The Annals of Probability
2001, Vol. 29, No. 3, 1111–1127

THE CONTROLLER-AND-STOPPER GAME FOR A LINEAR DIFFUSION

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Consider a process $X(\cdot) = \{X(t), 0 \le t < \infty\}$ with values in the interval $I = (0, 1)$, absorption at the boundary points of I, and dynamics

$$dX(t) = \beta(t)\,dt + \sigma(t)\,dW(t), \qquad X(0) = x.$$

The values $(\beta(t), \sigma(t))$ are selected by a *controller* from a subset of $\Re \times (0, \infty)$ that depends on the current position $X(t)$, for every $t \ge 0$. At any stopping rule τ of his choice, a second player, called a *stopper,* can halt the evolution of the process $X(\cdot)$, upon which he receives from the controller the amount $e^{-\alpha\tau}u(X(\tau))$; here $\alpha \in [0, \infty)$ is a discount factor, and u: $[0, 1] \to \Re$ is a continuous "reward function." Under appropriate conditions on this function and on the controller's set of choices, it is shown that the two players have a saddlepoint of "optimal strategies." These can be described fairly explicitly by reduction to a suitable problem of optimal stopping, whose maximal expected reward V coincides with the value of the game,

$$V = \sup_{\tau} \inf_{X(\cdot)} \mathbf{E}\left[e^{-\alpha\tau}u(X(\tau))\right] = \inf_{X(\cdot)} \sup_{\tau} \mathbf{E}\left[e^{-\alpha\tau}u(X(\tau))\right].$$

1. Introduction. We describe in this section a zero-sum stochastic game that involves a linear diffusion process. The game takes place between two players, one called *controller* and the other called *stopper*. This game will be the object of study in the paper.

The diffusion process $X(\cdot)$ evolves in the state-space $I = (\ell, r)$, a nonempty, bounded, open interval of the real line. Its evolution is governed by the equation

$$dX(t) = \beta(t)\,dt + \sigma(t)\,dW(t), \qquad X(0) = x \in I \tag{1.1}$$

on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{P}), \mathbf{F} = \{\mathscr{F}(t), 0 \le t < \infty\}$. The process $W(\cdot)$ is a standard Brownian motion with respect to $\mathbf{F}$, and $(\beta(\cdot), \sigma(\cdot))$ is a pair of real-valued, $\mathbf{F}$-progressively measurable processes which satisfy almost surely

$$\int_0^t [|\beta(s)| + \sigma^2(s)]\,ds < \infty, \qquad (\beta(t), \sigma(t)) \in \mathscr{K}(X(t)) \tag{1.2}$$

for all $t \in [0, \infty)$. Whenever the process $X(\cdot)$ is in a given state $X(t) = \zeta \in [\ell, r]$, the *controller* can choose a local drift–local volatility pair $(\beta(t), \sigma(t))$

Received December 1999; revised October 2000.
[1]Supported in part by the NSF Grant DMS-97-32810.
[2]Supported in part by NSF Grant DMS-97-03285.
AMS 2000 *subject classifications*. Primary 93E20, 60G40; secondary 62L15, 60D60.
Key words and phrases. Stochastic game, optimal stopping, one-dimensional diffusions, generalized Itô rule, local time, excessive functions.

from a given subset $\mathscr{K}(\zeta)$ of $\Re \times (0, \infty)$. The family $\{\mathscr{K}(\zeta)/\zeta \in [\ell, r]\}$ is specified in advance, and we set $\mathscr{K}(\ell) = \mathscr{K}(r) = \{(0,0)\}$; this choice forces the process $X(\cdot)$ of (1.1) to become absorbed, whenever it reaches either one of the boundary points of the interval $I = (\ell, r)$. Given an initial condition $x \in I$, we shall denote by $\mathscr{A}(x)$ the class of all processes $X(\cdot)$ with $X(0) = x$ that can be constructed this way and are thus "available" to the controller with this starting position.

A second player, called *stopper*, can halt the evolution of the process $X(\cdot)$ by selecting a *stop-rule* τ: $C[0,\infty) \to [0,\infty]$. This is a mapping from the space $C[0,\infty)$ of continuous functions ξ: $[0,\infty) \to \Re$ into the extended real half-line, with the property

$$\{\xi \in C[0,\infty)/\tau(\xi) \le t\} \in \mathscr{B}_t := \varphi_t^{-1}(\mathscr{B}) \qquad \forall\ 0 \le t < \infty. \tag{1.3}$$

Here $\mathscr{B} := \mathscr{B}(C[0,\infty))$ is the Borel σ-algebra generated by the open sets in $C[0,\infty)$, and φ_t: $C[0,\infty) \to C[0,\infty)$ is the mapping

$$(\varphi_t \xi)(s) := \xi(t \wedge s), \qquad 0 \le s < \infty$$

[cf. Karatzas and Shreve (1988), page 60, Problem 2.4.2]. The significance of (1.3) should be clear: "whether the stop-rule τ has acted by any given time t to halt the trajectory ξ, or not, can be decided by observing the trajectory up to time t, and not beyond." We shall denote by $\mathscr{S}$ the class of all stop-rules τ: $C[0,\infty) \to C[0,\infty)$ as above.

Consider now a *continuous* "reward function" u: $[\ell, r] \to \Re$, and a real "discount factor" $\alpha \ge 0$. If the controller selects a diffusion $X(\cdot)$ from the class $\mathscr{A}(x)$ of processes available at the initial position $x \in I$, and if the stopper employs the stop-rule $\tau \in \mathscr{S}$, then the controller pays the stopper the amount $e^{-\alpha\tau(X)}u(X(\tau(X)))$ at the time $\tau(X)$. The objective of the controller (resp., the stopper) is to try and minimize (resp., maximize) this random quantity, at least in expectation ("on the average"). In this spirit, we denote by

$$\overline{V}(x) := \inf_{X(\cdot)\in\mathscr{A}(x)} \sup_{\tau\in\mathscr{S}} \mathbf{E}\big[e^{-\alpha\tau(X)}u(X(\tau(X)))\big] \tag{1.4}$$

the *upper value*, and by

$$\underline{V}(x) := \sup_{\tau\in\mathscr{S}} \inf_{X(\cdot)\in\mathscr{A}(x)} \mathbf{E}\big[e^{-\alpha\tau(X)}u(X(\tau(X)))\big] \tag{1.5}$$

the *lower value*, of the game between the controller and the stopper. Clearly, $\underline{V}(x) \le \overline{V}(x)$.

In (1.4), (1.5) and throughout the paper, we are employing the convention $\chi(\infty) := \limsup_{t\to\infty} \chi(t)$ for any $\mathbf{F}$-progressively measurable process $\chi(\cdot)$.

DEFINITION 1.1. We say that the game *has a value* if $\underline{V}(x) = \overline{V}(x)$.

