds + m(t)$$

where $a = 2\beta$, $f(s) = \alpha(s)$, $\tau = T$, $[m](t) = \int_0^t 4y^2(s)\alpha(s)\,ds = \int_0^t 4x(s)\alpha(s)\,ds$. Also, $b = \Gamma/2$, $c = \Gamma$. Hence by Corollary 2,

$$\mathbf{E}\Big[\sup_{t\le T} x(t)\Big] \le q^2 + 2\Gamma T\bigg(\exp\bigg(\frac{2\beta q^2}{2\Gamma}\bigg) - 1\bigg)^{-1}.$$

For $m \geq 1$, consider

$$y_m(t) = \int_0^t \exp(-\beta_m(t-s))\sqrt{\alpha_m(s)}\, dW_m(s)$$

where $\{W_m\}$ are $\mathscr{G}_t$ Brownian motions, $\{\alpha_m\}$ are $\mathscr{G}_t$-adapted, $\beta_m = m^2$ and $2\sup_{t\leq T}\alpha_m(t) \leq m^{1-\delta}$ for some $0 < \delta \leq 1$. Set $q^2 = m^{-(1+\delta/2)}$.

Then

$$\mathbf{E}\Big[\sup_{t\leq T} y_m^2(t)\Big] \leq m^{-(1+\delta/2)} + 2m^{1-\delta}\big(\exp\big(m^{\delta/2}\big) - 1\big)^{-1}$$

and

$$\mathbf{E}\Big[\sup_{t\leq T}\sum_{m=1}^{\infty} y_m^2(t)\Big] \leq \sum_{m=1}^{\infty}\mathbf{E}\Big[\sup_{t\leq T} y_m^2(t)\Big] < \infty.$$

One also can use Corollary 1 to show

$$\mathbf{P}\bigg\{\sup_{t\leq T} y_m^2(t) \geq \frac{1}{m^{1+\delta/2}} \quad \text{i.o.}\bigg\} = 0.$$

Each approach shows

$$\mathbf{P}\bigg\{\lim_{N\to\infty}\sum_{m=N}^{\infty}\sup_{t\leq T} y_m^2(t) = 0\bigg\} = 1.$$

In Kotelenez (1987) a maximal inequality for non-Gaussian convolution is developed using a different method.

3. Proofs.

PROOF OF THEOREM 9 (Maximal inequality). Define

$$\tau(q^2) = \inf\{t\colon x(t) \geq q^2\}.$$

Now note that F and its derivatives are positive and strictly increasing on $v > 0$. Then

$$\mathbf{P}\Big[\sup_{t\leq T} x(t\wedge\tau) \geq q^2\Big] = \mathbf{P}\big[x\big(T\wedge\tau\wedge\tau(q^2)\big) \geq q^2\big] \leq \frac{\mathbf{E}\big[F(x(T\wedge\tau\wedge\tau(q^2)))\big]}{F(q^2)}.$$

By utilizing the properties of $F(\cdot)$, we now estimate the expectation. Set $\sigma = \tau\wedge\tau(q^2)$ and introduce the martingale $r(t) = \int_0^{t\wedge\sigma} F'(x(s))\, dm(s)$. Then by the Itô lemma, for $t \leq T$,

$$\begin{aligned} F(x(t\wedge\sigma)) &= F(x(0)) + \int_0^{t\wedge\sigma} F'(x(s))[-g(s) + f(s)]\, ds \\ &\quad + \tfrac{1}{2}\int_0^{t\wedge\sigma} F''(x(s))k(s)\, ds + \int_0^{t\wedge\sigma} F'(x(s))\, dm(s) \\ &\leq F(x(0)) + \int_0^{t\wedge\sigma} F'(x(s))[-ax(s) + b]\, ds + \int_0^{t\wedge\sigma} F''(x(s))cx(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&+r(t\wedge\sigma)\\
&= F(x(0)) + \int_0^{t\wedge\sigma} \big[F'(x(s))(b-ax(s)) + cx(s)F''(x(s))\big]\,ds\\
&\quad +r(t\wedge\sigma)\\
&= F(x(0)) + (t\wedge\sigma) + r(t\wedge\sigma).
\end{aligned}
$$

To obtain the last equality, observe that for the particular form of F,

$$cvF''(v) + F'(v)[b-av] \equiv 1.$$

Taking expectations concludes the proof of Theorem 9. □

PROOF OF COROLLARY 1 OF THE MAXIMAL INEQUALITY. For $v > 0$,

$$
\begin{aligned}
F(v) &\geq a^{-1}\sum_{n=1}^{\infty}\frac{(av)^n}{n\Gamma^n n!}\\
&= a^{-1}\sum_{n=1}^{\infty}\frac{(av/\Gamma n^{1/n})^n}{n!}\\
&\geq a^{-1}\Big(\exp\Big(\frac{av}{2\Gamma}\Big)-1\Big)
\end{aligned}
$$

since $\sup_{n\geq 1} n^{1/n} \leq 2$. □

PROOF OF COROLLARY 2 OF THE MAXIMAL INEQUALITY. Let

$$z = \sup_{t\leq T} x(t\wedge\tau) \geq 0.$$

For $q_0^2 > 0$ one has

$$
\begin{aligned}
\mathbf{E}[z] &= \int_0^{\infty}\mathbf{P}(z\geq u)\,du\\
&= \int_0^{q_0^2}\mathbf{P}(z\geq u)\,du + \int_{q_0^2}^{\infty}\mathbf{P}(z\geq u)\,du\\
&\leq q_0^2 + \int_{q_0^2}^{\infty} aT\Big(\exp\Big(\frac{au}{2\Gamma}\Big)-1\Big)^{-1}du\\
&= q_0^2 + \int_{q_0^2}^{\infty} aT\exp\Big(\frac{-au}{2\Gamma}\Big)\Big(1-\exp\Big(\frac{-au}{2\Gamma}\Big)\Big)^{-1}du\\
&\leq q_0^2 + \Big(1-\exp\Big(\frac{-aq_0^2}{2\Gamma}\Big)\Big)^{-1} T\int_{q_0^2}^{\infty} a\exp\Big(\frac{-au}{2\Gamma}\Big)\,du\\
&= q_0^2 + 2T\Gamma\Big(1-\exp\Big(\frac{-aq_0^2}{2\Gamma}\Big)\Big)^{-1}\exp\Big(\frac{-aq_0^2}{2\Gamma}\Big)\\
&= q_0^2 + 2T\Gamma\Big(\exp\Big(\frac{aq_0^2}{2\Gamma}\Big)-1\Big)^{-1}.
\end{aligned}
$$

□

LEMMA 1. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_F$ *and* $T > 0$.

