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CONTINUUM LIMIT FOR SOME GROWTH MODELS II[1]

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We continue our investigations on a class of growth models introduced in a previous paper. Given a nonnegative function $v: \mathbb{Z}^d \to \mathbb{Z}$ with $v(0) = 0$, we define the space of configurations Γ to consist of functions $h: \mathbb{Z}^d \to \mathbb{Z}$ such that $h(i) - h(j) \le v(i-j)$ for all $i, j \in \mathbb{Z}^d$. We then take two sequences of independent Poisson clocks $(p^{\pm}(i,t): i \in \mathbb{Z}^d)$ of rates $\lambda^{\pm}$. We start with a possibly random configuration $h \in \Gamma$. The function h increases (respectively, decreases) by one unit at site i, when the clock $p^+(i,\cdot)$ [respectively, $p^-(i,\cdot)$] rings and the resulting configuration is still in Γ. Otherwise the change in h is suppressed. In this way we have a process $h(i,t)$ that after a rescaling $u^{\varepsilon}(x,t) = \varepsilon h([\frac{x}{\varepsilon}], \frac{t}{\varepsilon})$ is expected to converge to a function $u(x,t)$ that solves a Hamilton–Jacobi equation of the form $u_t + H(u_x) = 0$. We established this when λ^- or $\lambda^+ = 0$ in the previous paper, employing a strong monotonicity property of the process $h(i,t)$. Such property is no longer available when both λ^+, λ^- are nonzero. In this paper we initiate a new approach to treat the problem when the dimension is 1 and the set Γ can be described by local constraints on the configuration h. In higher dimensions, we can only show that any limit point of the processes u^{ε} is a process u that satisfies a Hamilton–Jacobi equation for a suitable (possibly random) Hamiltonian H.

1. Introduction. Perhaps the simplest example of a stochastic growth model is the Eden–Richardson model that was studied in a biological context. In this model each lattice site $i \in \mathbb{Z}^d$ represents the center of a cubical cell and the set $A(t)$ denotes the union of the infected cells, where a healthy cube outside $A(t)$ becomes infected with a rate proportional to the number of adjacent infected cells. Richardson shows that the set $A(t)$ grows linearly in t, and as ε goes to zero,

$$\varepsilon A\left(\frac{t}{\varepsilon}\right) \approx \{x \in \mathbb{R}^n \mid N(x) \le t\}$$

for a suitable norm $N(\cdot)$ associated with the model. The proof of Richardson's theorem and related *shape theorems* can be found in [4].

In the previous paper [14] we investigated a class of growth models that allows more complicated growth rules but growths can only occur in one direction, say, parallel to the last coordinate axis. More precisely, the set of infected cells is always of the form

$$A = \{(i,k) \mid h(i) \ge k\} \subseteq \mathbb{Z}^n = \mathbb{Z}^{d+1}, \tag{1.1}$$

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where h: $\mathbb{Z}^d \to \mathbb{Z}$ is regarded as a random height function. Our method of proof requires a simultaneous construction for our growth model for all initial configurations. Such a construction is possible for a family of models that was called *v-exclusion processes* in [14]. To define a v-exclusion process for a nonnegative function v: $\mathbb{Z}^d \to \mathbb{Z}$ with $v(0) = 0$, we set

$$\Gamma = \Gamma_v = \{h\colon \mathbb{Z}^d \to \mathbb{Z} \mid h(i) - h(j) \le v(i-j) \text{ for all } i, j \in \mathbb{Z}^d\}. \tag{1.2}$$

Let $\omega = (p^+(i,t), p^-(i,t)\colon i \in \mathbb{Z}^d)$ denote a sequence of independent Poisson processes with rates λ^+ and λ^-. Initially we start with a possibly random height function $h \in \Gamma$. The function h attempts to increase (respectively, decrease) at a site i by one unit when the Poisson clock $p^+(i,t)$ [respectively, $p^-(i,t)$] rings. The increase (respectively, decrease) takes place if $h^i \in \Gamma$ (respectively, $h_i \in \Gamma$) where

$$h^i(j) = \begin{cases} h(i)+1, & \text{if } i = j, \\ h(j), & \text{otherwise}, \end{cases} \qquad h_i(j) = \begin{cases} h(i)-1, & \text{if } i = j, \\ h(j), & \text{otherwise}. \end{cases} \tag{1.3}$$

The increase (respectively, decrease) is suppressed if $h^i \notin \Gamma$ (respectively, $h_i \notin \Gamma$). In this way, we construct a stochastic process $(h(i,t)\colon i \in \mathbb{Z}^d, t \ge 0)$, and as in (1.1), we would like to derive a macroscopic equation for the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ where $u^\varepsilon(x,t) = \varepsilon h([\frac{x}{\varepsilon}], \frac{t}{\varepsilon})$. This is done in this article provided that Γ can be expressed by some local constraints on the height function h. Given d pairs of *integers* (α_r, β_r) with $\alpha_r \le 0 \le \beta_r$, define

$$v(i_1, \dots, i_d) = \sum_{r=1}^{d} p_r(i_r), \tag{1.4}$$

where $p_r(k) = \beta_r k^+ - \alpha_r k^-$. [We also regard v as a function from $\mathbb{R}^d$ to $\mathbb{R}$ by simply allowing $i \in \mathbb{R}^d$ in the formula (1.4).] For such a function v, one can readily show

$$\Gamma_v = \{h \mid \alpha_r \le h(i+e_r) - h(i) \le \beta_r \text{ for every } i \in \mathbb{Z}^d \text{ and } r \in \{1,\dots,d\}\}, \tag{1.6}$$

where e_r is the rth unit vector. The main result of this paper asserts that when $d = 1$, the limit $u(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ exists and satisfies a Hamilton–Jacobi equation

$$u_t + H(u_x) = 0 \tag{1.7}$$

with a deterministic Hamiltonian $H(p) = H(p; v, \lambda^+, \lambda^-)$.

When $\lambda^- = 0$, the derivation of (1.7) was carried out in [16] for $d = 1$ and in [14] for general d. In this case H is always convex and its convex conjugate L coincides with $\lim_{\varepsilon\to 0} u^\varepsilon(0,1)$ provided that the initial configuration $h(i,0) = v(i)$. In [14] we may also allow a rate that is slowly varying with space. This requires multiplying $H(u_x)$ with a continuous function $\lambda(x)$ in (1.7). Both [16] and [14] use the following important *strong monotonicity* property of the process $h(i,t)$: if h, k, l are three v-exclusion processes with $h(i,0) = \min(k(i,0), l(i,0))$ for all i, then the same is true at later times.

The strong monotonicity is no longer true when λ^-, λ^+ are both nonzero. However, we still have the following weaker monotonicity: if $h(i,t)$ and $k(i,t)$ are two v-exclusion processes with $h(i,0) \le k(i,0)$ for all i, then the same is true at later times.

We have already mentioned that when $\lambda^- = 0$, then the convex conjugate of H can be defined using the process $h(i,t)$ with the initial condition $h(i,0) = v(i)$. The key observation is that there is another way of determining the Hamiltonian H that does not require any convexity. Evidently $w(x,t,p) = x \cdot p - tH(p)$ is always a solution of (1.7) with the initial condition $w(x,0,p) = x \cdot p$. This suggests looking at the process $h_p(i,t)$ with the initial condition $h_p(i,0) = [i \cdot p]$. Here and below, $[x]$ denotes the integer part of x. We will show in Section 4 that when $d = 1$,

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t;p) = \lim_{\varepsilon\to 0} \varepsilon h_p\left(\left[\frac{x}{\varepsilon}\right], \frac{t}{\varepsilon}\right) \tag{1.8}$$

exists and is of the form $x \cdot p - tH(p)$ for a suitable nonrandom function $H(p)$. For the existence of the limit in (1.8), we mostly follow [5]. Since this paper is unpublished, we provide a detailed proof of some results from [5].

In Section 3, we show that the process u^ε can be regarded as a semigroup as we vary the time and the initial configuration. This will be used to show that all the limit points of u^ε possess some of the basic properties of a semigroup associated with a Hamilton–Jacobi equation of the form (1.7). We then appeal to a result of [10] to deduce that the limit points of the law of the processes $(u^\varepsilon\colon \varepsilon > 0)$ are concentrated on the space of Hamilton–Jacobi semigroups. The results of Section 4 allow us to show that when $d = 1$, the limit points are all concentrated on a single semigroup, namely the one associated with the solutions of (1.7). In higher dimensions, we also show that the convex hull of the limiting Hamiltonian is always deterministic and uniquely defined.

There is only one example for which H is known explicitly. When $d = 1$ and $v(i) = i^+$, then the v-exclusion process is the celebrated simple exclusion process and

$$H(p) = (\lambda^- - \lambda^+)p(1-p).$$

The case of simple exclusion was treated in [13] using a different method and $H(p)$ was calculated by taking the average of

$$\mathbb{1}(h, h^i \in \Gamma) = \mathbb{1}\big(h(i-1) = h(i) < h(i+1)\big)$$

with respect to the probability measure ν^p on Γ that is uniquely defined by requiring $(h(i)\colon i \in \mathbb{Z})$ to be independent and that the event $h(i+1) - h(i) = 1$ occur with probability p. Such probability measures are invariant for the height difference process $\eta(i,t) = h(i+1,t) - h(i,t)$. Apparently the simple exclusion process is the only nontrivial v-exclusion process for which the invariant (equilibrium) measures are known and have a simple form.

It seems plausible that our method in this article can be applied to general *attractive* growth models. For example, we may consider a growth model with the same configuration space Γ that has the following growth rates: the height function h becomes h^i (respectively, h_i) with rate $A_i^+(h)$ [respectively, $A_i^-(h)$]. We assume that the rate functions $A_i^{\pm}(h)$ satisfy the following conditions:

1. $A_i^{\pm}(h)$ are local functions of the height differences $(h(i) - h(j)\colon j \in \mathbb{Z}^d)$.
2. We have $A_i^{\pm} = \tau_i A_0^{\pm}$ for every i, where τ_i denotes the shift operator.
3. $A_i^+(h) = 0$ [respectively, $A_i^-(h) = 0$] if $h^i \notin \overline{\Gamma}$ (respectively, $h_i \notin \overline{\Gamma}$).
4. $A_i^+(h)$ [respectively, $A_i^-(h)$] is a nonincreasing (respectively, nondecreasing) function of the height differences $(h(i) - h(j)\colon j \in \mathbb{Z}^d)$.

To apply our method to this model, the main challenge is the construction of the microscopic semigroup $S^\varepsilon(s, t; g)$ as in Section 3. As we mentioned earlier, the v-exclusion process h can be constructed simultaneously for all initial configurations. One should be able to use the idea behind the proof of Lemma 3.7 below to make sense of such simultaneous construction for general attractive growth models. We do not pursue the issue further in this paper and leave it for a future investigation.

Another possible generalization is a Ginzburg–Landau type process $\phi\colon \mathbb{Z}^d \times [0, \infty) \to \mathbb{R}$ that satisfies a system of stochastic differential equations of the form

$$d\phi(i, t) = A_i(\phi)\, dt + dB_i,$$

where the functions A_i enjoy the same properties as the growth rates A_i^+ of the previous paragraph, and $(B_i\colon i \in \mathbb{Z}^d)$ is a sequence of independent identically distributed Brownian motions. For this model, the construction of the microscopic semigroup is straightforward and can be carried out as in Section 3. In the case of v-exclusion processes, we have the equicontinuity of the macroscopic height function in space variable for free because of our choice of configuration space Γ. This is no longer the case for the ϕ process. However, we expect to have some type of equicontinuity of the macroscopic height function associated with ϕ process if we assume that the function A_0 is Lipschitz continuous. One should be able to prove this by standard arguments.

The organization of the paper is as follows. In the next section the main results are stated. In Section 3, the microscopic semigroups associated with the process u^ε are studied. In Section 4, the convergence of u^ε is shown when the initial configuration is $[i \cdot p]$ and $d = 1$. The continuum limit for the v-exclusion is carried out in Section 5.

2. Main results.

Given a function v of the form (1.4) and two nonnegative numbers λ^+, λ^-, we define a Markov process $h(i, t)$ with the infinitesimal generator $\mathscr{L} = \mathscr{L}^+ + \mathscr{L}^-$ where

$$\begin{aligned} \mathscr{L}^+ F(k) &= \sum_{i \in \mathbb{Z}^d} \lambda^+ \mathbb{1}(k^i \in \Gamma)(F(k^i) - F(k)), \\ \mathscr{L}^- F(k) &= \sum_{i \in \mathbb{Z}^d} \lambda^- \mathbb{1}(k_i \in \Gamma)(F(k_i) - F(k)), \end{aligned} \tag{2.1}$$

where k^i, k_i are defined as in (1.3), Γ was defined by (1.2), and $F\colon \Gamma \to \mathbb{R}$ is any cylindrical function [$F(k)$ depends on finitely many coordinates $k(j)$].

We also define

$$\begin{aligned}\overline{\Gamma} &= \{g\colon \mathbb{R}^d \to \mathbb{R} \mid g(x) - g(y) \le v(x-y) \text{ for all } x, y \in \mathbb{R}^d\}\\ &= \big\{g\colon \mathbb{R}^d \to \mathbb{R} \mid \alpha_r\theta \le g(x+\theta e_r) - g(x) \le \beta_r\theta \text{ for all } x \in \mathbb{R}^d,\\ &\qquad\qquad \theta \ge 0 \text{ and } r \in \{1, \dots, d\}\big\}.\end{aligned} \tag{2.2}$$

Set $u^\varepsilon(x,t) = \varepsilon h([\frac{x}{\varepsilon}], \frac{t}{\varepsilon})$ and assume that for some $g \in \overline{\Gamma}$,

$$\lim_{\varepsilon\to 0} E \sup_{x\in B_0} |u^\varepsilon(x,0) - g(x)| = 0 \tag{2.3}$$

for every bounded $B_0 \subset \mathbb{R}^d$, where E denotes the expectation.

THEOREM 2.1. Suppose $d = 1$. *There exists a continuous function* H: $[\alpha_1, \beta_1] \to \mathbb{R}$ *such that for every bounded set* $B \subset \mathbb{R} \times [0,\infty)$,

$$\lim_{\varepsilon\to 0} E \sup_{(x,t)\in B} |u^\varepsilon(x,t) - u(x,t)| = 0, \tag{2.4}$$

where u *is the unique viscosity solution of*

$$u_t + H(u_x) = 0, \quad u(x,0) = g(x) \tag{2.5}$$

with $u(\cdot, t) \in \overline{\Gamma}$ *for every* t.

As we mentioned in the Introduction, we can only prove a weaker version of the above theorem in higher dimensions. To this end, let us display the dependence of the height profile on its initial data and write $S^\varepsilon(0,t;g)(x)$ for $u^\varepsilon(x,t)$ when the initial height $h(i,0)$ is given by $[\varepsilon^{-1}g(\varepsilon i)]$. The law of the random process S^ε is a probability measure on the space of functions of the form $S(0,t;g)(x)$, and such a law is denoted by $\mathscr{P}^\varepsilon$. (See Sections 3 and 5 for more details.) Let us write Y for the set $\prod_{r=1}^d [\alpha_r, \beta_r]$. In Theorem 2.2 below, the dimension d is arbitrary.

THEOREM 2.2. *Every limit point of* $\mathscr{P}^\varepsilon$ *is concentrated on the space of paths* $S(0,t;g)$ *such that each path is a solution of* (2.5) *for some continuous function* H: $Y \to \mathbb{R}$.

Note that Theorem 2.2 does not rule out the possibility of a random Hamiltonian H. However, we will show that the convex hull of H is nonrandom and uniquely determined. For this see Theorem 5.1 of Section 5.

The question of the regularity of H is a challenging problem. We only establish some rather straightforward algebraic properties of H.

THEOREM 2.3. *Assume* $\lambda^+ \ge \lambda^-$. *With probability* 1, *the following statements are true for the function* H *of Theorem* 2.2:

(i) *For every* $p \in Y$, *we have* $H(p) = H(\bar{p} - p)$, *where* $\bar{p} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_d + \beta_d)$.
(ii) $H = 0$ *on the boundary of* Y.

(iii) *The function H is Lipschitz.*
(iv) $H(p) \le 0$ *for every* $p \in Y$.
(v) *If* $\alpha_r = \beta_r$ *for some* r, *then* H *is identically zero.*

We end this section with a construction of our v-exclusion process in terms of the independent Poisson processes $(p^+(i,t), p^-(i,t): i \in \mathbb{Z}^d)$. Let D denote the set of step functions $l^{\pm}: [0,\infty) \to \mathbb{Z}^+$ such that for an increasing sequence of numbers $\sigma_0(l^{\pm}) = 0, \sigma_1(l^{\pm}), \dots$, we have $l^{\pm}(t) = k$ for $t \in [\sigma_k(l^{\pm}), \sigma_{k+1}(l^{\pm}))$. We set $\Omega = D^{\mathbb{Z}^d} \times D^{\mathbb{Z}^d}$ and let Q denote the law of independent Poisson processes with rates λ^+ and λ^-. Given $\omega = ((l_i^+(\cdot): i \in \mathbb{Z}^d), (l_i^-(\cdot): i \in \mathbb{Z}^d))$, we set $p^{\pm}(i,t) = l_i^{\pm}(t)$ and

$$\tau_j \omega = (l_{i-j}^+(\cdot), l_{i-j}^-(\cdot): i \in \mathbb{Z}^d), \qquad \gamma_s \omega = (\gamma_s l_i^+(\cdot), \gamma_s l_i^-(\cdot): i \in \mathbb{Z}^d), \tag{2.6}$$

where $(\gamma_s l)(t) = l(t+s) - l(s)$. Since Poisson processes with constant rates have stationary and independent increments, the law Q is stationary and ergodic with respect to both τ_j and γ. More precisely, for a measurable set A, $Q(\tau_j A) = Q(\gamma_s A) = Q(A)$. Moreover, if $\tau_j A = A$ for all j or $\gamma_s A = A$ for all $s > 0$, then $Q(A) = 0$ or 1.

