The Annals of Probability
2001, Vol. 29, No. 2, 862–881

LIMIT DISTRIBUTIONS OF NORMS OF VECTORS OF POSITIVE I.I.D. RANDOM VARIABLES[1]

BY MARTIN SCHLATHER

Lancaster University

This paper aims to combine the central limit theorem with the limit theorems in extreme value theory through a parametrized class of limit theorems where the former ones appear as special cases. To this end the limit distributions of suitably centered and normalized $l_{cp(n)}$-norms of n-vectors of positive i.i.d. random variables are investigated. Here, c is a positive constant and $p(n)$ is a sequence of positive numbers that is given intrinsically by the form of the upper tail behavior of the random variables. A family of limit distributions is obtained if c runs over the positive real axis. The normal distribution and the extreme value distributions appear as the endpoints of these families, namely, for $c = 0+$ and $c = \infty$, respectively.

1. Introduction. Combining the sum and the maximum of identically distributed random variables has been of interest to various authors. The joint distribution of the sum and the maximum has been investigated by Chow and Teugels (1979), Anderson and Turkman (1995), Hsing (1995) and Ho and Hsing (1996). The central limit theorem has been combined with the limit theorems in extreme value theory by Greenwood and Hooghiemstra (1991) and Hooghiemstra and Greenwood (1997). They study random processes of the form $X_n = \max\{X_{n-1},\ \alpha X_{n-1} + Y_n\}$, where the Y_i are i.i.d. random variables and α is a constant in $[0, 1)$. Obviously, if the Y_n are positive random variables, then X_n is a partial maximum for $\alpha = 0$ and a partial sum for $\alpha = 1$. Here a different approach that combines the limit theorems is suggested.

Our starting point is a reformulation of the central limit theorem. Let

$$|x_{1n}|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^P\right)^{1/p}, & 0 < p < \infty, \\ \max\{x_1, \dots, x_n\}, & p = \infty, \end{cases} \qquad \text{where } x_{1n} = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

be the l_p-(quasi-)norm of x_{1n} and let $X_{1n} = (X_1, \dots, X_n)$ be an n-vector of i.i.d. *positive* random variables. Then, the central limit theorem says that under certain conditions the suitably centered and normalized l_1-norm of X_{1n} converges to a Gaussian variable as $n \to \infty$; that is, there exist constants $a(n)$

Received June 1999; revised September 2000.
[1]Research supported by EU TMR Network ERB-FMRX-CT96-0095.
AMS 2000 *subject classifications*. Primary 60F05; secondary 60E07, 60G50, 60G70.
Key words and phrases. Central limit theorem, extreme value theory, i.i.d. positive random variables, *lp*-norm, limit theorems, normal distribution.

and $b(n)$, such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|X_{1n}|_1 - b(n)}{a(n)} < x\right) = \Phi(x),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt.$$

Similarly, the distribution of the suitably centered and normalized l_∞-norm of X_{1n} converges under certain conditions to a generalized extreme value distribution GEV as $n \to \infty$; that is, there exist constants $a'(n)$ and $b'(n)$, such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|X_{1n}|_\infty - b'(n)}{a'(n)} < x\right) = \mathrm{GEV}(x).$$

Depending on the form of the upper tail behavior of the random variable X_1, the generalized extreme value distribution GEV is the Weibull distribution Ψ_α with parameter $\alpha > 0$,

$$\Psi_\alpha(x) = \exp\bigl(-(-x)^\alpha\bigr), \qquad x < 0,$$

the Fréchet distribution Φ_α with parameter $\alpha > 0$,

$$\Phi_\alpha(x) = \exp(-x^{-\alpha}), \qquad x > 0,$$

or the Gumbel distribution Λ,

$$\Lambda(x) = \exp(e^{-x}), \qquad x \in \mathbb{R}.$$

For further details see Leadbetter, Lindgren and Rootzén (1983), Resnick (1987) or Embrechts, Klüppelberg and Mikosch (1997), for example.