(a)

$$\mathbf{E}\left[\sup_{t\le T} \mathbf{X}(t, \mathbf{1})\right] \le \mathbf{E}[\mathbf{X}(0, \mathbf{1})] + 2\sqrt{T}\sqrt{\mathbf{EX}(0, \mathbf{1})}.$$

(b)

$$\mathbf{E}\left[\sup_{t\le T} |\hat{\mathbf{M}}(t, \theta)|^2\right] \le c(T) < \infty.$$

(c)

$$\mathbf{P}\big\{\mathbf{M} \in \mathbf{C}([0, T]\colon \mathbf{H}_\gamma\big\} = 1 \quad \textit{for } \gamma < -\frac{d}{2}.$$

PROOF OF LEMMA 1. (a) Note that

$$\mathbf{X}(t, \mathbf{1}) = \mathbf{X}(0, \mathbf{1}) + \mathbf{M}(t, \mathbf{1}) \quad \text{and} \quad [\mathbf{M}(\cdot, \mathbf{1})](t) = \int_0^t \mathbf{X}(s, \mathbf{1})\, ds.$$

Hence

$$\mathbf{E}[\mathbf{X}(t, \mathbf{1})] = \mathbf{E}[\mathbf{X}(0, \mathbf{1})] \quad \text{and} \quad \mathbf{E}[\mathbf{M}^2(t, \mathbf{1})] = t\mathbf{E}[\mathbf{X}(0, \mathbf{1})].$$

Using Doob's maximal inequality,

$$\begin{aligned}\mathbf{E}\left[\sup_{t\le T} \mathbf{X}(t, \mathbf{1})\right] &\le \mathbf{E}[\mathbf{X}(0, \mathbf{1})] + \sqrt{4\mathbf{E}[\mathbf{M}^2(T, \mathbf{1})]}\\ &\le \mathbf{E}[\mathbf{X}(0, \mathbf{1})] + 2\sqrt{T}\sqrt{\mathbf{E}[\mathbf{X}(0, \mathbf{1})]}.\end{aligned}$$

Note that

$$\mathbf{E}[|\hat{\mathbf{M}}(t, \theta)|^2] = \mathbf{E}\int_0^t \mathbf{X}(s, |e_{-\theta}|^2)\, ds = \mathbf{E}\int_0^t \mathbf{X}(s, \mathbf{1})\, ds \le C(t).$$

Now the conclusion follows by Doob's inequality.

(b) (c) For $s, t \le T$, one has

$$\|\mathbf{M}(t) - \mathbf{M}(s)\|_\gamma^2 = \mu_\gamma\big(|\hat{\mathbf{M}}(t, \cdot) - \hat{\mathbf{M}}(s, \cdot)|^2\big).$$

The integrand is dominated by $4 \sup_{t\le T} |\hat{\mathbf{M}}(t, \theta)|^2$, which is a.s. integrable w.r.t. μ_γ by (b).

By the dominated convergence theorem and continuity in t of $\hat{\mathbf{M}}(t, \theta)$ then

$$\mathbf{P}\left\{\lim_{t\to s} \|\mathbf{M}(t) - \mathbf{M}(s)\|_\gamma = 0, \quad s, t \le T\right\} = 1.$$

This concludes the proof of Lemma 1. □

To establish the path properties of the convolution integral with respect to the martingale measure $\mathbf{M}$, we apply the maximal inequality to the convolution term in (2.6).

LEMMA 2. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_F$, $\gamma < (\alpha - d)/2$ *and* $T > 0$. *Define*

$$\mathbf{Y}(t) := \int_0^t \mathbf{S}_\alpha(t-s)\, d\mathbf{M}(s). \tag{3.1}$$

(a) *If* $\rho < \infty$ *and* $\tau(\rho) := \inf\{t\colon \mathbf{X}(t, \mathbf{1}) \geq \rho\}$, *then*

$$\mathbf{P}\left\{ \lim_{N\to\infty} \sup_{t\leq T} \int_{|\theta|>N} |\hat{\mathbf{Y}}(t\wedge\tau(\rho), \theta)|^2 \mu_\gamma(d\theta) = 0 \right\} = 1.$$

(b)

$$\mathbf{P}\left\{ \lim_{N\to\infty} \sup_{t\leq T} \int_{|\theta|>N} |\hat{\mathbf{Y}}(t, \theta)|^2 \mu_\gamma(d\theta) = 0 \right\} = 1.$$

(c)

$$\mathbf{P}\{\mathbf{Y} \in \mathbf{C}([0, T]\colon \mathbf{H}_\gamma\} = 1.$$

PROOF. (a) Noting

$$\hat{\mathbf{Y}}(t, \theta) = \int_0^t \exp(-|\theta|^\alpha (t-s))\, d\hat{\mathbf{M}}(s, \theta), \tag{3.2}$$

we have

$$\hat{\mathbf{Y}}(t, \theta) = -\int_0^t |\theta|^\alpha \hat{\mathbf{Y}}(s, \theta)\, ds + \hat{\mathbf{M}}(t, \theta)$$

and

$$\begin{aligned} |\hat{\mathbf{Y}}(t, \theta)|^2 = {} & -2|\theta|^\alpha \int_0^t |\hat{\mathbf{Y}}(t, \theta)|^2\, ds + \int_0^t \mathbf{X}(s, \mathbf{1})\, ds \\ & + \int_0^t \hat{\mathbf{Y}}(s, -\theta)\, d\hat{\mathbf{M}}(s, \theta) + \int_0^t \hat{\mathbf{Y}}(s, \theta)\, d\hat{\mathbf{M}}(s, -\theta). \end{aligned} \tag{3.3}$$

If $B \subset R^d$ is a bounded Borel set then

$$\begin{aligned} \int_B |\hat{\mathbf{Y}}(t, \theta)|^2 \mu_\gamma(d\theta) = {} & -\int_0^t \int_B 2|\theta|^\alpha \hat{\mathbf{Y}}(s, \theta)|^2 \mu_\gamma(d\theta)\, ds \\ & + \mu_\gamma(B) \int_0^t \mathbf{X}(s, \mathbf{1})\, ds \\ & + \int_{[0, t]\times\mathbf{R}^d} \int_B \big[\hat{\mathbf{Y}}(s, \theta) e_{-\theta}(x) + \hat{\mathbf{Y}}(s, -\theta) e_\theta(x)\big] \\ & \times \mu_\gamma(d\theta) \mathbf{M}(ds, dx). \end{aligned}$$

Let us remark that to obtain the last term, we have used a stochastic Fubini theorem (Theorem 2.6) of Walsh (1986). If we replace t by $t\wedge\tau(\rho)$, then the assumptions of Walsh are satisfied if

$$\mathbf{E}\left[\int_B \int_0^{t\wedge\tau(\rho)} \int_{\mathbf{R}^d} |\hat{\mathbf{Y}}(s, \theta)|^2 \mathbf{X}(s, dx)\, ds\, \mu_\gamma(d\theta) \right] < \infty,$$

where the orthogonality of the martingale measure simplifies the integrand in Walsh's statement of the theorem. The expectation is dominated by

$$\rho \int_B \int_0^t \mathbf{E}[|\hat{\mathbf{Y}}(s,\theta)|^2]\, ds\, \mu_\gamma(d\theta) = \\ \rho \mathbf{E}[\mathbf{X}(0,\mathbf{1})] \int_B \int_0^t \frac{(1-\exp(-2|\theta|^\alpha s))}{2|\theta|^\alpha}\, ds\, \mu_\gamma(d\theta) < \infty,$$

since B is bounded. As subsequently shown in proving part (b), we can choose $\rho(n)$ such that $\mathbf{P}(\tau(\rho(n)) \le T \quad \text{i.o.}) = 0$, so we may replace $t \wedge \tau(\rho)$ by t.

