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RENORMALIZATION OF THE VOTER MODEL IN EQUILIBRIUM[1]

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We consider the d-dimensional voter model for $d \geq 3$. Our interest is the large scale limit of the equilibrium state of the voter model, where we prove the $d = 3$ results of [1] for $d \geq 4$, which turn out to be of a different nature than for $d = 3$. For this purpose we use the historical process. We establish some surprising facts about the Green's function of random walks in dimension $d \geq 4$, which lead to the different features in $d = 3$ versus $d \geq 4$. Secondly, we prove an analogous result for the voter model on the hierarchical group.

1. Introduction and the main result. In this paper we study the structure of invariant measures of the voter model (cf. [8] and [11]). In contrast to other situations (e.g., the exclusion process), where we have a good characterization of the invariant measures (e.g., product measure), we do not have an explicit representation of the invariant measures of the voter model. The dependence between components is only slowly decaying so that we do not expect simply classical fluctuation behavior. However, since dependence of the components in equilibrium is induced by a local interaction, we can try a renormalization scheme. Rescaling in other contexts was investigated, for example, by Holley and Stroock; see [9]. Bramson and Griffeath investigated renormalization of the voter model on $\mathbb{Z}^3$, [1]. They studied the discrete time voter model which is defined with respect to a local symmetric random walk with finite second moments. The proof of their renormalization result is based on the methods of moments.

Major gave another proof of their result in [12] based on the historical process in today's terminology. The idea of his proof is easy to grasp; however, essential parts of the proof are not correct. The claim in [12] is also that these results hold on $\mathbb{Z}^d$ with $d \geq 4$; however, this is based on wrong assumptions on the behavior of the Green's function of random walks. Major's idea works in case of finite moments of order $3d - 1$ of the underlying random walk, which is in fact not necessary.

The generalization to the cases $d \geq 4$ is more subtle and uses some observations on random walks recently made by Lawler [10]. This paper contains the right assumptions on the model and the right formulation of results for the continuous time voter model. Furthermore, we establish the result for the voter model on the hierarchical group. In a self-contained section we state the asymptotics of the Green's function of random walks. Similar questions and

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problems arise in branching models, which are studied by rescaling from a different point of view by Dawson, Gorostiza and Wakolbinger, see [4].

We hope that the techniques used here for the voter model can be refined in order to study limiting states of branching evolutions and interacting Fisher-Wright diffusions in randomly fluctuating media. In fact, more generally locally interacting systems for which a historical process can be defined should be accessible and shall be treated in a forthcoming paper.

1.1. *The model.* We consider the voter model $(\xi_t)_{t\geq 0}$ on a countable Abelian group S, which we shall later specialize to the two cases $\mathbb{Z}^d$ and the hierarchical group $\Xi^{(N)}$. (For a survey see [11], Chapter V.) It is an interacting particle system with state space $\{0,1\}^S$. Each site $j \in S$ is occupied by an individual. The value 0 or 1 denotes for instance the political opinion of the person (the individual). The transition mechanism is specified by the function

$$c(i,\xi) = \begin{cases} \sum_j p(i,j)\xi(j), & \text{if } \xi(\mathrm{i}) = 0, \\ \sum_j p(i,j)[1-\xi(j)], & \text{if } \xi(\mathrm{i}) = 1, \end{cases} \tag{1.1}$$

where $p(i,j) \geq 0$ for $i, j \in S$ and

$$\sum_{j\in S} p(i,j) = 1 \qquad \text{for } i \in S. \tag{1.2}$$

The function c represents the rate at which the coordinate $\xi(i)$ flips from 0 to 1 or from 1 to 0 when the system is in the state ξ. That means

$$\mathbf{P}^{\xi}[\xi_t(i) \neq \xi(i)] = c(i,\xi)t + o(t) \tag{1.3}$$

as $t \downarrow 0$ for each $i \in S$ and $\xi \in \{0,1\}^S$. Furthermore, we require that in each transition only one coordinate changes, that is,

$$\mathbf{P}^{\xi}[\xi_t(i) \neq \xi(i);\ \xi_t(j) \neq \xi(j)] = o(t) \tag{1.4}$$

as $t \downarrow 0$ for each $i, j \in S$ with $i \neq j$ and $\xi \in \{0,1\}^S$. An equivalent way of describing the rates of the voter model is to say that a site i waits an exponential time with parameter one; after that time it flips to the value it sees at that time at a site j which is chosen with probability $p(i,j)$.

Let us consider the translation invariant setting, that is, $p(i,j) = p(0, j-i)$. We define $p(i) = p(0,i)$. We will need the symmetrized kernel

$$\hat{p}(i) = \frac{p(i)+p(-i)}{2}. \tag{1.5}$$

For $0 < \lambda < 1$ let the initial distribution $\mathscr{L}[\xi_0] = \nu$ be a translation invariant, ergodic measure ν with intensity

$$\nu\{\eta \in \{0,1\}^S : \eta(i) = 1\} = \lambda \qquad \forall i \in S. \tag{1.6}$$

The following basic result can be found in [11], V.1.13 or [8], Section 5. Liggett only does this for $\mathbb{Z}^d$ but the same proof works in the general Abelian case. Holley and Liggett do not restrict to $\mathbb{Z}^d$.

BASIC ERGODIC THEOREM. *In the translation invariant setting of the voter model on a countable Abelian group there is the following dichotomy in* $\hat{p}$ *concerning the longtime behavior*:

(a) *If* $\hat{p}$ *is recurrent*:

$$\mathscr{L}^{\nu}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \lambda\delta_{\underline{1}} + (1-\lambda)\,\delta_{\underline{0}}, \tag{1.7}$$

where $\underline{0}$ (*resp*. $\underline{1}$) *denotes the configuration with all sites* 0 (*resp*. 1).

(b) *If* $\hat{p}$ *is transient, there exists a unique probability measure* μ_λ, *depending only on* λ, *such that*

$$\mathscr{L}^{\nu}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \mu_\lambda. \tag{1.8}$$

This μ_λ *has the following properties*:

(i) μ_λ *is an invariant measure of the process* (ξ_t).
(ii) μ_λ *is ergodic*.
(iii) *Let* ξ_∞ *have the distribution* μ_λ, *then for all* $i, j \in S$:

$$\begin{aligned} &\mathbf{E}[\xi_\infty(i)] = \lambda, \\ &\mathbf{E}[\xi_\infty(i)\xi_\infty(j)] \underset{\|i-j\|\to\infty}{\searrow} \lambda^2. \end{aligned} \tag{1.9}$$

The symbol $\Rightarrow$ denotes weak convergence.

REMARK 1. We want to mention that the convergence in (1.9) is polynomial and not exponential.

1.2. *Main result on* $\mathbb{Z}^d$. Our goal is to study the regime in the transient case by means of renormalization of the random field under the equilibrium distribution. Here renormalization means forming sums over spatial blocks and rescaling their size.

Let ξ be a random variable with distribution μ_λ given in (1.8). (We omit the index ∞ in ξ_∞.) Now we define the rescaled field. As mentioned at the beginning, we are interested in the group $\mathbb{Z}^d$ and the hierarchical group. We have to distinguish between these two cases.

If $S = \mathbb{Z}^d$ ($d \geq 3$), we define for a test function $\varphi \in \mathscr{S}$ (Schwartzian space of rapidly decreasing functions) the following random variable

$$F_\lambda(\varphi) = \sum_{i\in\mathbb{Z}^d} [\xi(i) - \mathbf{E}\xi(i)]\varphi(i) = \sum_{i\in\mathbb{Z}^d} [\xi(i) - \lambda]\varphi(i). \tag{1.10}$$

This random variable will be rescaled now. For sums of independent random variables one chooses the classical rescaling of the central limit theorem. There are results that this rescaling can also be used in the case of dependent random variables, if the correlations are weak enough, for example, if they are *exponentially decreasing*. This means that the correlation function ρ of the distribution μ_λ (recall that $\mathscr{L}[\xi] = \mu_\lambda$) defined by

$$\rho(A) = \mathbf{P}\big[\xi(i) = 1 \text{ for all } i \in A\big] \tag{1.11}$$

fulfills

$$\big|\rho(A \cup B) - \rho(A)\rho(B)\big| \leq C_1 e^{-C_2 d(A,B)}, \tag{1.12}$$

where C_1 and C_2 are constants depending only on the cardinality of $A \cup B$, and $d(A, B) = \min\{|i - j| : i \in A, j \in B\}$ is the distance between A and B. This result can be found in [13].

However, since the process has positive spatially slowly decreasing correlations (cf. Remark 1), a classical rescaling as for i.i.d. random fields does not work. We choose

$$F_{\lambda, r}(\varphi) = F_\lambda(\varphi_r), \tag{1.13}$$

with

$$\varphi_r(x) = h(r)\varphi\left(\frac{x}{r}\right), \tag{1.14}$$

where we have to choose the function $h(r)$ depending on the Green's function of the underlying random walk and decreasing much faster than the classical rescaling

$$\frac{1}{\sqrt{\#\{i \in \mathbb{Z}^d : |i| \leq r\}}}. \tag{1.15}$$

Let $Z^{(p)} = (Z_1^{(p)}, \dots, Z_d^{(p)})$ denote a random variable with distribution p. We assume:

(1.16) The group $\mathscr{G}$ generated by $\{i : p(i) > 0\}$ is d-dimensional,

(1.17) $Z^{(p)}$ has finite second moments; $\mathbf{E}[Z_l^{(p)} Z_k^{(p)}] = \sigma_{l,k}$; $l, k = 1, \dots, d$.

The first assumption ensures transience of $\hat{p}$. Let $Q = (Q_{l,k})_{l,k=1}^d$ denote the matrix of second moments, that is, $Q_{l,k} = \sigma_{l,k}$. Let $|Q|$ denote the determinant of Q. From (1.16) one can derive that Q is positive definite. Hence Q is invertible. Let $\bar{Q}(x)$ denote the following quadratic form

$$\bar{Q}(x) = x^{tr} Q^{-1} x, \qquad x \in \mathbb{R}^d. \tag{1.18}$$

Note that the bar above Q indicates that the quadratic form is defined in terms of the matrix Q^{-1} and not of Q.

Now we formulate the main result.

THEOREM 1. *Assume $S = \mathbb{Z}^d$ with $d \geq 3$. Then under assumptions* (1.16), (1.17) *and for $d > 3$ under the additional assumption*

$$\text{for } d = 4: \quad n^2 \mathbf{P}[|Z^{(p)}| \geq n] = o\left(\frac{1}{\log n}\right), \tag{1.19}$$

$$\text{for } d \geq 5: \quad \text{finite moments of order } d - 1, \tag{1.20}$$

and with the choice

$$h(r) = r^{-\frac{d+2}{2}} \tag{1.21}$$

we obtain weak convergence

$$\mathscr{L}[F_{\lambda,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathscr{N}(0, C_\lambda B(\varphi,\varphi)). \tag{1.22}$$

Here B is the bilinear functional

$$B(\varphi,\psi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\varphi(x)\psi(y)}{\bar{Q}(x-y)^{(d-2)/2}}\,dx\,dy, \tag{1.23}$$

with $\bar{Q}(x)$ defined in (1.18). *C_λ has the form*

$$C_\lambda = \lambda(1-\lambda)\frac{\gamma}{2\pi^{d/2}|Q|^{1/2}}\Gamma\left(\frac{d-2}{2}\right), \tag{1.24}$$

where Γ denotes the Gamma function and γ denotes the escape probability of the discrete time random walk Y with kernel $\hat{p}(i) = \frac{1}{2}(p(i)+p(-i))$, which starts in 0:

$$\gamma = \mathbf{P}[Y_n \neq 0\ \forall n \geq 1]. \tag{1.25}$$

One can formulate Theorem 1 in terms of generalized random fields. The concept of generalized random fields has a physical motivation. Every actual measurement is accomplished by means of an apparatus. It is often impossible to measure the value of the random variable $X(s)$ at the instant s. Instead of this one gets a certain averaged value $F(\varphi) = \int \varphi(s)X(s)\,ds$, where φ is a function characterizing the apparatus.

The distribution of a generalized random field is a probability measure on the σ-algebra of Borel subsets (with respect to the weak topology) of the dual space $\mathscr{S}'$ of $\mathscr{S}$ (cf. [7], Chapter III). Since $\mathscr{S}$ is a normed space the dual $\mathscr{S}'$ is a Banach space. The convergence statement can be easily extended to the whole random field and its rescaled versions $F_{\lambda,r}$. Then we have:

COROLLARY 1. *Let $d \geq 3$ and consider the generalized random field $F_{\lambda,r}$. Under the assumptions of Theorem* 1 *and with $h(r) = r^{-(d+2)/2}$:*

$$\mathscr{L}[F_{\lambda,r}] \underset{r\to\infty}{\Longrightarrow} \mathscr{L}[C_\lambda \Psi], \tag{1.26}$$

where $C_\lambda > 0$ is given in (1.24) *and Ψ is the Gaussian self-similar generalized random field with covariance functional*

$$\mathbf{E}[\Psi(\varphi)\Psi(\psi)] = B(\varphi,\psi), \tag{1.27}$$

with B given in (1.23).

The convergence in (1.26) is weak convergence of probability measures on $\mathscr{S}'$. It is well-known that the function class $\{\varphi(t) = e^{ixt}; x \in \mathbb{R}\}$ is separating for the probability measures on $\mathbb{R}$. In the case of probability measures on $\mathscr{S}'$

the class $\{\Phi(F) = e^{iF(\varphi)}; \varphi \in \mathscr{S}\}$ is separating. That means to get (1.26) one needs to show

$$\mathbf{E}\big[e^{iF_{\lambda,r}(\varphi)}\big] \underset{r\to\infty}{\longrightarrow} \exp\big\{-\tfrac{1}{2}C_\lambda B(\varphi,\varphi)\big\}. \tag{1.28}$$

This in turn is the assertion of Theorem 1.