This paper investigates the limit behavior of $|X_{1n}|_{cp(n)}$, where c can be chosen freely in $(0, \infty]$ and the sequence $p(n)$, $n \in \mathbb{N}$, is given intrinsically and depends on the form of the upper tail behavior of X_1. A family of limit distributions is obtained if c runs over the positive real axis. The normal distribution and the extreme value distributions appear as the endpoints of these families, namely, for $c = 0+$ and $c = \infty$, respectively. If $p(n) \to \infty$ as $n \to \infty$, then the central limit theorem can be interpreted as the limit theorem for a kind of diagonal sequence of the $l_{cp(n)}$-norms, namely, for constants $c = c(n)$ converging to 0 such that $c(n)p(n) = 1$. The investigation of l_p-norms is interesting on its own as such norms appear when self-normalized sums are considered; see Logan, Mallows, Rice and Shepp (1973), Griffin and Kuelbs (1991), Hahn and Weiner (1992), Horváth and Shao (1996) and Shao (1997), for example. In each of these papers, p is independent of n.

In Section 2, the limit theorems for random variables that belong to the domain of attraction of a Weibull distribution or a Fréchet distribution are presented. However, only conjectures are given if the random variables belong to the domain of attraction of the Gumbel distribution; a partial result is given for exponential random variables. Proofs are presented in Sections 3 to 7. In a brief final section some open problems are mentioned.

2. Main results. As in extreme value theory, the limit distributions of the $l_{cp(n)}$-norms depend on the form of the upper tail behavior of the random variable X_1. The cases where, with respect to the l_∞-norm, the random variable X_1 belongs to the domain of attraction of the Weibull distribution Ψ_α or the Fréchet distribution Φ_α are treated in Theorems 2.2 and 2.3, respectively. If X_1 belongs to the domain of attraction of the Gumbel distribution Λ, some conjectures are given; a partial result for exponential random variables is presented in Theorem 2.4. For a distribution function F, let $\overline{F} = 1 - F$ and let $F^{\leftarrow}$ be the pseudo-inverse of F. The domain of attraction of a distribution F with respect to the l_p-norm is denoted by $D_p(F)$.

The case where $p(n)$ is identically 1 and $X_1^c \in D_1(\Phi)$ is considered first.

THEOREM 2.1. *Let c be a positive constant and let $X_1, X_2, \dots$ be positive i.i.d. random variables with distribution function F, where $X_1^c \in D_1(\Phi)$. If*

$$a_c(n) = c^{-1} A(n)(n\mathbb{E}X_1^c)^{1/c-1} \quad \text{and} \quad b_c(n) = (n\mathbb{E}X_1^c)^{1/c},$$

where the sequence $A(n)$ satisfies

$$\frac{n}{A(n)^2}\int_{-\mathbb{E}X_1^c}^{A(n)} t^2\, dF\big((t+\mathbb{E}X_1^c)^{1/c}\big) \to 1, \tag{2.1}$$

then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|X_{1n}|_c - b_c(n)}{a_c(n)} < x\right) = \Phi(x).$$

If all the moments of X_1 exist, then the suitably centered and normalized l_c-norm of $|X_{1n}|$ converges to a normal variable for all positive, finite constants c.

THEOREM 2.2. *Let $X_1, X_2, \dots$ be positive i.i.d. random variables with distribution function $F \in D_\infty(\Psi_\alpha)$ and upper endpoint $x_F < \infty$. If $p(n) = x_F \times (x_F - F^{\leftarrow}(1-1/n))^{-1}$, then, for any positive constant c, there exist constants $a_c(n)$ and $b_c(n)$, $n \in \mathbb{N}$, such that the unique limit distribution*

$$F_c(x) = \lim_{n\to\infty} \mathbb{P}\left(\frac{|X_{1n}|_{cp(n)} - b_c(n)}{a_c(n)} < x\right) \tag{2.2}$$

exists and, furthermore,

$$F_c \to_w \Phi, \qquad c \to 0 \tag{2.3}$$

and

$$F_c \to_w \Psi_\alpha, \qquad c \to \infty. \tag{2.4}$$

For $c < 1$, the constants $a_c(n)$ and $b_c(n)$ may be chosen as

$$a_c(n) = x_F 2^{-\alpha/2} c^{\alpha/2-1} \Gamma(\alpha+1)^{-1/2} p(n)^{-1} \tag{2.5}$$

and

$$b_c(n) = x_F + x_F c^{-1} \log\bigl(\Gamma(\alpha+1)c^{-\alpha}\bigr)\, p(n)^{-1}, \tag{2.6}$$

respectively. For $c \geq 1$, *they may be chosen as*

$$a_c(n) = x_F\, p(n)^{-1} \quad \textit{and} \quad b_c(n) = x_F. \tag{2.7}$$

For $0 < c < \infty$, *the distribution function* F_c *equals*

$$F_c(x) = G_c(\exp(D_c x + d_c)), \tag{2.8}$$

where the Fourier transform of G_c *is given by*

$$G_c(u) = \exp\biggl(c^{-\alpha}\Gamma(\alpha+1) \sum_{k=1}^{\infty} \frac{(iu)^k}{k!k^\alpha}\biggr).$$