If we define

$$y(t) = \int_B |\hat{\mathbf{Y}}(t,\theta)|^2\, \mu_\gamma(d\theta)$$

then the equality (3.1) can be expressed as

$$y(t) = -\int_0^t g(s)\, ds + \int_0^t f(s)\, ds + m(t)$$

with

$$g(s) = 2\int_B |\theta|^\alpha |\hat{\mathbf{Y}}(s,\theta)|^2\, \mu_\gamma(d\theta) \ge 2 \inf_{\theta\in B} |\theta|^\alpha y(s),$$
$$f(s) = \mu_\gamma(B)\mathbf{X}(s,\mathbf{1})$$

and

$$[m](t) = \int_0^t \mathbf{X}\left(s, \left[\int_B \left[\hat{\mathbf{Y}}(s,\theta)e_{-\theta} + \hat{\mathbf{Y}}(s,-\theta)e_\theta\right]\mu_\gamma(d\theta)\right]^2\right) ds.$$

Now

$$\begin{aligned}
&\left[\int_B [\hat{\mathbf{Y}}(s,\theta)e_{-\theta} + \hat{\mathbf{Y}}(s,-\theta)e_\theta]\mu_\gamma(d\theta)\right]^2 \\
&\quad \le \mu_\gamma(B)\int_B [\hat{\mathbf{Y}}(s,\theta)e_{-\theta} + \hat{\mathbf{Y}}(s,-\theta)e_\theta]^2 \mu_\gamma(d\theta) \\
&\quad \le 2\mu_\gamma(B)\int_B [|\hat{\mathbf{Y}}(s,\theta)e_{-\theta}|^2 + |\hat{\mathbf{Y}}(s,-\theta)e_\theta|^2]\mu_\gamma(d\theta) \\
&\quad = 4\mu_\gamma(B)\int_B |\hat{\mathbf{Y}}(s,\theta)|^2\mu_\gamma(d\theta) = 4\mu_\gamma(B)y(s).
\end{aligned}$$

Hence, $k(t)$, the derivative of the quadratic variation of $m(t)$, is dominated above by

$$k(t) \le 4\mu_\gamma(B)\mathbf{X}(t,\mathbf{1})\int_B |\hat{\mathbf{Y}}(t,\theta)|^2\mu_\gamma(d\theta) = 4\mu_\gamma(B)\mathbf{X}(t,\mathbf{1})\, y(t).$$

Thus $h(t) = 4\mu_\gamma(B)\mathbf{X}(t,\mathbf{1})$.

Then, for $\tau = \tau(\rho)$,

$$\sup_{t\le T} f(t\wedge\tau) \le \rho\mu_\gamma(B) \quad \text{and} \quad \sup_{t\le T} h(t\wedge\tau) \le 4\rho\mu_\gamma(B).$$

Setting $b = \rho\mu_\gamma(B)$ and $c = 2\rho\mu_\gamma(B)$, we have identified the parameters needed to apply Corollary 1 of the maximal inequality. So we consider a special B.

Let $\underline{m} \in \mathbf{Z}^d$ be a multi-index. For $B(\underline{m}) := \prod_{i=1}^{n}[m_i - \frac{1}{2}, m_i + \frac{1}{2}]$ with $|\underline{m}| \neq 0$, consider

$$a(\alpha, B(\underline{m})) = 2 \inf_{\theta \in B(\underline{m})} |\theta|^\alpha \quad \text{and} \quad \Gamma = 2\max(b, c).$$

Then there are strictly positive constants $c_1(\alpha, d)$, $c_2(\alpha, d)$, $d_1(\gamma, d)$ and $d_2(\gamma, d)$ such that one has

$$c_1(\alpha, d)|\underline{m}|^\alpha \leq a(\alpha, B(\underline{m})) \leq c_2(\alpha, d)|\underline{m}|^\alpha$$

and

$$\rho d_1(\gamma, d)(1 + |\underline{m}|^{2\gamma}) \leq \Gamma \leq \rho d_2(\gamma, d)(1 + |\underline{m}|^{2\gamma}).$$

Define

$$y_{\underline{m}} := \sup_{t \leq T} \int_{B(\underline{m})} |\mathbf{Y}(t \wedge \tau(\rho), e_\theta)|^2 \mu_\gamma(d\theta).$$

Using Corollary 1 to the maximal inequality, obtain

$$\mathbf{P}\Big[y_{\underline{m}} \geq q^2\Big] \leq T\, c_2(\alpha, d)|\underline{m}|^\alpha \left[\exp\left(\frac{q^2 c_1(\alpha, d)|\underline{m}|^\alpha}{d_2(\gamma, d)\rho(1 + |\underline{m}|^{2\gamma})}\right) - 1\right]^{-1}.$$

If $\alpha - 2\gamma - d > 0$ then choose ϵ such that $0 < \epsilon < \alpha - 2\gamma - d$ and define

$$q^2(\underline{m}) := \frac{1}{|\underline{m}|^{d+\varepsilon}}.$$

Then $\mathbf{P}[y_{\underline{m}} \geq q^2(\underline{m})] \leq a^2(\underline{m})$ where

$$a^2(\underline{m}) = T\, c_2(\alpha, d)|\underline{m}|^\alpha \Big[\exp(c(\alpha, d, \rho, \gamma)|\underline{m}|^\delta) - 1\Big]^{-1}$$

with $c(\alpha, d, \rho, \gamma) > 0$ and $\delta = \alpha - d - \epsilon - 2\gamma > 0$.

Since $\sum_{|\underline{m}| \neq 0} a^2(\underline{m}) < \infty$, by the Borel–Cantelli lemma

$$\mathbf{P}\big[y_m \geq q^2(\underline{m}) \quad \text{i.o.}\big] = 0.$$

Since $\sup_{t \leq T} \int_{|\theta| > N} |\hat{\mathbf{Y}}(t \wedge \tau, \theta)|^2 \mu_\gamma(d\theta) \leq \sum_{|\underline{m}| > N} y_{\underline{m}}$ and $\sum_{|\underline{m}| \neq 0} q^2(\underline{m}) < \infty$ the conclusion (a) follows.

(b) By Lemma 1(a), $\mathbf{P}(\tau(n) \leq T \quad \text{i.o.}) = 0$. Let

$$B_0 = \{\tau(n) \leq T \quad \text{i.o.}\}^c,$$

and, for $n = 1, 2, \ldots$,

$$B_n = \left\{\lim_{N \to \infty} \sup_{t \leq T} \int_{|\theta| > N} |\hat{\mathbf{Y}}(t \wedge \tau(n), \theta)|^2 \mu_\gamma(d\theta) = 0\right\}.$$

Let

$$C = \bigcap_{n=0}^{\infty} B_n \quad \text{and note } \mathbf{P}(C) = 1.$$

For $w \in C$,

$$\lim_{N\to\infty} \sup_{t\le T} \int_{|\theta|>N} |\hat{\mathbf{Y}}(t\wedge\tau(n, w), \theta, w)|^2 \mu_\gamma(d\theta) = 0 \quad \text{for all } n.$$

But $w \in B_0$, so we can choose $n(w)$ such that $\tau(n(w), w) > T$ and the previous limit holds without the stopping time for $w \in C$.

(c)

$$\begin{aligned}\|\mathbf{Y}(s) - \mathbf{Y}(t)\|_\gamma^2 \le \int_{|\theta|\le N} |\hat{\mathbf{Y}}(s, \theta) - \hat{\mathbf{Y}}(t, \theta)|^2 \mu_\gamma(d\theta) \\ + 2 \sup_{u\le T} \int_{|\theta|>N} |\hat{\mathbf{Y}}(u, \theta)|^2 \mu_\gamma(d\theta).\end{aligned} \tag{3.4}$$

Because of part (b) a.s. the second term on the right hand side of the last inequality can be made small for large N. Fix such an N and consider the first term on the right hand side of (3.4). Now applying integration by parts to (3.2) obtain

$$\hat{\mathbf{Y}}(t, \theta) = \hat{\mathbf{M}}(t, \theta) - \int_0^t |\theta|^\alpha \exp(-|\theta|^\alpha (t-u)) \hat{\mathbf{M}}(u, \theta)\, du.$$

The above representation, in addition to the continuity of $\hat{\mathbf{M}}$ in t as in Lemma 1(c), allows us to conclude that this term tends to zero as $s \to t$. This concludes the proof of Lemma 2. □

PROOF OF THEOREM 1. (a) As noted in the introduction, a.s. $\mathbf{X} \in \mathbf{C}([0, T]: \mathbf{H}_\beta)$ for $\beta < -d/2$ and the same holds for $\mathbf{M}$ by Lemma 1. Since (2.5) is an identity for the Fourier transforms of all terms appearing in it, the regularity also holds for the integral term.