To construct our process, we first take a sequence of finite sets $Z_m \subseteq \mathbb{Z}^d$ such that

$$Z_m \subset Z_{m+1}, \qquad \bigcup_{m=1}^{\infty} Z_m = \mathbb{Z}^d. \tag{2.7}$$

We then define $\mathscr{L}_{m_1,m_2} = \mathscr{L}_{m_1}^+ + \mathscr{L}_{m_2}^-$ where

$$\begin{aligned} \mathscr{L}_{m_1}^+ F(k) &= \lambda^+ \sum_{i \in Z_{m_1}} \mathbb{1}(k^i \in \Gamma)(F(k^i) - F(k)), \\ \mathscr{L}_{m_2}^- F(k) &= \lambda^- \sum_{i \in Z_{m_2}} \mathbb{1}(k_i \in \Gamma)(F(k_i) - F(k)). \end{aligned} \tag{2.8}$$

Let Ω_0 be the set of ω for which all $\sigma_r(l_i^{\pm})$ are distinct. It is not hard to show $Q(\Omega_0) = 1$. Given $k \in \Gamma$ and $\omega \in \Omega_0$, we define $h^{m_1,m_2}(i,t;k,\omega)$ to be a Markov process with the generator $\mathscr{L}_{m_1,m_2}$ and the initial data $h^{m_1,m_2}(i,0;k,\omega) = k(i)$. It is not hard to show that h^{m_1,m_2} is nondecreasing in m_1 and nonincreasing in m_2. This is a straightforward consequence of the fact that

$$\begin{aligned} &k^1, k^2 \in \Gamma, k^1(i) = k^2(i), k^1 \le k^2, (k^1)^i \in \Gamma \quad \Rightarrow \quad (k^2)^i \in \Gamma, \\ &k^1, k^2 \in \Gamma, k^1(i) = k^2(i), k^1 \le k^2, (k^2)_i \in \Gamma \quad \Rightarrow \quad (k^1)_i \in \Gamma. \end{aligned} \tag{2.9}$$

We then set

$$h(i,t;k,\omega) = \inf_{m_2} \sup_{m_1} h^{m_1,m_2}(i,t;k,\omega) \tag{2.10}$$

for $\omega \in \Omega_0$. One can readily verify that such $h(i,t)$ is a Markov process with the generator $\mathscr{L}$.

Observe that if $\hat{k}(i) = k(i) + \ell$, for some integer ℓ, then

$$h(i, t; \hat{k}, \omega) = h(i, t; k, \omega) + \ell \tag{2.11}$$

for every $\omega \in \Omega_0$. Finally, we define $\tau_j h$ by $\tau_j(i) = h(i - j)$ for all $i \in \mathbb{Z}^d$. From our construction, it is not hard to see

$$h(i - j, t; k, \omega) = h(i, t; \tau_j k, \tau_j \omega), \tag{2.12}$$

$$h(i, t; k, \omega) = h(i, t - s; h(\cdot, s; k, \omega), \gamma_s \omega), \tag{2.13}$$

for every $\omega \in \Omega_0$ and every s and t with $0 \le s \le t$.

REMARK 2.4. Observe that if $\tilde{k}(i) = k(i) - \sum_{r=1}^{d} \alpha_r i_r$ and $\tilde{v}(i) = \sum_{r=1}^{d} (\beta_r - \alpha_r) i_r^+$, then $k \in \Gamma_v$ if and only if $\tilde{k} \in \Gamma_{\tilde{v}}$. From this, it is not hard to deduce that if $\tilde{h}$ is a $\tilde{v}$-exclusion and h is a v-exclusion process with $\tilde{h}(i, 0) = h(i, 0) - \sum_{r=1}^{d} \alpha_r i_r$ initially, then $\tilde{h}(i, t) = h(i, t) - \sum_{r=1}^{d} \alpha_r i_r$ at later times. As a result, we may assume α_r's are zero for $r = 1, \dots, d$ in our theorems, with no loss of generality.

3. The microscopic semigroup. In this section we show that any limit point of the sequence u^ε constitutes a semigroup as we vary the time and the initial data. Let $g \in \overline{\Gamma}$ and define

$$k_g^\varepsilon(i) = [\varepsilon^{-1} g(\varepsilon i)]. \tag{3.1}$$

By (2.2),

$$\varepsilon^{-1} g(\varepsilon i) + \alpha_r \le \varepsilon^{-1} g(\varepsilon i + \varepsilon e_r) \le \varepsilon^{-1} g(\varepsilon i) + \beta_r. \tag{3.2}$$

Recall that α_r, β_r are two integers. From this we deduce $k_g^\varepsilon \in \Gamma$.

Set

$$\varepsilon\mathbb{Z}^d := \{x \mid x = \varepsilon a \text{ for some } a \in \mathbb{Z}^d\}.$$

Given $\omega \in \Omega_0$, $g \in \Gamma$, $x \in \varepsilon\mathbb{Z}^d$ and $s, t \in [0, \infty)$ with $s \le t$, we define

$$S_0^\varepsilon(s, t; g, \omega)(x) := \varepsilon h\Big(\frac{x}{\varepsilon}, \frac{t - s}{\varepsilon}; k_g^\varepsilon, \gamma_{s/\varepsilon}\omega\Big).$$

We then define S^ε as an appropriate extension of S_0^ε. For $x \in \mathbb{R}^d$ and $s \le t$,

$$S^\varepsilon(s, t; g, \omega)(x) = \inf_{y \in \varepsilon\mathbb{Z}^d} \big(S_0^\varepsilon(s, t; g, \omega)(y) + v(x - y)\big). \tag{3.3}$$

When $t \le s$, we set

$$S^\varepsilon(s, t; g, \omega) = g.$$

From $h(\cdot, t; k, \omega) \in \Gamma$, one can readily show that if $x \in \varepsilon\mathbb{Z}^d$, then

$$S^\varepsilon(s, t; g, \omega)(x) = S_0^\varepsilon(s, t; g, \omega)(x).$$

Moreover, since $v \in \overline{\Gamma}$ and $\overline{\Gamma}$ is closed with respect to translation and infimum, we have $S^\varepsilon(s, t; g, \omega) \in \overline{\Gamma}$. Note that the property (2.13) implies

$$S_0^\varepsilon(s_1, t; g, \omega)(x) = S_0^\varepsilon\big(s_2, t; S^\varepsilon(s_1, s_2; g, \omega), \omega\big)(x)$$

whenever $x \in \varepsilon\mathbb{Z}^d$, $s_1 \le s_2 \le t$. This in turn implies

$$S^\varepsilon(s_1, t; g, \omega) = S^\varepsilon\big(s_2, t; S^\varepsilon(s_1, s_2; g, \omega), \omega\big)\big). \tag{3.4}$$

The main result of this section asserts that all the limit points of the processes (S^ε: $\varepsilon > 0$) are concentrated on the space of semigroups which possess some of the basic properties of a semigroup associated with a Hamilton–Jacobi equation of the form (1.7). For our purposes, it is more convenient to regard S^ε as a random function of (s, t) with values in the space of functions F: $\overline{\Gamma} \to \overline{\Gamma}$. To this end, let us equip $\overline{\Gamma}$ with the uniform topology on compact subsets,

$$\|g_1 - g_2\|_\ell := \sup_{|x|\le \ell} |g_1(x) - g_2(x)|, \qquad d(g_1, g_2) := \sum_{\ell=1}^{\infty} 2^{-\ell}\|g_1 - g_2\|_\ell. \tag{3.5}$$

Here $|x| = |(x_1, \dots, x_d)| = (\sum_r x_r^2)^{1/2}$. One can readily show that there exists a constant c_0 such that for every $g_1, g_2 \in \overline{\Gamma}$, we have

$$\|g_1 - g_2\|_\ell \le |g_1(0) - g_2(0)| + c_0\ell. \tag{3.6}$$

Moreover, $d(g_n, g) \to 0$ if and only if $\|g_n - g\|_\ell \to 0$ for every ℓ. It is not hard to show that the space $\overline{\Gamma}$ with the metric d is a Polish (separable and complete metric) space.

Define

$$\overline{\Gamma}_\ell = \{g \in \overline{\Gamma} \mid |g(0)| \le \ell\}.$$

Let $\widehat{\mathscr{E}}$ denote the space of functions F: $\overline{\Gamma} \to \overline{\Gamma}$ with the following properties:

1. $F(g + m) = F(g) + m$ for every constant m.
2. $F(g_1) \le F(g_2)$ whenever $g_1 \le g_2$.
3. $\|F\|_0 := \sup_{g\in\overline{\Gamma}} |F(g)(0) - g(0)| < \infty$.

   We also define $\mathscr{E}_q$ to be set of $F \in \widehat{\mathscr{E}}$ such that
4. If $g_1(x) = g_2(x)$ for all $|x| \le R$ and $R > q$, then $F(g_1)(x) = F(g_2)(x)$ for all x with $|x| \le R - q$.

Set $\mathscr{E}_\infty = \cup_{q=1}^\infty \mathscr{E}_q$. Evidently, if $F \in \widehat{\mathscr{E}}$ and $g \in \overline{\Gamma}_0$, then by (3.6),

$$\|F(g)\|_\ell \le |F(g)(0)| + c_0\ell \le \|F\|_0 + |g(0)| + c_0\ell \le \|F\|_0 + c_0\ell. \tag{3.7}$$

Given $F_1, F_2 \in \widehat{\mathscr{E}}$, set

$$D(F_1, F_2) = \sum_{\ell=1}^{\infty} 2^{-\ell}\frac{\|F_1 - F_2\|_\ell}{1 + \|F_1 - F_2\|_\ell},$$

where

$$\|F_1 - F_2\|_\ell := \sup_{g\in\overline{\Gamma}} \|F_1(g) - F_2(g)\|_\ell = \sup_{g\in\overline{\Gamma}_0} \|F_1(g) - F_2(g)\|_\ell.$$

The second equality follows from $F(g) = F(g - g(0)) + g(0)$. From (3.7) we deduce that $\|F_1 - F_2\|_\ell < \infty$ for every ℓ. Moreover, it is not hard to see that D defines a metric on $\widehat{\mathscr{E}}$. The topological closure of $\mathscr{E}_\infty$ in $\widehat{\mathscr{E}}$ will be denoted by $\mathscr{E}$.

LEMMA 3.1. *Suppose $F \in \mathscr{E}_q$ and $g_1, g_2 \in \overline{\Gamma}$. Then*

$$\|F(g_1) - F(g_2)\|_\ell \le \|g_1 - g_2\|_{\ell+q}.$$

PROOF. Put $m = \|g_1 - g_2\|_{\ell+q}$ and let g_3 denote the maximum of the functions g_1 and $g_2 + m$. We certainly have $g_3(x) = g_2(x) + m$ for every x with $|x| \le \ell+q$. Hence, by the condition (4), $F(g_3)(x) = F(g_2+m)(x)$ for every x with $|x| \le \ell$. On the other hand, the condition (2) implies $F(g_1) \le F(g_3)$. Thus,

$$F(g_1)(x) \le F(g_3)(x) = F(g_2 + m)(x) = F(g_2)(x) + m$$

for every x with $|x| \le \ell$. In the same way, one can show

$$F(g_1)(x) \ge F(g_2)(x) - m,$$

for every x with $|x| \le \ell$. This and the previous inequality complete the proof. □

Define

$$\mathscr{E}_q^r = \{F \in \mathscr{E}_q | \ \|F\|_0 \le r\}.$$

LEMMA 3.2. *The space $\mathscr{E}$ is a Polish space. The space $\mathscr{E}_q$ is closed in $\mathscr{E}$. The space $\mathscr{E}_q^r$ is compact in $\mathscr{E}$.*

PROOF. We first show that the space $\widehat{\mathscr{E}}$ is complete. Let F_n be a Cauchy sequence in $\widehat{\mathscr{E}}$. Then the sequence $F_n(g)(x)$ is Cauchy for every $g \in \overline{\Gamma}$ and every x. Define $F(g)(x)$ to be the limit of $F_n(g)(x)$ as n goes to infinity. It is straightforward to check that $F \in \widehat{\mathscr{E}}$ and that $D(F_n, F)$ converges to zero, as $n \to \infty$. It is also straightforward to check that if $\{F_n | n \in \mathbb{N}\}$ is a subset of $\mathscr{E}_q$ (respectively, $\mathscr{E}_q^r$), then its limit points belong to $\mathscr{E}_q$ (respectively, $\mathscr{E}_q^r$). To complete the proof, it suffices to show that each $\mathscr{E}_q^r$ is compact. Note that the compactness of $\mathscr{E}_q^r$ implies that the space $\mathscr{E}_q$ is separable because $\mathscr{E}_q = \cup_{r=1}^\infty \mathscr{E}_q^r$.

We establish the compactness by showing that each $\mathscr{E}_q^r$ is totally bounded. Observe that if the functions in the set $\overline{\Gamma}_0$ are restricted to a compact subset of $\mathbb{R}^d$, then we obtain an equicontinuous family of functions. Hence, for every positive ℓ, there exists a finite subset A_ℓ of $\overline{\Gamma}_0$ such that for every $g \in \overline{\Gamma}_0$, there exists $a \in A_\ell$ with $\|g - a\|_\ell \le 1/\ell$. Let Σ_ℓ^r denote the set of all functions $\sigma = (\sigma^1, \sigma^2)$ with

$$\sigma^1 \colon A_{\ell+q} \to A_\ell, \quad \sigma^2 \colon A_{\ell+q} \to \{s \in \mathbb{Z} \mid |s| \le r\ell\}.$$

Given $\sigma, \in \Sigma^r_\ell$, we define $\mathscr{A}_\sigma$ to be the set of $F \in \mathscr{E}^r_q$ such that

$$\|F(a) - F(a)(0) - \sigma^1(a)\|_\ell \le 1/\ell, \qquad F(a)(0) \in \left[\frac{\sigma^2(a)}{\ell}, \frac{\sigma^2(a)+1}{\ell}\right),$$

for every $a \in A_{\ell+q}$. When the set $\mathscr{A}_\sigma$ is nonempty, pick some F_σ from $\mathscr{A}_\sigma$. Let $\mathscr{B} = \mathscr{B}(\ell_1)$ denote the set of all such F_σ, as we vary $\sigma \in \Sigma^r_\ell$ and $\ell \in \{1, 2, \dots, \ell_1\}$. Evidently the set $\mathscr{B}$ is finite. We now claim that $\mathscr{B}$ is a δ-net in $\mathscr{E}^r_q$ for

$$\delta = \frac{2}{\ell_1 + q} + \frac{3}{\ell_1} + 2^{-\ell_1}.$$

More precisely, for every $F \in \mathscr{E}^r_q$, there exists $\overline{F} \in \mathscr{B}$ such that $D(F, \overline{F}) \le \delta$.

Observe that for every positive integer ℓ_1 and every function $F \in \mathscr{E}^r_q$, there exists some $\bar\sigma = \sigma_{\ell_1}(F) \in \Sigma^r_{\ell_1}$ such that $F \in \mathscr{A}_{\bar\sigma}$. Moreover, for every $g \in \overline{\Gamma}_0$, there exists $a \in A_{\ell_1+q}$ with $\|g - a\|_{\ell_1+q} \le (\ell_1+q)^{-1}$. This and Lemma 3.1 imply that $\|F(g) - F(a)\|_{\ell_1} \le (\ell_1 + q)^{-1}$. For the same reason, $\|F_{\bar\sigma}(g) - F_{\bar\sigma}(a)\|_{\ell_1} \le (\ell_1 + q)^{-1}$. Also,

$$\begin{aligned}\|F(a) - F_{\bar\sigma}(a)\|_{\ell_1} &\le \|F(a) - F(a)(0) - \bar\sigma^1(a)\|_{\ell_1} \\ &\quad + \|F_{\bar\sigma}(a) - F_{\bar\sigma}(a)(0) - \bar\sigma^1(a)\|_{\ell_1} + |F(a)(0) - F_{\bar\sigma}(a)(0)| \\ &\le \frac{3}{\ell_1},\end{aligned}$$

for every $a \in A_{\ell_1+q}$. As a result,

$$\sup_{g\in\overline{\Gamma}_0} \|F(g) - F_{\bar\sigma}(g)\|_\ell \le \sup_{g\in\overline{\Gamma}_0} \|F(g) - F_{\bar\sigma}(g)\|_{\ell_1} \le \frac{2}{\ell_1 + q} + \frac{3}{\ell_1},$$

for every $\ell \le \ell_1$. This in turn implies

$$D(F, F_{\bar\sigma}) \le \sum_{\ell=1}^{\ell_1} 2^{-\ell}\left(\frac{2}{\ell_1+q} + \frac{3}{\ell_1}\right) + \sum_{\ell=\ell_1+1}^{\infty} 2^{-\ell} \le \delta.$$

Since $\delta \to 0$ as $\ell_1 \to \infty$, we have the total boundedness of $\mathscr{E}^r_q$. This implies that each $\mathscr{E}^r_q$ is compact. □

Set

$$\mathscr{D}_T = \mathscr{D}([0, T] \times [0, T]; \mathscr{E}) := \mathscr{D}\big([0, T]; \mathscr{D}([0, T]; \mathscr{E})\big)$$

for the Skorohod space of functions $S : [0, T] \times [0, T] \to \mathscr{E}$. The space $\mathscr{D}_T$ is a Polish space, because $\mathscr{E}$ is a Polish space (see, e.g., Theorem 5.6 in Chapter 3 of [6] for a proof).

Because of the discretization, the function S^ε does not belong to $\mathscr{D}_T$. However, a simple modification of S^ε would belong to $\mathscr{D}_T$. To this end, let us define

$$\widehat{S}^\varepsilon(s, t; g) = S^\varepsilon(s, t; g - g(0)) + g(0).$$

In this way, we always have

$$\widehat{S}^{\varepsilon}(s,t;g+m)=\widehat{S}^{\varepsilon}(s,t;g)+m, \tag{3.8}$$

for every constant m. On the other hand, by (2.11),

$$S^{\varepsilon}(s,t;g)+m-\varepsilon\le S^{\varepsilon}(s,t;g+m)\le S^{\varepsilon}(s,t;g)+m+\varepsilon,$$

for every constant m. From this we deduce,

$$-\varepsilon\le \widehat{S}^{\varepsilon}(s,t;g)-S^{\varepsilon}(s,t;g)\le\varepsilon. \tag{3.9}$$

LEMMA 3.3. *We have* $\widehat{S}^{\varepsilon}(\omega)=\widehat{S}^{\varepsilon}(\cdot,\cdot;\cdot,\omega)\in\mathscr{D}_T$, *for almost all* ω.

To prepare for the proof of Lemma 3.3, let us define a Markov process $G(t)\subseteq\mathbb{Z}^d$ with the infinitesimal generator

$$\mathscr{A}F(G)=(\lambda^{+}+\lambda^{-})\sum_{i\in\mathbb{Z}^d}d_i(G)(F(G^i)-F(G)),$$

where G^i is the set $G\cup\{i\}$ and

$$d_i(G)=\mathbb{1}(i+e_r\in G \text{ or } i-e_r\in G \text{ for some } r=1,\dots,d).$$

The process $G(t)=G(t;\omega)$ can be constructed with the aid of the Poisson clocks $\omega=(p^{+}(i,t),\,p^{-}(i,t)\colon i\in\mathbb{Z}^d)$; when the clock p^{+} or p^{-} rings at site i, the site i is added to the growing set G provided that at least one of the adjacent sites is already in G. The process $G(t;\omega)$ can be compared to the Eden–Richardson model that was introduced in Section 1. It is not hard to see that if $k_1=k_2$ on the set $\mathbb{Z}^d-G(0)$, then

$$h(i,t;k_1,\omega)=h(i,t;k_2,\omega) \tag{3.10}$$

for every $i\notin G(t;\omega)$.