The constants d_c *and* D_c *equal*

$$d_c = \log\bigl(\Gamma(\alpha+1)c^{-\alpha}\bigr) \quad \textit{and} \quad D_c = 2^{-\alpha/2}c^{\alpha/2}\Gamma(\alpha+1)^{-1/2} \quad \textit{if } c < 1, \tag{2.9}$$

and

$$d_c = 0 \quad \textit{and} \quad D_c = c \quad \textit{if } c \geq 1.$$

The distinction between $c \geq 1$ and $c < 1$ in the theorem requires comment. The constants chosen in (2.7) are suitable to establish (2.2) for all positive constants c, but the density functions shift to the right as $c \to 0+$. Thus, the constants in (2.7) are not suitable to establish (2.3). For simplicity, a distinction in the norming and centering constants is made for $c \geq C$ and $c < C$, where C is chosen arbitrarily to be 1. Indeed, $a_c(n)$ and $b_c(n)$ might also be chosen as smooth functions in c for fixed n in order to establish (2.3).

THEOREM 2.3. *Let* $X_1, X_2, \ldots$ *be positive i.i.d. random variables with distribution function* $F \in D_\infty(\Phi_\alpha)$. *Then, for any positive constant* c, *there exist constants* $a_c(n)$ *and* $b_c(n)$, $n \in \mathbb{N}$, *such that the unique limit distribution*

$$F_c(x) = \lim_{n\to\infty} \mathbb{P}\biggl(\frac{|X_{1n}|_c - b_c(n)}{a_c(n)} < x\biggr)$$

exists and, furthermore,

$$F_c = \Phi, \qquad c \leq \alpha/2$$

and

$$F_c \to_w \Phi_\alpha, \qquad c \to \infty. \tag{2.10}$$

The constants $a_c(n)$ and $b_c(n)$ can be chosen according to the following table:

c	$a_c(n)$	$b_c(n)$
$c < \frac{\alpha}{2}$	$\frac{n^{1/c-1/2}\operatorname{Var}(X_1^c)^{1/2}}{c(\mathbb{E}X_1^c)^{1-1/c}}$	$(n\mathbb{E}X_1^c)^{1/c}$
$c = \frac{\alpha}{2}$	$\frac{A(n)}{c(n\mathbb{E}X_1^c)^{1-1/c}}$	$(n\mathbb{E}X_1^c)^{1/c}$
$\frac{\alpha}{2} < c < \alpha$	$\frac{\lvert\Gamma(1-\alpha/c)\rvert^{c/\alpha}\left(F^{\leftarrow}(1-1/n)\right)^c}{c\left(n\mathbb{E}X_1^c\right)^{1-1/c}}$	$(n\mathbb{E}X_1^c)^{1/c}$
$c = \alpha$	$\frac{(F^{\leftarrow}(1-1/n))^c}{c\left(nB(n)\right)^{1-1/c}}$	$\left(nB(n)\right)^{1/c}$
$\alpha < c$	$\Gamma(1-\alpha/c)^{1/\alpha}F^{\leftarrow}(1-1/n)$	0

Here, $A(n) = n^{1/2}(\operatorname{Var} X_1^c)^{1/2}$, *if* $\operatorname{Var} X_1^c$ *is finite, and satisfies*

$$\frac{2n}{A(n)^2}\int_0^{A(n)} L(x^{1/c})x^{-1}\,dx \to 1, \qquad n \to \infty,$$

otherwise. The function L is the slowly varying function given by the representation $\overline{F}(x) = x^{-\alpha}L(x)$. The function $B(n)$ equals

$$B(n) = \left(F^{\leftarrow}\left(1-\frac{1}{n}\right)\right)^c \int_0^{\infty} \sin\left(x\left[F^{\leftarrow}\left(1-\frac{1}{n}\right)\right]^{-c}\right) dF(x^{1/c}).$$