(b) Using (3.1), rewrite (2.6) as

$$\mathbf{X}(t) = \mathbf{S}_\alpha(t)\mathbf{X}(0) + \mathbf{Y}(t). \tag{3.5}$$

However, $|\widehat{\mathbf{S}_\alpha(t)\mathbf{X}}(0, \theta)| = |\exp(-t|\theta|^\alpha) \hat{\mathbf{X}}(0, \theta)| \le \mathbf{X}(0, \mathbf{1}) \exp(-t|\theta|^\alpha)$.

Thus $\mathbf{S}_\alpha(t)\mathbf{X}(0) \in \mathbf{C}((0, T]: \mathbf{H}_\gamma) \cap \mathbf{C}([0, T]: \mathbf{H}_\beta)$ for $\beta < -d/2$ and any $\gamma \in \mathbf{R}$. This observation and the conclusion from Lemma 2(c) complete the proof of Theorem 1. □

PROOF OF THEOREM 2. Note, $(\mathbf{I} - \mathbf{A}_\alpha)^{-1}: \mathbf{H}_\gamma \to \mathbf{H}_{\gamma+\alpha}$ is an isomorphism. Thus (2.7) in Theorem 2 is algebraically equivalent to (2.5), and the regularity of $\mathscr{O}$ follows from the regularity of $\mathbf{X}$ and $\mathbf{M}$. This also proves (b) and (d).

The equation (2.8) in Theorem 2 follows from integrating (2.5) and using variation of constants. An application of Fourier transforms gives

$$|\hat{\mathscr{O}}(t, \theta)| \le \Big(\frac{1 - \exp(-|\theta|^\alpha t)}{|\theta|^\alpha}\Big)\Big(\mathbf{X}(0, \mathbf{1}) + \sup_{s\le t} |\hat{\mathbf{M}}(s, \theta)|\Big).$$

Regularity then follows from Lemma 1, which also shows $\sup_{t\le T}|\hat{\mathscr{O}}(t,\theta)|$ is a.s. integrable $d\theta$ for $d = 1$ and $\alpha > 1$. Thus (c) follows from the inverse Fourier transform. □

REMARK. The next six lemmas are needed for the proof of Theorems 3 and 4.

LEMMA 3. *For* $x \in \mathbf{R}^d$, *one has*:

(a) *For* $\frac{d}{2} < p$, $0 < \alpha < 2$, *and* $q = \min(p, (d+\alpha)/2)$

$$\int_{|y-x|\ge 1} \phi_p(y)\frac{1}{|y-x|^{d+\alpha}}\,dy \le C(p,\alpha)\phi_q(x).$$

(b)

$$\int_{|y|<1} |y|(2\wedge|\theta||y|)\frac{1}{|y|^{d+\alpha}}\,dy \le C(p,\alpha)\begin{cases} 1, & \text{if } 0<\alpha<1,\\ 1, & \text{if } 1\le\alpha<2, |\theta|\le 2,\\ \ln|\theta|, & \text{if } \alpha=1, |\theta|>2,\\ |\theta|^{\alpha-1}, & \text{if } 1<\alpha<2, |\theta|>2.\end{cases}$$

PROOF.

(a)
$$\begin{aligned}\int_{|y-x|\ge 1}\frac{1}{(1+|y|^2)^p}\frac{1}{|y-x|^{d+\alpha}}\,dy &= \int_{|y-x|\ge 1, |y|\le|x|/2}\frac{1}{(1+|y|^2)^p}\frac{1}{|y-x|^{d+\alpha}}\,dy\\ &\quad+\int_{|y-x|\ge 1, |y|>|x|/2}\frac{1}{(1+|y|^2)^p}\frac{1}{|y-x|^{d+\alpha}}\,dy\\ &\le \frac{C}{(1+|x|)^{d+\alpha}}\int_{|y-x|\ge 1, |y|\le|x|/2}\frac{1}{(1+|y|^2)^p}\,dy\\ &\quad+\frac{C}{(1+|x|^2)^p}\int_{|y-x|\ge 1, |y|>|x|/2}\frac{1}{|y-x|^{d+\alpha}}\,dy\\ &\le \frac{C}{(1+|x|^2)^q}.\end{aligned}$$

This concludes the proof of part (a).

(b) Let

$$I = \int_{|y|<1}|y|(2\wedge|\theta||y|)\frac{1}{|y|^{d+\alpha}}\,dy.$$

Then

$$I \le \begin{cases} 2\int_{|y|<1}|y|^{1-(d+\alpha)}\,dy = C\int_0^1 r^{1-(d+\alpha)}r^{d-1}\,dr, & \text{if } 0<\alpha<1,\\ 2\int_{|y|<1}|y|^{1-(d+\alpha)}|y|\,dy = C\int_0^1 r^{2-(d+\alpha)}r^{d-1}\,dr, & \text{if } 1\le\alpha<2, |\theta|\le 2,\\ \int_{|y|<1}|y|^{1-(d+\alpha)}[2\wedge|\theta||y|]\,dy, & \text{if } 1\le\alpha<2, |\theta|>2.\end{cases}$$

Now consider the last case when $|\theta| > 2$ and $1 \le \alpha < 2$.

$$\begin{aligned}\int_{|y|<1} |y|(2\wedge|\theta||y|)\frac{1}{|y|^{d+\alpha}}\,dy &= C\int_0^{\frac{2}{|\theta|}} r^{1-(d+\alpha)}(2\wedge|\theta|r)r^{d-1}\,dr \\ &\quad + C\int_{\frac{2}{|\theta|}}^1 r^{1-(d+\alpha)}(2\wedge|\theta|r)r^{d-1}\,dr \\ &\le C|\theta|\int_0^{\frac{2}{|\theta|}} r^{1-\alpha}dr + 2C\int_{\frac{2}{|\theta|}}^1 r^{-\alpha}\,dr.\end{aligned}$$

Thus

$$I \le \begin{cases} C\ln|\theta|, & \text{if } \alpha = 1, |\theta| > 2, \\ C|\theta|^{\alpha-1}, & \text{if } 1 < \alpha < 2, |\theta| > 2. \end{cases}$$

This completes the proof of Lemma 3. □

LEMMA 4. *Assume* $p > 0$. *Then*:

(a) *For* $n = 2m,\ m \ge 1$,

$$\mathbf{D}_i^n \phi_p(x) = \phi_{p+n/2}(x)\sum_{l=0}^{n/2} c(p,l)\phi_l(x)x_i^{2l}.$$

For $n = 2m - 1,\ m \ge 1$,

$$\mathbf{D}_i^n \phi_p(x) = \phi_{p+n/2}(x)\sum_{l=0}^{(n-1)/2} c(p,l)\phi_{l+1/2}(x)x_i^{2l+1}.$$

(b) *For* $m = (m_1, \ldots, m_d) \in \mathbf{Z}_+^d$,

$$\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d}\phi_p(x) = \phi_q(x)g_{m,p}(x),$$

where $q = p + (\sum_1^d m_i/2)$ *and* $|g_{m,p}(x)| \le c(m,p)$.