DEFINITION 1.2. A pair $(Z(\cdot), \rho) \in (\mathscr{A}(x) \times \mathscr{S})$ is called a *saddlepoint* of the game, if we have

$$
\begin{aligned}
\mathbf{E}\big[e^{-\alpha\tau(Z)}\, u(Z(\tau(Z)))\big] &\le \mathbf{E}\big[e^{-\alpha\rho(Z)}\, u(Z(\rho(Z)))\big] \\
&\le \mathbf{E}\big[e^{-\alpha\rho(X)}\, u(X(\rho(X)))\big]
\end{aligned}
\tag{1.6}
$$

for every $X(\cdot) \in \mathscr{A}(x)$, $\tau \in \mathscr{S}$.

As is straightforward to verify, (1.6) implies that the game has a value, namely,

$$
\underline{V}(x) = \overline{V}(x) = \mathbf{E}\big[e^{-\alpha\rho(Z)}\, u(Z(\rho(Z)))\big]. \tag{1.7}
$$

After some preliminary results from optimal stopping theory, we shall identify a saddlepoint for the undiscounted game ($\alpha = 0$) in some generality in Section 4, and for the discounted game ($\alpha > 0$) under a special assumption in Section 6.

The controller-and-stopper game has been studied in discrete time by Maitra and Sudderth (1996), who showed that the game has a value when the reward function is Borel-measurable and the state-space is Polish. Perhaps the same is true in continuous time, but here we consider only real-valued processes and a *continuous* reward function. In this much more concrete context we obtain not just the existence of a value but also that of a saddle-point for the game, which we are able to describe fairly exactly.

2. Reduction to optimal stopping. Let us consider now two Borel-measurable functions b: $I \to \Re$, s: $I \to \Re$ that satisfy

$$
\begin{aligned}
&s^2(x) > 0, \qquad (b(x), s(x)) \in \mathscr{K}(x), \\
&\int_{x-\varepsilon}^{x+\varepsilon} s^{-2}(y)[1 + |b(y)|]\, dy < \infty \qquad \text{for some } \varepsilon > 0
\end{aligned}
\tag{2.1}
$$

at every $x \in I$. We always assume that there exist functions b, s with these properties. It is then well known [cf. Karatzas and Shreve (1988), Theorem 5.5.15, page 341] that the stochastic differential equation

$$
dZ(t) = b(Z(t))\, dt + s(Z(t))\, dW(t), \qquad Z(0) = x \in I \tag{2.2}
$$

has a weak solution, which is unique in the sense of the probability law. The resulting process $Z(\cdot)$ is a *one-dimensional, time-homogeneous diffusion* on the interval I, with local drift function b, local volatility function s, and absorption at the boundary points of the interval [cf. Karatzas and Shreve, pages 348–350, for conditions about the attainability or nonattainability of the boundary points]. Furthermore, the processes $\beta(\cdot) \equiv b(Z(\cdot))$, $\sigma(\cdot) \equiv s(Z(\cdot))$ satisfy the requirements of (1.2), and thus the diffusion $Z(\cdot)$ of (2.2) belongs to the class $\mathscr{A}(x)$ of processes available to the controller at the initial position $x \in I$.

If the controller has selected the diffusion process $Z(\cdot) \in \mathscr{A}(x)$, then the best that the stopper can do is to select a stop-rule which attains the supremum

$$G(x) := \sup_{\tau \in \mathscr{S}} \mathbf{E}\big[e^{-\alpha\tau(Z)} u(Z(\tau(Z)))\big] \left(= \sup_{\mu \in \mathscr{M}} \mathbf{E}\big[e^{-\alpha\mu} u(Z(\mu))\big]\right). \tag{2.3}$$

In this last expression we have denoted by $\mathscr{M}$ the class of $\mathbf{F}$-stopping times, namely those random variables $\mu\colon \Omega \to [0, \infty]$ that satisfy $\{\mu \le t\} \in \mathscr{F}(t)$, for every $t \in [0, \infty)$. It is well known that this *optimal reward function* $G\colon I \to \Re$ is *α-excessive*; that is,

$$\begin{aligned} &G(x) \ge \mathbf{E}\big[e^{-\alpha t} G(Z(t))\big] \qquad \forall\ 0 \le t < \infty \quad \text{and} \\ &G(x) = \lim_{t \downarrow 0} \mathbf{E}\big[e^{-\alpha t} G(Z(t))\big] \end{aligned} \tag{2.4}$$

hold for every $x \in I$. The function $G(\cdot)$ clearly also *majorizes* $u(\cdot)$; that is,

$$G(x) \ge u(x), \qquad \forall\ x \in I \tag{2.5}$$

and is in fact the *smallest α-excessive majorant of* $u(\cdot)$. Furthermore, the stop-rule $\rho\colon C[0, \infty) \to C[0, \infty)$ given by

$$\rho(\xi) := \inf\{t \ge 0 / G(\xi(t)) = u(\xi(t))\} = \inf\{t \ge 0 / \xi(t) \in \Sigma\} \tag{2.6}$$

attains the supremum in (2.3), namely,

$$G(x) = \mathbf{E}\big[e^{-\alpha\rho(Z)} u(Z(\rho(Z)))\big]. \tag{2.7}$$

We are denoting by

$$\Sigma := \{x \in I / G(x) = u(x)\}, \qquad \mathscr{C} := \{x \in I / G(x) > u(x)\} \tag{2.8}$$

the "optimal stopping" and "optimal continuation" regions, respectively. For these classical results, the reader may wish to consult Shiryaev (1978) or Salminen (1985).

In terms of the optimal reward function $G(\cdot)$ of (2.3), (2.7) and of the stop-rule ρ in (2.6), the saddlepoint property (1.6) amounts to

$$\begin{aligned} \mathbf{E}\big[e^{-\alpha\rho(Z)} u(Z(\rho(Z)))\big] = G(x) \le \mathbf{E}\big[e^{-\alpha\rho(X)} u(X(\rho(X)))\big]& \\ \forall\ X(\cdot) \in \mathscr{A}(x).& \end{aligned} \tag{2.9}$$

If (2.9) holds, then the pair $(Z(\cdot), \rho)$ is a saddlepoint of the stochastic game, whose value is thus given by

$$\underline{V}(x) = \overline{V}(x) = G(x). \tag{2.10}$$

The whole problem, then, becomes to identify a diffusion process $Z(\cdot) \in \mathscr{A}(x)$ as in (2.2), with the properties (2.1) and (2.9). We shall carry out this program in Section 4 for the undiscounted case ($\alpha = 0$) and in Section 6 for the discounted case ($\alpha > 0$), under appropriate conditions.