For $\alpha/2 < c \le \alpha$, the derivative of F_c equals the stable density

$$f_c(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(ck\alpha^{-1}+1)}{k!}x^{k-1}\sin\frac{k\pi c}{\alpha} \qquad \textit{if } \frac{\alpha}{2} < c < \alpha, \tag{2.11}$$

and

$$f_c(x) = \frac{1}{\pi}\int_0^{\infty} e^{-xu}u^{-u}\sin(\pi u)\,du \quad \textit{if } c = \alpha. \tag{2.12}$$

For $\alpha < c < \infty$, it equals the (*nonstable*) *density*

$$f_c(x) = \frac{c}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(c^{-1}k\alpha+1)}{k!}(-1)^{k+1}x^{-(\alpha k+1)}\sin\left(\frac{k\pi\alpha}{c}\right)\mathbf{1}_{x\ge 0}. \tag{2.13}$$

If $F \in D_\infty(\Lambda)$, the behavior of the $l_{cp(n)}$-norms is more complex. In analogy to Theorems 2.2 and 2.3, where the norming and centering constants as well as $p(n)$ involve the norming and centering constants of the extreme value theorems, we conjecture the following. Denote the finite or infinite upper endpoint of F by x_F and let

$$A(n) = n\int_{B(n)}^{x_F} \overline{F}(t)\,dt \quad \text{and} \quad B(n) = F^{\leftarrow}\left(1-\frac{1}{n}\right);$$

cf. Theorem 3.3.26 in Embrechts, Klüppelberg and Mikosch (1997). As, for $B(n) \neq 0$,

$$\frac{|X_{1n}|_{cp(n)} - B(n)}{A(n)} \leq x \Leftrightarrow |B(n)^{-1}X_{1n}|_{cp(n)} \leq A(n)B(n)^{-1}x + 1 \tag{2.14}$$

we define

$$p(n) = B(n)A(n)^{-1}$$

and conjecture that $p(n) \to \infty$ as $n \to \infty$. In the limit, the right-hand side of (2.14) is equivalent to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(B(n)^{-1}X_i\right)^{cp(n)} \leq e^{cx}. \tag{2.15}$$

We conjecture that the left-hand side of (2.15) has a proper distribution. Then $A(n)^{-1}(|X_{1n}|_{cp(n)} - B(n))$ converges to a random variable, Y_c say. Furthermore, we conjecture that Y_c converges to a Gumbel variable as $c \to \infty$, and a recentered and renormalized version of Y_c converges to a normal variable as $c \to 0$.

As an example of a random variable that belongs to the domain of attraction of the Gumbel distribution, an exponential variable is considered. The following theorem provides a partial result.

THEOREM 2.4. *Assume that the X_i are i.i.d. exponential random variables with distribution function $F(x) = (1-e^{-x})\mathbf{1}_{x\geq 0}$. Then the suitably centered and normalized $l_{c\log n}$-norms of X_{1n} converge in distribution to a normal variable if $0 < c < 1/2$, but not if $1/2 < c \leq 1/(2\log 2)$.*

3. Auxiliary results. The following theorem and the subsequent lemma are needed to prove convergence to the normal distribution in the theorems of Section 2. Namely, by combining both assertions, we get that, under certain conditions, $|X_{1n}|_{p_n}$ belongs to the domain of attraction of the Gaussian distribution, if $\sum_{i=1}^{n} X_i^{p_n}$ does.

THEOREM 3.1 [cf. Theorem 3.27 in Kallenberg (1997)]. *Let I be an interval in $\mathbb{R}$. Assume that the functions $h_n : I \to \mathbb{C}$ converge uniformly on compact sets to a continuous function h. If the real random variables X_n converge in distribution to a random variable X as $n \to \infty$ and $X \in I$ a.s., then $h_n(X_n)$ converges in distribution to $h(X)$.*

LEMMA 3.1. *Let p_n, $n \in \mathbb{N}$, be a nondecreasing sequence of positive numbers. If the positive numbers e_n and σ_n, $n \in \mathbb{N}$, are such that $\sigma_n e_n^{-1} \to 0$ as $n \to \infty$, then the complex valued functions*

$$h_n(x) = \frac{(\sigma_n x + e_n)^{1/p_n} - e_n^{1/p_n}}{p_n^{-1}\sigma_n e_n^{1/p_n - 1}}, \qquad n \in \mathbb{N},\ x \in \mathbb{R},$$

converge uniformly on compact sets to the identity as $n \to \infty$. So do the pseudo-inverses $h_n^{\leftarrow}$ of h_n.