We want to interpret the rescaling factor $h(r)$. If the $\{\xi(i); i \in \mathbb{Z}^d\}$ were independent random variables, one would have to choose the classical rescaling one over the root of the volume of the ball with radius r. But the $\xi(i)$ are not independent. We will define a subdivision in families of the $\{\xi(i); i \in \mathbb{Z}^d\}$, to which 0's or 1's are assigned independently. To be more precise, all members of a family are assigned the same value and the values of different families are independent. Supposing that there are N_r families in the ball with radius r and a typical family has a size of order M_r, we have to choose the following rescaling term

$$h(r) = \frac{1}{M_r\sqrt{N_r}} = \frac{1}{\sqrt{M_r}} \cdot \frac{1}{\sqrt{N_r M_r}}. \tag{1.29}$$

The second factor is the classical rescaling. The first factor is the correction term. We observe that the correction factor is one over the root of the size of a typical family. For a more precise explanation we refer to Remark 2.

The main idea of cluster decomposition and the interpretation of the correction factor in terms of the historical process can be tested on other groups. An interesting candidate for that is the hierarchical group. We are able to establish the analogous result to Theorem 1 on the hierarchical group.

1.3. *Main result on the hierarchical group.* The hierarchical group plays an important role in spatial models in population genetics. It was introduced by Sawyer [14] and has appeared recently, for example, in [3], [6] and [5]. The hierarchical group $\Xi^{(N)}$ is defined by

$$\begin{aligned}\Xi^{(N)} := \Big\{ i = \big(i^{(m)}\big)_{m\in\mathbb{N}} \colon i^{(m)} \in \{0,\dots,N-1\},\\ i^{(m)} \neq 0 \text{ only for finitely many } m \Big\}\end{aligned} \tag{1.30}$$

with addition componentwise modulo N and distance $\|i\| = \max\{k : i^{(k)} \neq 0\} \vee 0$. We are interested in transition kernels with the property that $p(i)$ depends only on $\|i\|$. Let r be a distribution on $\{0,1,\dots\}$. Define an associated probability law p on $\Xi^{(N)}$ by setting

$$p(i) = \frac{r_{\|i\|}}{R_{\|i\|}}, \tag{1.31}$$

where

$$R_k = \#\{i \in \Xi^{(N)} : \|i\| = k\} \tag{1.32}$$

is the number of elements in the kth level set $\Xi_k^{(N)} = \{i \in \Xi^{(N)} : \|i\| = k\}$.

We consider the geometric kernels p_c $(c > 1/N)$ of the form (1.31) with

$$r_k = \theta_c \cdot (Nc)^{-k} \tag{1.33}$$

with the normalizing constant $\theta_c = \frac{Nc-1}{Nc}$. We assume $c < 1$ to get transience of p_c. (One can easily verify that p_c is transient iff $c < 1$.)

The rescaled field is defined by

$$F_{\lambda, r} = h(r) \sum_{\substack{i \in \Xi^{(N)} \\ \|i\| < r}} \big[\xi(i) - \mathbf{E}\xi(i)\big] = h(r) \sum_{\substack{i \in \Xi^{(N)} \\ \|i\| < r}} \big[\xi(i) - \lambda\big]. \tag{1.34}$$

We formulate the main result on the hierarchical group.

THEOREM 2. *Assume the voter model on the hierarchical group with geometric transition kernel* p_c $(1/N < c < 1)$. *With the choice*

$$h(r) = N^{-r} c^{-r/2} \tag{1.35}$$

we obtain weak convergence

$$\mathscr{L}[F_{\lambda, r}] \underset{r \to \infty}{\Longrightarrow} \mathscr{N}(0, C_\lambda) \tag{1.36}$$

with

$$C_\lambda = \lambda(1-\lambda)\gamma \frac{(N^3c^2 - N^2c^2 - N^2c + Nc)}{(N^2c - 1)(1-c)} \frac{N-1}{N^2(Nc-1)}. \tag{1.37}$$

Here γ *is the escape probability of the discrete time random walk* Y *with kernel* p_c, *which starts in* 0:

$$\gamma = \mathbf{P}[Y_n \neq 0 \ \forall n \geq 1]. \tag{1.38}$$

The interpretation of the rescaling factor is the same as in the lattice case. We refer to Remark 3.

2. The asymptotic behavior of the Green's function on $\mathbb{Z}^d$. The basis of the proof of the main result is the asymptotics of the Green's function $G(x, y)$ of random walks. In particular the question arises under what conditions does $G(x, y)$ behave for $|x - y| \to \infty$ as the Green's function of Brownian motion which decays like $|x - y|^{-(d-2)}$ for $|x - y| \to \infty$. The first guess is that it should suffice that the random walk is in the domain of normal attraction. However, this is only true for $d = 3$. In $d \geq 4$ this is false; here one needs stronger moment conditions. Since these facts are of independent interest, we state them here.

We consider a discrete time random walk (Y_n) with kernel q which starts in the origin. We assume (1.16) and (1.17) (with q instead of p) and in addition $\mathbf{E}Z^{(q)} = 0$. That means Q is the covariance matrix of q. Let $G(x)$ be the expected number of visits in x, that is,

$$G(x) = \sum_{n=0}^{\infty} q_n(x), \tag{2.1}$$

where q_n denotes the n-step transition probability of the kernel q. G is called the Green's function of the random walk Y. We want to establish the asymptotics of $G(x)$ as $|x| \to \infty$.

First of all we consider the case $\mathscr{G} = \mathbb{Z}^d$. Y is an aperiodic random walk on $\mathbb{Z}^d$ (aperiodic means $\mathscr{G} = \mathbb{Z}^d$). We have to distinguish the three cases $d = 3$, $d = 4$ and $d \geq 5$, where only the first case is well-known.

CASE 1 ($d = 3$). Here we use [15], 26.P1, to obtain a statement about the asymptotic behavior of the Green's function [recall (1.18) for $\bar{Q}(x)$ and that $|Q|$ denotes the determinant of Q]:

$$\lim_{|x|\to\infty} \bar{Q}(x)^{1/2} G(x) = \frac{1}{2\pi |Q|^{1/2}}. \tag{2.2}$$

We want to generalize this result to dimension $d \geq 4$. First we consider the case where Y is a strongly aperiodic random walk. That means

$$\mathscr{G}\{y \in \mathbb{Z}^d:\ y = x + z,\ \text{where } p(z) > 0\} = \mathbb{Z}^d \qquad \text{for all } x \in \mathbb{Z}^d, \tag{2.3}$$

where $\mathscr{G}\{\cdots\}$ denotes the group generated by $\{\cdots\}$. For instance, the symmetric nearest-neighbor random walk is aperiodic but not strongly aperiodic.

CASE 2 ($d = 4$). Assuming only (1.16) and (1.17), the above asymptotics do not hold (cf. [10]).

In [10] we find the result that under the slightly stronger boundedness assumption (1.19) (with q instead of p) one can guarantee the following asymptotics

$$\lim_{|x|\to\infty} \bar{Q}(x) G(x) = \frac{1}{2\pi^2 |Q|^{1/2}}. \tag{2.4}$$

CASE 3 ($d \geq 5$). We obtain an analogous result under the stronger assumptions (1.20) suggested by Lawler. Since he did not write a paper about it, we want to establish this result.

THEOREM 3. *Let $d \geq 5$. Suppose $(Y_n)_{n\geq 0}$ is a mean zero, finite variance, strongly aperiodic random walk on $\mathbb{Z}^d$ with covariance matrix Q. Assume also that the moments of order $d-1$ are finite, that is,*

$$\sum_{n=1}^{\infty} n^{d-2} \mathbf{P}[|Y_1| \geq n] < \infty. \tag{2.5}$$

Then

$$\lim_{|x|\to\infty} \bar{Q}(x)^{\frac{d-2}{2}} G(x) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2} |Q|^{1/2}} \tag{2.6}$$

as $|x| \to \infty$.

Now we summarize the results of the three cases $d = 3$, $d = 4$ and $d \geq 5$ and we extend it to aperiodic random walks on $\mathscr{G}$.

COROLLARY 2. *Let Y be the random walk defined at the beginning of this chapter. Under assumptions* (1.16), (1.17) *and the additional assumptions* (1.19) *for $d = 4$ and* (1.20) *for $d \geq 5$* (*each with q instead of p*) *we have the following asymptotics of the Green's function*:

$$G(x) \sim \begin{cases} C \cdot \bar{Q}(x)^{-\frac{d-2}{2}}, & \text{as } |x| \to \infty;\ x \in \mathscr{G}, \\ 0, & \text{if } x \in \mathbb{Z}^d \backslash \mathscr{G}, \end{cases} \tag{2.7}$$

where

$$C = \frac{|\mathbb{Z}^d/\mathscr{G}|\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|Q|^{1/2}}, \tag{2.8}$$

and $\mathscr{G}$ is the group generated by $\{i \in \mathbb{Z}^d : q(i) > 0\}$.

PROOF OF THEOREM 3. The transition probabilities are given by q. Let q_n denote the n-step transition probabilities. The finiteness of the moments of order $d-1$ and the local central limit theorem ([2], Corollary 22.3) give us

$$\begin{aligned} \sup_{x \in \mathbb{Z}^d} \left(\left(\frac{|x|}{\sqrt{n}} \right)^{d-1} + 1 \right) \Bigg| q_n(x) - n^{-d/2} \varphi_{0,Q}\left(\frac{x}{\sqrt{n}} \right) & \\ \times \left(1 + \sum_{k=1}^{d-3} n^{-k/2} P_k\left(\frac{x}{\sqrt{n}} \right) \right) \Bigg| & \\ = o\big(n^{-(2d-3)/2}\big), & \end{aligned} \tag{2.9}$$

where P_k is a polynomial of degree $3k$, and $\varphi_{0,Q}$ is the density of the normal distribution with expectation 0 and covariance matrix Q,

$$\varphi_{0,Q}(y) = \frac{1}{(2\pi)^{d/2}|Q|^{1/2}} e^{-\bar{Q}(y)/2}. \tag{2.10}$$

Let

$$\bar{q}_n(x) = n^{-d/2} \varphi_{0,Q}\left(\frac{x}{\sqrt{n}} \right) = \frac{1}{(2\pi n)^{d/2}|Q|^{1/2}} \exp\left\{ -\frac{\bar{Q}(x)}{2n} \right\}. \tag{2.11}$$

To prove the asymptotics of the Green's function we write (consider $x \neq 0$)

$$\begin{aligned} G(x)\bar{Q}(x)^{(d-2)/2} = & \sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \bar{q}_n(x) \\ & + \sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} [q_n(x) - \bar{q}_n(x)]. \end{aligned} \tag{2.12}$$

We can calculate the first sum on the r.h.s. of (2.12) as follows (let $\Delta = \bar{Q}(x)^{-1}$):

$$\begin{aligned}
\sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \bar{q}_n(x) &= \frac{1}{(2\pi)^{d/2}|Q|^{1/2}} \sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \frac{1}{n^{d/2}} \exp\left\{-\frac{\bar{Q}(x)}{2n}\right\} \\
&= \frac{1}{(2\pi)^{d/2}|Q|^{1/2}} \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n\Delta}\right)^{d/2} e^{-1/(2n\Delta)} \\
&\to \frac{1}{(2\pi)^{d/2}|Q|^{1/2}} \int_0^{\infty} s^{-d/2} e^{-1/(2s)}\, ds \\
&= \frac{1}{2\pi^{d/2}|Q|^{1/2}} \Gamma\left(\frac{d-2}{2}\right)
\end{aligned} \tag{2.13}$$

as $|x| \to \infty$.

It remains to show that the second sum on the r.h.s. of (2.12) goes to 0 as $|x| \to \infty$. By (2.9) we know

$$\begin{aligned}
&\sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} [q_n(x) - \bar{q}_n(x)] \\
&\quad = \sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \Bigg[\frac{E(n,x)}{n^{(2d-3)/2}\left(\left(\frac{|x|}{\sqrt{n}}\right)^{d-1} + 1\right)} \\
&\qquad\qquad + \frac{1}{(2\pi n)^{d/2}|Q|^{1/2}} e^{-\frac{\bar{Q}(x)}{2n}} \sum_{k=1}^{d-3} n^{-k/2} P_k\left(\frac{x}{\sqrt{n}}\right)\Bigg],
\end{aligned} \tag{2.14}$$

where $E(n,x) \to 0$ as $n \to \infty$ uniformly in x. We investigate the second sum on the r.h.s. of (2.14). It suffices to consider

$$\sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \frac{1}{(2\pi n)^{d/2}|Q|^{1/2}} e^{-\frac{\bar{Q}(x)}{2n}} n^{-k/2} \frac{x^{\beta}}{n^{l/2}} \tag{2.15}$$

for multi-indices β with $|\beta| = l$ and $0 \le l \le 3k$; $1 \le k \le d-3$. Analogously to the calculation in (2.13) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \frac{1}{(2\pi n)^{d/2}|Q|^{1/2}} e^{-\frac{\bar{Q}(x)}{2n}} n^{-k/2} \frac{x^{\beta}}{n^{l/2}} \\
&\quad \sim O\left(x^{\beta} \bar{Q}(x)^{-\frac{k}{2}-\frac{l}{2}}\right) \int_0^{\infty} s^{-(d+k+l)/2} e^{-1/(2s)}\, ds \\
&\quad \le O(|x|^{-k}) \underset{|x|\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{2.16}$$

Here we used that Q^{-1} is positive definite and thus $\bar{Q}(x) \ge c|x|^2$, where c is a constant independent of x.