3. The undiscounted optimal stopping problem. In the undiscounted case of $\alpha = 0$, a prominent role is played by the *scale function*,

$$p(x) := \int_{x_0}^{x} \exp\left[-2\int_{x_0}^{\zeta} (b/s^2)(u)\,du\right] d\zeta, \qquad x \in I \tag{3.1}$$

of the diffusion process $Z(\cdot)$ in (2.2), for some arbitrary but fixed $x_0 \in I$. This function is strictly increasing, with absolutely continuous first derivative,

$$p'(x) = \exp\left\{-2\int_{x_0}^{x} (b/s^2)(u)\,du\right\} = 1 + \int_{x_0}^{x} p''(u)\,du > 0, \qquad x \in I, \tag{3.2}$$

where

$$p''(x) := -\frac{2b(x)}{s^2(x)} \cdot p'(x). \tag{3.3}$$

Under the assumptions

$$\tilde{\ell} := p(\ell+) > -\infty, \qquad \tilde{r} := p(r-) < \infty, \tag{3.4}$$

the strictly increasing function $p(\cdot)$ maps the bounded interval $I = (\ell, r)$ *onto* the bounded interval $\widetilde{I} = (\tilde{\ell}, \tilde{r})$, and has the inverse function q: $\widetilde{I} \to I$ that satisfies $p(q(y)) = y, \forall\ y \in \widetilde{I}$. Furthermore, the transformation

$$Y(t) := p(Z(t)), \qquad 0 \le t < \infty \tag{3.5}$$

of the process $Z(\cdot)$ in (2.2), is a diffusion in natural scale, namely,

$$dY(t) = \tilde{s}(Y(t))\,dW(t), \qquad Y(0) = y := p(x) \in \widetilde{I}, \tag{3.6}$$

with $\tilde{s}(\cdot) := ((p's) \circ q)(\cdot)$. Similarly, with a new reward function

$$\tilde{u}(\cdot) := (u \circ q)(\cdot) \quad \text{on } \widetilde{I}, \tag{3.7}$$

we have

$$G(x) = \widetilde{G}(p(x)), \tag{3.8}$$

where

$$\widetilde{G}(y) := \sup_{\tau \in \mathscr{S}} \mathbf{E}[\tilde{u}(Y(\tau(Y)))] \qquad \text{for } y \in \widetilde{I}.$$

The auxiliary optimal reward function $\widetilde{G}(\cdot)$ of (3.8) is defined on $\widetilde{I}$. It is concave on this interval, and in fact is the *smallest concave majorant of* $\tilde{u}(\cdot)$, namely,

$$\widetilde{G}(y) = \inf\{f(y)/f(\cdot) \text{ affine, } f(\cdot) \ge \tilde{u}(\cdot)\}. \tag{3.9}$$

This leads to the representation

$$G(x) = \inf\{\beta + \gamma p(x)/\beta \in \Re, \gamma \in \Re, \beta + \gamma p(\cdot) \ge u(\cdot)\} \tag{3.10}$$

of the optimal reward function $G(\cdot)$ in (2.3), as the lower envelope of all affine transformations of the scale function $p(\cdot)$, that dominate the reward function $u(\cdot)$.

By analogy with (2.8), the optimal stopping region and the optimal continuation region for the problem of (3.8), are given by

$$\widetilde{\Sigma} := \{y \in \widetilde{I}/\widetilde{G}(y) = \tilde{u}(y)\}, \qquad \widetilde{\mathscr{C}} := \{y \in \widetilde{I}/\widetilde{G}(y) > \tilde{u}(y)\}, \tag{3.11}$$

respectively. As a concave function on $\widetilde{I}$, $\widetilde{G}(\cdot)$ has a decreasing left-derivative $D^-\widetilde{G}(\cdot)$, which induces a *positive* measure $\tilde{\nu}$ on $\mathscr{B}(\widetilde{I})$ through the recipe

$$\tilde{\nu}([a,b)) = D^-\widetilde{G}(a) - D^-\widetilde{G}(b) \qquad \text{for } \tilde{\ell} < a < b < \tilde{r}. \tag{3.12}$$

The measure $\tilde{\nu}$ does not charge the optimal continuation region $\widetilde{\mathscr{C}}$ in (3.11); that is,

$$\tilde{\nu}(\widetilde{\mathscr{C}}) = 0; \tag{3.13}$$

this reflects the fact that $\widetilde{G}(\cdot)$ is affine on each of the (at most countably many) open disjoint intervals, whose union constitutes $\widetilde{\mathscr{C}}$.

For proofs of these statements, the reader is referred to Section 3 in Karatzas and Sudderth (1999), and to the references cited there.

4. The undiscounted game. Let us discuss now the stochastic game of Section 1 in the *undiscounted* case ($\alpha = 0$), under the assumptions

$$\inf\{\sigma^2/(\beta,\sigma) \in \mathscr{K}(x) \quad \text{for some } \beta \in \Re,\ x \in I\} > 0, \tag{4.1}$$

$$u^* := \sup_{x\in[\ell,r]} u(x) = u(m) \qquad \text{for exactly one } m \in I. \tag{4.2}$$

We will also assume that we can select pairs $(b_\ell, s_\ell), (b_r, s_r)$ of measurable functions with the properties (2.1) and (3.4), as well as

$$\frac{b_\ell(x)}{s_\ell^2(x)} = \inf\left\{\frac{\beta}{\sigma^2}\Big/(\beta,\sigma) \in \mathscr{K}(x)\right\}, \tag{4.3}$$

$$\frac{b_r(x)}{s_r^2(x)} = \sup\left\{\frac{\beta}{\sigma^2}\Big/(\beta,\sigma) \in \mathscr{K}(x)\right\} \tag{4.4}$$

for every $x \in I$. One can then construct diffusion processes $Z_\ell(\cdot), Z_r(\cdot)$ as in (2.2), namely,

$$dZ_\ell(t) = b_\ell(Z_\ell(t))\,dt + s_\ell(Z_\ell(t))\,dW(t), \qquad Z_\ell(0) = x \in I, \tag{4.5}$$

$$dZ_r(t) = b_r(Z_r(t))\,dt + s_r(Z_r(t))\,dW(t), \qquad Z_r(0) = x \in I \tag{4.6}$$

as well as the corresponding optimal reward functions of (2.3) with $\alpha = 0$, namely,

$$G_j(x) := \sup_{\tau\in\mathscr{S}} \mathbf{E}[u(Z_j(\tau(Z_j)))], \qquad x \in I \quad \text{for } j = \ell, r. \tag{4.7}$$

With these ingredients in place, we can formulate the solution of the stochastic game in the undiscounted case ($\alpha = 0$).