PROOF. Note: $\mathbf{D}_i\phi_p(x) = -2px_i\phi_{p+1}(x)$. (a) then follows by induction. Note: $\phi_{a+b}(x) = \phi_a(x)\phi_b(x)$, and let n_1 be even and n_2 be odd. Then from (a),

$$\begin{aligned}\mathbf{D}_i^{n_1}\mathbf{D}_j^{n_2}\phi_p(x) &= \phi_{p+\frac{n_1+n_2}{2}}(x)\left(\sum_{l=0}^{\frac{n_1}{2}} c_1(p,l)\phi_l(x)x_i^{2l}\right) \\ &\quad \times\left(\sum_{l=0}^{(n_2-1)/2} c_2(p,l)\phi_{l+\frac{1}{2}}(x)x_j^{2l+1}\right).\end{aligned}$$

The analogous calculations for $\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d}\phi_p(x)$ prove (b). □

LEMMA 5. *Let* $p > d/2$ *and* $0 < \alpha < 2$.

(a) *If* $f \in \mathbf{C}_p^2$, *then*

$$|\mathbf{A}_\alpha f(x)| \le C(p,\alpha)\Big[\Big|\sup_{|z|<1}\max_{i,j}|\mathbf{D}_i\mathbf{D}_j f(\cdot+z)|\Big|_p \phi_p(x) + |f|_p\phi_q(x)\Big],$$

where $q = \min(p, (d+\alpha)/2)$.

(b) *If* $f \in \mathbf{C}_b^k$ *with* $k = 2 + \sum_{i=1}^d m_i$ *then*

$$\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d}\mathbf{A}_\alpha f = \mathbf{A}_\alpha\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d} f.$$

PROOF. Denoting the Hessian matrix by $\mathbf{D}^2 f$, use symmetry along with an integral form of Taylor's theorem to obtain

$$\mathbf{A}_\alpha f(x) = \int_{|y|<1}\int_0^1\int_0^1 \big(y\cdot\mathbf{D}^2 f(x+sty)y\big)t\,ds\,dt|y|^{-(d+\alpha)}\,dy$$
$$+ \int_{|y|\ge 1} f(x+y)|y|^{-(d+\alpha)}\,dy - f(x)\int_{|y|\ge 1}|y|^{-(d+\alpha)}\,dy.$$

The second integral is dominated by $|f|_p\int_{|y|\ge 1}\phi_p(x+y)|y|^{-(d+\alpha)}\,dy$, so now apply Lemma 3(a). It is clear how to dominate the remaining terms and that $\mathbf{A}_\alpha$ commutes with the derivatives. □

Let us introduce the notation

$$I(\alpha,\theta) = \begin{cases} 1, & \text{if } 0<\alpha<1,\\ 1, & \text{if } 1\le\alpha<2,\ |\theta|\le 2,\\ \ln|\theta|, & \text{if } \alpha=1,\ |\theta|>2,\\ |\theta|^{\alpha-1}, & \text{if } 1<\alpha\le 2,\ |\theta|>2.\end{cases}$$

LEMMA 6. *Let* $p > d/2$. *For* $f \in \mathbf{C}_b^1$, *define*

$$\mathbf{D}_{\alpha,\theta} f(x) = \begin{cases} \int_{\mathbf{R}^d}(e_{-\theta}(y)-1) & \\ \quad\times(f(x+y)-f(x))|y|^{-(d+\alpha)}\,dy, & \textit{if } 0<\alpha<2,\\ -2i\theta\cdot\nabla f(x), & \textit{if } \alpha=2.\end{cases} \tag{3.6}$$

(a) *If* $0<\alpha<2$ *and* $f \in \mathbf{C}_p^1$, *then*

$$|\mathbf{D}_{\alpha,\theta} f(x)| \le C(p,\alpha)\Big[I(\alpha,\theta)\Big|\sup_{|z|<1}\max_i|\mathbf{D}_i f(\cdot+z)|\Big|_p \phi_p(x) + |f|_p\,\phi_q(x)\Big],$$

where $q = \min(p,(d+\alpha)/2)$.

(b) *If* $f \in \mathbf{C}_b^k$ *with* $k = 1+\sum_{i=1}^d m_i$ *then*

$$\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d}\mathbf{D}_{\alpha,\theta} f = \mathbf{D}_{\alpha,\theta}\mathbf{D}_1^{m_1}\cdots\mathbf{D}_d^{m_d} f.$$

(c) *If* $f \in \mathbf{C}_p^1$, *then*

$$|\mathbf{D}_{2,\theta} f(x)| \le C|\theta|\max_i|\mathbf{D}_i f|_p\,\phi_p(x).$$

PROOF. For $0 < \alpha < 2$,

$$\mathbf{D}_{\alpha,\theta}f(x) = \int_{|y|<1}(e_{-\theta}(y)-1)\int_0^1(\nabla f(x+ty))\cdot y)\,dt|y|^{-(d+\alpha)}\,dy + \int_{|y|\geq 1}(e_{-\theta}(y)-1)(f(x+y)-f(x))|y|^{-(d+\alpha)}\,dy.$$

The first term is dominated by

$$C\left|\sup_{|z|<1}\max_i|\mathbf{D}_i f(\cdot+z)|\right|_p \phi_p(x)\int_{|y|<1}|y|(2\wedge|\theta||y|)|y|^{-(d+\alpha)}\,dy,$$

and now Lemma 3(b) can be applied to the integral. The remaining term can be bounded as in the proof of Lemma 5. The proof of parts (b) and (c) are now clear. □

For $f \in \mathbf{C}_b^2$, note that

$$\mathbf{R}_{\alpha,\theta}f = \mathbf{A}_\alpha f + \mathbf{D}_{\alpha,\theta}f.$$

With $n = 1$ and a Schwarz function in place of ϕ_p, the analogue of Lemma 7(a) is proved in Dawson and Gorostiza (1988).

LEMMA 7. *If $p > d/2$ and $\alpha = 2$, or $d/2 < p \leq (d+\alpha)/2$ and $0 < \alpha < 2$, then*:

(a) $\left|\mathbf{A}_\alpha^n\phi_p\right|_p \leq C(\alpha, p, n)$;
(b) $\left|\mathbf{D}_{\alpha,\theta}^n\phi_p\right|_p \leq C(\alpha, p, n)I^n(\alpha,\theta)$;
(c) $\left|\mathbf{R}_{\alpha,\theta}^n\phi_p\right|_p \leq C(\alpha, p, n)I^n(\alpha,\theta)$.

PROOF. The proof follows from Lemmas 4, 5, 6 and basic computations. For $f \in \mathbf{C}_b^2$, note

$$\mathbf{A}_\alpha(e_{-\theta}f) = (\mathbf{A}_\alpha e_{-\theta})f + e_{-\theta}\mathbf{R}_{\alpha,\theta}f.$$

Putting $\mathbf{R}_{\alpha,\theta}^n\phi_p$ in place of f shows, for $n = 0, 1, 2, \dots$

$$\mathbf{A}_\alpha(e_{-\theta}\mathbf{R}_{\alpha,\theta}^n\phi_p) = -|\theta|^\alpha e_{-\theta}\mathbf{R}_{\alpha,\theta}^n\phi_p + e_{-\theta}\mathbf{R}_{\alpha,\theta}^{n+1}\phi_p. \tag{3.7}$$

For $\nu \in \mathscr{M}_F$ define $\mathbf{G}_{\alpha,p,n}\nu$ by

$$\left(\widehat{\mathbf{G}_{\alpha,p,n}\nu}\right)(\theta) = \nu(e_{-\theta}\phi_p^{-1}\mathbf{R}_{\alpha,\theta}^n\phi_p). \tag{3.8}$$

Similarly

$$\left(\widehat{\mathbf{G}_{\alpha,p,n}\mathbf{M}_p(t)}\right)(\theta) = \mathbf{M}_p(t, e_{-\theta}\phi_p^{-1}\mathbf{R}_{\alpha,\theta}^n\phi_p). \tag{3.9}$$

LEMMA 8. *Assume* $\mathbf{EX}(0) \in \mathscr{M}_p$.