PROOF. Let M be a positive number. Under the previous conditions the Taylor expansion of h_n yields

$$h_n(x) = x + O\biggl(\frac{\sigma_n(1-p_n)}{e_n p_n}\biggr) \quad \text{for} \quad |x| \le M \text{ and } n \to \infty. \qquad \square$$

The next lemma gives a convergence criterion for sequences of numbers with double indices. It can be proved readily by contradiction, for example. A function g is involved to allow for direct application in the following theorems.

LEMMA 3.2. *Let $v_{t,n}$ be real numbers for $n \in \mathbb{N}$ and $t \in \mathbb{R}$ and let $g : \mathbb{N} \to \mathbb{R}$ be a function that converges to ∞ as the argument tends to ∞. Assume that*

$$w_t = \lim_{n\to\infty} v_{t,n}$$

exists for all $t \in \mathbb{R}$. If $\lim_{n\to\infty} v_{\xi(n),n}$ exists and equals z for all functions $\xi : \mathbb{N} \to \mathbb{R}$ that are less than g, but that tend to ∞ as $n \to \infty$, then

$$\lim_{t\to\infty} w_t = z.$$

4. Proof of Theorem 2.1. Theorems XVII.5.3 and VIII.4.1 in Feller (1971) and Theorem 3.1 yield

$$\mathbb{P}\biggl(\frac{\sum_{i=1}^n X_i^c - n\mathbb{E}X_1^c}{A(n)} < x\biggr) \to \Phi(x), \qquad n \to \infty,$$

where $A(n)$ is given by (2.1). If $e_n = n\mathbb{E}X_1^c$ and $\sigma_n = A(n)$, then

$$\frac{\sigma_n}{e_n} \sim (\mathbb{E}X_1^c)^{-1} A(n)^{-1} \int_{-\mathbb{E}X_1^c}^{A(n)} t^2\, dF\bigl((t + \mathbb{E}X_1^c)^{1/c}\bigr) \tag{4.1}$$

by (2.1). The right-hand side of (4.1) tends to 0 as $n \to \infty$, since $A(n)$ converges to ∞ by (2.1) and $A \mapsto \int_{-\mathbb{E}X_1^c}^{A} t^2\, dF((t + \mathbb{E}X_1^c)^{1/c})$ is a slowly varying function by Theorem XVII.5.2 in Feller (1971). Thus, Lemma 3.1 and Theorem 3.1 can be applied and yield the assertion of the theorem.

5. Proof of Theorem 2.2. The following theorem recalls Potter's theorem in part (a) [cf. Theorem 1.5.6 in Bingham, Goldie and Teugels (1987)] and gives a modification thereof in part (b).

THEOREM 5.1 (Potter's theorem). *Let L be a slowly varying function.*

(a) *For any constants $A > 1$ and $\delta > 0$, there exists a constant $x_0 = x_0(A, \delta)$ such that*

$$\frac{L(y)}{L(x)} \le A \max\left\{\left(\frac{y}{x}\right)^{\delta}, \left(\frac{x}{y}\right)^{\delta}\right\} \qquad \text{for } x \ge x_0 \text{ and } y \ge x_0.$$

(b) *Let $h(x)$ be a positive function that tends to ∞ as $x \to \infty$. Then there exist two positive functions $\varepsilon_h(x)$ and $\eta_h(x)$, both tending to 0 as $x \to \infty$, such that for all real functions g_1 and g_2 with $g_1 \ge h$ and $g_2 \ge h$ the following inequality holds*:

$$\left|\frac{L(g_1(x))}{L(g_2(x))}\left(\frac{g_2(x)}{g_1(x)}\right)^{\xi(x)} - 1\right| \le \eta_h(x).$$

Here ξ is a suitable real function depending on g_1 and g_2 and satisfying the inequality $|\xi| \le \varepsilon_h$.

(c) *Let $q(x)$ be a positive function that tends to ∞ as $x \to \infty$. Then there exists a function $\zeta(x)$, tending to 0 as $x \to \infty$, and*

$$\frac{L(\zeta(x)q(x))}{L(q(x))} \to 1, \qquad x \to \infty.$$

PROOF. (b) The function L has the representation [see, e.g., Theorem 1.3.1 in Bingham, Goldie and Teugels (1987)]

$$L(x) = c(x)\exp\left(\int_z^x u^{-1}\delta(u)\,du\right) \qquad \text{for } x \ge z, \tag{5.1}$$

where $c(x) \to c \in (0, \infty)$ and $\delta(x) \to 0$ as $x \to \infty$, and $z > 0$.