THEOREM 4.1. *With the above assumptions and notation* (4.1)–(4.7), *we have the following*:

(i) *For each* $x \in [\ell, m]$, *a saddle point for the stochastic game is given by* $(Z_\ell(\cdot), \rho_\ell)$, *in the notation of* (4.5) *and of*

$$\rho_\ell(\xi) := \inf\{t \geq 0 / G_\ell(\xi(t)) = u(\xi(t))\}, \qquad \xi \in C[0, \infty). \tag{4.8}$$

(ii) *For each* $x \in [m, r]$, *a saddlepoint for the stochastic game is given by* $(Z_r(\cdot), \rho_r)$, *in the notation of* (4.6) *and of*

$$\rho_r(\xi) := \inf\{t \geq 0 / G_r(\xi(t)) = u(\xi(t))\}, \qquad \xi \in C[0, \infty). \tag{4.9}$$

(iii) *The stochastic game has a value*, *given by*

$$\underline{V}(x) = \overline{V}(x) = \begin{cases} G_\ell(x), & x \in [\ell, m), \\ G_r(x), & x \in (m, r], \\ u^*, & x = m. \end{cases} \tag{4.10}$$

In words: the *controller* (minimizer) tries to "get away as effectively as he can from the point m," the position of the global maximum of $u(\cdot)$ on $[\ell, r]$, by minimizing (resp., maximizing) the mean-variance, or "signal-to-noise" ratio β/σ^2, when to the left (resp., to the right) of m. This same policy maximizes the probability of reaching the left boundary point ℓ (resp., the right boundary point r), as Pestien and Sudderth (1985) have demonstrated. It is not a priori clear that the controller should follow such a strategy (i.e., that his notion of "effectiveness" should be the same as trying to reach a goal with maximal probability), even in the vicinity of local minima for the function $u(\cdot)$.

On the other hand, the *stopper* (maximizer) finds it best, when to the left of the point m, to halt the controlled process $X(\cdot)$ at the time $\rho_\ell(X)$ of its first entrance into the optimal stopping region $\Sigma_\ell = \{x \in I / G_\ell(x) = u(x)\}$, that corresponds to the problem (4.7) for the diffusion process $Z_\ell(\cdot)$ in (4.5). And when to the right of m, the stopper finds it best to halt the controlled process $X(\cdot)$ at the time $\rho_r(X)$ of its first entrance into the optimal stopping region $\Sigma_r = \{x \in I / G_r(x) = u(x)\}$ of the problem (4.7) for the diffusion process $Z_r(\cdot)$ in (4.6). Clearly, the point m belongs to both stopping regions Σ_ℓ and Σ_r; so, under optimal play on the part of his opponent, the controller will find himself using only one of the two regimes (4.5), (4.6)—he will never have to switch from one regime to the other.

PROOF. It suffices to deal with the case $x \in [\ell, m]$ (the other case is then treated similarly). In order to simplify notation, we shall set

$$\begin{gathered} m = r, \quad s(\cdot) \equiv s_\ell(\cdot), \quad b(\cdot) \equiv b_\ell(\cdot), \\ Z(\cdot) \equiv Z_\ell(\cdot), \quad G(\cdot) \equiv G_\ell(\cdot) \end{gathered} \tag{4.11}$$

and try to establish the saddlepoint property

$$G(x) \le \mathbf{E}[u(X(\rho(X)))] \qquad \forall\ X(\cdot) \in \mathscr{A}(x) \tag{4.12}$$

of (2.9), for the stop-rule $\rho(\xi) = \inf\{t \ge 0 / G(\xi(t)) = u(\xi(t))\}$ of (4.8).

For any given $X(\cdot) \in \mathscr{A}(x)$, we claim that

$$\vartheta(\cdot) := p(X(\cdot)) \quad \text{is a submartingale} \tag{4.13}$$

with values in the bounded interval $\widetilde{I} = (\tilde{\ell}, \tilde{r})$. Indeed, we have

$$\begin{aligned} d\vartheta(t) &= p'(X(t)) \cdot [\beta(t)\,dt + \sigma(t)\,dW(t)] - \left(p' \frac{b}{s^2}\right)(X(t)) \cdot \sigma^2(t)\,dt \\ &= p'(X(t)) \left[\frac{\beta(t)}{\sigma^2(t)} - \frac{b(X(t))}{s^2(X(t))}\right] \sigma^2(t)\,dt + p'(X(t)) \cdot \sigma(t)\,dW(t) \end{aligned} \tag{4.14}$$

from Itô's rule and (1.1), (3.3). Thanks to $p'(\cdot) > 0$ and the definition (4.3), the drift term is nonnegative. In other words, $\vartheta(\cdot)$ is a local submartingale, thus also a (true) submartingale, because it is bounded.

Now let us recall (3.8), (4.13) and look at the process

$$\eta(t) := G(X(t)) = \widetilde{G}(\vartheta(t)), \qquad 0 \le t < \infty. \tag{4.15}$$

From the generalized Itô rule for concave functions [e.g., Karatzas and Shreve (1988), Section 3.7] we obtain

$$\begin{aligned} \eta(T) &= G(x) + \int_0^T D^- \widetilde{G}(\vartheta(t)) \cdot p'(X(t)) \left[\frac{\beta(t)}{\sigma^2(t)} - \frac{b(X(t))}{s^2(X(t))}\right] \sigma^2(t)\,dt \\ &\quad + \int_0^T D^- \widetilde{G}(\vartheta(t)) \cdot p'(X(t)) \sigma(t)\,dW(t) - \int_{\widetilde{I}} L_T^{\vartheta}(\zeta) \tilde{\nu}(d\zeta), \\ &\qquad\qquad 0 \le T < \infty. \end{aligned} \tag{4.16}$$

Here $t \mapsto L_t^{\vartheta}(\zeta)$ is the *local time* of the semimartingale $\vartheta(\cdot)$ at the point $\zeta \in I$: a continuous, increasing and $\mathbf{F}$-adapted process, flat off the set $\{t \ge 0 / \vartheta(t) = \zeta\}$.

As we are assuming that the continuous function $u(\cdot)$ attains its maximum over $[\ell, r]$ at $x = r$, so in turn the continuous function $\tilde{u}(\cdot)$ attains its maximum over $[\tilde{\ell}, \tilde{r}]$ at $y = \tilde{r}$; this implies $D^- \widetilde{G}(\cdot) \ge 0$ and the increase of $\widetilde{G}(\cdot)$ on $\widetilde{I}$. From the positivity of $p'(\cdot)$ and the definition (4.3), we deduce that the first integral on the right-hand side of (4.16) defines an increasing process. The second (stochastic) integral defines a continuous, local martingale. The third integral defines a continuous, increasing process that starts at zero, and for which we have

$$\int_{\widetilde{I}} L_{\rho(X)}^{\vartheta}(\zeta) \tilde{\nu}(d\zeta) = \int_{\widetilde{\Sigma}} L_{\rho(X)}^{\vartheta}(\zeta) \tilde{\nu}(d\zeta) = 0,$$

almost surely. This is because the measure $\tilde{\nu}$ does not charge the continuation region $\widetilde{\mathscr{C}}$ [recall (3.12) and (3.13)] and because

$$L_{\rho(X)}^{\vartheta}(\zeta) = 0 \quad \text{a.s.}$$

for $\zeta \in \widetilde{\Sigma}$, since $\rho(X) = \inf\{t \geq 0/\vartheta(t) \in \widetilde{\Sigma}\}$ and $t \mapsto L_t^{\vartheta}(\zeta)$ is flat off the set $\{t \geq 0/\vartheta(t) = \zeta\}$.