(a) $$\mathbf{E}\Big[\sup\nolimits_{t\le T}\mathbf{X}_p(t,\mathbf{1})\Big] \le C(T)\mathbf{E}[\mathbf{X}_p(0,\mathbf{1})],$$

(b) $$\mathbf{E}\Big[\sup\nolimits_{t\le T}|\hat{\mathbf{M}}_p(t,\theta)|^2\Big] \le C(T) < \infty,$$

(c) $$\mathbf{P}\Big\{\mathbf{M}_p \in \mathbf{C}\big([0,T]:\mathbf{H}_\gamma\big)\Big\} = 1 \quad \textit{for } \gamma < -\frac{d}{2},$$

(d) $$\big|(\widehat{\mathbf{G}_{\alpha,p,n}\nu})(\theta)\big| \le \nu(\mathbf{1})C(\alpha,p,n)I^n(\alpha,\theta),$$

(e) $$\mathbf{E}\Big[\sup\nolimits_{s\le t}\big|(\widehat{\mathbf{G}_{\alpha,p,n}\mathbf{M}_p})(s,\theta)\big|^2\Big] \le C(t,\alpha,p,n)I^n(\alpha,\theta).$$

PROOF. Note that

$$\mathbf{X}_p(t,\mathbf{1}) = \mathbf{X}_p(0,\mathbf{1}) + \int_0^t \mathbf{X}(s,\mathbf{A}_\alpha\phi_p)\,ds + \mathbf{M}_p(t,\mathbf{1})$$

and

$$[\mathbf{M}_p(\cdot,\mathbf{1})](t) = \int_0^t \mathbf{X}_p(s,\phi_p)\,ds.$$

Using Lemma 7(a),

$$\mathbf{E}[\mathbf{X}_p(t,\mathbf{1})] \le \mathbf{E}(\mathbf{X}_p(0,\mathbf{1})) + c\int_0^t \mathbf{E}[\mathbf{X}_p(s,\mathbf{1})]\,ds$$

and

$$\mathbf{E}\Big[\sup_{t\le T}\mathbf{X}_p(t,\mathbf{1})\Big] \le \mathbf{E}[\mathbf{X}_p(0,\mathbf{1})] + c\int_0^T \mathbf{E}[\mathbf{X}_p(s,\mathbf{1})]\,ds + 2\sqrt{\mathbf{E}[\sup_{t\le T}\mathbf{M}_p^2(t,\mathbf{1})]}.$$

Using Gronwall's inequality, the proof of (a), (b) and (c) can be completed with calculations similar to the proof of Lemma 1.

Proof of (d) follows from Lemma 7 and the proof of (e) follows from Lemma 7, Doob's inequality and part (a). □

PROOF OF THEOREM 3. From (2.12) we obtain (2.13) which holds in $\mathbf{C}([0,T]:\mathbf{H}_\gamma)$ for $\gamma < -d/2$ since this fact is true for all but the integral term and the equation is an identity.

Applying variation of constants gives (2.15) which we will also write as

$$\mathbf{X}_p(t) = \mathbf{S}_\alpha(t)\mathbf{X}_p(0) + \mathbf{Y}_p(t) + \mathbf{Z}_p(t). \tag{3.10}$$

Now the fact $\mathbf{P}\{\mathbf{Y}_p \in \mathbf{C}([0, T] : \mathbf{H}_\gamma)\} = 1$ for $\gamma < (\alpha - d)/2$ holds by applying the proof of Lemma 2 to $\mathbf{Y}_p$ in place of $\mathbf{Y}$ with minor notational changes such as using $\mathbf{M}_p$ and $\mathbf{X}_p(s, \mathbf{1})$ in place of $\mathbf{M}$ and $\mathbf{X}(s, \mathbf{1})$.

The regularity given for (2.14) holds with $\mathbf{S}_\alpha(t)\mathbf{X}_p(0) + \mathbf{Y}_p(t)$ in place of $\mathbf{X}_p$ and we need only to consider $\mathbf{Z}_p(t)$ where

$$\hat{\mathbf{Z}}_p(t, \theta) = \int_0^t \exp(-|\theta|^\alpha (t - s))(\widehat{\mathbf{G}_{\alpha, p, 1}\mathbf{X}_p})(s, \theta)\, ds$$

and, by Lemma 8,

$$|\widehat{\mathbf{Z}}_p(t, \theta)| \le C(\alpha, p)\left(\sup_{s \le t} \mathbf{X}_p(s, \mathbf{1})\right) I(\alpha, \theta) \frac{(1 - \exp(-|\theta|^\alpha t))}{|\theta|^\alpha}.$$

This gives the regularity for $\mathbf{X}_p$ by Lemma 8 and the definition of $I(\alpha, \theta)$. This completes the proof. □

PROOF OF THEOREM 4. From (2.9), (3.8) and (3.9) we have

$$\begin{aligned}\mathbf{G}_{\alpha, p, n}\mathbf{X}_p(t) &= \mathbf{G}_{\alpha, p, n}\mathbf{X}_p(0) + \int_0^t \mathbf{A}_\alpha \mathbf{G}_{\alpha, p, n}\mathbf{X}_p(s)\, ds \\ &\quad + \int_0^t \mathbf{G}_{\alpha, p, n+1}\mathbf{X}_p(s)\, ds + \mathbf{G}_{\alpha, p, n}\mathbf{M}_p(t).\end{aligned}$$

Integrating this and using the variation of constants we have

$$\begin{aligned}\int_0^t \mathbf{G}_{\alpha, p, n}\mathbf{X}_p(s)\, ds &= \int_0^t \mathbf{S}_\alpha(t - s)\mathbf{G}_{\alpha, p, n}[\mathbf{X}_p(0) + \mathbf{M}_p(s)]\, ds \\ &\quad + \int_0^t \mathbf{S}_\alpha(t - s) \int_0^s \mathbf{G}_{\alpha, p, n+1}\mathbf{X}_p(u)\, du\, ds.\end{aligned} \tag{3.11}$$

Noting $\mathbf{G}_{\alpha, p, 1} = \mathbf{G}_{\alpha, p}$ and setting $n = 0$ in (3.11) yields (2.15).