Finally we investigate the first error sum on the r.h.s. of (2.14)

$$\sum_{n=1}^{\infty} \bar{Q}(x)^{(d-2)/2} \frac{E(n,x)}{n^{(2d-3)/2}\left(\left(\frac{|x|}{\sqrt{n}}\right)^{d-1}+1\right)} \le O(|x|^{-1}) \sum_{n=1}^{\infty} \frac{E(n,x)}{n^{(d-2)/2}} \tag{2.17}$$
$$= O(|x|^{-1}) \underset{|x|\to\infty}{\longrightarrow} 0.$$

The sum on the r.h.s. of the first inequality is uniformly bounded since the error term $E(n, x)$ goes to 0 as $n \to \infty$ uniformly in x.

Equations (2.13), (2.16) and (2.17) lead to the assertion. This completes the proof. □

PROOF OF COROLLARY 2. First note that $G(x) = 0$ for $x \notin \mathscr{G}$.

At the last step of the proof of [15], 26.P1, we find an argument for the extension of the result from strongly aperiodic random walks to aperiodic random walks on $\mathbb{Z}^d$. This argument works also in dimension $d > 3$. Hence we obtain (2.7) for aperiodic random walks on $\mathbb{Z}^d$.

It remains to consider the case of an aperiodic random walk on $\mathscr{G}$. By assumption (1.16) $\mathscr{G}$ is d-dimensional. Hence there exists a bijective linear mapping $A: \mathscr{G} \to \mathbb{Z}^d$. We know that $|A^{-1}| = |\mathbb{Z}^d/\mathscr{G}|$.

We are given a random walk on $\mathscr{G}$ with transition kernel q. We define the following transition kernel $\tilde{q}$ on $\mathbb{Z}^d$:

$$\tilde{q}(x) = q(A^{-1}x). \tag{2.18}$$

For $y \in \mathscr{G}$ we obtain $G(y) = \tilde{G}(Ay)$, where $\tilde{G}$ is the Green's function of the kernel $\tilde{q}$.

Since $\tilde{q}$ is aperiodic on $\mathbb{Z}^d$ we know from the first part of this proof that

$$\tilde{G}(x) \sim \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|\tilde{Q}|^{1/2}} \tilde{Q}(x)^{-\frac{d-2}{2}}, \tag{2.19}$$

where $\tilde{Q}(x) = x^{tr}\tilde{Q}^{-1}x$. Obviously $\tilde{Q} = AQA^{tr}$, hence $|\tilde{Q}| = |A|^2|Q|$ and $\tilde{Q}(x) = \bar{Q}(A^{-1}x)$. Thus by (2.19),

$$G(y) = \tilde{G}(Ay) \sim \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|\tilde{Q}|^{1/2}} \tilde{Q}(Ay)^{-\frac{d-2}{2}} = \frac{|\mathbb{Z}^d/\mathscr{G}|\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|Q|^{1/2}} \bar{Q}(y)^{-\frac{d-2}{2}}. \tag{2.20}$$

This completes the proof. □

3. Proof of Theorem 1 (voter model on $\mathbb{Z}^d$). The proof is based on the characterization of the equilibrium μ_λ as the limit of $\mathscr{L}[\xi_t]$ as $t \to \infty$. From [11], V.1.13, we know that the limiting distribution μ_λ is the same for all initial distributions ν which are translation invariant and ergodic and have property (1.6). Thus it is enough to consider the special initial distribution ν_λ being the product measure on $\{0,1\}^{\mathbb{Z}^d}$ with intensity λ.

The proof is split into four parts. The first part contains the basic idea. We construct the historical process of the voter model, that is, we define a richer structure containing a family structure, which explains all dependencies of the components. This allows a cluster decomposition of the equilibrium state. Namely we can view it as an infinitely old system and decompose the components into clusters belonging to the same family. Then 0's and 1's are assigned in an i.i.d. fashion to the families, and we can apply the central limit theorem. In the second part we check the assumptions of the central limit theorem. The crucial quantities for this, as moments and covariances, can be expressed in terms of random walk quantities. The lemmas of the second part are proved in the third part, and some more technical facts are collected in the fourth part.

PART 1 (Representation via historical process). Our goal here is to write the random variable $F_{\lambda,r}(\varphi)$ as a functional of the historical process associated with the voter model.

First we formulate a graphical representation of the voter model which allows the definition of the law of the historical process in a very natural way. Let $\{X(j); j \in \mathbb{Z}^d\}$ be a system of coalescing continuous time random walks. They move according to the transition kernel

$$p_t(i, j) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!}\, p^{(n)}(i, j), \tag{3.1}$$

where $p^{(n)}$ denotes the n-step transition probability of the kernel p. The random walk $X(j)$ starts in j. Any random walk $X(j)$ evolves independently of each other except for the following collision rule. Whenever two or more random walks attempt to occupy the same site at the same time they merge into one.

Moreover let $\{\alpha(j); j \in \mathbb{Z}^d\}$ be i.i.d. random variables, which are independent of the random walks $\{X(j); j \in \mathbb{Z}^d\}$ and have marginal distribution

$$\mathbf{P}[\alpha(j) = 1] = 1 - \mathbf{P}[\alpha(j) = 0] = \lambda. \tag{3.2}$$

To determine the "opinion" $\xi_t(j)$ at site j at time t we follow the sites where the "opinion" came from. Define

$$\hat{\xi}_t(j) := \alpha(X_t(j)) \tag{3.3}$$

for $j \in \mathbb{Z}^d$. The following duality equation is valid for $j_1, \dots, j_k \in \mathbb{Z}^d;\ k \in \mathbb{N}$:

$$\mathbf{P}\big[\xi_t(j_1) = 1, \dots, \xi_t(j_k) = 1\big] = \mathbf{P}\big[\hat{\xi}_t(j_1) = 1, \dots, \hat{\xi}_t(j_k) = 1\big], \tag{3.4}$$

which means that the common distribution of $\{\xi_t(j); j \in \mathbb{Z}^d\}$ and the common distribution of $(\hat{\xi}_t(j); j \in \mathbb{Z}^d\}$ are equal. (For a treatment of duality of a voter model or more general of a spin system see [11], Section III.4.) Note that the process in (3.3) can be defined for all $t \geq 0$.

In the ^-process we can define "one"-opinions of the same family which come from the same ancestor at time 0. We are even able to define the depth of the relationship of two "one's."

At time t we partition $\mathbb{Z}^d$ in families of components which have the same value at time t and where the values in different families are independent. The configurations in one family are dependent. These families are determined by the coalescing random walks.

We define the time t and the equilibrium decomposition of the state in family clusters. This is obtained by partitions $\mathbb{B}(t), \mathbb{B}(\infty)$. Two sites j and j' belong to the same family cluster, that is, to the same element of the partition $\mathbb{B}(t)$ if the random walks $X(j)$ and $X(j')$ coalesce by time t, that is, if $X_t(j) = X_t(j')$. Analogously we can define partition $\mathbb{B}(\infty)$ in the equilibrium. Two sites j and j' belong to the same element of the partition $\mathbb{B}(\infty)$ if the random walks $X(j)$ and $X(j')$ coalesce eventually, i.e., if there exists a t such that $X_t(j) = X_t(j')$.

Now we are going to analyze the rescaled process by means of the cluster decomposition. First of all we consider the case of a test function φ with compact support, that is, $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$. Let $D(r)$ be the d-dimensional ball

$$D(r) = \{j \in \mathbb{Z}^d : |j| < r\}. \tag{3.5}$$

For φ there exists an $A > 0$ such that $\operatorname{supp}(\varphi) \subset D(A)$, hence $D(Ar)$ contains the support of φ_r.

Since φ vanishes outside $D(Ar)$, we are only interested in those elements of the partition which intersect $D(Ar)$. The partition $\mathbb{B}(r,t)$ [resp. $\mathbb{B}(r,\infty)$] is generated by $\mathbb{B}(t)$ [resp. $\mathbb{B}(\infty)$] by taking the intersection of the elements of $\mathbb{B}(t)$ [resp. $\mathbb{B}(\infty)$] with $D(Ar)$. These elements are denoted by $B_1(r,t), \dots, B_{K(r,t)}(r,t)$ [resp. $B_1(r,\infty), \dots, B_{K(r,\infty)}(r,\infty)$]. The number of elements is random. (Capital letters stand for random variables, small letters stand for fixed quantities.) For abbreviation we omit the sign ∞.

With the duality equation we establish the following equation:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^d} [\xi_t(j) - \lambda]\varphi_r(j) &\overset{d}{=} \sum_{j\in\mathbb{Z}^d} [\hat{\xi}_t(j) - \lambda]\varphi_r(j) \\
&= \sum_{k=1}^{K(r,t)} \sum_{j\in B_k(r,t)} [\hat{\xi}_t(j) - \lambda]\varphi_r(j) \\
&= \sum_{k=1}^{K(r,t)} [\alpha_k - \lambda] \sum_{j\in B_k(r,t)} \varphi_r(j),
\end{aligned} \tag{3.6}$$

where $\{\alpha_k;\ k \in \mathbb{N}\}$ are i.i.d. random variables being independent of the partition and satisfying $\mathbf{P}[\alpha_k = 1] = 1 - \mathbf{P}[\alpha_k = 0] = \lambda$. The last equality in (3.6) is valid since $\hat{\xi}_t(j) = \hat{\xi}_t(j')$ if j and j' are of the same element of the partition, and since $\hat{\xi}_t(j)$ and $\hat{\xi}_t(j')$ are independent if j and j' are of different elements of the partition. Define

$$\varphi(B, r) = \sum_{j \in B} \varphi_r(j). \tag{3.7}$$

Thus

$$\sum_{j \in \mathbb{Z}^d} [\xi_t(j) - \lambda]\varphi_r(j) \stackrel{d}{=} \sum_{k=1}^{K(r,t)} [\alpha_k - \lambda]\varphi(B_k(r, t), r). \tag{3.8}$$

Let $\mathscr{P}_{r,t}$ be the distribution of $\mathbb{B}(r, t)$ and let $\mathscr{P}_r$ be the distribution of $\mathbb{B}(r)$. Since $\mathbb{B}(r, t) \underset{t \to \infty}{\longrightarrow} \mathbb{B}(r)$ a.s. we obtain weak convergence

$$\mathscr{P}_{r,t} \underset{t \to \infty}{\Longrightarrow} \mathscr{P}_r. \tag{3.9}$$

By (3.9) and by $\mathscr{L}[\xi_\infty] = \lim_{t \to \infty} \mathscr{L}[\xi_0 | \xi_{-t}$ i.i.d. configuration], we derive from (3.8) the main result of part 1, namely

$$F_{\lambda, r}(\varphi) \stackrel{d}{=} \sum_{k=1}^{K(r)} [\alpha_k - \lambda]\varphi(B_k(r), r). \tag{3.10}$$

REMARK 2. We observe that our rescaling is classical rescaling times a correction term, namely

$$\frac{1}{\sqrt{r^d}} \cdot \frac{1}{\sqrt{r^2}} = r^{-\frac{d+2}{2}}. \tag{3.11}$$

The correction factor to the classical rescaling is 1 over the root of the expected size of the family which contains the origin and which lies in the ball with radius r. The expected size of that family is $\sum_{|i|<r} \hat{G}(i)$, where $\hat{G}$ is the Green's function of the symmetrized kernel $\hat{p}$. It turns out later that $\sum_{|i|<r} \hat{G}(i) =$ const r^2.

PART 2 (Assumptions of the CLT). The idea is to fix a partition, to condition on this partition and to apply the following version of the central limit theorem. That makes sense because given a particular partition we have a sum of independent random variables in (3.10).

CENTRAL LIMIT THEOREM. *Let $\{Z_{n,k}; n \in \mathbb{N}, 1 \le k \le k_n\}$ be a system of random variables where the $Z_{n,k}$, $k = 1, \dots, k_n$, are independent for each n. Let $s_n^2 = \mathrm{Var}[\sum_{k=1}^{k_n} Z_{n,k}]$. If $s_n^2 \to c$ and if the system $\{Z_{n,k}\}$ satisfies the Lyapunov condition*

$$\exists \delta > 0 : \lim_{n \to \infty} \sum_{k=1}^{k_n} \mathbf{E}[|Z_{n,k} - \mathbf{E}Z_{n,k}|^{2+\delta}] = 0, \tag{3.12}$$

then

$$\mathscr{L}\left[\sum_{k=1}^{k_n} (Z_{n,k} - \mathbf{E}Z_{n,k})\right] \underset{n \to \infty}{\Longrightarrow} \mathscr{N}(0, c). \tag{3.13}$$

We consider the expectation and the variance, and we check the Lyapunov condition for $\delta = 2$.