We deduce from all this, that the process $\eta(\cdot \wedge \rho(X))$ is a local submartingale; because it takes values in the bounded interval $\widetilde{I} = [\tilde{\ell}, \tilde{r}]$, the process

$$\eta(\cdot \wedge \rho(X)) \text{ is actually a (bounded) submartingale.} \tag{4.17}$$

Now from (4.17) and the optional sampling theorem, we obtain

$$\begin{aligned} G(x) = \mathbf{E}\eta(0) &\leq \mathbf{E}[\eta(\rho(X))] = \mathbf{E}[G(X(\rho(X)))] \\ &= \mathbf{E}[u(X(\rho(X)))] \qquad \forall\ X(\cdot) \in \mathscr{A}(x), \end{aligned} \tag{4.18}$$

and (4.12) is proved. Note that we have used the assumption (4.1) to guarantee

$$\rho(X) < \infty \qquad \text{a.s. } \forall\ X(\cdot) \in \mathscr{A}(x). \tag{4.19}$$

The proof of Theorem 4.1 is complete. □

REMARK 4.1. We cannot expect (4.10) to hold, in the absence of condition (4.1).

To see this, consider a reward function $u(\cdot)$ which is continuous on $[\ell, r]$, strictly decreasing on (ℓ, x_*) and strictly increasing on (x_*, r), for some $x_* \in I$. Furthermore, in accordance with the simplifying assumption (4.11), we take $u(r) > u(\ell)$, which amounts to $m = r$. Suppose also that $\mathscr{K}(x) = \{(-1, 1)\} \cup \{(0, \varepsilon)/\varepsilon > 0\}$ for every $x \in I = (\ell, r)$. Then

$$\frac{b(x)}{s^2(x)} = \frac{b_\ell(x)}{s_\ell^2(x)} = -1$$

in the notation of (4.3), so that the process $Z(\cdot) \equiv Z_\ell(\cdot)$ of (4.5), (4.11) is Brownian motion with negative drift

$$Z(t) = x - t + W(t). \tag{4.20}$$

The point x_*, and a small open interval $\mathscr{N}_*$ containing it, belong to the continuation region for the *stopper's* (maximizer's) problem $G(x) = \sup_{\tau \in \mathscr{S}} \mathbf{E}[u(Z(\tau(Z)))]$ for the process $Z(\cdot)$ of (4.20). On the other hand, the *controller* (minimizer) can effectively halt the process $X(\cdot)$ near any point $x \in I$, by choosing controls of the type

$$(\beta(t), \sigma(t)) = (0, \varepsilon(t)) \tag{4.21}$$

for some deterministic function ε: $(0, \infty) \to (0, \infty)$ with $\int_0^\infty \varepsilon^2(t)\, dt$ very small.

Therefore, if $X(0) = x_*$ and the controller uses controls as in (4.21) so as to keep the process $X(\cdot)$ in a sufficiently small neighborhood $\mathscr{N}_*$ containing x_*, then the controller can secure a payoff very near $u(x_*)$, in particular, smaller than $G(x_*)$, so that (4.10) fails. On the other hand, we have in this case $\rho(X) = \infty$ a.s., so (4.19) fails, and the argument in (4.18) fails too, as it should.

It would be interesting to study the stochastic game of Section 2, and try to find a saddlepoint of optimal strategies for the two players, in the absence of condition (4.1).

5. The discounted optimal stopping problem. Our objective in this section will be to study in some detail the optimal reward function of (2.3) in the *discounted case* $\alpha > 0$ after the manner of Salminen (1985), culminating with the following result.

THEOREM 5.1. *The function $G(\cdot)$ of* (2.3) *with $\alpha > 0$ can be written as the difference of two convex functions, and the measure*

$$\nu(dx) := [\alpha G(x) - b(x) \cdot D^- G(x)]\, dx - (s^2(x)/2) \cdot D^2 G(dx) \tag{5.1}$$

is positive and does not charge the optimal continuation region $\mathscr{C}$ of (2.8):

$$\nu(\mathscr{C}) = 0. \tag{5.2}$$

Here and in the sequel, we are denoting by $D^- G(\cdot), D^+ G(\cdot)$ the left- and right-derivatives of the function $G(\cdot)$, and by $D^2 G(dx)$ the second-derivative *signed measure* defined on the σ-algebra $\mathscr{B}(I)$ of Borel sets of I, through

$$D^2 G([a, b)) := D^- G(b) - D^- G(a), \qquad \ell < a < b < r. \tag{5.3}$$

These derivatives exists because, as asserted in Theorem 5.1, we have $G = G_1 - G_2$ for two convex functions G_1, G_2; from well known theory [see, e.g., Karatzas and Shreve (1988), page 213], these have right- and left-derivatives $D^\pm G_i$ which exist, are nondecreasing, and satisfy $D^- G_i \le D^+ G_i$ on I (with strict inequality on a set which is at most countable) for $i = 1, 2$.

The proof of Theorem 5.1 occupies the remainder of this section and follows very closely the work of Salminen (1985). It is rather technical, so readers may wish to skip it on first reading and proceed directly to Section 6. There we treat, with the help of Theorem 5.1, the stochastic game of (1.2)–(1.6) in the discounted case $\alpha > 0$.

In order to make headway with the proof, let us recall from Dynkin (1969), Theorem 10.1 or Salminen (1985), Sections 3 and 4, that the optimal reward function $G(\cdot)$ of (2.3) admits an *integral representation* of the type

$$G(x) = \int_{[\ell, r]} K_y(x; x_0) \lambda(dy), \qquad x \in I. \tag{5.4}$$

Here, the point $x_0 \in I$ is arbitrary but fixed, and λ is some *positive, finite measure* on $\mathscr{B}(I)$ that does not charge the optimal continuation region $\mathscr{C}$ of (2.8), namely,

$$\lambda(\mathscr{C}) = 0. \tag{5.5}$$

[cf. Salminen (1985), 4.2 and 4.5]. We have set $K_y(x; x_0) = k_y(x)/k_y(x_0)$, where

$$k_y(x) := \left\{ \begin{array}{ll} \Phi^\uparrow(x)/\Phi^\uparrow(y); & x < y \\ \Phi^\downarrow(x)/\Phi^\downarrow(y); & x \ge y \end{array} \right\} = \mathbf{E}\big[e^{-\alpha T_y}\big] \tag{5.6}$$

and $T_y := \inf\{t \geq 0/Z(t) = y\}$ is the first hitting time of the point $y \in [\ell, r]$ by the diffusion process $Z(\cdot)$ of (2.2). We are using here the notation $\Phi^{\uparrow}(\cdot)$ (resp., $\Phi^{\downarrow}(\cdot)$) for a positive, strictly increasing (resp., decreasing) function that is *α-harmonic* [meaning that it solves the equation $(s^2/2)\Phi'' + b\Phi' = \alpha\Phi$ in the generalized sense of (5.10) below]. The functions $\Phi^{\uparrow}(\cdot), \Phi^{\downarrow}(\cdot)$ are linearly independent, and the Wronskian

$$B = \frac{1}{p'(x)}\big[D^{\pm}\Phi^{\uparrow}(x)\cdot\Phi^{\downarrow}(x) - D^{\pm}\Phi^{\downarrow}(x)\cdot\Phi^{\uparrow}(x)\big] \tag{5.7}$$

is a positive constant.