In the following, assume that $x \ge z$ and, w.l.o.g., $\inf_{x \ge z} c(x) > 0$. Then

$$\frac{L(g_1(x))}{L(g_2(x))} = \frac{c(g_1(x))}{c(g_2(x))}\exp\left(\xi(x)\log\frac{g_1(x)}{g_2(x)}\right) \tag{5.2}$$

for some number $\xi(x)$ satisfying

$$\inf_{u \ge \min\{g_1(x),\, g_2(x)\}} \delta(u) \le \xi(x) \le \sup_{u \ge \min\{g_1(x),\, g_2(x)\}} \delta(u).$$

Thus,

$$\frac{L(g_1(x))}{L(g_2(x))}\left(\frac{g_2(x)}{g_1(x)}\right)^{\xi(x)} - 1 = \frac{c(g_1(x))}{c(g_2(x))} - 1.$$

and the assertion of part (b) holds if

$$\varepsilon_h(x) = \sup_{u \geq h(x)} |\delta(u)| \quad \text{and} \quad \eta_h(x) = \frac{\sup_{u \geq h(x)} c(u)}{\inf_{u \geq h(x)} c(u)} - 1.$$

(c) Assume w.l.o.g. that $q(x) > 1$ for all $x \in \mathbb{R}$. Let δ be given by (5.1), $h(x) = \sqrt{q(x)}$,

$$\zeta(x) = \max\left\{h(x)^{-1}, \sup_{u \geq h(x)} |\delta(u)|\right\},$$

$g_1(x) = \zeta(x)q(x)$ and $g_2(x) = q(x)$. Then, both g_1 and g_2 are bounded from below by h. The functions $\zeta(x)$ and $g_1(x)$ tend to 0 and ∞, respectively, as $x \to \infty$. If ε_h and ξ are given by part (b) and $\varepsilon_h(x) < 1$, then

$$\begin{aligned}\left|\xi(x) \log \frac{g_1(x)}{g_2(x)}\right| &= |\xi(x)| \min\{|\log(h(x)^{-1})|, |\log \varepsilon_h(x)|\} \\ &\leq \varepsilon_h(x) |\log \varepsilon_h(x)| \to 0, \qquad x \to \infty.\end{aligned}$$

Equality (5.2) provides the assertion. □

The proof of Theorem 2.2 is mainly based on the following theorem.

THEOREM 5.2 [Theorem 4.1.3 and the references in Bingham, Goldie and Teugels (1987)]. *Let L be a slowly varying, locally bounded function of order $O(1)$ at $0+$. Let k be a real function and let δ be a positive constant such that $\int_0^\infty t^\sigma |k(t)|\, dt$ converges for $-\delta \leq \sigma \leq \delta$. Then,*

$$\int_0^\infty k(t) L(xt)\, dt \sim L(x) \int_0^\infty k(t)\, dt, \qquad x \to \infty.$$

Let $k(t) = t^{-2-\alpha} e^{-1/t}$, $\alpha > 0$, and $U(t) = t^{-\alpha} L(t)$, where L is a slowly varying, locally bounded function of order $O(1)$ at $0+$. Then, Theorem 5.2 yields

$$\int_0^\infty \frac{e^{-1/t}}{t^2} U(\beta t)\, dt = \beta^{-\alpha} \int_0^\infty k(t) L(\beta t)\, dt \sim \Gamma(\alpha+1) U(\beta), \quad \beta \to \infty. \tag{5.3}$$

LEMMA 5.1. *Let X be a positive random variable with distribution function $F \in D_\infty(\Psi_\alpha)$ and assume that the upper endpoint x_F equals 1.*

(a) $\mathbb{E} X^\beta \sim \Gamma(\alpha+1) \overline{F}(1 - 1/\beta)$ *as* $\beta \to \infty$; *that is,* $\beta \mapsto \mathbb{E} X^\beta$ *is a regularly varying function with index* $-\alpha$.