PROOF. To show $\widehat{S}^{\varepsilon}\in\mathscr{D}_T$, it suffices to check $\widehat{S}^{\varepsilon}(s,t;\cdot)\in\mathscr{E}$. The condition (1) is simply (3.8). The condition (2) follows from Lemma 3.4 below. The condition (3) follows from

$$-\varepsilon p^{-}\left(0,\frac{t}{\varepsilon}\right)\le S^{\varepsilon}(s,t;g-g(0))(0)\le\varepsilon p^{+}\left(0,\frac{t}{\varepsilon}\right).$$

It remains to verify the condition (4). Without loss of generality, we may assume that $s=0$. Fix t and ε. Recall the process $G(t,\omega)$ that was defined right after the statement of Lemma 3.3. Let us define the initial condition to be the set $G(0)=B_n:=\{i\mid |\varepsilon i|>n\}$. This set grows to the set $G_n^{\varepsilon}(\theta;\omega)$ at a later time θ. Define $A(n,\ell)$ to be the set of ω for which

$$G_n^{\varepsilon}(\varepsilon^{-1}t;\omega)\cap B'_{\ell}\ne\varnothing,$$

where $B'_{\ell}=\{i\mid|\varepsilon i|\le\ell\}$. We first claim that

$$\lim_{n\to\infty}Q\big(A(n,\ell)\big)=0, \tag{3.11}$$

for every ℓ. The proof of this is standard and we only sketch it. For each $i, j \in \mathbb{Z}^d$, define $\tau(i, j) = \tau(i, j; \omega)$ to be the smallest θ for which if $G(0) = \{i\}$; then $j \in G(\theta; \omega)$. It is well known that for some positive constants c_1, c_2 and c_3,

$$Q(\tau(i, j) < c_1|i - j|) \le c_2 \exp(-c_3|i - j|), \tag{3.12}$$

for all i and j. (See for example Chapter 1 of [4].) If $\omega \in A(n, \ell)$, then $\tau(i, j) \le \varepsilon^{-1}t$ for some (i, j) with i on the boundary of B_n and j on the boundary of B'_ℓ. From this and (3.12), one can readily derive (3.11) by standard arguments.

Let us write $F(g; \omega)$ for $\widehat{S}^\varepsilon(0, t; g, \omega)$. Define γ_n: $\overline{\Gamma} \to \overline{\Gamma}$ by

$$\gamma_n(g)(x) = \inf_{|y| \le n} (g(y) + v(x - y)).$$

Evidently $\gamma_n(g)(x) = g(x)$ for every x with $|x| \le n$. Set

$$F_n(g; \omega) = F(\gamma_n(g); \omega).$$

We have $F_n \in \mathscr{E}_n$. This is because if $R \ge n$ and $g_1(x) = g_2(x)$ for every x with $|x| \le R$, then $\gamma_n(g_1) = \gamma_n(g_2)$, which in turn implies that $F_n(g_1) = F_n(g_2)$. From (3.10) we learn that if $\omega \notin A(n, \ell)$ and $g_1(x) = g_2(x)$ for all x with $|x| \le n$, then $F(g_1; \omega)(x) = F(g_2; \omega)(x)$ for all x with $|x| \le \ell$. As a result, if $\omega \notin A(n, \ell)$, then $\|F_n(\cdot; \omega) - F(\cdot; \omega)\|_\ell = 0$ because $g(x) = \gamma_n(g)(x)$ for every $g \in \overline{\Gamma}$ and every x with $|x| \le n$. This in turn implies

$$D(F_n(\cdot; \omega), F(\cdot; \omega)) \le \sum_{\ell_1 = \ell + 1}^{\infty} 2^{-\ell_1} = 2^{-\ell}.$$

Given ω, if there exists a sequence (n_j, ℓ_j) such that $n_j, \ell_j \to \infty$ as $j \to \infty$, and $\omega \notin A(n_j, \ell_j)$ for every j, then we have $F(\cdot; \omega) \in \mathscr{E}$. This is because $D(F_{n_j}(\cdot; \omega), F(\cdot; \omega)) \to 0$ as $j \to \infty$. Using (3.10), it is not hard to show that such a sequence exists for almost all ω. This completes the verification of the condition (4). □

The proof of Lemma 3.4 when $\lambda^- = 0$ can be found in [14]. The case $\lambda^- \ne 0$ can be treated likewise.

LEMMA 3.4. *If $k_1, k_2 \in \Gamma$ and $k_1 \le k_2$, then*

$$h(i, t; \omega, k_1) \le h(i, t; \omega, k_2)$$

for every $i \in \mathbb{Z}^d$, $t \ge 0$ and $\omega \in \Omega_0$.

Define the probability measures $\mathscr{P}^\varepsilon$ on $\mathscr{D}_T$ by

$$\int F \, d\mathscr{P}^\varepsilon = \int F(\widehat{S}^\varepsilon(\omega)) Q(d\omega).$$

Let us also define $\mathscr{D}_T(q) \subseteq \mathscr{D}_T$ to be the set of functions $S(s, t; g)(x)$ with the following properties:

1. $S(s, t; g) = g$ whenever $t \le s$.

2. $g(x) - \lambda^-|t-s| \le S(s,t;g)(x) \le g(x) + \lambda^+|t-s|$.
3. The function $S(s,t;g)(x)$ is Lipschitz continuous in (s,t) with a Lipschitz constant that is uniformly bounded in x.
4. If $g_1(x) = g_2(x)$ for all x with $|x - x_0| \le R$ and $R > q(t-s)$, then $S(s,t;g_1)(x) = S(s,t;g_2)(x)$ for all (x,s,t) with $|x-x_0| \le R - q(t-s)$ and $t \in [s,T]$.
5. $S(s_1,s_3;g) = S(s_2,s_3;S(s_1,s_2;g))$ if $0 \le s_1 \le s_2 \le s_3 \le T$.

We are now ready to state the main result of this section.

THEOREM 3.5. *The sequence* $\{\mathscr{P}^\varepsilon\}$ *is precompact. Moreover, there exists a constant* q_0 *such that any limit point of* $\mathscr{P}^\varepsilon$ *is concentrated on the set* $\mathscr{D}_T(q_0)$.

We state and prove several lemmas that will prepare us for the proof of Theorem 3.5. Recall that the expectation with respect to the probability measure Q is denoted by E.

LEMMA 3.6. *There exists a constant* $C_1(T)$ *such that if* $\varepsilon, \delta > 0$ *and* $\varepsilon + \delta \le 1$, *then*

$$\sup_{k\in\Gamma} E \sup_{|x|\le T} \sup_{s\le T} \sup_{\substack{|t_1-t_2|\le\delta \\ t_1,t_2\in[s,T]}} |u^\varepsilon(x,t_1;k,\gamma_{s/\varepsilon}\omega) - u^\varepsilon(x,t_2;k,\gamma_{s/\varepsilon}\omega)| \le C_1(T)(\delta^{1/2} + \varepsilon^{d+\frac{1}{2}}\delta^{-d}).$$

When $\lambda^- = 0$ and the s-supremum is outside the expectation, Lemma 3.6 is established in [14] as Lemma 3.1. The proof of Lemma 3.6 is omitted because it is just a matter of modifying the proof of Lemma 3.1 in [14] in an obvious way.

For our purposes, we would like to move the k-supremum inside the expectation.

LEMMA 3.7. *There exist constants* $C_2(T)$ *and* $C_3(T)$ *such that for every* $\delta, \eta > 0$,

$$\limsup_{\varepsilon\to 0} E \sup_{g\in\overline{\Gamma}_T} \sup_{|x|\le T} \sup_{s\le T} \sup_{\substack{|t_1-t_2|\le\delta \\ t_1,t_2\in[s,T]}} \left|S^\varepsilon(s,t_1;g,\omega)(x) - S^\varepsilon(s,t_2;g,\omega)(x)\right| \le \eta + C_2(T)\delta^{1/2}\exp(C_3(T)\eta^{-d}).$$

Before proving Lemma 3.7, let us state three more lemmas. The first two lemmas concern the speed of propagation of the process h. A variant of Lemma 3.9 below appeared in [14] as Lemma 6.4.

Fix $x_0 \in \mathbb{R}^d$ and let us write $\Gamma^\varepsilon(R)$ for the set of pairs $(k_1,k_2) \in \Gamma$ such that

$$k_1(i) = k_2(i)$$

for all i with $|\varepsilon i - x_0| \le R$. We also write $\mathscr{B}^\varepsilon(T,R,C,\delta)$ for the set of $\omega \in \Omega_0$ such that

$$u^\varepsilon(x,t-s;k_1,\gamma_{s/\varepsilon}\omega) \ne u^\varepsilon(x,t-s;k_2,\gamma_{s/\varepsilon}\omega)$$

for some (x, s, t, k_1, k_2) with $|x - x_0| \le R - \delta - CT$, $s \in [0, T]$, $t \in [s, T]$ and $(k_1, k_2) \in \Gamma^\varepsilon(R)$.

LEMMA 3.8. *Suppose* $\omega \notin \mathscr{B}^\varepsilon(T, R, C, \delta)$, $k_1, k_2 \in \Gamma$, *and* $\varepsilon|k_1(i) - k_2(i)| \le l$ *for all* i *with* $|\varepsilon i - x_0| \le R$. *Then*

$$|u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) - u^\varepsilon(x, t - s; k_2, \gamma_{s/\varepsilon}\omega)| \le l,$$

for all x *with* $|x - x_0| \le R - \delta - CT$ *and every* (s, t) *with* $0 \le s \le t \le T$.

PROOF. The following claim is a straightforward consequence of Lemma 3.3 and the definition of $\mathscr{B}^\varepsilon$. If $\omega \notin \mathscr{B}^\varepsilon(T, R, C, \delta)$, $k_1, k_3 \in \Gamma$, and $k_1(i) \le k_3(i)$ for all i with $|\varepsilon i - x_0| \le R$, then

$$u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) \le u^\varepsilon(x, t - s; k_3, \gamma_{s/\varepsilon}\omega) \tag{3.13}$$

for all x with $|x - x_0| \le R - \delta - CT$ and every (s, t) with $0 \le s \le t \le T$. To see this, observe that $(k_3, \max(k_1, k_3)) \in \Gamma^\varepsilon(R)$, and if $\omega \notin \mathscr{B}^\varepsilon(T, R, C, \delta)$, then

$$u^\varepsilon(x, t - s; k_3, \gamma_{s/\varepsilon}\omega) = u^\varepsilon(x, t - s; \max(k_1, k_3), \gamma_{s/\varepsilon}\omega) \tag{3.14}$$

for every x with $|x - x_0| \le R - \delta - CT$. On the other hand, by Lemma 3.3,

$$u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) \le u^\varepsilon(x, t - s; \max(k_1, k_3), \gamma_{s/\varepsilon}\omega).$$

From this and (3.14) we deduce (3.13).

We next apply (3.13) with k_1 and $k_3 = k_2 + [l\varepsilon^{-1}]$ and use (2.11) to deduce

$$u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) \le u^\varepsilon(x, t - s; k_2, \gamma_{s/\varepsilon}\omega) + l.$$

The proof of

$$u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) \ge u^\varepsilon(x, t - s; k_2, \gamma_{s/\varepsilon}\omega) - l$$

is similar. □

LEMMA 3.9. *There exists a constant* C_4 *such that for every* $T, R, \delta > 0$,

$$\lim_{\varepsilon \to 0} Q\big(\mathscr{B}^\varepsilon(T, R, C_4, \delta)\big) = 0.$$

PROOF. Recall the Markov process $G(t) \subseteq \mathbb{Z}^d$ that was defined right after the statement of Lemma 3.3. Using the shape theorem (1.1), it is not hard to show that if

$$G(0) = G^\varepsilon(0) = \{i \mid \varepsilon i \in \overline{G}(0)\}$$

for some open set $\overline{G}(0)$, then

$$\lim_{\varepsilon \to 0} Q\left(\varepsilon G\left(\frac{t}{\varepsilon}; \omega\right) \subseteq \overline{G}(T) + B(0, \delta) \text{ for all } t \in [0, T]\right) = 1,$$

for every positive δ, where

$$\begin{aligned} B(z, \delta) &= \{x \mid |x - z| \le \delta\}, \\ \overline{G}(T) &= \{x \mid N(x - y) \le T \text{ for some } y \in \overline{G}(0)\}, \end{aligned}$$

where N is a suitable nondegenerate norm. Hence, for some constant C_4,

$$\lim_{\varepsilon\to 0} Q\left(\varepsilon G\left(\frac{t}{\varepsilon};\omega\right) \subseteq \overline{G}(0) + B(0, C_4T + \delta) \text{ for all } t \in [0, T]\right) = 1. \tag{3.15}$$

We now choose $\overline{G}(0) = \mathbb{R}^d - B(x_0, R)$. It is not hard to see that if $k_1 = k_2$ on the set $\mathbb{Z}^d - G^\varepsilon(0)$, then

$$u^\varepsilon(x, t; k_1, \omega) = u^\varepsilon(x, t; k_2, \omega)$$

for $x \notin \varepsilon G(\frac{t}{\varepsilon};\omega)$. We now argue that in fact

$$u^\varepsilon(x, t-s; k_1, \gamma_{s/\varepsilon}\omega) = u^\varepsilon(x, t-s; k_2, \gamma_{s/\varepsilon}\omega) \tag{3.16}$$

if $x \notin \varepsilon G(\frac{T}{\varepsilon};\omega)$, $s \in [0, T]$, $t \in [s, T]$. This is because the set

$$\{i \mid k_1(i) \neq k_2(i)\}$$

is contained in the set $G(\frac{s}{\varepsilon};\omega)$, and this set grows to at most $G^\varepsilon(\frac{T}{\varepsilon};\omega)$ after $\frac{T-s}{\varepsilon}$ seconds. Evidently (3.15) and (3.16) complete the proof. □

The next lemma gives us an upper bound on $|u^\varepsilon|$. Let $h^\pm(i, t)$ denote the processes with the infinitesimal generators

$$\mathscr{L}^+ F(k) = \lambda^+ \sum_{i\in\mathbb{Z}^d} (F(k^i) - F(k)),$$

$$\mathscr{L}^- F(k) = \lambda^- \sum_{i\in\mathbb{Z}^d} (F(k_i) - F(k))$$

and the initial conditions $h^\pm(\cdot, 0) \equiv 0$, respectively.

LEMMA 3.10. *For every $k \in \Gamma$, $i \in \mathbb{Z}^d$ and $t \geq 0$,*

$$h^-(i, t) + k(i) \leq h(i, t; k) \leq h^+(i, t) + k(i).$$

Moreover,

$$\lim_{\varepsilon\to 0} E\left|\varepsilon h^\pm\left(\left[\frac{x}{\varepsilon}\right], \frac{t}{\varepsilon}\right) \mp \lambda^\pm t\right|^2 = 0.$$

The proof of Lemma 3.10 is straightforward and omitted. We are now ready to prove Lemma 3.7.

PROOF OF LEMMA 3.7.

Step 1. Let Γ_ℓ denote the set of $k \in \Gamma$ such that $|k(0)| \leq \ell$. We first construct a set $G^\varepsilon_\eta(T) \subseteq \Gamma_{T\varepsilon^{-1}}$ such that for every $k \in \Gamma_{T\varepsilon^{-1}}$, there exists a $k' \in G^\varepsilon_\eta(T)$ with

$$\sup_{|x|\leq T}\left|\varepsilon k\left(\left[\frac{x}{\varepsilon}\right]\right) - \varepsilon k'\left(\left[\frac{x}{\varepsilon}\right]\right)\right| \leq \frac{1}{2}\eta. \tag{3.17}$$

To construct $G^{\varepsilon}_{\eta}(T)$, we divide $\mathbb{R}^{d+1}$ into disjoint cubes of the form

$$A = \prod_{j=1}^{d} \left[a_j, a_j + \frac{\eta}{8\bar{\alpha}} \right) \times \left[a_{d+1}, a_{d+1} + \frac{\eta}{4} \right)$$

where $\bar{\alpha} = \sum_r (\beta_r - \alpha_r)$. If two cubes A, A' share the same base set $\prod_{j=1}^{d} [a_j, a_j + \frac{\eta}{8\bar{\alpha}})$, we say that they belong to the same *column*. From such a partition, we can build a partition for the lattice $\mathbb{Z}^{d+1}$ by taking sets of the form

$$\widehat{A} = \{ a \in \mathbb{Z}^{d+1} \mid \varepsilon a \in A \}.$$

Let $Z(\varepsilon)$ denote the set of such $\widehat{A}$. We also write $\mathscr{A}(\varepsilon)$ for the set of finite subsets of $Z(\varepsilon)$. The columns of $\widehat{A}$ are defined as before. Note that the graph of $k \in \Gamma$ intersects at most two elements of each column in $Z(\varepsilon)$. Set

$$F^{\varepsilon}_{T}(k) = \{ \widehat{A} \in Z(\varepsilon) \mid (i, k(i)) \in \widehat{A} \text{ for some } i \text{ with } |\varepsilon i| \le T \}.$$

Clearly F^{ε}_{T} is a transformation from Γ into $\mathscr{A}(\varepsilon)$. Observe

$$F^{\varepsilon}_{T}(k) = F^{\varepsilon}_{T}(k') \Rightarrow \sup_{|\varepsilon i| \le T} |\varepsilon k(i) - \varepsilon k'(i)| \le \tfrac{1}{2}\eta. \tag{3.18}$$

Because of this, we choose $G^{\varepsilon}_{\eta}(T)$ to be a subset of $\Gamma_{T\varepsilon^{-1}}$ that has exactly one element in every level set of F^{ε}_{T}. More precisely, let $\overline{\mathscr{A}}(\varepsilon) = F^{\varepsilon}_{T}(\Gamma_{T\varepsilon^{-1}}) \subseteq \mathscr{A}(\varepsilon)$ and choose $G^{\varepsilon}_{\eta}(T)$ so that

$$G^{\varepsilon}_{\eta}(T) \cap \left(F^{\varepsilon}_{T}\right)^{-1}\{F\}$$

consists of one element for every $F \in \overline{\mathscr{A}}(\varepsilon)$. Because of (3.18), the set $G^{\varepsilon}_{\eta}(T)$ does satisfy the property associated with (3.17) that was described in the beginning of the proof.