Let us concentrate on the interval $[\ell, x_0]$ (a similar analysis can be carried out for $[x_0, r]$). On this interval, Salminen (1985) computes the left-derivative of the optimal reward function in (2.3), (5.4), as

$$\begin{aligned} D^-G(x) &= \frac{D^-\Phi^{\downarrow}(x)}{\Phi^{\downarrow}(x_0)}\cdot\lambda([\ell, x)) + \frac{D^-\Phi^{\uparrow}(x)}{\Phi^{\uparrow}(x_0)}\cdot\lambda((x_0, r]) \\ &\quad + \frac{D^-\Phi^{\uparrow}(x)}{\Phi^{\downarrow}(x_0)}\int_{[x, x_0]}\frac{\Phi^{\downarrow}(y)}{\Phi^{\uparrow}(y)}\cdot\lambda(dy), \qquad x \leq x_0. \end{aligned} \tag{5.8}$$

A similar expression can also be computed for the right-derivative, leading to the property

$$D^-G(x) - D^+G(x) = \frac{Bp'(x)}{\Phi^{\downarrow}(x_0)\Phi^{\uparrow}(x)}\cdot\lambda(\{x\}) \geq 0, \qquad x \leq x_0.$$

From (5.4), written for $x \leq x_0$ in the more suggestive form

$$G(x) = \frac{\Phi^{\downarrow}(x)}{\Phi^{\downarrow}(x_0)}\cdot\lambda([\ell, x)) + \frac{\Phi^{\uparrow}(x)}{\Phi^{\uparrow}(x_0)}\cdot\lambda((x_0, r]) + \frac{\Phi^{\uparrow}(x)}{\Phi^{\downarrow}(x_0)}\int_{[x, x_0]}\frac{\Phi^{\downarrow}(y)}{\Phi^{\uparrow}(y)}\cdot\lambda(dy)$$

and (5.8), we obtain

$$\begin{aligned} \alpha G(x) - b(x)\cdot D^-G(x) &= \frac{\alpha\Phi^{\downarrow}(x) - b(x)\cdot D^-\Phi^{\downarrow}(x)}{\Phi^{\downarrow}(x_0)}\cdot\lambda([\ell, x)) \\ &\quad + \frac{\alpha\Phi^{\uparrow}(x) - b(x)\cdot D^-\Phi^{\uparrow}(x)}{\Phi^{\uparrow}(x_0)}\cdot\lambda((x_0, r]) \\ &\quad + \frac{\alpha\Phi^{\uparrow}(x) - b(x)\cdot D^-\Phi^{\uparrow}(x)}{\Phi^{\downarrow}(x_0)} \\ &\quad \times \int_{[x, x_0]}\frac{\Phi^{\downarrow}(y)}{\Phi^{\uparrow}(y)}\cdot\lambda(dy), \qquad x \leq x_0. \end{aligned} \tag{5.9}$$

Now each of the functions $\Phi^{\uparrow}(\cdot), \Phi^{\downarrow}(\cdot)$ is *α-harmonic*, which means in particular that its second derivative measure

$$D^2\Phi([a, b)) := D^-\Phi(b) - D^-\Phi(a), \qquad \ell < a < b < r$$

exists and is given by

$$D^2\Phi(dx) = \frac{2}{s^2(x)}\big[\alpha\Phi(x) - b(x)\cdot D^-\Phi(x)\big]\,dx; \tag{5.10}$$

see Salminen [(1985), equation (2.4), page 88] and Revuz and Yor [(1991), Theorem 3.12 and Exercise 3.20, pages 285–289]. This way, we may rewrite (5.9) in the more compact form

$$\begin{aligned}
&\frac{2}{s^2(x)}\big[\alpha G(x) - b(x)\cdot D^-G(x)\big]\,dx\\
&\quad = \frac{D^2\Phi^{\downarrow}(dx)}{\Phi^{\downarrow}(x_0)}\cdot\lambda([\ell, x)) + \frac{D^2\Phi^{\uparrow}(dx)}{\Phi^{\uparrow}(x_0)}\cdot\lambda((x_0, r])\\
&\qquad + \frac{D^2\Phi^{\uparrow}(dx)}{\Phi^{\downarrow}(x_0)}\int_{[x,\,x_0]}\frac{\Phi^{\downarrow}(y)}{\Phi^{\uparrow}(y)}\cdot\lambda(dy), \qquad x \le x_0.
\end{aligned} \tag{5.11}$$

On the other hand, we may differentiate the expression of (5.8) to see that the second derivative of $G(\cdot)$ exists as a signed measure, namely,

$$\begin{aligned}
D^2G(dx) &= \frac{D^2\Phi^{\downarrow}(dx)}{\Phi^{\downarrow}(x_0)}\cdot\lambda([\ell, x)) + \frac{D^2\Phi^{\uparrow}(dx)}{\Phi^{\uparrow}(x_0)}\cdot\lambda((x_0, r]) + \frac{D^-\Phi^{\downarrow}(x)}{\Phi^{\downarrow}(x_0)}\cdot\lambda(dx)\\
&\quad + \frac{D^2\Phi^{\uparrow}(dx)}{\Phi^{\downarrow}(x_0)}\int_{[x,\,x_0]}\frac{\Phi^{\downarrow}(y)}{\Phi^{\uparrow}(y)}\cdot\lambda(dy) - \frac{D^-\Phi^{\uparrow}(x)}{\Phi^{\downarrow}(x_0)}\cdot\frac{\Phi^{\downarrow}(x)}{\Phi^{\uparrow}(x)}\cdot\lambda(dx)\\
&= \big[\alpha G(x) - b(x)D^-G(x)\big]\frac{2dx}{s^2(x)}\\
&\quad - \frac{\Phi^{\downarrow}(x)D^-\Phi^{\uparrow}(x) - \Phi^{\uparrow}(x)D^-\Phi^{\downarrow}(x)}{\Phi^{\uparrow}(x)\Phi^{\downarrow}(x_0)}\cdot\lambda(dx)
\end{aligned}$$

in conjunction with (5.11), so that

$$\big[\alpha G(x) - b(x)\cdot D^-G(x)\big]\frac{2dx}{s^2(x)} - D^2G(dx) = \frac{Bp'(x)}{\Phi^{\downarrow}(x_0)\Phi^{\uparrow}(x)}\cdot\lambda(dx) \quad \text{on } (\ell, x_0)$$

or equivalently,

$$\nu(dx) = \frac{Bp'(x)s^2(x)}{2\Phi^{\downarrow}(x_0)\Phi^{\uparrow}(x)}\cdot\lambda(dx) \qquad \text{on } (\ell, x_0) \tag{5.12}$$

in the notation of (5.1), where $B > 0$ is the Wronskian of (5.7).