(b) *Let* $p(n) = (1 - F^{\leftarrow}(1 - 1/n))^{-1}$ *and let* c_0 *be a fixed positive number. Then, there exist a constant C and positive functions $\varepsilon(n)$, $\eta(n)$ and $\delta(n)$, $n \in \mathbb{N}$, all of them converging to 0 as $n \to \infty$, such that, for all $c \geq c_0$,*

$$\left| n \mathbb{E} X^{cp(n)} - \frac{\Gamma(\alpha+1)}{c^\alpha} \right| \leq \delta(n) + C\big[\exp\big(|\log c| \varepsilon(n)\big)\big(1 + \eta(n)\big) - 1\big].$$

(c) *Let* $p(n) = (1 - F^{\leftarrow}(1-1/n))^{-1}$. *If* $c(n)$, $n \in \mathbb{N}$, *is a sequence of numbers, such that* $c(n) \to 0$ *and* $c(n)p(n) \to \infty$ *as* $n \to \infty$, *then*

$$n\mathbb{E}X^{c(n)p(n)} \to \infty, \qquad n \to \infty.$$

PROOF. (a) As F belongs to the domain of attraction of Ψ_α, it has the representation $\overline{F}(1 - 1/x) = x^{-\alpha}L(x)$, where L is a slowly varying function. Thus, it follows directly from the asymptotic equivalence that $\beta \mapsto \mathbb{E}X^\beta$ is regularly varying with index $-\alpha$.

For $\beta > 0$, the expectation of X^β equals

$$\begin{aligned}\mathbb{E}X^\beta &= \beta\int_0^1 x^{\beta-1}\overline{F}(x)\,dx = \beta\int_1^\infty \left(1 - \frac{1}{x}\right)^{\beta-1} x^{-2}\overline{F}\left(1 - \frac{1}{x}\right)dx \\ &= \int_0^\infty \frac{e^{-1/x}}{x^2}\overline{F}^*(1 - \beta^{-1}x^{-1})\,dx \\ &\quad + \beta\int_1^\infty \frac{(1-1/x)^{\beta-1} - e^{-\beta/x}}{x^2}\overline{F}\left(1 - \frac{1}{x}\right)dx,\end{aligned} \tag{5.4}$$

where $\overline{F}^*(1 - 1/y) = \overline{F}(1 - 1/y)\mathbf{1}_{y\geq 1}$. The first term on the right-hand side of (5.4) is asymptotically equivalent to $\Gamma(\alpha+1)\overline{F}(1-1/\beta)$ by (5.3). The second term is asymptotically negligible since

$$\begin{aligned}&\left|\overline{F}\left(1 - \frac{1}{\beta}\right)^{-1}\beta\int_1^\infty \frac{(1-1/x)^{\beta-1} - e^{-\beta/x}}{x^2}\overline{F}\left(1 - \frac{1}{x}\right)dx\right| \\ &\quad = \left|\frac{\beta^{\alpha+1}}{L(\beta)}\int_1^{\sqrt{\beta}} \frac{(1 - \frac{1}{x})^{\beta-1} - e^{-\beta/x}}{x^2}\overline{F}\left(1 - \frac{1}{x}\right)dx\right. \\ &\qquad \left. + \int_0^{\sqrt{\beta}}\left[\left(1 - \frac{x}{\beta}\right)^{\beta-1} - e^{-x}\right]\frac{x^\alpha L(\beta/x)}{L(\beta)}\,dx\right|.\end{aligned}$$

As the functions $x \mapsto (1 - 1/x)^{\beta-1}$ and $x \mapsto e^{-\beta/x}$ take their maxima within $[1, \sqrt{\beta}]$ at $\sqrt{\beta}$ the right-hand side is less than or equal to

$$\begin{aligned}&\frac{\beta^{\alpha+1}}{L(\beta)}\left((1 - \beta^{-1/2})^{\beta-1} + e^{-\sqrt{\beta}}\right) \\ &\qquad + \int_0^{\sqrt{\beta}} e^{-x}\left|\exp\left[x + (\beta - 1)\log\left(1 - \frac{x}{\beta}\right)\right] - 1\right|\frac{x^\alpha L(\beta/x)}{L(\beta)}\,dx.\end{aligned}$$

The first term tends to 0 as $\beta \to \infty$. So does the second, since, by Taylor's expansion,

$$\sup_{x\in[0,\sqrt{\beta}]}\left|\exp\left[x + (\beta - 1)\log\left(1 - \frac{x}{\beta}\right)\right] - 1\right| \to 0, \qquad \beta \to \infty,$$

and $\int_0^{\sqrt{\beta}} e^{-x}x^\alpha L(\beta/x)/L(\beta)\,dx$ is uniformly bounded by Potter's theorem [Theorem 5.1(a)] for β sufficiently large.