STEP 1 (Expectation). Since $\mathbf{E}[\alpha_k] = \lambda$ and since $\{\alpha_k;\ k \in \mathbb{N}\}$ are independent of the partition we have

$$\mathbf{E}\left[\sum_{k=1}^{K(r)} (\alpha_k - \lambda)\varphi(B_k(r), r) \,\middle|\, \mathbb{B}\right] = \sum_{k=1}^{K(r)} \varphi(B_k(r), r)\mathbf{E}[(\alpha_k - \lambda)] = 0 \qquad \text{a.s.} \tag{3.14}$$

STEP 2 (Variance).

$$\operatorname{Var}\left[\sum_{k=1}^{K(r)} (\alpha_k - \lambda)\varphi(B_k(r), r) \,\middle|\, \mathbb{B}\right] = \lambda(1-\lambda) \sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2. \tag{3.15}$$

We have to investigate the expression $\sum \varphi(B_k(r), r)^2$. In Lemma 1 we prove

$$\operatorname{Var}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] \underset{r\to\infty}{\longrightarrow} 0. \tag{3.16}$$

With Chebychev's inequality and with (3.16) we can then conclude

$$\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2 - \mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] \xrightarrow[r\to\infty]{\mathbf{P}} 0, \tag{3.17}$$

where "$\xrightarrow{\mathbf{P}}$" denotes convergence in probability. Lemma 2 provides

$$\mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] \underset{r\to\infty}{\longrightarrow} \frac{\gamma}{2|Q|^{\frac{1}{2}}\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right)B(\varphi, \varphi). \tag{3.18}$$

Thereby we have

$$\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2 \xrightarrow[r\to\infty]{\mathbf{P}} \frac{\gamma}{2|Q|^{\frac{1}{2}}\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right)B(\varphi, \varphi). \tag{3.19}$$

From (3.15) and (3.19) we can conclude

$$\operatorname{Var}\left[\sum_{k=1}^{K(r)} (\alpha_k - \lambda)\varphi(B_k(r), r) \,\middle|\, \mathbb{B}\right] \xrightarrow[r\to\infty]{\mathbf{P}} C_\lambda B(\varphi, \varphi). \tag{3.20}$$

STEP 3 (Lyapunov condition). In Lemma 3 we show

$$\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^4 \xrightarrow[r\to\infty]{\mathbf{P}} 0. \tag{3.21}$$

Hence with the independence of $\{\alpha_k\}$ and $\mathbb{B}$ again we get

$$\begin{aligned}&\mathbf{E}\left[\sum_{k=1}^{K(r)}(\alpha_k-\lambda)^4\varphi(B_k(r),r)^4\,\middle|\,\mathbb{B}\right]\\ &\quad=\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^4\mathbf{E}\big[(\alpha_k-\lambda)^4\,|\,\mathbb{B}\big]\\ &\quad=\big[(1-\lambda)^4\lambda+\lambda^4(1-\lambda)\big]\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^4\xrightarrow[r\to\infty]{\mathbf{P}}0.\end{aligned}\tag{3.22}$$

STEP 4 (Application of the CLT). Now we are able to deal with the expression

$$\sum_{k=1}^{K(r)}(\alpha_k-\lambda)\varphi(B_k(r),r),\tag{3.23}$$

since it satisfies (3.14), (3.20) and (3.22). We want to show

$$\mathbf{E}\left[\exp\left\{i\sum_{k=1}^{K(r)}(\alpha_k-\lambda)\varphi(B_k(r),r)\right\}\right]\xrightarrow[r\to\infty]{}\exp\left(-\frac{C_\lambda B(\varphi,\varphi)}{2}\right).\tag{3.24}$$

First of all we prove

$$\mathbf{E}\left[\exp\left\{i\sum_{k=1}^{K(r)}(\alpha_k-\lambda)\varphi(B_k(r),r)\right\}\,\middle|\,\mathbb{B}\right]\xrightarrow[r\to\infty]{\mathbf{P}}\exp\left(-\frac{C_\lambda B(\varphi,\varphi)}{2}\right).\tag{3.25}$$

For that purpose we fix a subsequence (r_l) of (r) with $r_l\to\infty$. Then there exists a subsequence (r'_l) of (r_l), such that by (3.20)

$$\operatorname{Var}\left[\sum_{k=1}^{K(r'_l)}(\alpha_k-\lambda)\varphi(B_k(r'_l),r'_l)\,\middle|\,\mathbb{B}\right]\xrightarrow[l\to\infty]{\text{a.s.}}C_\lambda B(\varphi,\varphi)\tag{3.26}$$

and by (3.22)

$$\mathbf{E}\left[\sum_{k=1}^{K(r'_l)}(\alpha_k-\lambda)^4\varphi(B_k(r'_l),r'_l)^4\,\middle|\,\mathbb{B}\right]\xrightarrow[l\to\infty]{\text{a.s.}}0.\tag{3.27}$$

By (3.14), (3.26) and (3.27) we obtain that for $\mathbf{P}_{\mathbb{B}}$-a.e. $\mathbb{B}$,

$$\left\{\sum_{k=1}^{k(r'_l)}(\alpha_k-\lambda)\varphi(b_k(r'_l),r'_l),\ l\in\mathbb{N}\right\}\tag{3.28}$$

is an array of independent random variables, which satisfies the assumptions of the central limit theorem. If we apply this theorem we get

$$\text{(3.29)}\quad \mathbf{E}\left[\exp\left\{i\sum_{k=1}^{K(r'_l)}(\alpha_k-\lambda)\varphi(B_k(r'_l),r'_l)\right\}\middle|\,\mathbb{B}\right]\xrightarrow[l\to\infty]{\text{a.s.}}\exp\left(-\frac{C_\lambda B(\varphi,\varphi)}{2}\right).$$

That means for every subsequence there exists an a.s. convergent subsequence such that all limits are the same and deterministic. Hence we obtain (3.25). We observe that the following set of random variables

$$\text{(3.30)}\qquad \left\{\left|\mathbf{E}\left[\exp\left\{i\sum_{k=1}^{K(r)}(\alpha_k-\lambda)\varphi(B_k(r),r)\right\}\middle|\,\mathbb{B}\right]\right|;\ r\geq 0\right\}$$

is uniformly integrable. Hence (3.24) holds.

We finally proved for $\varphi\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ that $F_{\lambda,r}(\varphi)$ converges in distribution to a normally distributed random variable with expectation 0 and variance $C_\lambda B(\varphi,\varphi)$.

This statement is also true for $\varphi\in\mathscr{S}$ by Lemma 6. Thus the proof of Theorem 1 is complete. □

PART 3 (Proofs of the lemmas). We establish the lemmas used in part 2. For notation recall the definition of the partitions of $\mathbb{Z}^d$ as well as the definitions of $\varphi(B,r)$ given in (3.7). Recall also that $D(r)$ is the ball with radius r.

The lemmas in this part are used to check the assumptions of the central limit theorem.

LEMMA 1 (Variance estimate). *Under assumptions* (1.16), (1.17), (1.19) *and* (1.20),

$$\text{(3.31)}\qquad \operatorname{Var}\left[\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^2\right]\xrightarrow[r\to\infty]{}0$$

holds for $\varphi\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}(\varphi)\subset D(A)$.

PROOF. With the help of the hitting probability of the random walks and the notation

$$\text{(3.32)}\qquad V_{j_1,\dots,j_k}=\{X(j_1),\dots,X(j_k)\text{ all coalesce eventually}\}$$

we can write

$$\text{(3.33)}\qquad \mathbf{E}\left[\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^2\right]=\sum_{i,\,j\in D(Ar)}\mathbf{P}[V_{i,j}]\,\varphi_r(i)\varphi_r(j).$$

Hence

$$\left(\mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right]\right)^2 = \sum_{j_1,\dots,j_4 \in D(Ar)} \mathbf{P}[V_{j_1,j_2}]\mathbf{P}[V_{j_3,j_4}]\varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4). \tag{3.34}$$

Analogously we write

$$\mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right]^2 = \sum_{j_1,\dots,j_4 \in D(Ar)} \mathbf{P}[V_{j_1,j_2} \cap V_{j_3,j_4}]\varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4). \tag{3.35}$$

Then by (3.34) and by (3.35),

$$\begin{aligned} \operatorname{Var}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] &= \sum_{j_1,\dots,j_4 \in D(Ar)} \left(\mathbf{P}[V_{j_1,j_2} \cap V_{j_3,j_4}] - \mathbf{P}[V_{j_1,j_2}]\,\mathbf{P}[V_{j_3,j_4}]\right) \\ &\qquad \times \varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4). \end{aligned} \tag{3.36}$$

Let $\tilde{V}_{j_1,\dots,j_4}$ denote the event that the random walks $X(j_1)$ and $X(j_2)$ coalesce and so do the random walks $X(j_3)$ and $X(j_4)$ but the random walks $X(j_1)$ and $X(j_2)$ do not meet the random walks $X(j_3)$ and $X(j_4)$. Then

$$\mathbf{P}[V_{j_1,j_2} \cap V_{j_3,j_4}] = \mathbf{P}[\tilde{V}_{j_1,\dots,j_4}] + \mathbf{P}[V_{j_1,\dots,j_4}]. \tag{3.37}$$

Let $\{X'(j); j \in \mathbb{Z}^d\}$ be a system of independent continuous time random walks evolving as $\{X(j); j \in \mathbb{Z}^d\}$ but running independent also after hitting. Let $\tau_{i,j}$ denote the first hitting time of the two random walks $X'(i), X'(j)$. With this notation we write

$$\begin{aligned} \mathbf{P}[\tilde{V}_{j_1,\dots,j_4}] &= \mathbf{P}[\tau_{j_1,j_2} < \infty; \tau_{j_3,j_4} < \infty; \tau_{j_1,j_3} = \infty; \tau_{j_1,j_4} > \tau_{j_3,j_4}; \\ &\qquad\qquad \tau_{j_2,j_3} > \tau_{j_1,j_2}; \tau_{j_2,j_4} > \tau_{j_1,j_2}] \\ &\le \mathbf{P}[\tau_{j_1,j_2} < \infty;\ \tau_{j_3,j_4} < \infty] = \mathbf{P}[\tau_{j_1,j_2} < \infty]\mathbf{P}[\tau_{j_3,j_4} < \infty] \\ &= \mathbf{P}[V_{j_1,j_2}]\mathbf{P}[V_{j_3,j_4}]. \end{aligned} \tag{3.38}$$

From (3.37) and from (3.38) we conclude

$$\mathbf{P}[V_{j_1,j_2} \cap V_{j_3,j_4}] \le \mathbf{P}[V_{j_1,j_2}]\,\mathbf{P}[V_{j_3,j_4}] + \mathbf{P}[V_{j_1,\dots,j_4}]. \tag{3.39}$$

Hence applying (3.39) to (3.36) we obtain

$$
\begin{aligned}
&\operatorname{Var}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] \\
&\quad \le \sum_{j_1,\dots,j_4 \in D(Ar)} \mathbf{P}[V_{j_1,\dots,j_4}]\varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4) \\
&\quad \le \|\varphi\|_\infty^4 r^{-2(d+2)} \sum_{j_1,\dots,j_4 \in D(Ar)} \mathbf{P}[V_{j_1,\dots,j_4}],
\end{aligned} \tag{3.40}
$$

since φ is bounded. From Lemma 5 below we know $\sum \mathbf{P}[V_{j_1,\dots,j_4}] = O(r^{d+6})$. Hence $\operatorname{Var}[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2] = O(r^{-d+2})$. Since $d \ge 3$ this leads to the assertion. □

LEMMA 2 (Variance convergence). *Recall C_λ from Theorem 1. Under the assumptions of Lemma 1 and for $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi) \subset D(A)$ we have*

$$
\mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] \xrightarrow[r\to\infty]{} \frac{\gamma}{2|Q|^{\frac{1}{2}}\pi^{\frac{d}{2}}}\, \Gamma\left(\frac{d-2}{2}\right) B(\varphi, \varphi). \tag{3.41}
$$

PROOF. With the help of the hitting probability we can write [cf. (3.33)]

$$
\mathbf{E}\left[\sum_{k=1}^{K(r)} \varphi(B_k(r), r)^2\right] = \sum_{i,\, j \in \mathbb{Z}^d} \mathbf{P}[V_{i,\, j}]\, r^{-(d+2)}\, \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right). \tag{3.42}
$$

We want to show that

$$
\begin{aligned}
&\sum_{i,\, j \in \mathbb{Z}^d} \mathbf{P}[V_{i,\, j}] r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right) \\
&\qquad \xrightarrow[r\to\infty]{} \operatorname{const} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\varphi(x)\varphi(y)}{\bar{Q}(x-y)^{(d-2)/2}}\, dx\, dy.
\end{aligned} \tag{3.43}
$$

Note that the integral on the r.h.s. is well defined.

Fix $\varepsilon > 0$. In order to employ the asymptotics on $\mathbf{P}[V_{i,\, j}]$ for $|i-j|$ large, we split the sum into two parts depending on whether $|i-j| < M$ or $|i-j| \ge M$. Here $M = M(\varepsilon)$ is a constant that depends only on ε and will be chosen below in (3.46).