Step 2. We next show that there exists a constant $k_0 = k_0(T)$ that is independent of (ε, η), and the cardinality of $G^{\varepsilon}_{\eta}(T)$ is at most $\exp(k_0 \eta^{-d})$. Since the set $G^{\varepsilon}_{\eta}(T)$ has the same cardinality as $\overline{\mathscr{A}}(\varepsilon)$, it suffices to show that $\overline{\mathscr{A}}(\varepsilon)$ has at most $\exp(k_0 \eta^{-d})$ elements for some constant k_0. This follows from two properties of the elements in $\overline{\mathscr{A}}(\varepsilon)$. First, each column in $Z(\varepsilon)$ can have at most two elements intersecting the graph of k. Second, there exists a universal constant ℓ_0 such that if the intersection of $F^{\varepsilon}_{T}(k)$ with a column is known and if we take an adjacent column, then there are at most ℓ_0 many choices for the intersection of $F^{\varepsilon}_{T}(k)$ with the adjacent column. We now start with the column above the origin. Since $k \in \Gamma_{T\varepsilon^{-1}}$, the number of choices for the intersection of the graph of k and the column above the origin is $O(T\eta^{-1})$. Then by going to an adjacent column we encounter ℓ_0 choices. In this way we can readily find an exponential bound of the form $\exp(k_0 \eta^{-d})$ on the cardinality of $\overline{\mathscr{A}}(\varepsilon)$.

Final step. Let C_4 be as in Lemma 3.9 and set $G = G^{\varepsilon}_{\eta}\big((1+C_4)T+1\big)$. From Lemma 3.6 and the previous steps we deduce

$$\begin{aligned}(3.19)\qquad & E\sup_{k\in G}\sup_{|x|\le T}\sup_{s\le T}\sup_{\substack{|t_1-t_2|\le\delta\\ t_1,t_2\in[s,T]}}|u^{\varepsilon}(x,t_1;k,\gamma_{s/\varepsilon}\omega)-u^{\varepsilon}(x,t_2;k,\gamma_{s/\varepsilon}\omega)|\\ &\quad\le C_1(T)\Big(\delta^{1/2}+\varepsilon^{d+\frac12}\delta^{-d}\Big)\exp\Big[k_0\big((1+C_4)T\big)\eta^{-d}\Big].\end{aligned}$$

Choose $x_0 = 0$ in the definition of $\mathscr{B}^{\varepsilon}$ and set $\Omega^{\varepsilon} = \Omega^{\varepsilon}(T) = \Omega_0 - \mathscr{B}^{\varepsilon}(T, (1+C_4)T+1, C_4, 1)$. From Lemma 3.9 we know that

$$(3.20)\qquad \lim_{\varepsilon\to 0} Q(\Omega^{\varepsilon}) = 1.$$

Moreover, (3.19) and Lemma 3.8 imply

$$\begin{aligned}(3.21)\qquad & E\sup_{g\in\overline{\Gamma}_T}\sup_{|x|\le T}\sup_{s\le T}\sup_{\substack{|t_1-t_2|\le\delta\\ t_1,t_2\in[s,T]}}\Big|u^{\varepsilon}(x,t_1;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)-u^{\varepsilon}(x,t_2;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)\Big|\mathbb{1}(\omega\in\Omega^{\varepsilon})\\ &\quad\le \eta+C_1(T)\Big(\delta^{1/2}+\varepsilon^{d-\frac12}\delta^{-d}\Big)\exp\Big[k_0\big((1+C_4)T\big)\eta^{-d}\Big].\end{aligned}$$

This is because replacing $u^{\varepsilon}(\cdot;k)$ with $u^{\varepsilon}(\cdot;k')$ for some $k'\in G$ results in an error that, by Lemma 3.8, is bounded in absolute value by $\frac{\eta}{2}$. On the other hand, if we replace Ω^{ε} in (3.21) with its complement $(\Omega^{\varepsilon})^c$, then we can apply Lemma 3.10 and (3.20) to deduce that the left-hand side of (3.21) goes to zero as $\varepsilon\to 0$: For some constant c_1,

$$\begin{aligned}& E\sup_{g\in\overline{\Gamma}_T}\sup_{|x|\le T}\sup_{s\le T}\sup_{\substack{|t_1-t_2|\le\delta\\ t_1,t_2\in[s,T]}}\Big|u^{\varepsilon}(x,t_1;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)-u^{\varepsilon}(x,t_2;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)\big|\mathbb{1}(\omega\notin\Omega^{\varepsilon})\\ &\quad\le 2E\sup\big\{|u^{\varepsilon}(x,t;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)|\mathbb{1}(\omega\notin\Omega^{\varepsilon})|\ g\in\overline{\Gamma}_T,\ s,|x|\le T,\ t\in[s,T]\big\}\\ &\quad\le 2E\sup\big\{\big[|u^{\varepsilon}(0,t;k^{\varepsilon}_g,\gamma_{s/\varepsilon}\omega)|+c_1T\big]\mathbb{1}(\omega\notin\Omega^{\varepsilon})\ |\ g\in\overline{\Gamma}_T,\ s\le T,\ t\in[s,T]\big\}\\ &\quad\le 2E\big[\varepsilon h^{+}(0,\varepsilon^{-1}T)-\varepsilon h^{-}(0,\varepsilon^{-1}T)+\varepsilon k^{\varepsilon}_g+c_1T\big]\mathbb{1}(\omega\notin\Omega^{\varepsilon})\to 0.\end{aligned}$$

In this and (3.21), it is not hard to replace u^{ε} with S^{ε}, completing the proof of lemma. □

We are now ready to establish Theorem 3.5.

PROOF OF THEOREM 3.5.

Step 1. We choose

$$(3.22)\qquad \eta = (4C_3(T))^{\frac1d}\left(\log\frac1\delta\right)^{-\frac1d}$$

in Lemma 3.7. As a result

$$\begin{aligned}(3.23)\qquad & \limsup_{\varepsilon\to 0} E\sup_{g\in\overline{\Gamma}_T}\sup_{s\le T}\sup_{|x|\le T}\sup_{\substack{|t_1-t_2|\le\delta\\ t_1,t_2\in[s,T]}}|S^{\varepsilon}(s,t_1;g,\omega)(x)-S^{\varepsilon}(s,t_2;g,\omega)(x)|\\ &\quad\le (4C_3(T))^{\frac1d}\left(\log\frac1\delta\right)^{-\frac1d}+C_2(T)\delta^{\frac14}.\end{aligned}$$

Let us write

$$\widehat{T} = (1 + C_4)T + 1,$$

where C_4 appeared in Lemma 3.9. Let $\Omega_1^\varepsilon(T)$ denote the set of $\omega \in \Omega_0$ such that

$$u^\varepsilon(x, t - s; k_1, \gamma_{s/\varepsilon}\omega) = u^\varepsilon(x, t - s; k_2, \gamma_{s/\varepsilon}\omega)$$

for all $(k_1, k_2) \in \Gamma^\varepsilon(\widehat{T})$ and all (x, s, t) with $|x| \le T$, $s \in [0, T]$, $t \in [s, T]$. From Lemma 3.9 we know that

$$\lim_{\varepsilon \to 0} Q(\Omega_1^\varepsilon(T)) = 1. \tag{3.24}$$

Evidently, if $\varepsilon|i| \le \widehat{T}$, then

$$\varepsilon|k_{g_1}^\varepsilon(i) - k_{g_2}^\varepsilon(i)| \le \|g_1 - g_2\|_{\widehat{T}} + c_1\varepsilon,$$

for some constant c_1. From this and Lemma 3.8 we deduce that for every $\omega \in \Omega_1^\varepsilon(T)$,

$$\|S^\varepsilon(s, t; g_1, \omega) - S^\varepsilon(s, t; g_2, \omega)\|_T \le \|g_1 - g_2\|_{\widehat{T}} + c_1\varepsilon. \tag{3.25}$$

Moreover, for every $\omega \in \Omega_0$,

$$S^\varepsilon(s, t; g, \omega)(0) \le g(0) + \varepsilon p^+\left(0, \frac{t}{\varepsilon}\right) + \varepsilon p^-\left(0, \frac{t}{\varepsilon}\right), \tag{3.26}$$

$$S^\varepsilon(s, t; g, \omega)(x) - S^\varepsilon(s, t; g, \omega)(0) \le v(x). \tag{3.27}$$

From (3.24)–(3.27) we can readily deduce,

$$\limsup_{\varepsilon \to 0} E \sup_{0 \le s \le t \le T} \sup_{\|g_1 - g_2\|_{\widehat{T}} \le \delta} \|S^\varepsilon(s, t; g_1, \omega) - S^\varepsilon(s, t; g_2, \omega)\|_T \le \delta, \tag{3.28}$$

$$\lim_{\varepsilon \to 0} E \sup_{s, t \le T} \sup_{g \in \overline{\Gamma}_T} \|S^\varepsilon(s, t; g, \omega)\|_T < \infty. \tag{3.29}$$

Moreover, (3.23) implies

$$\limsup_{\varepsilon \to 0} E \sup_{g \in \overline{\Gamma}_T} \sup_{s \in [0, T]} \sup_{\substack{|t_1 - t_2| \le \delta \\ t_1, t_2 \in [s, T]}} \|S^\varepsilon(s, t_1; g) - S^\varepsilon(s, t_2; g)\|_T \le c_2 \left(\log \frac{1}{\delta}\right)^{-1/d}, \tag{3.30}$$

for some constant $c_2 = c_2(T)$.

Step 2. In the previous step, we discussed the equicontinuity in the t-variable (3.30) and in the g-variable (3.28)–(3.29). We now turn to the equicontinuity in the s-variable. Let $\Omega_1^\varepsilon(T)$ be as in the previous step. If $\omega \in \Omega_1^\varepsilon(T)$ and $0 \le s_1 \le s_2 \le t \le T$, then by (3.25),

$$\begin{aligned} &\|S^\varepsilon\big(s_2, t; S^\varepsilon(s_1, s_2; g, \omega), \omega\big) - S^\varepsilon(s_2, t; g, \omega)\|_T \\ &\qquad \le \|S^\varepsilon(s_1, s_2; g, \omega) - g\|_{\widehat{T}} + c_1\varepsilon. \end{aligned} \tag{3.31}$$

This and the semigroup property (3.4) imply that for $\omega \in \Omega_1^\varepsilon(T)$,

$$\begin{aligned} &\|S^\varepsilon(s_1, t; g, \omega) - S^\varepsilon(s_2, t; g, \omega)\|_T \\ &\quad \le \|S^\varepsilon(s_1, s_2; g, \omega) - g\|_{\widehat{T}} + c_1\varepsilon. \end{aligned} \tag{3.32}$$

We then apply Lemma 3.7 with η as in (3.22) to conclude

$$\begin{aligned} &\limsup_{\varepsilon\to 0} E \sup_{g\in\overline{\Gamma}_{\widehat{T}}} \sup_{\substack{s_1-s_2\le\delta \\ s_1, s_2\in[0,T]}} \sup_{t\in[s_2,T]} \|S^\varepsilon(s_1, t; g) - S^\varepsilon(s_2, t; g)\|_T \mathbb{1}(\omega \in \Omega_1^\varepsilon(T)) \\ &\quad \le c_3\left(\log\frac{1}{\delta}\right)^{-\frac{1}{d}} \end{aligned} \tag{3.33}$$

for some constant c_3. As in the previous step, we can use (3.26)–(3.27) to drop $\mathbb{1}(\omega \in \Omega_1^\varepsilon(T))$ in (3.33).

Step 3. Let us write $\mathscr{X}$ for $\mathscr{D}([0, T]; \mathscr{E})$. With this abbreviation, we may write

$$\mathscr{D}_T = \mathscr{D}([0, T]; \mathscr{X}).$$

From (3.28)–(3.30), (3.33) and (3.9) we would like to deduce that the family $(\mathscr{P}^\varepsilon\colon\ \varepsilon > 0)$ is precompact in $\mathscr{D}_T$. So far we have an approximate equicontinuity of the sequence $\widehat{S}^\varepsilon$. For the precompactness, we need to verify that for each s, the probability that the process $\widehat{S}^\varepsilon(s, \cdot; \cdot)$ does not belong to a small neighborhood of a compact subset of $\mathscr{X}$ is small. Since we have an approximate equicontinuity of $\widehat{S}^\varepsilon$ in the (t, g) variable, we need to show that for each (s, t), the probability that the process $\widehat{S}^\varepsilon(s, t; \cdot)$ does not belong to a small neighborhood of a compact subset of $\mathscr{E}$ is small. Given q and R with $R \ge q$, let us define $\mathscr{E}^r_{R,q}$ to be the set of $F \in \widehat{\mathscr{E}}$ such that $\|F(g)\|_0 \le r$, and if $g_1(x) = g_2(x)$ for every x with $|x| \le R$, then $F(g_1)(x) = F(g_2)(x)$ for every x with $|x| \le R - q$. Lemma 3.9 and the fact that

$$|\widehat{S}^\varepsilon(s, t; g)(0) - g(0)| \le \varepsilon p^+\left(0, \frac{T}{\varepsilon}\right) + \varepsilon p^-\left(0, \frac{T}{\varepsilon}\right),$$

imply that

$$\lim_{\varepsilon\to 0} Q\left(\widehat{S}^\varepsilon(s, t; \cdot) \notin \mathscr{E}^r_{R,q}\right) = 0 \tag{3.34}$$

for every $R \in (q, \infty)$, provided that $q > C_4$ and r is sufficiently large. We now argue that the set $\mathscr{E}^r_{R,q}$ is contained in a small neighborhood of the set $\mathscr{E}^r_R$ if R is large. (Recall that by Lemma 3.2, the set $\mathscr{E}^r_R$ is compact.) As in the proof of Lemma 3.2, define $F_R(g) = F(\gamma_R(g))$ where

$$\gamma_R(g)(x) = \inf_{|y|\le R}(g(y) + v(x - y)).$$

As in the proof of Lemma 3.3, we can readily show that if $F \in \mathscr{E}^r_{R,q}$, then $F_R \in \mathscr{E}^r_R$ and $D(F_R, F) \le 2^{-R+q+1}$. As a result, the set $\mathscr{E}^r_{R,q}$ is contained in the η_R-neighborhood of the compact set $\mathscr{E}^r_R$ for $\eta_R = 2^{-R+q+1}$. Hence (3.34) implies that

$$\lim_{\varepsilon\to 0} Q\big(\widehat{S}^{\varepsilon}(s,t;\cdot)\notin (\mathscr{E}_R^r)^{\hat{\eta}_R}\big)=0,$$

where $\hat{\eta}_R = 2^{-R+C_4+2}$ and we are using the notation

$$A^{\eta}=\big\{F\in\widehat{\mathscr{E}}\mid D(F,G)\le\eta \text{ for some } G\in A\big\}.$$

From this, (3.28)–(3.30), (3.33) and (3.9), one can readily deduce the precompactness of the family $(\mathscr{P}^{\varepsilon}:\varepsilon>0)$ by standard arguments.

Step 4. In this step, we are going to use some well-known facts about the Skorohod topology that can be found in Chapter 3 of [6]. Recall that we write $\mathscr{X}$ for $\mathscr{D}([0,T];\mathscr{E})$.

Define $\mathscr{F}_1$ to be the set of $S\in\mathscr{D}_T$ for which the condition (5) of the definition $\mathscr{D}_T(q)$ is satisfied. If we show that the set $\mathscr{F}_1$ is closed in $\mathscr{D}_T$, then we can use $\mathscr{P}^{\varepsilon}(\mathscr{F}_1)=1$ to deduce

$$\mathscr{P}(\mathscr{F}_1)=1. \tag{3.35}$$

Suppose that the sequence $S_n\in\mathscr{F}_1$ converges to S with respect to the Skorohod topology of $\mathscr{D}([0,T];\mathscr{X})$. We would like to show that $S\in\mathscr{F}_1$. For this, we need to establish

$$S(s_1,s_3;g)=S\big(s_2,s_3;S(s_1,s_2;g)\big), \tag{3.36}$$

for every (s_1,s_2,s_3) with $s_1\le s_2\le s_3$ and every $g\in\overline{\Gamma}$. As we vary s in $S(s,t;g)$, we have a Skorohod function $S(\cdot)\in\mathscr{D}([0,T];\mathscr{X})$ that, by definition, is continuous at all but countably many points s. Moreover, for a fixed point s, we have a Skorohod function $S(s,\cdot)\in\mathscr{X}$ that is also continuous at all but countably many points t. From these considerations, we deduce that it suffices to establish (3.36) for $(s_1,s_2,s_3)\in A$ where A is defined to be the set of points (s_1,s_2,s_3) such that $s_1\le s_2\le s_3$, $S\in\mathscr{D}([0,T];\mathscr{X})$ is continuous at $s=s_1$ and $s=s_2$, $S(s_1,\cdot)\in\mathscr{X}$ is continuous at $t=s_2$ and $t=s_3$ and $S(s_2,\cdot)\in\mathscr{X}$ is continuous at $t=s_3$. By assumption,

$$S_n(s_1,s_3;g)=S_n\big(s_2,s_3;S_n(s_1,s_2;g)\big), \tag{3.37}$$

for every n. Since the convergence in Skorohod topology implies the pointwise convergence at every continuity point of the limit S, we can readily show that the left-hand side of (3.37) converges to the left-hand side of (3.36). For the same reason, $\lim_{n\to\infty} g_n = S(s_1,s_2;g)$, where $g_n=S_n(s_1,s_2;g)$. Again, since $(s_1,s_2,s_3)\in A$, we have $F_n=S_n(s_2,s_3)\in\mathscr{E}$ converges to $F=S(s_2,s_3)\in\mathscr{E}$. As a result,

$$\lim_{n\to\infty} F_n(g_n)=F\big(S(s_1,s_2;g)\big),$$

because

$$\|F_n(g_n)-F(g_n)\|_{\ell}\le\sup_{g\in\overline{\Gamma}}\|F_n(g)-F(g)\|_{\ell}=\|F_n-F\|_{\ell}\to 0,$$

as $n\to\infty$. This completes the proof of (3.35).