Applying (3.11) to the last integral in (2.15), we obtain

$$\begin{aligned}\mathscr{O}_p(t) &= \int_0^t \mathbf{S}_\alpha(t - s)[\mathbf{X}_p(0) + \mathbf{M}_p(s)]\, ds \\ &\quad + \int_0^t \int_0^s \mathbf{S}_\alpha(t - u)\mathbf{G}_{\alpha, p, 1}[\mathbf{X}_p(0) + \mathbf{M}_p(u)]\, du\, ds \\ &\quad + \int_0^t \int_0^s \mathbf{S}_\alpha(t - u) \int_0^u \mathbf{G}_{\alpha, p, 2}\mathbf{X}_p(v)\, dv\, du\, ds.\end{aligned}$$

Thus,

$$\begin{aligned}|\hat{\mathscr{O}}_p(t, \theta)| &\le C[\mathbf{X}_p(0, \mathbf{1}) + \sup_{s \le t} |\hat{\mathbf{M}}_p(s, \theta)|] \frac{(1 - \exp(-|\theta|^\alpha t))}{|\theta|^\alpha} \\ &\quad + C\left[I(\alpha, \theta)\mathbf{X}_p(0, \mathbf{1}) + \sup_{s \le t} |\widehat{\mathbf{G}_{\alpha, p, 1}\mathbf{M}_p}(s, \theta)| + I^2(\alpha, \theta) \sup_{s \le t} \mathbf{X}_p(s, \mathbf{1}) t \right] \\ &\quad \times \exp(-|\theta|^\alpha t) \frac{(\exp(|\theta|^\alpha t) - 1 - |\theta|^\alpha t)}{|\theta|^{2\alpha}}.\end{aligned}$$

By Lemma 8, (a) and (b) hold.

The previous estimate shows that almost surely $\sup_{t\le T}|\hat{\mathscr{O}}_p(t,\theta)|$ is integrable with respect to $d\theta$ if $d=1$ and $1<\alpha\le 2$. Then (c) follows by applying the inverse Fourier transform. Finally (d) follows from (b). □

LEMMA 9. *Assume that* $\mathbf{EX}(0)$ *has a bounded density with respect to Lebesgue measure. Then*:

(a) *For each* $g\in\mathbf{H}_0$, $\mathbf{M}(\cdot,g)$ *is a.s. continuous and*

$$\mathbf{E}\Big[\sup_{t\le T}\mathbf{M}(t,g)^2\Big]\le c(T)\|g\|_0^2;$$

(b) $$\sup_{t\le T}\mathbf{E}\big[\mathbf{Y}(t,g)^2\big]\le c(T)\|g\|_{-\alpha/2}^2.$$

(c) *For each* $g\in\mathbf{H}_{-\alpha/2}$,

$$\mathbf{E}\bigg[\Big(\int_0^t|\mathbf{Y}(s,g)|\,ds\Big)^2\bigg]\le c(t)\|g\|_{-\alpha/2}^2,$$

and $\int_0^{\cdot}\mathbf{Y}(s,g)\,ds$ *is a.s. continuous.*

(d) *For each* $g\in\mathbf{H}_{\alpha/2}$, *a.s.*

$$\mathbf{Y}(\cdot,g)=\int_0^{\cdot}\mathbf{Y}(s,\mathbf{A}_\alpha g)\,ds+\mathbf{M}(\cdot,g)$$

and is continuous. Also,

$$\mathbf{E}\Big[\sup_{t\le T}\mathbf{Y}(t,g)^2\Big]\le c(T)\|g\|_{\alpha/2}^2.$$

PROOF. If $\mathbf{EX}(0)$ has a bounded density $f(x)$, then, for $t>0$, $\mathbf{E}[\mathbf{X}(t)]$ has a density $f(t,x)=\int_{\mathbf{R}^d}p_\alpha(t,x-y)f(y)\,dy$ where $p_\alpha(t,\cdot)$ is the transition density of the symmetric stable process of index α. Thus $\sup_x f(t,x)\le\sup_x f(x)$.

For a Schwarz function g, note that

$$\mathbf{E}[\mathbf{M}(t,g)^2]=\int_0^t\mathbf{EX}(s,g^2)\,ds\le ct\|g\|_0^2.$$

Since $\mathbf{M}(\cdot,g)$ is continuous, Doob's inequality and a standard approximation argument prove (a).

Consider

$$\begin{aligned}\mathbf{E}[\mathbf{Y}(t,g)^2]&=\int_0^t\mathbf{EX}\big(s,(\mathbf{S}_\alpha(t-s)g)^2\big)\,ds\\&\le c\int_0^t\|\mathbf{S}_\alpha(t-s)g\|_0^2\,ds\\&=c\int_{\mathbf{R}^d}|\hat g(\theta)|^2\Big(\frac{1-\exp(-2|\theta|^\alpha t)}{2|\theta|^\alpha}\Big)\,d\theta.\end{aligned}$$

This proves (b).

If $g \in \mathbf{H}_{-\alpha/2}$, then using (b) and Cauchy–Schwarz,

$$\mathbf{E}\left[\left(\int_0^t |\mathbf{Y}(s,g)|\,ds\right)^2\right] \le \mathbf{E}\left[t\int_0^t |\mathbf{Y}(s,g)|^2\,ds\right] \le c(t)\|g\|^2_{-\alpha/2}.$$

Thus a.s. $|\mathbf{Y}(s,g)|$ is locally integrable with respect to ds, which proves (c).

Now (d) holds for a Schwarz function g so that the result holds as stated by taking limits and using (a), (b) and (c). □

LEMMA 10.

(a) *If* $f \in \mathbf{C}_b^1$, *then*

$$\|\mathbf{B}_\alpha(f,g)\|_0 \le c(f)\|\mathbf{A}_{\alpha/2}g\|_0.$$

(b)
$$\int_{\mathbf{R}^d} \mathbf{B}_\alpha(g,g)(x)\,dx = c\|\mathbf{A}_{\alpha/2}g\|_0^2.$$

(c) *If* $f \in \mathbf{C}_b^2$, *then*

$$\|\mathbf{B}_{\alpha,2}(f,g)\|_0 \le c(f)\|\mathbf{A}_\alpha g\|_0.$$

(d)
$$\int_{\mathbf{R}^d} \mathbf{B}_{\alpha,2}(g,g)(x)\,dx = c\|\mathbf{A}_\alpha g\|_0^2.$$

PROOF. For $\alpha = 2$ the proof is straightforward, so we assume $\alpha < 2$. The proofs of (c) and (d) are more difficult and make clear how to prove (a) and (b). So we only complete the proofs of (c) and (d).

Note that for fixed x, $\mathbf{B}_\alpha(f,g)(x)$ and $\mathbf{B}_{\alpha,2}(f,g)(x)$ define nonnegative bilinear forms, and for f, g real, Cauchy–Schwarz implies

$$|\mathbf{B}_\alpha(f,g)(x)|^2 \le \mathbf{B}_\alpha(f,f)(x)\mathbf{B}_\alpha(g,g)(x);$$
$$|\mathbf{B}_{\alpha,2}(f,g)(x)|^2 \le \mathbf{B}_{\alpha,2}(f,f)(x)\mathbf{B}_{\alpha,2}(g,g)(x).$$

Thus it suffices to show $\sup_x \mathbf{B}_\alpha(f,f)(x) < \infty$, $\sup_x \mathbf{B}_{\alpha,2}(f,f)(x) < \infty$, $\int_{\mathbf{R}^d} \mathbf{B}_\alpha(g,g)(x)dx = c\|\mathbf{A}_{\frac{\alpha}{2}}g\|_0^2$, and $\int_{\mathbf{R}^d} \mathbf{B}_{\alpha,2}(g,g)(x)dx = c\|\mathbf{A}_\alpha g\|_0^2$.