The right-hand side of (5.12) is a positive, finite measure that *does not charge the optimal continuation region* $\mathscr{C}$ [recall (5.5)]. Thus ν is also a positive measure, and (5.2) holds. On the other hand, the representation (5.12) for the measure ν of (5.1) allows us to write the second-derivative measure of $G(\cdot)$ as

$$D^2G = \lambda_1 - \lambda_2, \tag{5.13}$$

the difference of two positive, finite measures

$$\lambda_1(dx) := f^+(x)\,dx, \qquad \lambda_2(dx) := f^-(x)\,dx + \frac{2}{s^2(x)}\cdot\nu(dx),$$

with $f(x) := 2[\alpha G(x) - b(x) \cdot D^- G(x)]/s^2(x)$, $f^{\pm}(x) := \max(\pm f(x), 0)$. This means that

$$D^- G(x) = D^- G(x_0) + \lambda_2([x, x_0]) - \lambda_1([x, x_0])$$

is the difference of two increasing functions on $[\ell, x_0]$, or in other words that $G(\cdot)$ is the difference of two convex functions. A similar argument establishes this same property on $[x_0, r]$, and completes the proof of Theorem 5.1. □

6. The discounted game. Let us take up again the stochastic game of Section 1, now in the *discounted case* $\alpha > 0$. We shall assume that the continuous function

$$u: [\ell, r] \to [0, \infty) \quad \text{is increasing} \tag{6.1}$$

(and nonnegative) and that there exists a pair of real-valued, measurable functions (b, s) on I, with the properties (2.1), (3.4) as well as

$$\frac{b(x)}{s^2(x)} = \inf\left\{ \frac{\beta}{\sigma^2} \Big/ (\beta, \sigma) \in \mathscr{K}(x) \right\}, \tag{6.2}$$

$$s^2(x) = \inf\{\sigma^2/(\beta, \sigma) \in \mathscr{K}(x) \quad \text{for some } \beta \in \Re\} \tag{6.3}$$

at every $x \in I$. For this pair (b, s) of local drift and volatility functions, let us consider the diffusion process $Z(\cdot)$ of (2.2) and recall the optimal reward function $G(\cdot)$ of (2.3). We have the following result.

THEOREM 6.1. *With the above assumptions and notation* (6.1)–(6.3), *the pair* $(Z(\cdot), \rho)$ *with*

$$\rho(\xi) := \inf\{t \geq 0/G(\xi(t)) = u(\xi(t))\}, \qquad \xi \in C[0, \infty) \tag{6.4}$$

is a saddlepoint for the stochastic game of Section 1, *and the value of this game is given by*

$$\underline{V}(x) = \overline{V}(x) = G(x), \qquad x \in I. \tag{6.5}$$

In other words, since the reward function $u(\cdot)$ is increasing, the *controller* (minimizer) always tries to "get as close to the left boundary point ℓ as possible," by minimizing the mean-variance or signal-to-noise ratio β/σ^2. At the same time, because of the discount factor $\alpha > 0$, he also tries to minimize the local variance (noise) σ^2, thus "slowing the process down and allowing the positive discount to reduce his expected cost." For his part, the *stopper* (maximizer) halts the game at the stopping time

$$\rho(X) = \inf\{t \geq 0/X(t) \in \Sigma\} \in \mathscr{M} \tag{6.6}$$

when the controlled process $X(\cdot)$ first enters into the optimal stopping region $\Sigma = \{x \in I/G(x) = u(x)\}$ of (2.8).

Conditions like (6.2) and (6.3) are also imposed by Sudderth and Weerasinghe (1989), who studied the problem of controlling a diffusion process to a goal on a fixed, finite time horizon. We shall conclude this section with a discussion of some examples, for which it is indeed possible to minimize simultaneously the variance σ^2 and the mean-variance ratio β/σ^2, as mandated by (6.2) and (6.3).

PROOF. We need to show the saddlepoint property

$$G(x) \le \mathbf{E}\big[e^{-\alpha\rho(X)} u\big(X(\rho(X))\big)\big] \qquad \forall\ X(\cdot) \in \mathscr{A}(x), \tag{6.7}$$

and all the claims of the theorem will follow. To this end, let us apply the generalized Itô rule for convex functions of semimartingales [e.g., Karatzas and Shreve (1988), Section 3.7] to the process

$$e^{-\alpha t} G(X(t)), \qquad 0 \le t < \infty.$$

[This is possible because, as we showed in Theorem 5.1, the function $G(\cdot)$ can be written as the difference of two convex functions.] The result is

$$\begin{aligned} e^{-\alpha T} G(X(T)) - G(x) = \int_0^T & e^{-\alpha t} D^- G(X(t))[\beta(t)\,dt + \sigma(t)\,dW(t)] \\ & - \alpha \int_0^T e^{-\alpha t} G(X(t))\,dt \\ & + \int_0^T \int_I e^{-\alpha t} D^2 G(dy) \cdot d_t\big(L_t^X(y)\big), \qquad 0 \le T < \infty. \end{aligned} \tag{6.8}$$

Here $t \mapsto L_t^X(y)$ is the *local time* of the semimartingale $X(\cdot)$ at the point $y \in I$: a continuous, increasing and $\mathbf{F}$-adapted process, flat off the set $\{t \ge 0/X(t) = y\}$. We may rewrite (6.8) as

$$\begin{aligned} & e^{-\alpha T} G(X(T)) - G(x) - \int_0^T e^{-\alpha t}\ D^- G(X(t))\sigma(t) dW(t) \\ & \quad = \int_0^T \int_I e^{-\alpha t} \left[D^2 G(dy) + \frac{2\beta(t)}{\sigma^2(t)} \cdot D^- G(y)\,dy - \frac{2\alpha}{\sigma^2(t)} G(y)\,dy \right] \cdot d_t\big(L_t^X(y)\big) \\ & \quad = \int_0^T \int_I e^{-\alpha t} \left[D^2 G(dy) + \frac{2b(y)}{s^2(y)} \cdot D^- G(y)\,dy - \frac{2\alpha}{s^2(y)} G(y)\,dy \right] \cdot d_t\big(L_t^X(y)\big) \\ & \qquad + 2 \int_0^T \int_I e^{-\alpha t} \left[\frac{\beta(t)}{\sigma^2(t)} - \frac{b(y)}{s^2(y)} \right] D^- G(y)\,dy \cdot d_t\big(L_t^X(y)\big) \\ & \qquad + 2\alpha \int_0^T \int_I e^{-\alpha t} \left(\frac{\sigma^2(t)}{s^2(y)} - 1 \right) \frac{G(y)\,dy}{\sigma^2(t)} \cdot d_t\big(L_t^X(y)\big) \end{aligned}$$