Define $\mathscr{F}_2$ to be the set of $S \in \mathscr{D}_T$ for which the condition (1) of the definition $\mathscr{D}_T(q)$ is satisfied. As in the previous paragraph, we can show that the set $\mathscr{F}_2$ is closed. This can be used to deduce

$$\mathscr{P}(\mathscr{F}_2) = 1. \tag{3.38}$$

Fix a set of points $(s_1, t_1, x_1), (s_2, t_2, x_2), \ldots, (s_r, t_r, x_r)$ and a positive number η. Let $\mathscr{F}_3$ denote the set of $S \in \mathscr{D}_T$ such that for every $g \in \overline{\Gamma}$ and i,

$$g(x_i) - (\lambda^- + \eta)|t_i - s_i| \le S(s_i\pm, t_i\pm; g)(x_i) \le g(x_i) + (\lambda^+ + \eta)|t_i - s_i|.$$

Here by $S(s_i\pm, \cdot; \cdot)$ we mean the right and left limits of $S(\cdot) \in \mathscr{D}([0, T]; \mathscr{X})$ at the point s_i, and by $S(s_i\pm, t_i\pm; \cdot)$ we mean the right and left limits of $S(s_i\pm, \cdot) \in \mathscr{X}$ at the point t_i. It is not hard to show that the set $\mathscr{F}_3$ is closed, and by Lemma 3.10,

$$\lim_{\varepsilon \to 0} \mathscr{P}^\varepsilon(\mathscr{F}_3) = 1.$$

This implies that for any limit point $\mathscr{P}$,

$$\mathscr{P}(\mathscr{F}_3) = 1. \tag{3.39}$$

Fix $x_0 \in \mathbb{R}^d$, $R, \delta > 0$ and $(s_0, t_0) \in [0, T]$ with $s_0 < t_0$. Let $\mathscr{F}(s_0, t_0, x_0, R, \delta, q)$ be the set of $S \in \mathscr{D}_T$ such that if $g_1(x) = g_2(x)$ for every x with $|x - x_0| \le R$, then $S(s, t; g_1)(x) = S(s, t; g_2)(x)$ for every x with $|x - x_0| \le R - \delta - q(t_0 - s_0)$ and (s, t) with $s_0 \le s \le t \le t_0$. It is not hard to show that the set $\mathscr{F}(s_0, t_0, x_0, R, \delta, q)$ is closed. Lemma 3.9 implies

$$\lim_{\varepsilon \to 0} \mathscr{P}^\varepsilon\big(\mathscr{F}(s_0, t_0, x_0, R, \delta, C_4)\big) = 1.$$

This in turn implies that for any limit point $\mathscr{P}$,

$$\mathscr{P}\big(\mathscr{F}(s_0, t_0, x_0, R, \delta, C_4)\big) = 1.$$

We then send $\delta \to 0$ and vary (s_0, t_0, x_0, R) in a countable dense set. From this, (3.35), (3.38) and (3.39) we deduce that any limit point $\mathscr{P}$ is concentrated on the set of S for which the conditions (1)–(2) and (4)–(5) of the definition $\mathscr{D}_T(C_4)$ are satisfied.

Final step. It remains to show that every limit point of $\mathscr{P}^\varepsilon$ is concentrated on the set of (s, t)-Lipschitz functions with a Lipschitz constant that is uniformly bounded in x.

Define

$$R^\varepsilon(s, t; g, \omega)(x) = \lambda^+ \mathbb{1}\left(h\left(\cdot, \frac{t-s}{\varepsilon}; k_g^\varepsilon, \gamma_{s/\varepsilon}\omega\right)^i \in \Gamma\right)$$
$$-\lambda^- \mathbb{1}\left(h\left(\cdot, \frac{t-s}{\varepsilon}; k_g^\varepsilon, \gamma_{s/\varepsilon}\omega\right)_i \in \Gamma\right),$$

$$M^\varepsilon(s, t; g, \omega)(x) = S^\varepsilon(s, t; g, \omega)(x) - g(x) - \int_s^t R^\varepsilon(s, \theta; g, \omega)(x)\, d\theta,$$

for (s,t) with $s \le t$ and $i = [x/\varepsilon]$. It is well known that for any local function f, the processes

$$M_{t_2} = f(h(t_2)) - f(h(t_1)) - \int_{t_1}^{t_2} \mathscr{L} f(h(\theta))\, d\theta,$$

$$N_{t_2} = M_{t_2}^2 - \int_{t_1}^{t_2} (\mathscr{L} f^2 - 2f \mathscr{L} f)(h(\theta))\, d\theta$$

are martingales for $t_2 \ge t_1$. By choosing $f(h) = \varepsilon h(i)$, $(t_1, t_2) = (s/\varepsilon, t/\varepsilon)$, and using

$$\begin{aligned} EM_{t_2}^2 &= E \int_{t_1}^{t_2} (\mathscr{L} f^2 - 2f \mathscr{L} f)(h(\theta))\, d\theta \\ &= E \int_{s/\varepsilon}^{t/\varepsilon} \varepsilon^2 \big(\lambda^+ \mathbb{1}(h(\theta)^i \in \Gamma) + \lambda^- \mathbb{1}\big(h(\theta)_i \in \Gamma)\big) d\theta, \end{aligned}$$

we see that for every $x \in \varepsilon\mathbb{Z}^d$,

$$E[M^\varepsilon(s,t;g,\omega)(x)^2] \le c_1 |t-s| \varepsilon,$$

with $c_1 = \lambda^+ + \lambda^-$. We then apply Doob's inequality to deduce

$$E \sup_{0 \le s \le t \le T} [M^\varepsilon(s,t;g,\omega)(x)^2] \le 4c_1 T\varepsilon,$$

for every $x \in \varepsilon\mathbb{Z}^d$. From this we deduce that for some constant c_2,

$$\sup_x \sup_{g \in \overline{\Gamma}} E \sup_{0 \le s \le t \le T} |M^\varepsilon(s,t;g,\omega)(x)| \le c_2 \sqrt{\varepsilon}. \tag{3.40}$$

Set $R^\varepsilon(s,t;g,\omega) = 0$ whenever $s > t$. For each ε, g and ω, the function $R^\varepsilon(s,t;g,\omega)(x)$ is a bounded function of (s,t,x) with values in the interval $[-\lambda^-, \lambda^+]$. Let Σ denote the set of measurable functions of $(s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d$ with values in $[-\lambda^-, \lambda^+]$. We equip this set with the weak topology (the set Σ is a compact subset of the space of signed measures equipped with the weak topology). A sequence $R_n \in \Sigma$ converges to $R \in \Sigma$ if and only if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \int_0^T \int_0^T \phi R_n\, dx\, ds\, dt = \int_{\mathbb{R}^d} \int_0^T \int_0^T \phi R\, dx\, ds\, dt$$

for every continuous function ϕ of compact support. We fix g and the map

$$\omega \mapsto (S^\varepsilon(s,t;g), R^\varepsilon(s,t;g))$$

induces a probability measure $\widetilde{\mathscr{P}}_g^\varepsilon$ on the space

$$\mathscr{D}\big([0,T]; \mathscr{D}([0,T]; \overline{\Gamma})\big) \times \Sigma.$$

Since the space Σ is compact, we may study the limit points of $\widetilde{\mathscr{P}}_g^\varepsilon$. Let $\widetilde{\mathscr{P}}_g$ be a limit point and fix a continuous function $\phi\colon \mathbb{R}^d \to \mathbb{R}$ of compact support.

Define

$$\Phi(S,R)=\sup_{0\le s\le t\le T}\left|\int S(s,t;g)(x)\phi(x)\,dx-\int g(x)\phi(x)\,dx\right.$$
$$\left.-\int_s^t\int R(s,\theta;g)(x)\phi(x)\,dx d\theta\right|.$$

From (3.40) we deduce

$$\lim_{\varepsilon\to 0}\int\Phi(S,R)\widetilde{\mathscr{P}}_g^{\varepsilon}(dS,dR)=0.$$

This implies

$$\int\Phi(S,R)\widetilde{\mathscr{P}}_g(dS,dR)=0.$$

By picking ϕ from a countable dense subset of $C_c(\mathbb{R}^d)$, we deduce that the measure $\widetilde{\mathscr{P}}_g$ is concentrated on the set of (S,R) such that

$$S(s,t;g)(x)=g(x)+\int_s^t R(s,\theta,g)(x)\,d\theta,$$

for every (s,t) with $s\le t$. Since $R\in\Sigma$, we deduce the Lipschitzness of S in t with the Lipschitz constant at most $\max(\lambda^-,\lambda^+)$. For the Lipschitzness in s-variable, observe that if $s_1<s_2<t$, then we apply (3.36) to assert that for almost all S in the support of $\mathscr{P}$,

$$S(s_2,t;g)-S(s_1,t;g)=S(s_2,t;g)-S(s_2,t;S(s_1,s_2;g)).$$

From this, the condition (5) of the definition $\mathscr{D}_T$, and Lemma 3.1 we deduce that for constant $c_3=c_3(\ell)$,

$$\|S(s_2,t;g)-S(s_1,t;g)\|_\ell\le\|g-S(s_1,s_2;g)\|_{\ell+q_0}$$
$$=\left\|\int_{s_1}^{s_2}R(\theta,t;g)\,d\theta\right\|_{\ell+q_0}\le c_3|s_2-s_1|$$

almost surely with respect to the measure $\mathscr{P}$. This completes the proof. □

For the sake of definiteness, we chose the initial height functions of the form $k_g^\varepsilon(i)=[\varepsilon^{-1}g(\varepsilon i)]$ in the definition of our microscopic semigroup S^ε. In Lemma 3.11 below, we show that the initial height function k_g^ε can be replaced with any height function that has the same macroscopic profile g. We only sketch the proof of Lemma 3.11, which is a straightforward consequence of Lemmas 3.8 and 3.9.

LEMMA 3.11. *Assume that for some* $g\in\overline{\Gamma}$,

$$\lim_{\varepsilon\to 0}E\sup_{x\in B_0}|u^\varepsilon(x,0)-g(x)|=0$$

for every bounded $B_0 \subset \mathbb{R}^d$. *Then*

$$\lim_{\varepsilon\to 0} E \sup_{(x,t)\in B} |u^\varepsilon(x,t) - S^\varepsilon(0,t;g)| = 0$$

for every bounded set $B \subset \mathbb{R} \times [0,\infty)$.

PROOF. Let us use $\omega_0 \in \Omega^0$ for the randomness coming from the initial data and display this randomness in our notations by writing $u^\varepsilon(x,0;\omega_0)$ for the initial data. We also write $u^\varepsilon(x,t;\omega_0;\omega)$ for the resulting rescaled height function at a later time t. Given $R, \eta > 0$, we can find a positive number $\varepsilon_0 = \varepsilon_0(R,\eta)$ and a set $\Omega^1 = \Omega^1(R,\eta,\varepsilon) \subseteq \Omega^0$ such that

$$P(\Omega^0 - \Omega^1) \le \eta \tag{3.41}$$

and

$$|u^\varepsilon(x,0;\omega_0) - g(x)| \le \eta, \tag{3.42}$$

for every x with $|x| \le R$, every $\varepsilon \in (0,\varepsilon_0)$ and every $\omega_0 \in \Omega^1$. From this and Lemma 3.8 we deduce that

$$|u^\varepsilon(x,t;\omega_0;\omega) - S_0^\varepsilon(0,t;g,\omega)(x)| \le \eta, \tag{3.43}$$

for x with $|x| \le R - \delta - C_4T$, $\omega_0 \in \Omega^1$, $t \in [0,T]$ and $\omega \notin \mathscr{B}^\varepsilon(T,R,C_4,\delta)$. The rest of the proof is straightforward and follows from (3.41)–(3.43) and Lemma 3.9. □

4. Equilibrium measures. Throughout this section we assume $d = 1$, $\alpha_1 = 0$ and $\beta_1 > 0$. Recall that by Remark 2.4, we may assume $\alpha_1 = 0$ without loss of generality.

Define $k_\rho(i) = [i\rho]$ for $\rho \in [0,\beta_1]$ and $i \in \mathbb{Z}$. We now state the main result of this section.

THEOREM 4.1. *There exists a function* H: $[0,\beta_1] \to \mathbb{R}$ *such that*

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le T} E\big|u^\varepsilon(0,t;\omega,k_\rho) + tH(\rho)\big| = 0. \tag{4.1}$$

In this section, we mostly follow [5]. Our proof of Lemma 4.2 below uses the same ideas as in [5], but our presentation is somewhat different and is inspired by [12]. The arguments that are used for Lemmas 4.5 and 4.6 are identical to those that appeared in [5]. For the rest of the section after Lemma 4.7, we use a rather different approach to complete the proof of Theorem 4.1.

Recall $\eta(i,t) = h(i+1,t) - h(i,t)$. Note that η is also a Markov process with the generator

$$\begin{aligned}\mathscr{A}f(\eta) = \lambda^+ \sum_i b(\eta(i),\eta(i-1))(f(\eta^{i,i-1}) - f(\eta))\\ + \lambda^- \sum_i b(\eta(i),\eta(i+1))(f(\eta^{i,i+1}) - f(\eta)),\end{aligned}$$

where

$$\eta^{i,j}(\ell) = \begin{cases} \eta(i) - 1, & \text{if } \ell = i, \\ \eta(j) + 1, & \text{if } \ell = j, \\ \eta(\ell), & \text{otherwise }, \end{cases}$$

$$b(n, m) = \mathbb{1}(n > 0, m < \beta_1).$$

Evidently b is nondecreasing in n and nonincreasing in m. Moreover, if

$$\eta(i, t) = h(i+1, t; k_1) - h(i, t; k_1),$$
$$\zeta(i, t) = h(i+1, t, k_2) - h(i, t; k_2),$$

then both processes η and ζ evolve according to $\mathscr{A}$ and they are *coupled* in such a way that when

$$b(\eta(i, t), \eta(j, t)) = b(\zeta(i, t), \zeta(j, t)) = 1$$

with $j = i+1$ or $i-1$, then an η-particle and a ζ-particle at site i jump to site j simultaneously. The generator of the coupled process (η, ζ) will be denoted by $\widetilde{\mathscr{A}}$. Let us define the shift operator τ_i by

$$(\tau_i \eta)(j) = \eta(i - j).$$

Initially we assume that the process $(\eta(\cdot, 0), \zeta(\cdot, 0))$ is distributed according to a measure $\tilde{\mu}_0(d\eta, d\zeta)$ that is translation invariant and ergodic with respect to the shift operator τ. The distribution of $(\eta(\cdot, t), \zeta(\cdot, t))$ will be denoted by $\tilde{\mu}_t(d\eta, d\zeta)$. It is well known that the probability measure $\tilde{\mu}$ is also ergodic (see, e.g., [8], page 38). We write $(\mu_t^1(d\eta), \mu_t^2(d\zeta))$ for the marginals of $\tilde{\mu}_t$, and ρ_1, ρ_2 for the average particle density of the marginals,

$$\rho_1 = \int \eta(0)\mu_t^1(d\eta), \qquad \rho_2 = \int \zeta(0)\mu_t^2(d\zeta).$$

From the translation invariance of $\tilde{\mu}_t$ and the conservation of the number of particles, it is not hard to deduce that the numbers ρ_1 and ρ_2 are independent of t.

LEMMA 4.2. *Let $\tilde{\mu}_t$ be as above and $\rho_1 \le \rho_2$. Then*

$$\lim_{t\to\infty} \int (\eta(0) - \zeta(0))^+ \tilde{\mu}_t(d\eta, d\zeta) = 0. \tag{4.2}$$

The next lemma can be found in [5] or [3] and its proof is omitted.

LEMMA 4.3. *For every $i \ne j$,*

$$\lim_{t\to\infty} \int \mathbb{1}(\eta(i) - \zeta(i) > 0, \eta(j) - \zeta(j) < 0)\tilde{\mu}_t(d\eta, d\zeta) = 0.$$

PROOF OF LEMMA 4.2.

Step 1. Let us assume that there exist a positive number δ and a sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that,

$$\int (\eta(0) - \zeta(0))^{\pm} \tilde{\mu}_{t_n}(d\eta, d\zeta) \geq \delta. \tag{4.3}$$

From such an assumption, we eventually arrive at a contradiction.

As the first step, we tag particles in such a way that we can differentiate between three types of particles. Define $\gamma(i) = \eta(i) \wedge \zeta(i)$ and regard $\gamma(i)$ as the occupation number of the *neutral* particles. When $\eta(i) > \zeta(i)$, the discrepancy $\eta(i) - \zeta(i)$ is regarded as the occupation number of *positive* particles and when $\zeta(i) > \eta(i)$, the discrepancy $\zeta(i) - \eta(i)$ is regarded as the occupation number of *negative* particles. We label the positive and negative particles and write $x_r^{\pm}$ for the location of the rth positive and negative particle, where r varies in the set of integers. The labeling is done in such a way that we always have

$$r_1 < r_2 \quad \Rightarrow \quad x_{r_1}^{\pm} \leq x_{r_2}^{\pm}.$$

Let us write $\mathbf{x}^{\pm}$ for the collection of the particle locations $x_r^{\pm}$'s with r in a suitable subset of $\mathbb{Z}$. It is possible to define a Markov process $(\mathbf{x}^{-}(t), \mathbf{x}^{+}(t), \gamma(\cdot, t))$ so that we always have

$$\sum_r \mathbb{1}(x_r^{\pm}(t) = i) = (\eta(i,t) - \zeta(i,t))^{\pm}, \qquad \gamma(i,t) = \eta(i,t) \wedge \zeta(i,t).$$

We omit the tedious and straightforward definition of the generator of the process $(\mathbf{x}^{+}, \mathbf{x}^{-}, \gamma)$ and only make some remarks about it. There are several cases to consider when we examine the jump rates for the process $(\mathbf{x}^{-}, \mathbf{x}^{+}, \gamma)$:

Case (i) If $b(\eta(i,t), \eta(i \pm 1, t)) = b(\zeta(i,t), \zeta(i \pm 1, t)) = 1$, then $b(\gamma(i,t), \gamma(i \pm 1, t)) = 1$ and a γ-particle jumps from the site i to the site $i \pm 1$ with rate $\lambda^{\mp}$.

Case (ii) If $b(\eta(i,t), \eta(i \pm 1, t)) = 1$; $b(\zeta(i,t), \zeta(i \pm 1, t)) = 0$ and $b(\gamma(i,t), \gamma(i \pm 1, t)) = 1$, then a γ-particle jumps from the site i to the site $i \pm 1$ with rate $\lambda^{\mp}$, and simultaneously a negative particle jumps from $i \pm 1$ to i.

Case (iii) If $b(\eta(i,t), \eta(i \pm 1, t)) = 1$; $b(\zeta(i,t), \zeta(i \pm 1, t)) = 0$ and $b(\gamma(i,t), \gamma(i \pm 1, t)) = 0$, then a positive particle jumps from the site i to the site $i \pm 1$ with rate $\lambda^{\mp}$. Moreover, if there is a negative particle at the site $i \pm 1$, then the jumped positive particle and the negative particle are annilated and replaced with a neutral particle.

Case (iv) In the previous Cases (ii)–(iv), interchange the role of η with ζ, and the role of positive particle with negative particle.