In order to show that the sum over $|i-j| < M$ is negligible the crude estimate $\mathbf{P}[V_{i,\, j}] \le 1$ is sufficient to get (recall $\operatorname{supp}(\varphi) \subset D(A)$)

$$
\left|\sum_{|i-j|<M} \mathbf{P}[V_{i,\, j}] r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right)\right| \le r^{-2}(2M)^d (2A)^d \|\varphi\|_\infty^2. \tag{3.44}
$$

It remains to investigate the sum over $|i-j| \geq M$ on the r.h.s. of (3.42). Obviously for $(i-j) \notin \mathscr{G}$, $\mathbf{P}[V_{i,j}] = 0$. Hence we get

$$\begin{aligned}\sum_{|i-j|\geq M} \mathbf{P}[V_{i,j}] r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right) \\ = \sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} \mathbf{P}[V_{i,j}] r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right).\end{aligned} \tag{3.45}$$

Let $\tilde{\varepsilon} > 0$. By Lemma 4 there exists a constant $M(\tilde{\varepsilon})$ such that for $(i-j) \notin \mathscr{G}; |i-j| > M(\tilde{\varepsilon})$

$$\left| \frac{\mathbf{P}[V_{i,j}]}{\frac{C}{\bar{Q}(i-j)^{(d-2)/2}}} - 1 \right| < \tilde{\varepsilon}, \tag{3.46}$$

where

$$C = \frac{\gamma |\mathbb{Z}^d/\mathscr{G}|}{2|Q|^{\frac{1}{2}} \pi^{\frac{d}{2}}} \Gamma\left(\frac{d-2}{2}\right). \tag{3.47}$$

We write

$$\begin{aligned}\sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} \mathbf{P}[V_{i,j}] r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right) \\ = \sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} \left(\mathbf{P}[V_{i,j}] - \frac{C}{\bar{Q}(i-j)^{(d-2)/2}}\right) r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right) \\ + \sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} \left(\frac{C}{\bar{Q}(i-j)^{(d-2)/2}}\right) r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right).\end{aligned} \tag{3.48}$$

We investigate the two sums on the r.h.s. of (3.49). For the second sum we observe that

$$\begin{aligned}\sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} \frac{1}{\bar{Q}(i-j)^{(d-2)/2}} r^{-(d+2)} \varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right) \\ = \sum_{\substack{i-j\in\mathscr{G};\\ |i-j|\geq M}} r^{-2d} \frac{\varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right)}{\bar{Q}\left(\frac{i}{r} - \frac{j}{r}\right)^{(d-2)/2}} \underset{r\to\infty}{\longrightarrow} \frac{1}{|\mathbb{Z}^d/\mathscr{G}|} B(\varphi, \varphi).\end{aligned} \tag{3.49}$$

For the first sum on the r.h.s. of (3.49) we get by (3.46)

$$\begin{aligned}&\left|\sum_{\substack{i-j\in\mathscr{G};\\|i-j|\geq M}}\left(\mathbf{P}[V_{i,j}]-\frac{C}{\bar{Q}(i-j)^{(d-2)/2}}\right)r^{-(d+2)}\varphi\left(\frac{i}{r}\right)\varphi\left(\frac{j}{r}\right)\right|\\ &\qquad\leq\tilde{\varepsilon}C\sum_{\substack{i-j\in\mathscr{G}\\|i-j|\geq M}}\frac{1}{\bar{Q}(i-j)^{(d-2)/2}}r^{-(d+2)}\left|\varphi\left(\frac{i}{r}\right)\right|\left|\varphi\left(\frac{j}{r}\right)\right|\end{aligned}\tag{3.50}$$

for $M\geq M(\tilde{\varepsilon})$. The sum on the r.h.s. of (3.50) converges to $B(|\varphi|,|\varphi|)/|\mathbb{Z}^d/\mathscr{G}|$ as $r\to\infty$ [analogously to (3.49)]. For the given ε we choose $\tilde{\varepsilon}$ such that $\tilde{\varepsilon}\,C\,B(|\varphi|,|\varphi|)/|\mathbb{Z}^d/\mathscr{G}|<\varepsilon$. Hence the l.h.s. of (3.50) is less than ε for $M>M(\varepsilon)=M(\tilde{\varepsilon}(\varepsilon))$. Combining this fact with (3.49) we obtain

$$\mathbf{E}\left[\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^2\right]\underset{r\to\infty}{\longrightarrow}\frac{C}{|\mathbb{Z}^d/\mathscr{G}|}B(\varphi,\varphi).\tag{3.51}$$

This leads to assertion (3.41). □

LEMMA 3 (For the Lyapunov condition). *Under the assumptions of Lemma* 1,

$$\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^4\xrightarrow[r\to\infty]{\mathbf{P}}0\tag{3.52}$$

for $\varphi\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}(\varphi)\subset D(A)$.

PROOF. Similarly as in (3.33) we write

$$\mathbf{E}\left[\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^4\right]=\sum_{j_1,\dots,j_4}\mathbf{P}[V_{j_1,\dots,j_4}]\varphi_r(j_1)\dots\varphi_r(j_4).\tag{3.53}$$

We estimate as in (3.40), hence by Lemma 5,

$$\mathbf{E}\left[\sum_{k=1}^{K(r)}\varphi(B_k(r),r)^4\right]=O(r^{-d+2}).\tag{3.54}$$

Since $d\geq 3$ this completes the proof. □

PART 4 (Some tools). The following lemmas are auxiliary lemmas for the previous results.

LEMMA 4 (Used in Lemma 2). *Recall* $V_{i,j}$ *from* (3.32). *Under the assumptions of Lemma* 1 *there exists a constant* $C>0$ *such that*

$$\mathbf{P}[V_{i,j}]\sim\begin{cases}C\cdot\left(\bar{Q}(i-j)\right)^{-\frac{d-2}{2}}, & i-j\in\mathscr{G},\\ 0, & i-j\notin\mathscr{G},\end{cases}\tag{3.55}$$

for $|i-j| \to \infty$. *Moreover* C *has the following form*:

$$C = \frac{\gamma|\mathbb{Z}^d/\mathscr{G}|}{2|Q|^{\frac{1}{2}}\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right), \tag{3.56}$$

with γ *defined in* (1.25).

PROOF. Recall that $\{X'(j); j \in \mathbb{Z}^d\}$ is a system of independent continuous time random walks evolving as $\{X(j); j \in \mathbb{Z}^d\}$ but running independent also after hitting. Recall that $V_{i,j}$ is the hitting probability of $X(i)$ and $X(j)$. Note that the random walks $X(i)$ and $X(j)$ coalesce if and only if the difference random walk $X'(i) - X'(j)$ hits the origin. The difference random walk runs according to the symmetrized kernel $\hat{p}$ defined in (1.5) at double speed. Due to the symmetry we have $\mathbf{P}[V_{i,j}] = \mathbf{P}[\exists t : \hat{X}_t = i-j]$, where $\hat{X}$ starts in the origin and it runs according to $\hat{p}$ at double speed, that is,

$$\mathbf{P}[\hat{X}_t = x] = \sum_{n=0}^{\infty} e^{-2t}\frac{(2t)^n}{n!}\hat{p}^{(n)}(0,x). \tag{3.57}$$

Using the Markov property it is easy to verify $\mathbf{P}[\exists t : \hat{X}_t = i-j] = \hat{G}(i-j)/\hat{G}(0)$, where $\hat{G}$ is the Green's function of $\hat{p}$. Hence $\mathbf{P}[V_{i,j}] = \hat{G}(i-j)/\hat{G}(0)$. It is well-known that $\hat{G}(0) = \gamma^{-1}$. This relation immediately implies that in order to prove the asymptotics of the hitting probability we need the asymptotics of the Green's function. However such an asymptotics was established in Corollary 2, namely [observe that the covariance matrix of $\hat{p}$ is Q given in (1.17)]

$$\hat{G}(x) \sim \begin{cases} C\cdot\bar{Q}(x)^{-(d-2)/2}, & \text{as } |x|\to\infty;\ x\in\mathscr{G},\\ 0, & \text{if } x\in\mathbb{Z}^d\backslash\mathscr{G}, \end{cases} \tag{3.58}$$

where

$$C = \frac{|\mathbb{Z}^d/\mathscr{G}|\Gamma(\frac{d-2}{2})}{2|Q|^{1/2}\pi^{d/2}}. \tag{3.59}$$

This completes the proof. □

LEMMA 5 (Used in Lemma 1 and Lemma 3). *Recall* $V_{j_1,\dots,j_4}$ *from* (3.32) *and* $D(Ar)$ *from* (3.5). *Under the assumptions of Lemma* 1 *the following holds*:

$$\sum_{j_1,\dots,j_4\in D(Ar)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(r^{d+6}). \tag{3.60}$$

PROOF. In order to apply Lemma 9 we have to check assumptions (5.1) and (5.2).

Concerning the first assumption we observe the following. Recall that $\hat{G}$ is the Green's function of the symmetrized kernel $\hat{p}$. By Corollary 2 we find a constant C' such that for all $i \in \mathbb{Z}^d$

$$\hat{G}(i) \le \frac{C'}{(|i|+1)^{d-2}}. \tag{3.61}$$

Hence $\hat{G}(i) = O(r^{-(d-2)})$ for all $|i| \ge r$. That means condition (5.1) is fulfilled with $f(r) = r^{-(d-2)}$.

The second assumption is satisfied with $g(r) = r^2$. Namely we perform the following short calculation. Define

$$D(r,k) := \left\{ i \in \mathbb{Z}^d : \frac{r}{2^{k+1}} \le |i| < \frac{r}{2^k} \right\}. \tag{3.62}$$

We can estimate

$$\begin{aligned} \sum_{i\in D(r)} \hat{G}(i) &\le O(1) \sum_{i \in D(r)} \frac{1}{(|i|+1)^{d-2}} \\ &= O(1) \sum_{k=0}^{\infty} \sum_{i\in D(r,k)} \frac{1}{(|i|+1)^{d-2}} + 1 \\ &\le O(1) \sum_{k=0}^{\infty} |D(r,k)| \frac{1}{(\frac{r}{2^{k+1}}+1)^{d-2}} + 1 \\ &= O(r^2) \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k, \end{aligned} \tag{3.63}$$

where we used (3.61) in the first inequality.

By Lemma 9 we obtain [note $|D(r)| = O(r^d)$]

$$\sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(r^{d+6}). \tag{3.64}$$

This completes the proof. □

Now we have to generalize the result for $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ to the result for $\varphi \in \mathscr{S}$.

LEMMA 6 (For general test functions). *If*

$$\mathscr{L}[F_{\lambda,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathscr{N}(0, C_\lambda B(\varphi,\varphi)) \tag{3.65}$$

is valid for $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, *then the statement is true for* $\varphi \in \mathscr{S}$.

PROOF. Let $\varphi \in \mathscr{S}$. For $\varepsilon > 0$ there exists a $\varphi_\varepsilon \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ such that

$$B\big(|\varphi - \varphi_\varepsilon|, |\varphi - \varphi_\varepsilon|\big) < \varepsilon. \tag{3.66}$$

Recall

$$\mathbf{E}F_{\lambda,r}(\varphi)^2 = \sum_{i,\,j\in\mathbb{Z}^d} \mathbf{P}[V_{i,j}]\varphi_r(i)\varphi_r(j). \tag{3.67}$$

Using Lemma 4 we find a constant K such that

$$\mathbf{P}[V_{i,j}] \leq K \cdot \left(\bar{Q}(i-j)\right)^{-\frac{d-2}{2}} \tag{3.68}$$

for all $i, j \in \mathbb{Z}^d$. Now we can estimate as in the proof of (3.43). Note that we do not have to take the constants M and $|\mathbb{Z}^d/\mathscr{G}|$ into account since we are only interested in an estimate of the limit

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{E}[F_{\lambda,r}(\varphi)]^2 &\leq \limsup_{r\to\infty} \sum_{i,j} \mathbf{P}[V_{i,j}]\, |\varphi_r(i)|\, |\varphi_r(j)| \\ &\leq K \limsup_{r\to\infty} \sum_{i,j} r^{-2d} \left(Q\left(\frac{i}{r} - \frac{j}{r}\right)\right)^{-\frac{d-2}{2}} \left|\varphi\left(\frac{i}{r}\right)\right| \left|\varphi\left(\frac{j}{r}\right)\right| \\ &\leq KB(|\varphi|, |\varphi|).\end{aligned} \tag{3.69}$$

By the last equation applied to $\varphi - \varphi_\varepsilon$ we can estimate

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{E}[F_{\lambda,r}(\varphi) - F_{\lambda,r}(\varphi_\varepsilon)]^2 &= \limsup_{r\to\infty} \mathbf{E}[F_{\lambda,r}(\varphi - \varphi_\varepsilon)]^2 \\ &\leq KB\big(|\varphi - \varphi_\varepsilon|, |\varphi - \varphi_\varepsilon|\big) \leq K\,\varepsilon.\end{aligned} \tag{3.70}$$

From this and the assumption

$$\mathscr{L}[F_{\lambda,r}(\varphi_\varepsilon)] \underset{r\to\infty}{\Longrightarrow} \mathscr{N}(0, C_\lambda B(\varphi_\varepsilon, \varphi_\varepsilon)), \tag{3.71}$$

we can conclude

$$\mathscr{L}[F_{\lambda,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathscr{N}(0, C_\lambda B(\varphi, \varphi)). \tag{3.72}$$

This completes the proof. □

4. Proof of Theorem 2 (The voter model on $\Xi^{(N)}$). Now we come to the proof of Theorem 2. Note first that the reason behind the nonclassical scaling is as before. Namely:

REMARK 3. As in the case of the group $\mathbb{Z}^d$ we observe that the correction factor to the classical rescaling is 1 over the root of the expected size of the family which contains the origin and which lies in the ball with radius r. The expected size of that family is $\sum_{|i|<r} G_c(i)$. One can verify that $\sum_{|i|<r} G_c(i) = \mathrm{const}(Nc)^r$. Hence the correction term together with the classical rescaling factor is given by

$$\frac{1}{\sqrt{N^r}} \cdot \frac{1}{\sqrt{N^r c^r}} = N^{-r} c^{-r/2}. \tag{4.1}$$

This is exactly the rescaling we chose in (1.35).

First of all we want to mention that is suffices to consider $r \in \mathbb{N}$, since the distance between two points in $\Xi^{(N)}$ is always integral.