Let $\mu(dz) = |z|^{-(d+\alpha)}dz$ and consider

$$\mathbf{B}_{\alpha,2}(f,f)(x) = \int_{\mathbf{R}^d\times\mathbf{R}^d} \left[f(x+y+z)-f(x+y)-\big(f(x+z)-f(x)\big)\right]^2 \mu(dy)\mu(dz).$$

If $f \in \mathbf{C}_b^2$, the integrand is dominated by $c\min(1,|y|^2,|z|^2,|y|^2|z|^2)$. Thus

$$\begin{aligned}\mathbf{B}_{\alpha,2}(f,f)(x) &\le c\iint_{|z|<1,\,|y|<1} |z|^2|y|^2\mu(dz)\mu(dy) + c\iint_{|z|\ge 1,\,|y|\ge 1}\mu(dz)\mu(dy)\\ &\quad + c\iint_{1\le|z|<\infty,|y|<1} |y|^2\mu(dy)\mu(dz)\\ &< \infty.\end{aligned}$$

Applying Fubini's theorem and the Fourier transform show

$$\begin{aligned}\int_{\mathbf{R}^d} \mathbf{B}_{\alpha,2}(g,g)(x)\,dx &= \int_{\mathbf{R}^d\times\mathbf{R}^d}\int_{\mathbf{R}^d} |e_\theta(z)-1|^2|e_\theta(y)-1|^2|\hat{g}(\theta)|^2\,d\theta\,\mu(dz)\,\mu(dy)\\ &= 4\int_{\mathbf{R}^d}|\theta|^{2\alpha}\,|\hat{g}(\theta)|^2\,d\theta\\ &= 4(2\pi)^{d/2}\|\mathbf{A}_\alpha g\|_0^2.\end{aligned}$$

Note that

$$\int_{\mathbf{R}^d} |e_\theta(z)-1|^2\mu(dz) = \mathbf{B}_\alpha(e_{-\theta}, e_\theta)(0) = 2|\theta|^\alpha$$

by using the Leibnitz formula. □

PROOF OF THEOREM 5. Let $g \in \cap_\gamma \mathbf{H}_\gamma$. Then $(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g \in \cap_\gamma\mathbf{H}_\gamma$, and $\mathbf{A}_\alpha(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g \in \cap_\gamma\mathbf{H}_\gamma$, and applying the Leibnitz formula (2.10) to $\mathbf{A}_\alpha\big(f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g\big)$ with Lemma 10 shows $f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g \in \mathbf{H}_\alpha$. By Lemma 9 (d),

$$\mathbf{Y}\big(t, f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g\big) = \int_0^t \mathbf{Y}\big(s, \mathbf{A}_\alpha(f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g\big)\,ds + \mathbf{M}\big(t, f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g\big).$$

Expanding $\mathbf{A}_\alpha(f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)$ by the Leibnitz formula (2.10), then adding to both sides of the previous equation the term $\int_0^t \mathbf{Y}(s, f(\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)\,ds$ and using basic algebra prove (2.17) with $g \in \cap_\gamma\mathbf{H}_\gamma$. Note that the algebraic manipulations involve functions in $\mathbf{H}_0$ and are well defined.

Again using Lemma 10, the Leibnitz formula (2.16) applied to $\mathbf{A}_\alpha\mathbf{B}_\alpha(f, (\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)$, and Lemma 9(d), one obtains

$$\begin{aligned}\mathbf{Y}\big(t, \mathbf{B}_\alpha(f, (\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)\big) &= \int_0^t \mathbf{Y}\big(s, \mathbf{A}_\alpha\mathbf{B}_\alpha(f, (\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)\big)\,ds\\ &\quad+\mathbf{M}\big(t, \mathbf{B}_\alpha(f, (\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)\big).\end{aligned}$$

Now applying the Leibnitz formula (2.16) and basic algebra gives (2.18) with $g \in \cap_\gamma\mathbf{H}_\gamma$. The proof is now complete using Lemma 9 and Lemma 10 and by the denseness of $\cap_\gamma\mathbf{H}_\gamma$ in $\mathbf{H}_{-\alpha}$. □

PROOF OF THEOREM 6. Consider $\mathbf{X}(t) = \mathbf{S}_\alpha(t)\mathbf{X}(0) + \mathbf{Y}(t)$ with $\mathbf{X}(0) \in \mathbf{H}_0$. Letting $\mathbf{V}(t) = \mathbf{S}_\alpha(t)\mathbf{X}(0)$, note that

$$\|\mathbf{V}(t)\|_0 = \|\mathbf{X}(0)\|_0 \quad\text{and}\quad \mathbf{V}(t,g) = \mathbf{X}(0,g) + \int_0^t \mathbf{V}(s, \mathbf{A}_\alpha g)\,ds$$

for $g \in \mathbf{H}_\alpha$. A simpler version of the argument given for $\mathbf{Y}$ shows, for $g \in \cap_\gamma\mathbf{H}_\gamma$,

$$\int_0^t \mathbf{V}\big(s, fg\big)\,ds = \mathbf{F}_{\alpha,1}(\mathbf{V}, \mathbf{0}, f, (\mathbf{I}-\mathbf{A}_\alpha)^{-1}g)(t)$$

and

$$\int_0^t \mathbf{V}\big(s, \mathbf{B}_\alpha(f,g)\big)\,ds = \mathbf{F}_{\alpha,2}(\mathbf{V}, \mathbf{0}, f, (\mathbf{I}-\mathbf{A}_\alpha)^{-2}g)(t).$$

Now

$$\sup_{t\leq T}\mathbf{E}\bigg[\bigg(\int_0^t \mathbf{V}(s, fg)\,ds\bigg)^2\bigg] \leq C(T, f)\|g\|_{-\alpha}^2$$

holds by using Lemma 10 and the bound on $\|\mathbf{V}(t)\|_0$. Since $\mathbf{X}(t) = \mathbf{V}(t) + \mathbf{Y}(t)$, the theorem easily follows by applying Theorem 5 to $\mathbf{Y}$. □

PROOF OF THEOREM 7. Since $\mathbf{S}_\alpha(t)(\mathbf{1}) = \mathbf{1}$, one has

$$\int_0^t \mathbf{X}(s, fh)\,ds = t\mathbf{X}(0, fh) + \int_0^t \mathbf{Y}(s, fh)\,ds$$

for $h \in \mathbf{C}_c^\infty$. If $g \in \mathbf{H}_{-\alpha}$ with compact support, we can choose $h_n \in \mathbf{C}_c^\infty$ with $\|g - h_n\|_{-\alpha} \to 0$. Since f is locally in $\mathbf{H}_2$, (a) then follows from

$$\mathbf{X}(0, fh_n) = \int_{\mathbf{R}^d} f(x)h_n(x)\,dx \to \langle g, f\rangle,$$

together with Theorem 5 applied to $\mathbf{Y}$. Note that (b) is a special case of (a). □

PROOF OF THEOREM 8. Part (a) follows from Lemma 9(d) and the invariance of the Lebesgue measure.

To prove (b), we need to show $\mathbf{X}(s, g_i g_j)$ is a.s. locally integrable with respect to ds. The result then follows by the Itô formula, (a) and Lemma 10. By Fubini's theorem, it suffices to show $\mathbf{E}[\mathbf{X}(t, f)] = \mathbf{X}(0, f)$ for $f \in \mathbf{L}^1(\mathbf{R}^d)$ and $f \geq 0$. Let $f_n(x) = (f(x)\wedge n)\mathbf{I}_{[0,n]}(x)$. Then $f_n(x) \uparrow f(x)$ for all x and $f_n \in \mathbf{L}^2(\mathbf{R}^d)$. Using Lemma 9, $\mathbf{E}[\mathbf{X}(t, f_n)] = \mathbf{X}(0, f_n) + \mathbf{E}[\mathbf{Y}(t, f_n)] = \mathbf{X}(0, f_n)$, and the proof is completed by the monotone convergence theorem. □

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