In summary, the $\mathbf{x}^{\pm}$ and γ particles jump with the exclusion rules but the γ particles have *priority* to $\mathbf{x}^{\pm}$ particles, and when an $\mathbf{x}^{\pm}$ particle jumps to a site that has a particle of the opposite sign, then both particles are annihilated and replaced with one neutral particle. As a result, the $\mathbf{x}^{\pm}$ particles may have a finite *lifetime* and each $x_r^\pm(t)$ is defined for $t < \sigma_r^\pm$, where σ_r^- and σ_r^+ are two random times. One can readily verify

$$r_1 < r_2, \qquad t < \min\left(\sigma_{r_1}^\pm, \sigma_{r_2}^\pm\right) \quad \Rightarrow \quad x_{r_1}^\pm(t) \le x_{r_2}^\pm(t). \tag{4.4}$$

We say a particle is *immortal* if its lifetime is infinite, otherwise such a particle is called *mortal*. Define

$$\eta^\pm(i,t,\hat{t}\,) := \sum_r \mathbb{1}(x_r^\pm(t) = i, \sigma_r^\pm > \hat{t}\,), \qquad \eta^\pm(i,t) := \eta^\pm(i,t,t),$$

for $\hat{t} \ge t$. The distribution of the pair $(\eta^-(\cdot,t,\hat{t}\,), \eta^+(\cdot,t,\hat{t}\,))$ will be denoted by $\nu_{t,\hat{t}}$. Evidently, the measure $\nu_{t,\hat{t}}$ is translation invariant.

Step 2. Define

$$X^\pm(t,\hat{t}\,) = \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{i=1}^{\ell} \eta^\pm(i,t,\hat{t}\,).$$

The existence of such a (possibly random) limit follows from the ergodic theorem. Since the speed of propagation is bounded on average, it is not hard to show

$$X^\pm(t,\hat{t}\,) = X^\pm(\hat{t},\hat{t}\,). \tag{4.5}$$

In fact, it is not hard to construct a sequence of random variables $L_\ell = L_\ell(\omega) \in [0,\ell]$ such that

$$\sup_\ell EL_\ell < \infty \tag{4.6}$$

and

$$\sum_{i=-L_\ell}^{\ell+L_\ell} \eta^\pm(i,t,\hat{t}\,) \ge \sum_{i=1}^{\ell} \eta^\pm(i,\hat{t}\,) \ge \sum_{i=L_\ell}^{\ell-L_\ell} \eta^\pm(i,t,\hat{t}\,). \tag{4.7}$$

From this and (4.6), we can readily deduce (4.5). On the other hand, from the ergodicity of the process (η,ζ) with respect to the measure $\tilde{\mu}_{t_n}$, and our assumption (4.3), we deduce

$$X^\pm(t_n,t_n) = \int (\eta(0)-\zeta(0))^\pm \tilde{\mu}_{t_n}(d\eta,d\zeta) \ge \delta.$$

From this and (4.5) we conclude that if $t < t_n$, then

$$X^\pm(t,t_n) \ge \delta, \tag{4.8}$$

almost surely with respect to ν_{t,t_n}.

Step 3. Let us write π_t for the probability measure $\nu_{t,\infty}$. In other words, the measure π_t is the distribution of the occupation numbers $(\eta^-(\cdot,t,\infty), \eta^+(\cdot,t,\infty))$ of the immortal particles. Given $\ell > 0$, define

$$X^\pm(t) = \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{i=1}^{\ell} \eta^\pm(i,t,\infty).$$

Again, the existence of the (possibly random) limit follows from the ergodic theorem. From the same theorem we also know that $EX^{\pm}(t) = E\eta^{\pm}(0, t, \infty)$. Note that we always have

$$X^{\pm}(t) \le X^{\pm}(t, t_n),$$

which implies $X^{\pm}(t) \le \widehat{X}^{\pm}(t) := \inf_n X^{\pm}(t, t_n)$. It turns out that $\widehat{X}^{\pm} = X^{\pm}$ and this can be shown by verifying

$$EX^{\pm}(t) \ge E\widehat{X}^{\pm}(t).$$

To see this, observe that

$$EX^{\pm}(t) = E\eta^{\pm}(0, t, \infty) = \lim_{n\to\infty} E\eta^{\pm}(0, t, t_n) = \lim_{n\to\infty} EX^{\pm}(t, t_n) \ge E\widehat{X}^{\pm}(t).$$

From $X^{\pm} = \widehat{X}^{\pm}$ and (4.8) we deduce that

$$X^{\pm}(t) \ge \delta, \tag{4.9}$$

almost surely with respect to the measure π_t. Note that (4.9) implies that there are infinitely many immortal particles of both types on both sides of the origin. Let $\xi(i, t)$ denote the indicator function of the event that there exists at least one immortal particle at the site i and the next immortal particle on the right of i is of the opposite sign. We first claim

$$q_t := \int \xi(i, t)\, d\pi_t > 0. \tag{4.10}$$

This is because if $q_t = 0$, then the translation invariance of π_t implies that whenever there is an immortal particle at some site i then the first immortal particle on the right side of i is of the same type, which contradicts (4.9). We next claim that q_t is independent of t. The proof of this claim can be established in the same way we showed (4.5). By translation invariance,

$$q_t = E\left(\frac{1}{\ell}\sum_{i=1}^{\ell} \xi(i, t)\right). \tag{4.11}$$

Moreover, since the speed of propagation is bounded on average, and the immortal particles are neither created, nor annilated, and they do not cross each other (the order is preserved), we can construct a sequence of the random variables L_ℓ such that (4.6) is true and

$$\sum_{i=-L_\ell}^{\ell+L_\ell} \xi(i, 0) \ge \sum_{i=1}^{\ell} \xi(i, t) \ge \sum_{i=L_\ell}^{\ell-L_\ell} \xi(i, 0).$$

From this and (4.6), it is not hard to deduce

$$\lim_{\ell\to\infty} \frac{1}{\ell}\sum_{i=1}^{\ell} \xi(i, t) = \lim_{\ell\to\infty} \frac{1}{\ell}\sum_{i=1}^{\ell} \xi(i, 0).$$

This and (4.11) imply that q_t is independent of t.

Final step. Let us write A for the event $\xi(0,t)=1$. We decompose A into two disjoint sets B_N and C_N where B_N consists of those configurations in A for which the first immortal particle to the right of the origin is located somewhere in the interval $(0,N]$. For any configuration that has an immortal particle at the origin, we define a sequence $\cdots < y_{-2} < y_{-1} < y_0 = 0 < y_1 < y_2 \ldots$ such that y_j is the location of the first immortal particle to the right of y_{j-1}. It is well known that

$$\int (y_{j+1}-y_j)\xi(0,t)\,d\pi_t \le 1.$$

See, for example, [9], Theorem B47 for a proof. Hence

$$\int_{C_N} (y_{j+1}-y_j)\,d\pi_t \le 1. \tag{4.12}$$

Moreover, if a configuration belongs to C_N, then there will be no immortal particle in $(0,N]$, which means $y_1 - y_0 > N$. From this and (4.12) we deduce

$$\pi_t(C_N) \le \frac{1}{N}.$$

From this we conclude that for sufficiently large N,

$$\pi_t(B_N) \ge \frac{1}{2}\pi_t(A) = \frac{1}{2}q_t = \frac{1}{2}q_0,$$

for every $t \ge 0$. Moreover, if a configuration belongs to B_N, then we have two particles of opposite sign in the interval $[0,N]$. In other words, for all $t \ge 0$,

$$\begin{aligned}&\tilde{\mu}_t\big(\big\{(\eta(i)-\zeta(i))(\eta(j)-\zeta(j)) < 0 \text{ for some } i,j \in [0,N] \text{ with } i \ne j\big\}\big)\\ &\quad\ge \pi_t(B_N) \ge \tfrac{1}{2}q_0.\end{aligned}$$

But this contradicts Lemma 4.3. Hence, (4.3) cannot be true. Since the number of positive (respectively, negative) particles does not increase with time, we know

$$\frac{d}{dt}\int (\eta(0,t)-\zeta(0,t))^{\pm}\,d\tilde{\mu}_t \le 0.$$

As a result, either (4.2) must be true or

$$\lim_{t\to\infty}\int (\eta(0,t)-\zeta(0,t))^{-}\,d\tilde{\mu}_t = 0. \tag{4.13}$$

If the latter occurs, then any limit point $\hat{\mu}$ of the sequence $\tilde{\mu}_t$ must be concentrated on the set of configurations (η,ζ) with $\eta \ge \zeta$. On the other hand, since $\hat{\mu}$ is translation invariant and

$$\int \eta(0)\,d\hat{\mu} = \rho_1 \le \rho_2 = \int \zeta(0)\,d\hat{\mu},$$

we deduce that if (4.13) is true, then the measure $\hat{\mu}$ is concentrated on the set $\eta = \zeta$. This again implies (4.2). □

Given two probability measures μ_1 and μ_2, we write $\mu_1 \le \mu_2$ if there exists a measure $\tilde{\mu}$ with marginals μ_1 and μ_2 that is concentrated on the set

$$\{(\eta, \zeta) \mid \eta \le \zeta\}.$$

If initially the configuration η (respectively (η, ζ)) is distributed according to μ (respectively $\tilde{\mu}$), then the distribution of η (respectively (η, ζ)) at a later time t will be denoted by $\mathscr{S}(t)\mu$ (respectively $\widetilde{\mathscr{S}}(t)\tilde{\mu}$).

COROLLARY 4.4. *Let μ_1 and μ_2 be two translation invariant ergodic measures with*

$$\int \eta(0)\, d\mu_1 =: \rho_1 \le \rho_2 := \int \eta(0)\, d\mu_2.$$

If $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \mathscr{S}(t_n)\mu_1 = \bar{\mu}_1$, $\lim_{n\to\infty} \mathscr{S}(t_n)\mu_2 = \bar{\mu}_2$ exist, then $\bar{\mu}_1 \le \bar{\mu}_2$. If $\rho_1 = \rho_2$ and $\lim_{n\to\infty} \mathscr{S}(t_n)\mu_1 = \bar{\mu}_1$ exists, then $\lim_{n\to\infty} \mathscr{S}(t_n)\times \mu_2 = \bar{\mu}_1$.

PROOF. Let $\tilde{\mu}$ be any ergodic coupling of μ_1 and μ_2, and let $\hat{\mu}$ be a limit point of the sequence $\tilde{\mu}_{t_n} = \widetilde{\mathscr{S}}(t_n)\tilde{\mu}$ as $n \to \infty$. Then the marginals of $\hat{\mu}$ are $\bar{\mu}_1$ and $\bar{\mu}_2$, and by Lemma 4.2, the measure $\hat{\mu}$ is concentrated on the set $\{(\eta, \zeta) \mid \eta \le \zeta\}$. This implies the first claim of the lemma. If $\rho_1 = \rho_2$ and $\bar{\mu}_2$ is a limit point of $\mathscr{S}(t_n)\mu_2$, then $\bar{\mu}_1 \le \bar{\mu}_2$ and $\bar{\mu}_1 \ge \bar{\mu}_2$. This clearly implies $\bar{\mu}_2 = \bar{\mu}_1$. As a result, $\lim_{n\to\infty} \mathscr{S}(t_n)\mu_2 = \bar{\mu}_1$. □

Let $\mathscr{J}$ denote the space of translation invariant equilibrium probability measures. More precisely, $\nu \in \mathscr{J}$ if and only if

$$\int \mathscr{A} f(\eta)\nu(d\eta) = 0,$$

$$\int \tau_i f(\eta)\nu(d\eta) = \int f(\eta)\nu(d\eta)$$

for every $i \in \mathbb{Z}$ and every local function f. The space $\mathscr{J}$ is a convex compact set with respect to the weak topology, and the set of extreme points of $\mathscr{J}$ will be denoted by $\mathscr{J}_{\text{ex}}$. As a result, for every $\nu \in \mathscr{J}$, there exists a measure θ on $\mathscr{J}_{\text{ex}}$ such that

$$\nu = \int_{\mathscr{J}_{\text{ex}}} \alpha\theta(d\alpha). \tag{4.14}$$

We also write $\mathscr{J}_{\text{erg}}$ for the space of the translation invariant ergodic measures.

For a measure $\nu \in \mathscr{J}$, let us write P_ν for the law of $\eta(\cdot, \cdot)$ when the random variable $\eta(\cdot, t)$ is distributed according to the measure ν. Consider the operators $(\tau_i, \hat{\tau}_s)$ acting on the process $\eta(\cdot, \cdot)$ where τ_i is the space translation and $\hat{\tau}_s \eta(i, t) = \eta(i, t+s)$. Note that since $\nu \in \mathscr{J}$, we can define the process $\eta(i, t)$ for all $(i, t) \in \mathbb{Z} \times \mathbb{R}$. By standard arguments, it is not hard to show that the law P_ν is ergodic with respect to the shift operators $(\tau_i, \hat{\tau}_s)$ if and only if $\nu \in \mathscr{J}_{\rm ex}$. As a result, if $b(\eta)$ is a local function and $\nu \in \mathscr{J}_{\rm ex}$, then

$$\lim_{\ell\to\infty} \frac{1}{\ell^2} \int_0^\ell \sum_{i=0}^{\ell} \tau_i b(\eta(\cdot, s))\, ds = \int b\, d\nu, \tag{4.15}$$

P_ν-almost surely.

Define

$$\chi = \left\{ \int \eta(0)\, d\nu \mid \nu \in \mathscr{J}_{\rm ex} \right\}.$$

As we will see below, the space $\mathscr{J}_{\rm ex}$ is parametrized by the set χ.

LEMMA 4.5. *Suppose $\nu \in \mathscr{J}_{\rm ex}$ and $\int \eta(0)\, d\nu = \rho$. Then*

$$X(\eta) := \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{j=1}^{\ell} \eta(j) = \rho,$$

ν-almost surely.

PROOF. Since $\nu \in \mathscr{J}_{\rm ex}$, it suffices to show that the conditional measures $\nu(\cdot \mid X = c)$ belong to $\mathscr{J}$. Define

$$X_\ell(\eta) := \frac{1}{\ell} \sum_{j=1}^{\ell} \eta(j).$$

Let f be a local function and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. It is not hard to show that

$$\lim_{\ell\to\infty} \mathscr{A}(g(X_\ell) f)(\eta) = \lim_{\ell\to\infty} g(X_\ell(\eta)) \mathscr{A} f(\eta) = g(X(\eta)) \mathscr{A} f(\eta),$$

ν-almost surely. From this and $\int \mathscr{A}(g(X_\ell) f)(\eta)\, d\nu = 0$ we deduce

$$\int g(X) \mathscr{A} f\, d\nu = 0.$$

From this we conclude that $\nu(\cdot \mid X = c) \in \mathscr{J}$, completing the proof of the lemma. □

LEMMA 4.6. (i) *For every $\rho \in \chi$, there exists a unique measure $\nu^\rho \in \mathscr{J}_{\rm ex}$ such that $\int \eta(0)\, d\nu^\rho = \rho$.*

(ii) *If μ is an ergodic measure with*

$$\int \eta(0)\, d\mu = \rho, \tag{4.16}$$

then every limit point of $\mathscr{S}(t)\mu$ as $t \to \infty$ belongs to $\mathscr{J}$. Moreover, if $\rho \in \chi$, then $\lim_{t\to\infty} \mathscr{S}(t)\mu = \nu^\rho$.

PROOF. (i) The uniqueness of ν^ρ follows from Lemma 4.5 and Corollary 4.4. To see this, fix an ergodic measure μ with $\int \eta(0)\,d\mu = \rho$ and choose a sequence t_n for which $\lim_{n\to\infty} \mathscr{S}(t_n)\mu =: \nu^\rho$ exists. Let ν be a probability measure in $\mathscr{J}_{\text{ex}}$ with density ρ. Let θ be a probability measure on the set $\mathscr{J}_{\text{erg}}$ such that

$$\nu = \int_{\mathscr{J}_{\text{erg}}} \alpha\theta(d\alpha).$$

By Lemma 4.5, each α in the support of θ must have density ρ. From this and Corollary 4.4 we deduce that

$$\lim_{n\to\infty} \mathscr{S}(t_n)\alpha = \nu^\rho,$$

for every α in the support of θ. As a result,

$$\nu = \lim_{n\to\infty} \mathscr{S}(t_n)\nu = \lim_{n\to\infty} \int \mathscr{S}(t_n)\alpha\ \theta(d\alpha) = \nu^\rho.$$

This completes the proof of uniqueness. This also shows that if the measure μ is ergodic with $\int \eta(0)\,d\mu = \rho \in \chi$, then $\lim_{t\to\infty} \mathscr{S}(t)\mu = \nu^\rho$.

(ii) Let μ be an arbitrary ergodic measure for which (4.16) is true. Let $\bar{\mu}$ be a limit point of $\mathscr{S}(t)\mu$ as $t \to \infty$. Then we have $\lim_{n\to\infty} \mathscr{S}(t_n)\mu = \bar{\mu}$, for some sequence t_n with $\lim_{n\to\infty} t_n = \infty$. The measures μ and $\mathscr{S}(s)\mu$ are both ergodic with the same density,

$$\int \eta(0)\,d\mu = \int \eta(0)\,d\mathscr{S}(s)\mu.$$

By Corollary 4.4,

$$\lim_{n\to\infty} \mathscr{S}(t_n + s)\mu = \lim_{n\to\infty} \mathscr{S}(t_n)\mu.$$

From this and the semigroup property of $\mathscr{S}$ we deduce $\mathscr{S}(s)\bar{\mu} = \bar{\mu}$. Hence $\bar{\mu} \in \mathscr{J}$ and $\int \eta(0)\,d\bar{\mu} = \rho$. $\square$

An immediate consequence of Lemma 4.6 is

$$\mathscr{J}_{\text{ex}} = \{\nu^\rho \mid \rho \in \chi\}. \tag{4.17}$$

LEMMA 4.7. *The set χ is closed. Moreover, if μ is a translation invariant ergodic measure with $\rho = \int \eta(0)\,d\mu \notin \chi$, then*

$$\lim_{t\to\infty} \mathscr{S}(t)\mu = \nu^\rho := \alpha\nu^{\bar{\rho}} + (1-\alpha)\nu^{\underline{\rho}}, \tag{4.18}$$

where $\alpha = \frac{\rho - \underline{\rho}}{\bar{\rho} - \underline{\rho}}$ and

$$\bar{\rho} = \inf\{m \in \chi \mid m > \rho\}, \qquad \underline{\rho} = \sup\{m \in \chi \mid m < \rho\}.$$

PROOF. Suppose $\rho \notin \chi$ and let $\bar{\mu}$ be any limit point of $\mathscr{S}(t)\mu$. By (4.17), (4.14) and Lemma 4.6, there exists a measure θ on χ such that

$$\bar{\mu} = \int_\chi \nu^m\,\theta(dm).$$

Suppose $\theta([m_2, \beta_1]) \neq 0$ for some $m_2 > \bar{\rho}$ and choose $m_1 \in \chi$ with $\bar{\rho} \leq m_1 < m_2$. Then by Lemma 4.5,

$$\bar{\mu}(\{\eta \mid X(\eta) \geq m_2\}) > 0. \tag{4.19}$$

However, Corollary 4.4 implies that $\bar{\mu} \leq \nu^{m_1}$, which in particular implies $X(\eta) \leq m_1$ with probability 1 with respect to $\bar{\mu}$. This is in contradiction with (4.19). Hence $\theta([m_2, \beta_1]) = 0$ for all such a number m_2. From this we deduce that $\theta(\{\bar{\rho}\}) = \theta([\bar{\rho}, \beta_1])$. Similarly $\theta(\{\underline{\rho}\}) = \theta([0, \underline{\rho}])$. Hence, $\underline{\rho}, \bar{\rho} \in \chi$. As a result, the set χ is closed and the measure θ is concentrated on the set $\{\underline{\rho}, \bar{\rho}\}$. □

Define

$$b_i(\eta) := \lambda^+ b(\eta(i), \eta(i-1)) - \lambda^- b(\eta(i-1), \eta(i)).$$

Also define

$$H(\rho) := -\int b_i(\eta)\nu^\rho(d\eta), \tag{4.20}$$

where ν^ρ was defined in Lemma 4.7. Note that $H(\rho) = \alpha H(\bar{\rho}) + (1-\alpha)H(\underline{\rho})$ when $\rho \notin \chi$.