The proof of Theorem 2 is similar to the one of Theorem 1 except the random walk estimates and the coalescing random walk estimates. Hence only the Green's function, which will have another asymptotics, and the coalescing probabilities have to be analyzed. We want to sketch the modifications of the arguments needed in order to transfer the argument from $\mathbb{Z}^d$ to the hierarchical group. Analogously to (3.10) we obtain

$$F_{\lambda, r} = \sum_{k=1}^{K(r)} (\alpha_k - \lambda) h(r) |B_k(r)|. \tag{4.2}$$

The term $\varphi(B_k(r), r)$ has here the form $h(r)|B_k(r)|$. We have to prove the versions of the lemmas of Section 3 part 3 on the hierarchical group. That means we have to deal with Lemma 1, Lemma 2 and Lemma 3. Let $D(r)$ be the ball with radius r

$$D(r) = \{i \in \Xi^{(N)} : \|i\| < r\}. \tag{4.3}$$

4.1. *Modification of Lemma* 1. The proof of this lemma for the hierarchical group works exactly the same as in the lattice case until equation (3.40). Then we can use Lemma 8. We obtain

$$\operatorname{Var}\left[\sum_{k=1}^{K(r)} \varphi(r)^2 |B_k(r)|^2\right] \le N^{-4r} c^{-2r} \sum_{j_1, \dots, j_4 \in D(r)} \mathbf{P}[V_{j_1, \dots, j_4}] = O(c^r). \tag{4.4}$$

Since $c < 1$ this leads to the assertion

$$\operatorname{Var}\left[\sum_{k=1}^{K(r)} \varphi(r)^2 |B_k(r)|^2\right] \underset{r \to \infty}{\longrightarrow} 0. \tag{4.5}$$

4.2. *Modification of Lemma* 2. By Lemma 7 we know that

$$\mathbf{P}[V_{i, j}] = \gamma\, C(N, c)\, c^{\|i-j\|}, \tag{4.6}$$

where

$$C(N, c) = \frac{N^3 c^2 - N^2 c^2 - N^2 c + N c}{(N^2 c - 1)(1 - c)}, \tag{4.7}$$

since $\mathbf{P}[V_{i, j}] = \gamma G_c(j - i)$. Hence

$$\begin{aligned} &\mathbf{E}\left[\sum_{k=1}^{K(r)} h(r)^2 |B_k(r)|^2\right] \\ &\quad = N^{-2r} c^{-r} \sum_{i, j \in D(r)} \mathbf{P}[V_{i, j}] \\ &\quad = \gamma C(N, c) \cdot N^{-2r} c^{-r} \sum_{i, j \in D(r)} c^{\|i-j\|} \end{aligned} \tag{4.8}$$

$$= \gamma C(N,c) N^{-2r} c^{-r} \sum_{k=0}^{r-1} \sum_{\substack{i,\, j \in D(r),\\ \|i-j\|=k}} c^k$$

$$= \gamma C(N,c) N^{-2r} c^{-r} \left[(N-1) N^{r-1} \sum_{k=1}^{r-1} c^k N^{k-1} + N^{r-1} \right]$$

$$= \gamma C(N,c)(N-1) N^{-r-1} c^{-r+1} \frac{(Nc)^{r-1} - 1}{Nc - 1} + \gamma C(N,c) N^{-r-1} c^{-r}.$$

This leads to

$$\mathbf{E}\left[\sum_{k=1}^{K(r)} (h(r)|B_k(r)|)^2 \right] \underset{r\to\infty}{\longrightarrow} \gamma C(N,c) \frac{N-1}{N^2(Nc-1)}. \tag{4.9}$$

4.3. *Modification of Lemma* 3. By Lemma 8 we know

$$\sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(N^{4r} c^{3r}). \tag{4.10}$$

Hence

$$\mathbf{E}\left[\sum_{k=1}^{K(r)} h(r)^4 |B_k(r)|^4 \right] = h(r)^4 \sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(c^r). \tag{4.11}$$

This leads to

$$\sum_{k=1}^{K(r)} h(r)^4 |B_k(r)|^4 \underset{r\to\infty}{\overset{\mathbf{P}}{\longrightarrow}} 0. \tag{4.12}$$

4.4. *Tools.* We want to investigate the asymptotics of the Green's function as the argument tends to infinity. We need the following formula:

LEMMA 7. *The Green's function of a random walk on the hierarchical group with geometric transition kernel given in* (1.31) *and* (1.33) *has the following form*:

$$G_c(i) = c^{\|i\|} \frac{N^3c^2 - N^2c^2 - N^2c + Nc}{(N^2c-1)(1-c)}, \qquad i \neq 0. \tag{4.13}$$

PROOF. From [6] (2.10) we know

$$p_c^{(n)}(i) = -[1 - \delta_0(i)] N^{-\|i\|} (f_{\|i\|})^n + (N-1) \sum_{k > \|i\|} N^{-k} (f_k)^n, \tag{4.14}$$

where

$$f_k = r_0 + r_1 + \dots + r_{k-1} - \frac{r_k}{N-1} \tag{4.15}$$

and (r_l) given by (1.33). A short calculation shows

$$1 - f_k = \left(\frac{1}{Nc}\right)^k \frac{N^2c - 1}{(N-1)Nc}. \tag{4.16}$$

One can verify that $|f_k| < 1$. Hence we obtain for $i \neq 0$

$$\begin{aligned} G_c(i) &= \sum_{n=1}^{\infty} p_c^{(n)}(i) = \sum_{n=1}^{\infty} \left[(N-1)\sum_{k>\|i\|} N^{-k}(f_k)^n - N^{-\|i\|}(f_{\|i\|})^n\right] \\ &= (N-1)\sum_{k>\|i\|} N^{-k}\left(\frac{1}{1-f_k} - 1\right) - N^{-\|i\|}\left(\frac{1}{1-f_{\|i\|}} - 1\right) \qquad (4.17) \\ &= (N-1)\sum_{k>\|i\|}\left(c^k \frac{(N-1)Nc}{N^2c-1} - N^{-k}\right) - \left(c^{\|i\|}\frac{(N-1)Nc}{N^2c-1} - N^{-\|i\|}\right). \end{aligned}$$

By a straightforward calculation we get (4.13). This completes the proof. □

LEMMA 8. *Recall* $V_{j_1,\dots,j_4}$ *from* (3.32). *Under the assumptions of Theorem* 2 *the following holds*:

$$\sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(N^{4r}c^{3r}). \tag{4.18}$$

PROOF. In order to apply Lemma 9 we have to check assumptions (5.1) and (5.2).

Concerning the first assumption we observe the following. By Lemma 7 we know

$$G_c(i) = O(c^{\|i\|}). \tag{4.19}$$

Hence $G_c(i) = O(c^r)$ for all $\|i\| \geq r$. That means condition (5.1) is fulfilled with $f(r) = c^r$.

The second assumption is satisfied with $g(r) = N^r c^r$, since one can easily check

$$\sum_{i \in D(r)} c^{\|i\|} = O(N^r c^r). \tag{4.20}$$

Furthermore note $|D(r)| = O(N^r)$. Hence by Lemma 9,

$$\sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O(N^{4r}c^{3r}). \tag{4.21}$$

This completes the proof. □

5. A coalescent random walk estimate. Let S be a countable Abelian group with a metric $|\cdot|$. Let $D(r)$ denote the ball with radius r in S.

We consider a system $\{X(j); j \in S\}$ of coalescing random walks with continuous time kernel p and $X_0(j) = j$ for all $j \in S$. We define $V_{j_1,\dots,j_k}$ for a $k \in \mathbb{N}$ as the event that the random walks $X(j_1), \dots, X(j_k)$ coalesce eventually.

LEMMA 9. *Assume for the Green's function of the symmetrized kernel* $\hat{p}$ *that*

$$\hat{G}(i) = O(f(r)) \qquad \text{for all } |i| \geq r \tag{5.1}$$

for some function f. *Furthermore assume*

$$\sum_{i \in D(r)} \hat{G}(i) = O(g(r)) \tag{5.2}$$

for some other function g. *Then*

$$\sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]^2\Big). \tag{5.3}$$

PROOF. First of all we want to distinguish whether the indices $j_1, \dots, j_4$ are different or not. We split the sum

$$\begin{aligned} \sum_{j_1,\dots,j_4 \in D(r)} \mathbf{P}[V_{j_1,\dots,j_4}] = & \sum_{\substack{j_1,\dots,j_4 \in D(r)\\ j_1 \neq \dots \neq j_4}} \mathbf{P}[V_{j_1,\dots,j_4}] \\ & + 6 \sum_{\substack{j_1,j_2,j_3 \in D(r)\\ j_1 \neq j_2 \neq j_3}} \mathbf{P}[V_{j_1,j_2,j_3}] \\ & + 7 \sum_{\substack{j_1,j_2 \in D(r)\\ j_1 \neq j_2}} \mathbf{P}[V_{j_1,j_2}] + \sum_{j_1 \in D(r)} 1. \end{aligned} \tag{5.4}$$

In step 1 we will show

$$\sum_{\substack{j_1,\dots,j_4 \in D(r)\\ j_1 \neq \dots \neq j_4}} \mathbf{P}[V_{j_1,\dots,j_4}] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]^2\Big). \tag{5.5}$$

In step 5 we prove

$$\sum_{\substack{j_1,j_2,j_3 \in D(r)\\ j_1 \neq j_2 \neq j_3}} \mathbf{P}[V_{j_1,j_2,j_3}] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]\Big). \tag{5.6}$$

For the sum with two different indices we observe the following. It is well known that

$$\sum_{\substack{j_1,j_2 \in D(r)\\ j_1 \neq j_2}} \mathbf{P}[V_{j_1,j_2}] \leq O(1) \sum_{j_1,j_2 \in D(r)} \hat{G}(j_2 - j_1). \tag{5.7}$$

The assumption (5.2) then justifies

$$\sum_{\substack{j_1, j_2 \in D(r) \\ j_1 \neq j_2}} \mathbf{P}[V_{j_1, j_2}] = O\Big(|D(r)|\, g(r)\Big). \tag{5.8}$$

Since g is obviously increasing in r, (5.5), (5.6) and (5.8) lead to assertion (5.3). It remains to prove equations (5.5) and (5.6).

STEP 1. In order to establish (5.5) we consider in which order do the random walks meet each other. If two of the four random walks have coalesced we have two general possibilities for the second coalescing event. Either the two other walks coalesce or the two coalesced walks meet one of the remaining walks. That means there are the following two general coalescing orders:

$$\begin{aligned} \text{First order:}\quad &\text{(i)}\ \ X(j_1) \leftrightarrow X(j_2),\\ &\text{(ii)}\ \ X(j_1) \leftrightarrow X(j_3),\\ &\text{(iii)}\ \ X(j_1) \leftrightarrow X(j_4). \end{aligned} \tag{5.9}$$

$$\begin{aligned} \text{Second order:}\quad &\text{(i)}\ \ X(j_1) \leftrightarrow X(j_2),\\ &\text{(ii)}\ \ X(j_3) \leftrightarrow X(j_4),\\ &\text{(iii)}\ \ X(j_1) \leftrightarrow X(j_3). \end{aligned} \tag{5.10}$$

We denote the first (resp. the second) event with $V^{(1)}_{j_1,\dots,j_4}$ resp. $V^{(2)}_{j_1,\dots,j_4}$. We can change the roles of the j_l in each case. There are twelve combinations of the j_l to obtain the first coalescing order and six for the second one. Since we sum over all $j_1, j_2, j_3, j_4 \in D(r)$; $j_1 \neq \dots \neq j_4$ and we have symmetry we can write

$$\begin{aligned} \sum_{\substack{j_1,\dots,j_4 \in D(r) \\ j_1 \neq \dots \neq j_4}} \mathbf{P}[V_{j_1,\dots,j_4}] = 12 &\sum_{\substack{j_1,\dots,j_4 \in D(r) \\ j_1 \neq \dots \neq j_4}} \mathbf{P}\left[V^{(1)}_{j_1,\dots,j_4}\right] \\ &+ 6 \sum_{\substack{j_1,\dots,j_4 \in D(r) \\ j_1 \neq \dots \neq j_4}} \mathbf{P}\left[V^{(2)}_{j_1,\dots,j_4}\right]. \end{aligned} \tag{5.11}$$

In step 3 we will prove

$$\sum_{\substack{j_1,\dots,j_4 \in D(r) \\ j_1 \neq \dots \neq j_4}} \mathbf{P}\left[V^{(1)}_{j_1,\dots,j_4}\right] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]^2\Big) \tag{5.12}$$

and in step 4 we will show

$$\sum_{\substack{j_1,\dots,j_4 \in D(r) \\ j_1 \neq \dots \neq j_4}} \mathbf{P}[V^{(2)}_{j_1,\dots,j_4}] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]^2\Big). \tag{5.13}$$

This leads to the assertion (5.5). It remains to prove (5.12) and (5.13).

STEP 2. Here we want to establish the basic techniques and the basic estimates we will use in step 3 and 4 repeatedly. The basic technique which will be used is the following. Assume that we have to deal with an expression of the form

$$\sum_{i,\, j\in S} \hat{G}(j-i)\psi(i,\, j), \tag{5.14}$$

for some function ψ. We want to distinguish between the cases "large distance between i and j" and "small distance between i and j." In the first case we want to exploit the assumption (5.1) on the Green's function of $(i,\, j)$. On the other hand there are not too many pairs $(i,\, j)$ fulfilling the second condition.