or equivalently in the form

$$\begin{aligned}
&e^{-\alpha T}G(X(T)) - G(x) - \int_0^T e^{-\alpha t} D^- G(X(t))\sigma(t)\,dW(t)\\
&\quad = \alpha \int_0^T e^{-\alpha t}\left(\frac{\sigma^2(t)}{s^2(X(t))} - 1\right) G(X(t))\,dt\\
&\qquad + \int_0^T e^{-\alpha t}\left[\frac{\beta(t)}{\sigma^2(t)} - \frac{b(X(t))}{s^2(X(t))}\right] D^- G(X(t))\cdot \sigma^2(t)\,dt\\
&\qquad + \int_0^T \int_I \frac{2e^{-\alpha t}}{s^2(y)}\big[(s^2(y)/2)\cdot D^2 G(dy)\\
&\qquad\qquad + \{b(y)D^- G(y) - \alpha G(y)\}\,dy\big]\cdot d_t\big(L_t^X(y)\big).
\end{aligned} \tag{6.9}$$

[Recall Theorem 3.7.1 from Karatzas and Shreve (1988) and its proof, particularly the last equation on page 224 and the first equation of page 225; see also Exercise 1.15, page 216 in Revuz and Yor (1991).]

Let us look at the three integrals on the right-hand side of (6.9). The first is nonnegative, because of conditions (6.3) and (6.1), which implies $G(\cdot) \ge 0$. The second of these integrals is also nonnegative, this time because of conditions (6.2) and (6.1), which also implies $D^- G(\cdot) \ge 0$. The last integral of (6.9), on the other hand, is equal to

$$-\int_I \left(\int_0^T 2e^{-\alpha t}\, d_t\big(L_t^X(y)\big)\right)\frac{\nu(dy)}{s^2(y)} \le 0$$

in the notation of Theorem 5.1; but this integral is *identically equal to zero on the event* $\{T \le \rho(X)\}$, where $\rho(X)$ is the stopping time of (6.6), again thanks to Theorem 5.1. It develops from this discussion that the continuous process

$$e^{-\alpha(t\wedge\rho(X))}G(X((t\wedge\rho(X)))), \qquad 0 \le t < \infty \tag{6.10}$$

is a local submartingale; it is also bounded, so it is, in fact, a (bounded) submartingale. Applying the optional sampling theorem to the process of (6.10), and recalling the definition of the stop-rule ρ from (6.4), we obtain

$$\begin{aligned}
G(x) &\le \mathbf{E}\big[e^{-\alpha\rho(X)}G(X(\rho(X)))\big]\\
&= \mathbf{E}\big[e^{-\alpha\rho(X)}G(X(\rho(X)))\cdot 1_{\{\rho(X)<\infty\}}\big]\\
&= \mathbf{E}\big[e^{-\alpha\rho(X)}u(X(\rho(X)))\cdot 1_{\{\rho(X)<\infty\}}\big]\\
&= \mathbf{E}\big[e^{-\alpha\rho(X)}u(X(\rho(X)))\big] \qquad \forall\ X(\cdot)\in\mathscr{A}(x).
\end{aligned} \tag{6.11}$$

This establishes (6.7) and completes the proof of Theorem 6.1. □

Notice that in this discounted case ($\alpha > 0$) there is no need to impose the condition (4.1), because in turn there is no need to guarantee $\mathbf{P}[\rho(X) < \infty] = 1$ in (6.11).

REMARK 6.1. Suppose that the function $u(\cdot)$ is continuous, nonnegative, satisfies the condition (4.2) and is increasing on $[\ell, m]$, decreasing on $[m, r]$. Suppose also that we can select pairs (b_ℓ, s_ℓ), (b_r, s_r) of measurable functions with the properties (2.1), (3.4) and $s_\ell(\cdot) \equiv s_r(\cdot) \equiv s(\cdot)$ as in (6.3), which satisfy (4.3), (4.4) for every $x \in I$. Then, with the notation of (4.5), (4.6) and with

$$G_j(x) := \sup_{\tau \in \mathscr{S}} \mathbf{E}\big[e^{-\alpha\tau(Z_j)} u(Z_j(\tau(Z_j)))\big], \qquad x \in I$$

for $j = \ell, r$, we have all the conclusions (i)–(iii) of Theorem 4.1.

Let us conclude with a look at two special cases, which illustrate Theorem 6.1 and also reveal some open problems.

CASE I. Suppose that the control sets are equal to a fixed rectangle, say

$$\mathscr{K}(x) = [\beta_1, \beta_2] \times [\sigma_1, \sigma_2], \qquad x \in I,$$

where $\beta_1 < \beta_2$, $0 < \sigma_1 < \sigma_2$. Recall that the controller (minimizer) prefers that the process move to the left, because we are assuming that the reward function $u(\cdot)$ is increasing. Thus, we can regard the game as being "superfair" from the controller's viewpoint, if the local drift coefficient can be chosen to be negative. This means $\beta_1 < 0$ and, if so, Theorem 6.1 applies with optimal controls $\beta(t) \equiv \beta_1$, $\sigma(t) \equiv \sigma_1$.

In the "subfair" case $\beta_1 > 0$, Theorem 6.1 does not apply because no pair (β, σ) can satisfy both (6.2) and (6.3). It still seems clear that the controller should take $\beta(t) \equiv \beta_1$, but the choice of $s(\cdot)$ is more delicate. [The process is drifting in the "wrong direction"; large values of $s(\cdot)$ mollify the drift as reflected in the scale function, whereas small values of $s(\cdot)$ slow the process down and allow the discount factor to reduce the controller's expected loss.] We suggest this case as an interesting open problem.

CASE II. Assume $I = (0, 1)$ and that the controlled process is of the form

$$dX(t) = \pi(t)[\beta_0\, dt + \sigma_0\, dW(t)].$$

Here $\beta_0 \neq 0$ and $\sigma_0 > 0$ are given real constants, and the controller chooses the $\mathbf{F}$-progressively measurable process $\pi(\cdot)$ subject to the constraint

$$\varepsilon X(t) \le \pi(t) \le X(t), \qquad 0 \le t < \infty,$$

for some given $0 < \varepsilon < 1$. Then the control sets are

$$\mathscr{K}(x) = \{(\pi\beta_0, \pi\sigma_0)/\varepsilon x \le \pi \le x\}, \qquad x \in I.$$

Here "superfairness" to the controller means $\beta_0 < 0$, in which case Theorem 6.1 yields the optimal control $\pi(t) = \varepsilon X(t)$; that is, $\beta(t) = \varepsilon\beta_0 X(t)$ and $\sigma(t) = \varepsilon\sigma_0 X(t)$. If $\beta_0 > 0$, we do not know an optimal strategy for the controller.

Acknowledgments. We thank the referee and Professor Paavo Salminen for their helpful comments on our work.

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