LEMMA 4.8. *Let K: $[0, \beta_1] \to \mathbb{R}$ be any continuous function that coincides with the function H on the set χ. If initially the height-difference process $(\eta(i, 0)$: $i \in \mathbb{Z})$ is distributed according to an ergodic measure μ with $\int \eta(0)\,d\mu = \rho$, then*

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} E \sup_{0\leq t\leq T} \left| \varepsilon h(0, t\varepsilon^{-1}) - \varepsilon h(0,0) \right. \\ \left. + \int_0^t K\left(\frac{1}{\ell}\sum_1^\ell \eta(j, \theta\varepsilon^{-1})\right) d\theta \right| = 0. \tag{4.21}$$

Moreover, if $\rho \in \chi$, then

$$\lim_{\varepsilon\to 0} E \sup_{0\leq t\leq T} \left| \varepsilon h(0, t\varepsilon^{-1}) - \varepsilon h(0,0) + tH(\rho) \right| = 0. \tag{4.22}$$

PROOF. Without loss of generality, we may assume $h(0,0) = 0$. Since $h \in \Gamma$, we have

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} E \sup_{0\leq t\leq T} \left| \varepsilon h(0, t\varepsilon^{-1}) - \frac{\varepsilon}{\ell}\sum_{j=1}^\ell h(j, t\varepsilon^{-1}) \right| = 0. \tag{4.23}$$

Let $f(h) = \frac{\varepsilon}{\ell}\sum_{j=1}^\ell h(j)$. It is well known that if

$$M_s = f(h(s)) - f(h(0)) - \int_0^s \mathscr{L} f(h(\theta))\,d\theta, \tag{4.24}$$

then M_s is a martingale and

$$EM_s^2 = \int_0^s (\mathscr{L}f^2 - 2f\mathscr{L}f)(h(\theta))\,d\theta. \tag{4.25}$$

Recall that by Doob's inequality,

$$E \sup_{0\le s\le T\varepsilon^{-1}} M_s^2 \le 4EM_{T\varepsilon^{-1}}^2.$$

A straightforward calculation yields

$$\lim_{\varepsilon\to 0} EM_{T\varepsilon^{-1}}^2 = 0, \qquad \mathscr{L}f(h) = \frac{\varepsilon}{\ell}\sum_{j=1}^{\ell} b_j(\eta).$$

From this, Doob's inequality and (4.23)–(4.25) we deduce

$$\lim_{\ell\to\infty}\limsup_{\varepsilon\to 0} E \sup_{0\le t\le T}\left|\varepsilon h(0, t\varepsilon^{-1}) - \frac{1}{\ell}\int_0^t \sum_{j=1}^{\ell} b_j(\eta(\theta\varepsilon^{-1}))\,d\theta\right| = 0. \tag{4.26}$$

Define

$$F_\ell(\eta) = E^\eta\left|\frac{1}{\ell^2}\int_0^\ell \sum_{j=1}^{\ell} b_i(\eta(s))\,ds + K\left(\frac{1}{\ell}\sum_{0}^{\ell}\eta(j, 0)\right)\right|,$$

where E^η denotes the expectation when the process $\eta(\cdot, t)$ starts from the configuration η at time zero. One can use the finiteness of the speed of propagation (Lemma 3.8) to approximate F_ℓ by local functions and show that the function F_ℓ is continuous. Recall that the measure ν^ρ was defined in Lemma 4.7 for all values of $\rho \in [0, \beta_1]$. By Lemma 4.7,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathscr{S}(\theta)\mu\,d\theta = \lim_{t\to\infty}\mathscr{S}(t)\mu = \nu^\rho.$$

Note that

$$\lim_{\ell\to\infty}\ \limsup_{\varepsilon\to 0} E \sup_{0\le t\le T}\left|\frac{1}{\ell}\int_0^t \sum_{j=1}^{\ell} b_j(\eta(\theta\varepsilon^{-1}))\,d\theta \right.$$
$$\left. - \frac{1}{\ell^2}\int_0^t\int_0^\ell \sum_{j=1}^{\ell} b_j(\eta(\theta\varepsilon^{-1}+s))\,ds\,d\theta\right| = 0.$$

Hence,

$$\begin{aligned}
&\lim_{\ell\to\infty}\limsup_{\varepsilon\to 0} E \sup_{0\le t\le T}\left|\frac{1}{\ell}\int_0^t \sum_{j=1}^{\ell} b_j(\eta(\theta\varepsilon^{-1}))\,d\theta + \int_0^t K\left(\frac{1}{\ell}\sum_{0}^{\ell}\eta(j, \theta\varepsilon^{-1})\right)d\theta\right|\\
&\quad\le \lim_{\ell\to\infty}\limsup_{\varepsilon\to 0} E\int_0^T F_\ell\big(\eta(\theta\varepsilon^{-1})\big)\,d\theta\\
&\quad\le \lim_{\ell\to\infty}\limsup_{\varepsilon\to 0}\int_0^T\int F_\ell(\eta)\,d\mathscr{S}(\theta\varepsilon^{-1})\mu\,d\theta\\
&\quad= \lim_{\ell\to\infty} T\int F_\ell(\eta)\,d\nu^\rho = 0,
\end{aligned} \tag{4.27}$$

where for the last equality we used (4.15). This and (4.26) imply (4.21). If $\rho \in \chi$, then a repetition of (4.27) yields

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} E \sup_{0\le t\le T} \left| \frac{1}{\ell} \int_0^t \sum_{j=1}^{\ell} b_i(\eta(\theta\varepsilon^{-1}))\, d\theta + tH(\rho) \right| = 0.$$

This and (4.26) imply (4.22). □

PROOF OF THEOREM 4.1. By Lemma 3.11, we may assume that the initial height differences are distributed according to an ergodic measure with density ρ. We only establish (4.1) for $\rho \notin \chi$. The case $\rho \in \chi$ is a straightforward consequence of (4.22). Fix $\rho \notin \chi$. To take advantage of Lemma 4.8, we augment our sample space by defining the *Young measures* $\pi^{\ell,\varepsilon}(x, t, d\lambda; \omega)$ as

$$\int_0^{\beta_1} F(\lambda)\pi^{\ell,\varepsilon}(x, t, d\lambda; \omega) = F\left(\frac{1}{\ell} \sum_{j=1}^{\ell} \eta^{\rho}([x\varepsilon^{-1}] + j, t\varepsilon^{-1}, \omega) \right),$$

where F is any continuous function and

$$\eta^{\rho}(j, \theta, \omega) = h(j+1, \theta; k_\rho, \omega) - h(j, \theta; k_\rho, \omega)$$

with $k_\rho(j) = [\rho j]$. Let X_T denote the space of measurable maps from $\mathbb{R}\times[0, T]$ into $\mathscr{M}$ where $\mathscr{M} = \mathscr{M}([0, \beta_1])$ is the space of probability measures on the interval $[0, \beta_1]$. The space $\mathscr{M}$ is equipped with the weak topology and the space X_T is also equipped with the weak topology. More precisely, a sequence $\pi_n \in X_T$ converges to an element $\pi \in X_T$ if and only if

$$\begin{aligned} \lim_{n\to\infty} &\int_{\mathbb{R}} \int_0^T \int_0^{\beta_1} \Phi(x, t, \lambda)\pi_n(x, t, d\lambda)\, dt\, dx \\ &= \int_{\mathbb{R}} \int_0^T \int_0^{\beta_1} \Phi(x, t, \lambda)\pi(x, t, d\lambda)\, dx, \end{aligned} \tag{4.28}$$

for every continuous function Φ of compact support. Note that we may regard X_T as a subspace of $\mathscr{N}_T$, the space of positive measures $\gamma(dx, dt, d\lambda)$ on the set $A = \mathbb{R} \times [0, T] \times [0, \beta_1]$. More precisely, the set X_T is equal to the set of measures $\gamma \in \mathscr{N}_T$ such that

$$\int_{\mathbb{R}} \int_0^T \int_0^{\beta_1} \Phi(x, t) f(\lambda)\gamma(dx, dt, d\lambda) \le \int_{\mathbb{R}} \int_0^T |\Phi(x, t)|\, dx\, dt \max_{\lambda\in[0,\beta_1]} |f(\lambda)|,$$

for every continuous functions Φ and f of compact support. The space $\mathscr{N}_T$ is also equipped with the topology of weak convergence in the sense of (4.28). One can easily verify that X_T is a closed subset of $\mathscr{N}_T$. Since the space $\mathscr{N}_T$ is a compact metric space, the space X_T is also a compact metric space. The transformation

$$\omega \mapsto (\widehat{S}^{\varepsilon}, \pi^{\ell,\varepsilon})$$

induces a probability measure $\widetilde{\mathscr{P}}^{\varepsilon,\ell}$ on the product space $\mathscr{D}_T \times X_T$. It is not hard to show that the compactness of X_T and the tightness of $\mathscr{P}^{\varepsilon}$ imply the tightness of $\widetilde{\mathscr{P}}^{\varepsilon,\ell}$. We take convergent subsequences of $\widetilde{\mathscr{P}}^{\varepsilon,\ell}$ as $\varepsilon \to 0$. If $\{\widetilde{\mathscr{P}}^{\ell}\}$ is a sequence of such limit points, we can further take a convergent subsequence as $\ell \to \infty$. Let $\widetilde{\mathscr{P}}$ denote a limit point of the sequence $\widetilde{\mathscr{P}}^{\ell}$. Now (4.21) can be used to deduce

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} \int \sup_{0\le t\le T} \left| S(0,t;g_\rho)(0) + \int_0^t \int K(\lambda)\pi(0,\theta,d\lambda)d\theta \right| d\widetilde{\mathscr{P}}^{\varepsilon,\ell} = 0,$$

where $g_\rho(x) = x\rho$. After a translation and an integration, we deduce that for every continuous function J of compact support,

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} \int \sup_{0\le t\le T} \bigg| \int_{\mathbb{R}} \big(S(0,t;g_\rho)(x) - g_\rho(x)\big) J(x)\,dx + \int_0^t \int_{\mathbb{R}} J(x) \int K(\lambda)\pi(x,\theta,d\lambda)\,dx d\theta \bigg| d\widetilde{\mathscr{P}}^{\varepsilon,\ell} = 0.$$

This implies

$$\lim_{\ell\to\infty} \int \sup_{0\le t\le T} \bigg| \int_{\mathbb{R}} \big(S(0,t;g_\rho)(x) - g_\rho(x)\big) J(x)\,dx + \int_0^t \int_{\mathbb{R}} J(x) \int K(\lambda)\pi(x,\theta,d\lambda)\,dx d\theta \bigg| d\widetilde{\mathscr{P}}^{\ell} = 0.$$

This in turn implies

$$\int \sup_{0\le t\le T} \bigg| \int_{\mathbb{R}} \big(S(0,t;g_\rho)(x) - g_\rho(x)\big) J(x)\,dx + \int_0^t \int_{\mathbb{R}} J(x) \int K(\lambda)\pi(x,\theta,d\lambda)\,dx d\theta \bigg| d\widetilde{\mathscr{P}} = 0, \tag{4.29}$$

for every continuous function J of compact support and every continuous K that coincides with H on the set χ. We will show in Lemma 5.4 of the next section that any limit point of $\mathscr{P}^{\varepsilon}$ is concentrated on the set of semigroups S for which

$$S(0,t;g_\rho) = g_\rho - tH_S(\rho),$$

for some Lipschitz function H_S. From this and (4.29) we deduce that the measure $\widetilde{\mathscr{P}}$ is concentrated on the space of the pairs (S,π) such that

$$S(0,t;g_\rho)(x) = g_\rho(x) - \int_0^t \int K(\lambda)\pi(x,\theta,d\lambda)\,d\theta = g_\rho(x) - tH_S(\rho),$$

for almost all x, all $t \in [0,T]$ and every continuous K that coincides with H on the set χ. From this we can readily deduce that the measure $\pi(x,\theta,d\lambda)$ must be concentrated on the set $\{\bar\rho, \underline{\rho}\}$ and that

$$\int K(\lambda)\pi(x,\theta,d\lambda) = \int H(\lambda)\pi(x,\theta,d\lambda)$$

is independent of $(x, \theta) \in \mathbb{R} \times [0, T]$. On the other hand, a straightforward calculation yields

$$\int_0^x \int \lambda \pi^{\ell, \varepsilon}(y, t, d\lambda; \omega)\, dy = S^\varepsilon(0, t; g_\rho)(x) - S^\varepsilon(0, t; g_\rho)(0) + O(\varepsilon\ell),$$

for every $x > 0$. As a result, the measure $\widetilde{\mathscr{P}}$ is concentrated on the set of (S, π) for which,

$$\int_0^x \int \lambda\ \pi(y, t, d\lambda)\, dy = g_\rho(x) - g_\rho(0),$$

for every $x > 0$. Hence, for almost all $(x, \theta) \in \mathbb{R} \times [0, T]$,

$$\int \lambda\ \pi(x, \theta, d\lambda) = \rho.$$

Therefore,

$$H_S(\rho) = \int K(\lambda)\pi(x, \theta, d\lambda) = \alpha H(\underline{\rho}) + (1 - \alpha)H(\bar{\rho}),$$

for almost all $(x, \theta) \in \mathbb{R} \times [0, T]$, where α satisfies

$$\rho = \alpha\underline{\rho} + (1 - \alpha)\bar{\rho}.$$

From this we deduce

$$S(0, t; g_\rho) = g_\rho - tH(\rho),$$

for almost all S in the support of the measure $\widetilde{\mathscr{P}}$. Using this and again Lemma 4.8, it is not hard to deduce (4.1) because the functional

$$\mathscr{F}(S) = \sup_{0 \le t \le T} |S(0, t; g_\rho)(0) + tH(\rho)|$$

is continuous and for every limit point $\mathscr{P}$ of $\mathscr{P}^\varepsilon$ we have $\int \mathscr{F}\, d\mathscr{P} = 0$. □

5. Continuum limit. If $\alpha_r = \beta_r = 0$ for some $r \in \{1, \dots, d\}$, then $h(i_1, \dots, i_d)$ is independent of i_r for every $h \in \Gamma$. From this, it is not hard to deduce that if $h \in \Gamma$, then $h_i, h^i \notin \Gamma$ for every $i \in \mathbb{Z}^d$. As a result, $h(i, t; k) = k$ and we have a continuum limit with $H \equiv 0$. Recall that by Remark 2.4, we can always assume $\alpha_r = 0$ for all $r \in \{1, \dots, d\}$. For the rest of this section, we assume $\alpha_r = 0$ and $\beta_r > 0$ for every $r \in \{1, \dots, d\}$.

The main objective of this section is the identification of the semigroup $S \in \mathscr{D}_T(q_0)$ in the support of $\mathscr{P}$ with the semigroups of the Hamilton–Jacobi equations. Given a continuous function $H\colon Y \to \mathbb{R}$, we define $S^H(s, t; g)(x) = u(x, t - s)$ to be the unique viscosity solution of the Hamilton–Jacobi equation (2.5). This means that u is continuous, $u(x, 0) = g(x)$, and if $\phi \in C^1(\mathbb{R}^d \times (0, T))$, $(x_0, t_0) \in \mathbb{R}^d \times (0, T)$, $u(x_0, t_0) = \phi(x_0, t_0)$, $u \le \phi$ (respectively, $u \ge \phi$), then

$$\phi_t(x_0, t_0) + H(\phi_x(x_0, t_0)) \le 0, \tag{5.1}$$

$$(\phi_t(x_0, t_0) + H(\phi_x(x_0, t_0)) \ge 0, \text{respectively}). \tag{5.2}$$

(See [7] for more information on viscosity solutions.) The main result of this section is Theorem 5.1. This theorem implies Theorems 2.2 and 2.3 and will be used to prove Theorem 2.1. Define

$$\mathscr{H} = \{H \in C(Y) \mid H \text{ satisfies the conditions (i)–(iv) of Theorem 2.3}\}.$$

THEOREM 5.1. *Let $\mathscr{P}^\varepsilon$ be as in Theorem 3.5. There exists a convex function $\widetilde{H} \in \mathscr{H}$ such that every limit point $\mathscr{P}$ of the sequence $\mathscr{P}^\varepsilon$ is concentrated on the space of semigroups*

$$\{S^H \mid H \in \mathscr{H} \text{ and convex hull of } H \text{ equals } \widetilde{H}\}.$$

The main ingredients of the proof of Theorem 5.1 are Lemmas 5.3 and 5.4. Lemma 5.2 will be used for the proof of Lemma 5.3 and can be established in the same way we proved Lemma 3.1. The proof of Lemma 5.2 is omitted. The proof of Lemma 5.3 follows the semigroup identification result of [10]. Define $B(x_0, r) = \{x \in \mathbb{R}^d \mid |x - x_0| \le r\}$ and

$$\|g\|_{B(x_0,r)} = \sup\{|g(x)| \ \mid x \in B(x_0, r)\}.$$

LEMMA 5.2. *Suppose $S \in \mathscr{D}_T(q_0)$. Then*

$$|S(s, t; g_1)(x_0) - S(s, t; g_2)(x_0)| \le \|g_1 - g_2\|_{B(x_0,\,(t-s)q_0)}.$$

LEMMA 5.3. *Let $S \in \mathscr{D}_T(q_0)$. Suppose that there exists a continuous function $H\colon Y \to \mathbb{R}$ such that*

$$S(s, t; g_p) = g_p - (t - s)H(p), \tag{5.3}$$

for every $p \in Y$, where $g_p(x) = x \cdot p$. Then $S = S^H$.