Denote for $i \in S$:

$$D_i(r) = \{j \in S : |j-i| < r\}. \tag{5.15}$$

We use the following splitting technique for $(i,\, j)$

$$\begin{aligned} &\sum_{i,\, j\in S} \hat{G}(j-i)\psi(i,\, j) \\ &\quad = \sum_{i\in S}\sum_{j\notin D_i(r)} \hat{G}(j-i)\psi(i,\, j) + \sum_{i\in S}\sum_{j\in D_i(r)} \hat{G}(j-i)\psi(i,\, j). \end{aligned} \tag{5.16}$$

Then we treat the two sums separately. For the first sum we will use that $|j-i| \geq r$, thus by (5.1)

$$\hat{G}(j-i) = O(f(r)). \tag{5.17}$$

Hence we obtain

$$\sum_{i\in S}\sum_{j\notin D_i(r)} \hat{G}(j-i)\psi(i,\, j) \leq O(f(r)) \sum_{i,\, j\in S} \psi(i,\, j). \tag{5.18}$$

For the second sum we will use that $j-i \in D(r)$ for all $j \in D_i(r)$, hence

$$\sum_{i\in S}\sum_{j\in D_i(r)} \hat{G}(j-i)\psi(i,\, j) \leq \sum_{j\in D(r)} \hat{G}(j) \sum_{i\in S} \psi(i,\, j+i). \tag{5.19}$$

STEP 3a. In order to prove (5.12) we establish first of all the following estimate: for $j_1, \dots j_4$ different in pairs

$$\begin{aligned} &\mathbf{P}[V^{(1)}_{j_1,\dots,j_4}] \\ &\quad \leq O(1) \sum_{x,y,z,w\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\ &\qquad\qquad \times \int_0^\infty ds\, p_s(x,z)p_s(y,z)p_{t+s}(j_4,w)\int_0^\infty du\, \hat{p}_{2u}(z,w). \end{aligned} \tag{5.20}$$

This can be seen by the following argument. We have to subdivide the coalescing event in all possible coalescing times and locations.

As in the proof of Lemma 1 we introduce the following notation. Let $\{X'(j); j \in S\}$ be a system of independent continuous time random walks evolving as $\{X(j); j \in S\}$ but running independent also after hitting. First of all we turn our attention to the coalescing times. Let $\tau_{i,j}$ denote the first hitting time of the two random walks $X'(i)$ and $X'(j)$ and let $\varrho_{j_1,\dots,j_n} = \inf\{t \geq 0;\ \exists k \neq l : X'_t(j_k) = X'_t(j_l)\}$. We rewrite

$$
\begin{aligned}
\mathbf{P}\big[V^{(1)}_{j_1,\dots,j_4}\big] &= \mathbf{P}\big[\varrho_{j_1,j_2,j_3,j_4} = \tau_{j_1,j_2}; \varrho_{j_1,j_3,j_4} = \tau_{j_1,j_3}; \tau_{j_1,j_4} < \infty\big] \\
&= \int_0^\infty \mathbf{P}\big[\varrho_{j_1,j_2,j_3,j_4} = \tau_{j_1,j_2} \in dt; \varrho_{j_1,j_3,j_4} = \tau_{j_1,j_3}; \tau_{j_1,j_4} < \infty\big].
\end{aligned} \tag{5.21}
$$

Now we sum over all hitting locations. Due to the Markov property,

$$
\begin{aligned}
&\mathbf{P}\big[V^{(1)}_{j_1,\dots,j_4}\big] \\
&\quad = \sum_{x,y,z \in S} \int_0^\infty \mathbf{P}\big[\varrho_{j_1,j_2,j_3,j_4} = \tau_{j_1,j_2} \in dt; \varrho_{j_1,j_3,j_4} = \tau_{j_1,j_3}; \tau_{j_1,j_4} < \infty; \\
&\qquad\qquad X'_t(j_1) = X'_t(j_2) = x; X'_t(j_3) = y; X'_t(j_4) = z\big] \\
&\quad \leq \sum_{x,y,z \in S} \int_0^\infty \mathbf{P}\big[\tau_{j_1,j_2} \in dt; X'_t(j_1) = X'_t(j_2) = x; \\
&\qquad\qquad X'_t(j_3) = y; X'_t(j_4) = z\big] \\
&\qquad \times \mathbf{P}\big[\varrho_{x,y,z} = \tau_{x,y}; \tau_{x,z} < \infty\big].
\end{aligned} \tag{5.22}
$$

Here we estimate $\mathbf{P}[\tau_{j_1,j_2} \in dt; \dots]$ by $\mathbf{P}[\dots]\,dt$. This can be justified by the following argument. The event $\{\tau_{j_1,j_2} \in [s,t]\} \cap A$ (with $A = \{X'_t(j_1) = X'_t(j_2) = x;\ X'_t(j_3) = y;\ X'_t(j_4) = z\}$) is less likely than the event that there is a jump of one of the random walks $X'(j_1)$ and $X'(j_2)$ in the time-interval $[s,t]$ intersected with A. Since A is independent of the jump event this leads to

$$
\begin{aligned}
\mathbf{P}\big[\{\tau_{j_1,j_2} \in [s,t]\} \cap A\big] &\leq (1 - e^{-2(t-s)})\mathbf{P}[A] \\
&\leq 4(t-s)\mathbf{P}[A] \qquad \text{for all } 0 \leq (t-s) \leq 1/4.
\end{aligned} \tag{5.23}
$$

This estimate gives us

$$
\begin{aligned}
&\mathbf{P}\big[V^{(1)}_{j_1,\dots,j_4}\big] \\
&\quad \leq O(1) \sum_{x,y,z \in S} \int_0^\infty dt\, \mathbf{P}\big[X'_t(j_1) = X'_t(j_2) = x; X'_t(j_3) = y; X'_t(j_4) = z\big] \\
&\qquad\qquad \times \mathbf{P}\big[\varrho_{x,y,z} = \tau_{x,y}; \tau_{x,z} < \infty\big].
\end{aligned} \tag{5.24}
$$

We treat the remaining expression $\mathbf{P}[\dots]$ on the r.h.s. of (5.24) in the same way as the expression in the middle of (5.21). Hence we obtain

$$\begin{aligned}
&\mathbf{P}[V^{(1)}_{j_1,\dots,j_4}] \\
&\quad\le O(1) \sum_{x,y,z\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z) \\
&\qquad\qquad \times \sum_{v,w\in S} \int_0^\infty ds\, p_s(x,v)p_s(y,v)p_s(z,w) \int_0^\infty du\, \hat{p}_{2u}(v,w) \\
&\quad= O(1) \sum_{x,y,v,w\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\qquad\qquad \times \int_0^\infty ds\, p_s(x,v)p_s(y,v)p_{t+s}(j_4,w) \int_0^\infty du\, \hat{p}_{2u}(v,w).
\end{aligned} \tag{5.25}$$

This completes the proof of (5.20). □

STEP 3b. Return now to inequality (5.20). Estimating the integral over u by the Green's function we obtain

$$\begin{aligned}
\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\ne\dots\ne j_4}} \mathbf{P}[V^{(1)}_{j_1,\dots,j_4}] \le O(1) &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z,w\in S} \hat{G}(w-z) \\
&\times \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\; p_s(x,z)p_s(y,z)p_{t+s}(j_4,w).
\end{aligned} \tag{5.26}$$

Now we use the splitting technique given in (5.16) for (z,w). Hence

$$\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\ne\dots\ne j_4}} \mathbf{P}[V^{(1)}_{j_1,\dots,j_4}] \le O(1)\Big[I_1(r)+I_2(r)\Big], \tag{5.27}$$

where

$$\begin{aligned}
I_1(r) = &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \sum_{w\notin D_z(r)} \hat{G}(w-z) \\
&\times \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\; p_s(x,z)p_s(y,z)p_{t+s}(j_4,w)
\end{aligned} \tag{5.28}$$

and

$$\begin{aligned}
I_2(r) = &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \sum_{w\in D_z(r)} \hat{G}(w-z) \\
&\times \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\; p_s(x,z)p_s(y,z)p_{t+s}(j_4,w).
\end{aligned} \tag{5.29}$$

For $I_1(r)$ we use the estimate (5.18), hence

$$
\begin{aligned}
I_1(r) &\le O(f(r)) \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z,w\in S} \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\qquad \times \int_0^\infty ds\; p_s(x,z)p_s(y,z)p_{t+s}(j_4,w) \\
&\le O(f(r)) \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\qquad \times \int_0^\infty ds\; p_s(x,z)p_s(y,z),
\end{aligned}
\tag{5.30}
$$

where we performed the sum over w. There is no occurrence of j_4 in the sum any more. We can estimate the sum over j_4 by $|D(r)|$. Furthermore we perform the sum over z. We get

$$
\begin{aligned}
I_1(r) \le O(|D(r)|f(r)) \sum_{j_1,j_2,j_3\in D(r)} \sum_{x,y\in S} \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
\times \int_0^\infty ds\; \hat{p}_{2s}(x,y).
\end{aligned}
\tag{5.31}
$$

Return to (5.27). To $I_2(r)$ we apply (5.19), hence

$$
\begin{aligned}
I_2(r) \le O(1) \sum_{j_1,\dots,j_4,w\in D(r)} \sum_{x,y,z\in S} \hat{G}(w) \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
\times \int_0^\infty ds\; p_s(x,z)p_s(y,z)p_{t+s}(j_4,w+z).
\end{aligned}
\tag{5.32}
$$

Now we estimate $\sum_{j_4\in D(r)} p_{t+s}(j_4,w+z) \le 1$. We obtain

$$
\begin{aligned}
I_2(r) &\le O(1) \sum_{w\in D(r)} \hat{G}(w) \\
&\quad \times \sum_{j_1,j_2,j_3\in D(r)} \sum_{x,y,z\in S} \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\quad \times \int_0^\infty ds\; p_s(x,z)p_s(y,z).
\end{aligned}
\tag{5.33}
$$

Using (5.2) and performing the sum over z we end up with

$$
\begin{aligned}
I_2(r) \le O(g(r)) \sum_{j_1,j_2,j_3\in D(r)} \sum_{x,y\in S} \int_0^\infty dt\; p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
\times \int_0^\infty ds\; \hat{p}_{2s}(x,y).
\end{aligned}
\tag{5.34}
$$

We combine (5.31) and (5.34), thus

$$\begin{aligned}\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\neq\dots\neq j_4}} \mathbf{P}[V^{(1)}_{j_1,\dots,j_4}] &\\ \leq\ & O\Big(|D(r)|f(r)+g(r)\Big)\\ &\times \sum_{j_1,j_2,j_3\in D(r)}\sum_{x,y\in S}\int_0^\infty dt\ p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)\\ &\times\int_0^\infty ds\ \hat{p}_{2s}(x,y).\end{aligned}\tag{5.35}$$

In step 5, (5.75), we prove

$$\begin{aligned}\sum_{j_1,j_2,j_3\in D(r)}\sum_{x,y\in S}\int_0^\infty dt\ p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)\int_0^\infty ds\ \hat{p}_{2s}(x,y)&\\ = O\Big(|D(r)|\,g(r)\,[|D(r)|f(r)+g(r)]\Big).&\end{aligned}\tag{5.36}$$

Hence

$$\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\neq\dots\neq j_4}} \mathbf{P}\big[V^{(1)}_{j_1,\dots,j_4}\big] = O\Big(|D(r)|\,g(r)\,[|D(r)|f(r)+g(r)]^2\Big).\tag{5.37}$$

This completes the proof of assertion (5.12). □

STEP 4a. In order to prove (5.13) we can establish the following estimate analogously to step 3a:

$$\begin{aligned}\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\neq\dots\neq j_4}} \mathbf{P}[V^{(2)}_{j_1,\dots,j_4}] &\\ \leq\ & O(1)\\ &\times \sum_{j_1,\dots,j_4\in D(r)}\ \sum_{x,y,z,v,w\in S}\int_0^\infty dt\ p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\times\int_0^\infty ds\ p_s(y,v)p_s(z,v)p_s(x,w)\int_0^\infty du\ \hat{p}_{2u}(v,w).\end{aligned}\tag{5.38}$$

STEP 4b. On the r.h.s. of (5.38) we estimate the integral over u by the Green's function, thus

$$\begin{aligned}\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\neq\dots\neq j_4}} \mathbf{P}[V^{(2)}_{j_1,\dots,j_4}] \leq O(1) &\sum_{j_1,\dots,j_4\in D(r)}\ \sum_{x,y,z,v,w\in S}\hat{G}(w-v)\\ &\times\int_0^\infty dt\ p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\times\int_0^\infty ds\ p_s(y,v)p_s(z,v)p_s(x,w).\end{aligned}\tag{5.39}$$

Now we use the splitting technique given in (5.16) for (v, w). Hence

$$\sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\neq\dots\neq j_4}} \mathbf{P}[V^{(2)}_{j_1,\dots,j_4}] \le O(1)\Big[I_3(r)+I_4(r)\Big], \tag{5.40}$$

where

$$\begin{aligned} I_3(r) = &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z,v\in S} \sum_{w\notin D_v(r)} \hat{G}(w-v)\\ &\times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\times \int_0^\infty ds\, p_s(y,v)p_s(z,v)p_s(x,w) \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} I_4(r) = &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z,v\in S} \sum_{w\in D_v(r)} \hat{G}(w-v)\\ &\times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\times \int_0^\infty ds\, p_s(y,v)p_s(z,v)p_s(x,w). \end{aligned} \tag{5.42}$$

For $I_3(r)$ we use the estimate (5.18) to find

$$\begin{aligned} I_3(r) &\le O(f(r))\\ &\quad\times \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z,v,w\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\quad\times \int_0^\infty ds\, p_s(y,v)p_s(z,v)p_s(x,w)\\ &= O(f(r))\\ &\quad\times \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z)\\ &\quad\times \int_0^\infty ds\, \hat{p}_{2s}(y,z), \end{aligned} \tag{5.43}$$

where we performed the sums over v and w in the latter equality. Estimating the integral over s by the Green's function we get

$$\begin{aligned} I_3(r) \le O(f(r)) &\sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \hat{G}(z-y)\\ &\times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z). \end{aligned} \tag{5.44}$$