PROOF. It suffices to show that $u(x, t) = S(s_1, s_1 + t; g)(x)$ is a viscosity solution of (2.5). Without loss of generality, we assume $s_1 = 0$. We only verify (5.1) because the proof of (5.2) is similar.

Let $\phi \in C^1(\mathbb{R}^d \times (0, T))$ with $\phi(x_0, t_0) = u(x_0, t_0)$ and $u \le \phi$. Since $S \in \mathscr{D}_T(q_0)$, we may use the monotonicity and the semigroup property to assert

$$\begin{aligned}\phi(x_0, t_0) = u(x_0, t_0) &= S(t_0 - \delta, t_0; u(\cdot, t_0 - \delta))(x_0)\\ &\le S(t_0 - \delta, t_0; \phi(\cdot, t_0 - \delta))(x_0),\end{aligned} \tag{5.4}$$

for every $\delta > 0$. Define

$$T_1(\delta) := S(t_0 - \delta, t_0; \phi(\cdot, t_0 - \delta))(x_0) - S(t_0 - \delta, t_0; \phi(\cdot, t_0) - \delta\phi_t(\cdot, t_0))(x_0),$$
$$T_2(\delta) := S(t_0 - \delta, t_0; \phi(\cdot, t_0) - \delta\phi_t(\cdot, t_0))(x_0) - S(t_0 - \delta, t_0; \phi(\cdot, t_0))(x_0).$$

Again, since $S \in \mathscr{D}_T(q_0)$, we can apply Lemma 5.2 to deduce

$$|T_1(\delta)| \le \|\phi(\cdot, t_0 - \delta) - \phi(\cdot, t_0) + \delta\phi_t(\cdot, t_0)\|_{B(x_0,\,q_0\delta)}.$$

This implies

$$\lim_{\delta\to 0} \delta^{-1} T_1(\delta) = 0, \tag{5.5}$$

because ϕ is continuously differentiable. Moreover, from

$$T_2(\delta) = -\delta\phi_t(x_0, t_0) + S(t_0 - \delta, t_0; \phi(\cdot, t_0) - \delta\phi_t(\cdot, t_0) + \delta\phi_t(x_0, t_0))(x_0) \\ - S(t_0 - \delta, t_0; \phi(\cdot, t_0))(x_0),$$

and Lemma 5.2 we deduce

$$\delta^{-1}|T_2(\delta) + \delta\phi_t(x_0, t_0)| \le \|\phi_t(\cdot, t_0) - \phi_t(x_0, t_0)\|_{B(x_0, q_0\delta)}.$$

Hence,

$$\lim_{\delta\to 0} \delta^{-1}|T_2(\delta) + \delta\phi_t(x_0, t_0)| = 0$$

This and (5.5) imply

$$\lim_{\delta\to 0} \delta^{-1}\big|S(t_0 - \delta, t_0; \phi(\cdot, t_0 - \delta))(x_0) \\ -S(t_0 - \delta, t_0; \phi(\cdot, t_0))(x_0) + \delta\phi_t(x_0, t_0)\big| = 0. \tag{5.6}$$

Also, if we define $g(x) = \phi(x_0, t_0) + \phi_x(x_0, t_0) \cdot (x - x_0)$, then Lemma 5.2 implies

$$|S(t_0 - \delta, t_0; \phi(\cdot, t_0))(x_0) - S(t_0 - \delta, t_0; g)(x_0)| \le \|\phi(\cdot, t_0) - g(\cdot)\|_{B(x_0, q_0\delta)}.$$

This, the continuous differentiability of ϕ, and our assumption (5.3) imply

$$\lim_{\delta\to 0} \delta^{-1}\big|S(t_0 - \delta, t_0; \phi(\cdot, t_0))(x_0) - g(x_0) + \delta H(\phi_x(x_0, t_0))\big| = 0.$$

From this, (5.4) and (5.6) we deduce

$$\phi(x_0, t_0) \le \phi(x_0, t_0) - \delta\phi_t(x_0, t_0) - \delta H(\phi_x(x_0, t_0)) + o(\delta).$$

This evidently implies (5.1). □

The next lemma will allow us to apply Lemma 5.3 to semigroups S in the support of $\mathscr{P}$. Recall $(\tau_y g)(x) = g(x-y)$, $k_g^\varepsilon(i) = [\varepsilon^{-1}g(\varepsilon i)]$, and $g_p(x) = x\cdot p$.

LEMMA 5.4. *Let $\mathscr{P}$ be any limit point of $\mathscr{P}^\varepsilon$. Then there exists a Lipschitz function H_S for each semigroup $S \in \mathscr{D}_T(q_0)$ such that*

$$S(s, t; g_p) = g_p - (t - s)H_S(p),$$

$\mathscr{P}$-almost surely.

PROOF. Given a function $g \in \overline{\Gamma}$, consider the height function $k(i) = k_g^\varepsilon(i)$ and its translate $\tau_j k(i) = k(i - j)$ with $j = \big[\frac{y}{\varepsilon}\big]$. Since

$$\big|k^\varepsilon_{\tau_y g}(i) - \tau_j k_g^\varepsilon(i)\big| \le c_0,$$

for some constant c_0, one can readily show,

$$\big|u^\varepsilon(x, t; k^\varepsilon_{\tau_y g}, \omega)(x) - u^\varepsilon(x, t; \tau_j k_g^\varepsilon, \omega)\big| \le c_0\varepsilon.$$

We then use (2.12) to replace $u^\varepsilon(x, t; \tau_j k_g^\varepsilon, \omega)$ with

$$u^\varepsilon\left(x - \varepsilon\left[\frac{y}{\varepsilon}\right], t; k_g^\varepsilon, \tau_{-j}\omega\right).$$

As a result,

$$E\left|S^\varepsilon(s, t; \tau_y g, \omega)(x) - \tau_y S^\varepsilon(s, t; g, \tau_{-[y/\varepsilon]}\omega)\right| \le c_1\varepsilon,$$

for some constant c_1. From this and the translation invariance of the measure Q we deduce that for every bounded continuous function F: $\mathscr{D}_T \to \mathbb{R}$,

$$\lim_{\varepsilon\to 0}\left|\int F(\Phi_y S)\, d\mathscr{P}^\varepsilon - \int F(\Psi_y S)\, d\mathscr{P}^\varepsilon\right| = 0,$$

where

$$(\Phi_y S)(s, t; g)(x) = S(s, t; g)(x - y), \quad (\Psi_y S)(s, t; g)(x) = S(s, t, \tau_y g)(x).$$

As a result, for every bounded continuous function F: $\mathscr{D}_T \to \mathbb{R}$,

$$\int F(\Phi_y S)\, d\mathscr{P} = \int F(\Psi_y S)\, d\mathscr{P},$$

which is equivalent to saying

$$\Phi_y S = \Psi_y S, \tag{5.7}$$

$\mathscr{P}$-almost surely. By varying y in a countable dense set, we deduce that (5.7) is true for all $y \in \mathbb{R}^d$, $\mathscr{P}$-almost surely. This in particular implies

$$S(s,t;g_p)(x-y) = S(s,t;\tau_y g_p)(x) = S(s,t;g_p - p\cdot y)(x) = S(s,t;g_p)(x) - p\cdot y,$$

$\mathscr{P}$-almost surely. As a result,

$$S(s, t; g_p)(x) = x \cdot p + S(s, t; g_p)(0). \tag{5.8}$$

Moreover, by the semigroup property,

$$S(s_1, t; g_p) = S(s_2, t; S(s_1, s_2; g_p)) = S(s_2, t; g_p) + S(s_1, s_2; g_p)(0),$$

whenever $s_1 \le s_2 \le t$. This in turn implies

$$S(s_1, t; g_p)(0) = S(s_2, t; g_p)(0) + S(s_1, s_2; g_p)(0). \tag{5.9}$$

Furthermore, if

$$\Theta_\theta S(s, t; g) = S(s + \theta, t + \theta; g),$$

then the invariance of the measure Q with respect to the time-shift operator γ implies that for every bounded continuous function F: $\mathscr{D}_T \to \mathbb{R}$,

$$\int F(\Theta_\theta S)\, d\mathscr{P}^\varepsilon = \int F(S)\, d\mathscr{P}^\varepsilon.$$

Hence

$$\int F(\Theta_\theta S)\, d\mathscr{P} = \int F(S)\, d\mathscr{P}.$$

From this we deduce

$$S(s, t; g) = S(0, t - s; g),$$

$\mathscr{P}$-almost surely. This and (5.9) imply

$$S(0, t - s_1; g_p)(0) = S(0, t - s_2; g_p)(0) + S(0, s_2 - s_1; g_p)(0).$$

As a result, $S(s, t; g_p)(0) = -(t - s)H_S(p)$ for some constant $H_S(p)$.

It remains to verify the Lipschitzness of H_S. By Lemma 5.2,

$$\begin{aligned} H_S(p_2) - H_S(p_1) &= S(0, 1; g_{p_1})(0) - S(0, 1; g_{p_2})(0) \\ &\le \|g_{p_1} - g_{p_2}\|_{B(0,q_0)} \le |q_0||p_1 - p_2|. \end{aligned}$$

This completes the proof of the lemma. □

LEMMA 5.5. *There exists a convex function* $\overline{L} \in \overline{\Gamma}$ *such that*

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, t; v, \omega) = t\overline{L}\left(\frac{x}{t}\right)$$

almost surely.

When the initial data k coincides with the function v, the convergence of u^ε can be established with the aid of a subadditive ergodic theorem. Lemma 5.5 appeared as Theorem 4.1 in [14] when $\lambda^- = 0$. The case of $\lambda^- \neq 0$ can be treated likewise and the proof is omitted. The next lemma will allow us to relate the function $\overline{L}$ to H.

LEMMA 5.6. *For a continuous function* $H: Y \to \mathbb{R}$, *define*

$$H^*(q) = \sup_{p \in Y}(p \cdot q - H(p)).$$

Then

$$S^H(0, t; v)(x) = tH^*\left(\frac{x}{t}\right). \tag{5.10}$$

PROOF. Since the initial data v is convex, we may apply the Hopf's formula to assert

$$S^H(0, t; v) = (v^* + tH)^*,$$

where

$$v^*(p) = \sup_q(p \cdot q - v(q)) = \begin{cases} 0, & \text{if } p \in Y, \\ \infty, & \text{otherwise.} \end{cases}$$

See [1] for a proof of Hopf's formula. Hence,

$$S^H(0, t; v)(x) = \sup_{p \in Y}(p \cdot x - tH(p)).$$

This clearly implies (5.10). □

We now construct a reversible measure for the process h that is the analog of the so-called *blocking measure* for the simple exclusion process. A consequence of the existence of such a measure is $H \le 0$. We assume $\lambda^+ > \lambda^- > 0$. [When $\lambda^- = 0$, our blocking measure is trivial and is concentrated on a single height function $\tilde{v}(i) = -v(-i)$.] To this end, let us set

$$\Lambda = \{h \in \Gamma \mid h(i) + v(-i) \le 0 \ \text{ for every } \ i \in \mathbb{Z}^d\}$$

and define a measure ν which is concentrated on the set Λ, and for $h \in \Lambda$,

$$\nu(\{h\}) = \frac{1}{Z}\left(\frac{\lambda^+}{\lambda^-}\right)^{K(h)},$$

where

$$K(h) = \sum_{i \in \mathbb{Z}^d} (h(i) + v(-i)), \qquad Z = \sum_{h \in \Lambda} \left(\frac{\lambda^+}{\lambda^-}\right)^{K(h)}.$$

This measure is well defined if $Z < \infty$. Moreover, if $\nu(\{h\}) \ne 0$ then $h(i) + v(-i) = 0$ for all but finitely many i's. Hence ν has a countable support.

LEMMA 5.7. *The normalization constant Z is finite and the measure ν is reversible with respect to the generator $\mathscr{L}$.*

PROOF. Recall that by Remark 2.4, we may assume that $\alpha_r = 0$ for $r = 1, \dots d$. For each nonnegative ℓ, define

$$\Lambda_\ell = \{h \in \Lambda \mid K(h) = -\ell\}.$$

Given $h \in \Lambda$, we also define

$$p(h) = \{i \mid h(i) + v(-i) \ne 0\}.$$

We may regard $p(h)$ as a union of 2^d sets

$$p(h) = \cup\{p(h, \tau_1 \dots \tau_d) \mid \tau_1, \dots, \tau_d \in \{-1, 1\}\},$$

$$p(h, \tau_1, \dots, \tau_d) = p(h) \cap \{(i_1 \dots i_d) \mid \tau_r i_r \ge 0 \text{ for } r = 1, \dots, d\}.$$

To get a bound on $|\Lambda_\ell|$, we restrict each $h \in \Gamma_\ell$ to its corresponding set $p(h, \tau_1, \dots, \tau_d)$ and call the resulting configuration $h(\cdot, \tau_1, \dots, \tau_d)$. We would like to find an upper bound on the number of such configurations $h(\cdot, \tau_1, \dots, \tau_d)$. First we argue that there is a one-to-one correspondence between the configurations $h(\cdot, \tau_1, \dots, \tau_d)$ for different $(\tau_1, \dots, \tau_d)$. We only verify this for the case $\tau_1 = \tau_2 = \dots = \tau_d = 1$ and $\tau_1 = \tau_2 = \dots = \tau_d = -1$. Define

$$\hat{h}(i) = h(-i) + \sum_{r=1}^{d} \beta_r i_r.$$

It is not hard to see that $h \in \Lambda$ if and only if $\hat{h} \in \Lambda$. Evidently,

$$\hat{h}(i, -1, \dots, -1) = h(-i, 1, \dots, 1) + \sum_{r=1}^{d} \beta_r i_r.$$

This identity provides us with a one-to-one correspondence between the configurations $h \in \Lambda$ restricted to the set $\mathbb{Z}^d_+ = \{(i_1 \cdots i_d) \mid i_1, \ldots, i_d \geq 0\}$ and the configurations $h \in \Lambda$ restricted to the set $\mathbb{Z}^d_- = \{(i_1, \ldots, i_d) \mid i_1, \ldots, i_d \leq 0\}$. From now on we write $p_1(h)$ for $p(h, 1, 1, \ldots, 1)$ and count the number of possible configurations $h \in \Lambda_\ell$ that we may have when such a configuration h is restricted to $p_1(h)$. In fact, a result of [2] asserts that the number of $k\colon \mathbb{Z}^d_+ \to \mathbb{Z}_+$ with $k(i + e_r) \geq k(i)$ for $r = 1, \ldots, d$ and

$$\sum_{i \in \mathbb{Z}^d} k(i) = n,$$

is bounded above by $\exp(c_1 n^{\frac{d-1}{d}})$, for some constant c_1. From this we easily deduce $|\Lambda_\ell| \leq 2^d \exp(c_1 \ell^{\frac{d-1}{d}})$. This certainly implies the finiteness of Z.

The reversibility is straightforward and follows from

$$\begin{aligned} &h \in \Lambda, \qquad h^i \in \Gamma \quad \Rightarrow \quad h^i \in \Lambda,\\ &h \in \Lambda, \qquad h_i \in \Gamma \quad \Rightarrow \quad h_i \in \Lambda,\\ &\lambda^+ \nu(\{h\}) \mathbb{1}(h, h^i \in \Lambda) = \lambda^- \nu(\{h^i\}) \mathbb{1}(h, h^i \in \Lambda),\\ &\lambda^- \nu(\{h\}) \mathbb{1}(h, h_i \in \Lambda) = \lambda^+ \nu(\{h_i\}) \mathbb{1}(h, h_i \in \Lambda). \end{aligned} \qquad \square$$

PROOF OF THEOREM 5.1. Let $\mathscr{P}$ be any limit point of $\mathscr{P}^\varepsilon$. From Lemmas 5.3, 5.4 and Theorem 3.5 we learn that $S = S^H$ for some Lipschitz function H, for almost all S in the support of $\mathscr{P}$. (Note that H may depend on S.) Using Lemma 5.5, it is not hard to deduce that for all rational points (x, t), the measure $\mathscr{P}$ is concentrated on the set of S for which $S(0, t; v)(x) = t\overline{L}(\frac{x}{t})$. By the continuity of S, we have this for all (x, t). From this and Lemma 5.6 we deduce that $\overline{L}^* =: \widetilde{H}$ is the convex hull of H.

Pick a point p on the boundary of Y, say $p = (p_1, p_2, \ldots, p_d)$ with $p_1 \in \{0, \beta_1\}$. Let $k(i) = [i \cdot p]$. It is not hard to see that for such a choice of p, we always have $k^i, k_i \notin \Gamma$ for every $i \in \mathbb{Z}^d$. This implies that $h(i, t; k) = k(i)$. From this we deduce $S(0, t, g_p)(x) = g_p(x)$, for almost all S in the support of $\mathscr{P}$ and every p on the boundary of Y.

Lemma 5.7 can be used to deduce that $\mathscr{P}$ is concentrated on the set of $S = S^H$ with

$$S(0, t; \tilde{v}) = \tilde{v},$$

where $\tilde{v}(x) = -v(-x)$. If we choose $\phi(x, t) = x \cdot p$ for some $p \in Y$ and $x_0 = 0$ in (5.1), we deduce $H(p) \leq 0$.

The property $H(p) = H(\bar{p} - p)$ was established as a part of Theorem 7.1 in [14] when $\lambda^- = 0$. The proof of the general case is identical. This completes the proof of $H \in \mathscr{H}$, $\mathscr{P}$-almost surely. $\square$

PROOF OF THEOREM 2.1. From Theorem 4.1, it is not hard to deduce that any limit point of $\mathscr{P}^\varepsilon$ is concentrated on the set of S such that $S(0, t; g_p) = g_p - tH(p)$, where H is defined by (4.20). From this we deduce that H is non-random. By uniqueness, $\mathscr{P}$ is concentrated on a single semigroup S^H. Since the functional

$$\mathscr{F}(S) = \sup_{0 \le s \le t \le T} \sup_{|x| \le T} |S(s, t; g)(x) - S^H(s, t; g)(x)|$$

is continuous with respect to the topology of $\mathscr{D}_T$, we deduce

$$\lim_{\varepsilon \to 0} \int \mathscr{F}(S) \mathscr{P}^\varepsilon(dS) = 0.$$

This evidently completes the proof when the initial height function is k_g^ε. The general case follows from Lemma 3.11. □

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