Now we use the splitting technique given in (5.16) again but now for (y, z),

$$I_3(r) \le I_{3,1}(r) + I_{3,2}(r), \tag{5.45}$$

where

$$I_{3,1}(r) = O(f(r)) \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y\in S} \sum_{z\notin D_y(r)} \hat{G}(z-y) \times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z) \tag{5.46}$$

and

$$I_{3,2}(r) = O(f(r)) \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y\in S} \sum_{z\in D_y(r)} \hat{G}(z-y) \times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z). \tag{5.47}$$

For $I_{3,1}(r)$ we use the estimate (5.18), thus

$$\begin{aligned} I_{3,1}(r) &\le O(f^2(r)) \sum_{j_1,\dots,j_4\in D(r)} \sum_{x,y,z\in S} \\ &\quad \times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z) \\ &\le O(f^2(r)) \sum_{j_1,\dots,j_4\in D(r)} \int_0^\infty dt\, \hat{p}_{2t}(j_1,j_2), \end{aligned} \tag{5.48}$$

where we performed the sums over x, y and z in the latter equality. We estimate the integral over t by the Green's function and we apply (5.2), hence

$$\begin{aligned} I_{3,1}(r) &\le O\Big(|D(r)|^2 f^2(r)\Big) \sum_{j_1,j_2\in D(r)} \hat{G}(j_2-j_1) \\ &= O\Big(|D(r)|^3 f^2(r)\, g(r)\Big). \end{aligned} \tag{5.49}$$

Return to (5.45). To $I_{3,2}(r)$ we apply (5.19), hence

$$\begin{aligned} I_{3,2}(r) &\le O(f(r)) \sum_{j_1,\dots,j_4,z\in D(r)} \hat{G}(z) \\ &\quad \times \sum_{x,y\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z+y) \\ &\le O(f(r)) \sum_{j_1,j_2,j_3,z\in D(r)} \hat{G}(z) \\ &\quad \times \sum_{x,y\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y), \end{aligned} \tag{5.50}$$

where we estimated $\sum_{j_4} p_t(j_4, z+y) \le 1$. Performing the sums over x and y we get

$$\begin{aligned} I_{3,2}(r) &\le O(f(r)) \sum_{j_1,j_2,j_3,z\in D(r)} \hat{G}(z) \int_0^\infty dt\, \hat{p}_{2t}(j_1,j_2) \\ &\le O\Big(|D(r)|\, f(r)\Big) \sum_{z\in D(r)} \hat{G}(z) \sum_{j_1,j_2\in D(r)} \hat{G}(j_2-j_1). \end{aligned} \tag{5.51}$$

We apply (5.2), thus

$$I_{3,2}(r) = O\Big(|D(r)|^2 f(r)\, g^2(r)\Big). \tag{5.52}$$

Hence by (5.49) and by (5.52)

$$I_3(r) = O\Big(|D(r)|^2 f(r)\, g(r)\,[|D(r)|f(r) + g(r)]\Big). \tag{5.53}$$

Return to (5.40). For $I_4(r)$ we use the estimate (5.19), hence

$$\begin{aligned}
I_4(r) \le{}& O(1) \sum_{j_1,\dots,j_4,w\in D(r)} \hat{G}(w) \\
&\times \sum_{x,y,z,v\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y)p_t(j_4,z) \\
&\times \int_0^\infty ds\, p_s(y,v)p_s(z,v)p_s(x,w+v) \\
\le{}& O(1) \sum_{j_1,j_2,j_3,w\in D(r)} \hat{G}(w) \\
&\times \sum_{x,y,z,v\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\, p_s(y,v)p_s(z,v)p_s(x,w+v),
\end{aligned} \tag{5.54}$$

where we used $\sum_{j_4} p_t(j_4,z) \le 1$. Next perform the sum over z, thus

$$\begin{aligned}
I_4(r) \le{}& O(1) \sum_{j_1,j_2,j_3,w\in D(r)} \hat{G}(w) \sum_{x,y,v\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\, p_s(y,v)p_s(x-w,v) \\
\le{}& O(1) \sum_{j_1,j_2,j_3,w\in D(r)} \hat{G}(w) \sum_{x,y\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
&\times \int_0^\infty ds\, \hat{p}_{2s}(y,x-w),
\end{aligned} \tag{5.55}$$

where we performed the sum over v. Estimating the integral over s by the Green's function we end up with

$$\begin{aligned}
I_4(r) \le{}& O(1) \sum_{j_1,j_2,j_3,w\in D(r)} \hat{G}(w) \sum_{x,y\in S} \hat{G}(x-w-y) \\
&\times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\
={}& O(1) \sum_{j_1,j_2,j_3,w\in D(r)} \hat{G}(w) \sum_{x,y\in S} \hat{G}(x-y) \\
&\times \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y-w).
\end{aligned} \tag{5.56}$$

Then we use the splitting technique given in (5.16) for (x, y). We obtain

$$I_4(r) \le I_{4,1}(r) + I_{4,2}(r), \tag{5.57}$$

where

$$\begin{aligned} I_{4,1}(r) = O(1) & \sum_{j_1, j_2, j_3, w \in D(r)} \hat{G}(w) \sum_{x \in S} \sum_{y \notin D_x(r)} \hat{G}(x-y) \\ & \times \int_0^\infty dt\, p_t(j_1, x) p_t(j_2, x) p_t(j_3, y-w) \end{aligned} \tag{5.58}$$

and

$$\begin{aligned} I_{4,2}(r) = O(1) & \sum_{j_1, j_2, j_3, w \in D(r)} \hat{G}(w) \sum_{x \in S} \sum_{y \in D_x(r)} \hat{G}(x-y) \\ & \times \int_0^\infty dt\, p_t(j_1, x) p_t(j_2, x) p_t(j_3, y-w). \end{aligned} \tag{5.59}$$

For $I_{4,1}(r)$ we use the estimate (5.18), thus

$$\begin{aligned} I_{4,1}(r) & \le O(f(r)) \sum_{j_1, j_2, j_3, w \in D(r)} \hat{G}(w) \\ & \quad \times \sum_{x, y \in S} \int_0^\infty dt\, p_t(j_1, x) p_t(j_2, x) p_t(j_3, y-w) \\ & \le O(f(r)) \sum_{j_1, j_2, j_3, w \in D(r)} \hat{G}(w) \int_0^\infty dt\, \hat{p}_{2t}(j_1, j_2), \end{aligned} \tag{5.60}$$

where we performed the sums over x and y. We estimate the integral over t by the Green's function, thus, by assumption (5.2),

$$\begin{aligned} I_{4,1}(r) & \le O\Big(|D(r)| f(r)\Big) \sum_{w \in D(r)} \hat{G}(w) \sum_{j_1, j_2 \in D(r)} \hat{G}(j_2 - j_1) \\ & = O\Big(|D(r)|^2 f(r)\, g^2(r)\Big). \end{aligned} \tag{5.61}$$

To $I_{4,2}(r)$ we apply estimate (5.19), thus

$$\begin{aligned} I_{4,2}(r) & \le O(1) \sum_{j_1, j_2, j_3, w, y \in D(r)} \hat{G}(w) \hat{G}(y) \\ & \quad \times \sum_{x \in S} \int_0^\infty dt\, p_t(j_1, x) p_t(j_2, x) p_t(j_3, x-y-w) \\ & \le O(1) \sum_{j_1, j_2, w, y \in D(r)} \hat{G}(w) \hat{G}(y) \sum_{x \in S} \int_0^\infty dt\, p_t(j_1, x) p_t(j_2, x), \end{aligned} \tag{5.62}$$

where we used $\sum_{j_3} p_t(j_3, x-y-w) \le 1$. Next perform the sum over x, thus

$$(5.63)\qquad \begin{aligned} I_{4,2}(r) &\le O(1) \sum_{j_1, j_2, w, y \in D(r)} \hat{G}(w)\hat{G}(y) \int_0^\infty dt\, \hat{p}_{2t}(j_1, j_2) \\ &\le O(1) \sum_{w\in D(r)} \hat{G}(w) \sum_{y\in D(r)} \hat{G}(y) \sum_{j_1, j_2 \in D(r)} \hat{G}(j_2 - j_1). \end{aligned}$$

We apply (5.2) and obtain

$$(5.64)\qquad I_{4,2}(r) = O\Big(|D(r)|\, g^3(r)\Big).$$

Hence by (5.61) and to (5.64)

$$(5.65)\qquad I_4(r) = O\Big(|D(r)|\, g^2(r)\,[|D(r)|f(r) + g(r)]\Big).$$

Thus by (5.53) and (5.65)

$$(5.66)\qquad \sum_{\substack{j_1,\dots,j_4\in D(r)\\ j_1\ne\dots\ne j_4}} \mathbf{P}[V^{(2)}_{j_1,\dots,j_4}] = O\Big(|D(r)|\, g(r)\,[|D(r)|f(r) + g(r)]^2\Big).$$

That means we proved assertion (5.13).

STEP 5. In order to establish (5.6) we first observe that, analogously to (5.11),

$$(5.67)\qquad \sum_{\substack{j_1, j_2, j_3\in D(r)\\ j_1\ne j_2\ne j_3}} \mathbf{P}[V_{j_1,j_2,j_3}] = 3 \sum_{\substack{j_1, j_2, j_3\in D(r)\\ j_1\ne j_2\ne j_3}} \mathbf{P}[V^{(1)}_{j_1,j_2,j_3}],$$

where $V^{(1)}_{j_1,j_2,j_3}$ denotes the event that $X(j_1)$ and $X(j_2)$ coalesce at first. Analogously to (5.20),

$$(5.68)\qquad \begin{aligned} \mathbf{P}[V^{(1)}_{j_1,j_2,j_3}] \le O(1) \sum_{x,y\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\ \times \int_0^\infty ds\, \hat{p}_{2s}(x,y). \end{aligned}$$

Hence

$$(5.69)\qquad \begin{aligned} &\sum_{\substack{j_1, j_2, j_3\in D(r)\\ j_1\ne j_2\ne j_3}} \mathbf{P}[V_{j_1,j_2,j_3}] \\ &\quad\le\ O(1) \sum_{j_1, j_2, j_3\in D(r)} \sum_{x,y\in S} \int_0^\infty dt\, p_t(j_1,x)p_t(j_2,x)p_t(j_3,y) \\ &\qquad \times \int_0^\infty ds\, \hat{p}_{2s}(x,y). \end{aligned}$$

We estimate the integral over s by the Green's function, and then we use the splitting technique given in (5.16) for (x, y), thus

$$
\begin{aligned}
&\sum_{j_1, j_2, j_3 \in D(r)} \sum_{x, y \in S} \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y) \int_0^\infty ds\ \hat{p}_{2s}(x, y) \\
&\quad \le \sum_{j_1, j_2, j_3 \in D(r)} \sum_{x, y \in S} \hat{G}(y - x) \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y) \\
&\quad = I_5(r) + I_6(r),
\end{aligned}
\tag{5.70}
$$

where

$$
\begin{aligned}
I_5(r) = &\sum_{j_1, j_2, j_3 \in D(r)} \sum_{x \in S} \sum_{y \notin D_x(r)} \hat{G}(y - x) \\
&\times \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y)
\end{aligned}
\tag{5.71}
$$

and

$$
\begin{aligned}
I_6(r) = &\sum_{j_1, j_2, j_3 \in D(r)} \sum_{x \in S} \sum_{y \in D_x(r)} \hat{G}(y - x) \\
&\times \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y).
\end{aligned}
\tag{5.72}
$$

For $I_5(r)$ we use the estimate (5.18), hence

$$
\begin{aligned}
I_5(r) &\le O(f(r)) \sum_{j_1, j_2, j_3 \in D(r)} \sum_{x, y \in S} \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y) \\
&= O(f(r)) \sum_{j_1, j_2, j_3 \in D(r)} \int_0^\infty dt\ \hat{p}_{2t}(j_1, j_2) \\
&\le O\Big(|D(r)|\, f(r)\Big) \sum_{j_1, j_2 \in D(r)} \hat{G}(j_2 - j_1).
\end{aligned}
\tag{5.73}
$$

To $I_6(r)$ we apply the estimate given in (5.19), thus

$$
\begin{aligned}
I_6(r) &\le \sum_{j_1, j_2, j_3, y \in D(r)} \hat{G}(y) \sum_{x \in S} \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y + x) \\
&\le \sum_{j_1, j_2, y \in D(r)} \hat{G}(y) \sum_{x \in S} \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) \\
&\le \sum_{y \in D(r)} \hat{G}(y) \sum_{j_1, j_2 \in D(r)} \hat{G}(j_2 - j_1).
\end{aligned}
\tag{5.74}
$$

Now we apply (5.2) to (5.74) and to (5.75), hence

$$
\begin{aligned}
&\sum_{j_1, j_2, j_3 \in D(r)} \sum_{x, y \in S} \int_0^\infty dt\ p_t(j_1, x) p_t(j_2, x) p_t(j_3, y) \int_0^\infty ds\ \hat{p}_{2s}(x, y) \\
&\quad = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]\Big).
\end{aligned}
\tag{5.75}
$$

Thus by (5.69),

$$\sum_{\substack{j_1, j_2, j_3 \in D(r) \\ j_1 \neq j_2 \neq j_3}} \mathbf{P}[V_{j_1, j_2, j_3}] = O\Big(|D(r)|\, g(r)\, [|D(r)| f(r) + g(r)]\Big). \tag{5.76}$$

That means we proved assertion (5.6). This completes the proof. □

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