(b) Part (a) of the lemma states that

$$\frac{\mathbb{E}X^{\beta}}{\overline{F}(1-1/\beta)} = \frac{\beta^{\alpha}\mathbb{E}X^{\beta}}{L(\beta)} \to \Gamma(\alpha+1), \qquad \beta \to \infty. \tag{5.5}$$

The assumption that $p(n) = (1 - F^{\leftarrow}(1-1/n))^{-1}$ is equivalent to

$$p(n)^{\alpha} = nL\big(p(n)\big). \tag{5.6}$$

Let $\beta = cp(n)$; combining (5.6) with (5.5) yields first

$$\Gamma(\alpha+1) - nc^{\alpha}\frac{L\big(p(n)\big)}{L\big(cp(n)\big)}\mathbb{E}X^{cp(n)} \to 0, \qquad n \to \infty,$$

but then also

$$\delta(n) = c_0^{-\alpha}\sup_{c \ge c_0}\left|\Gamma(\alpha+1) - nc^{\alpha}\frac{L\big(p(n)\big)}{L\big(cp(n)\big)}\mathbb{E}X^{cp(n)}\right| \to 0, \quad n \to \infty, \tag{5.7}$$

since the convergence rate is determined by the lower bound c_0. An application of Potter's theorem [Theorem 5.1(a)] to $L(p(n))/L(cp(n))$ in equality (5.7) provides finite constants n_0 and $C = C(c_0, n_0)$, such that $n\mathbb{E}X^{cp(n)}$ is uniformly bounded by C for $n \ge n_0$ and $c \ge c_0$. Theorem 5.1(b) provides positive functions $\eta(n)$ and $\varepsilon(n)$, $n \in \mathbb{N}$, which tend to 0 as $n \to \infty$, such that

$$\exp\big(|\log[L\big(p(n)\big)/L\big(cp(n)\big)]|\big) \le \exp\big(|\log c|\varepsilon(n)\big)\big(1+\eta(n)\big).$$

Then, for $n \ge n_0$ and $c \ge c_0$, the triangular inequality yields

$$\begin{aligned}\delta(n) &\ge \left|\frac{\Gamma(\alpha+1)}{c^{\alpha}} - n\mathbb{E}X^{cp(n)}\right| - n\left|\frac{L\big(p(n)\big)}{L\big(cp(n)\big)} - 1\right|\mathbb{E}X^{cp(n)} \\ &\ge \left|\frac{\Gamma(\alpha+1)}{c^{\alpha}} - n\mathbb{E}X^{cp(n)}\right| - C\Big(\exp\big(|\log c|\varepsilon(n)\big)\big(1+\eta(n)\big) - 1\Big).\end{aligned}$$

The second inequality is true as, for arbitrary $u > 0$, the inequality $|u-1| \le |e^{|\log u|} - 1|$ holds.

(c) Part (a) of the lemma and equality (5.6) give

$$\begin{aligned}n\mathbb{E}X^{c(n)p(n)} &\sim \Gamma(\alpha+1)nc(n)^{-\alpha}p(n)^{-\alpha}L\big(c(n)p(n)\big) \\ &= \Gamma(\alpha+1)c(n)^{-\alpha}\frac{L\big(c(n)p(n)\big)}{L\big(p(n)\big)},\end{aligned} \tag{5.8}$$

and the assertion follows from Potter's theorem, Theorem 5.1(a). □

PROOF OF THEOREM 2.2. As $p(n) = x_F(x_F - F^{\leftarrow}(1-1/n))^{-1}$ is invariant to any rescaling of X_1 and

$$\frac{|X_{1n}|_{cp(n)} - b(n)}{a(n)} = \frac{|X_{1n}/x_F|_{cp(n)} - b(n)/x_F}{a(n)/x_F}, \tag{5.9}$$

it can be assumed w.l.o.g. that $x_F = 1$. Subsequently, we first show that

$$Y_{c,n} = \frac{|X_{1n}|_{cp(n)} - 1}{1 - F^{\leftarrow}(1 - 1/n)} = p(n)\big(|X_{1n}|_{cp(n)} - 1\big) \tag{5.10}$$

has a nondegenerate limit distribution H_c for all nonnegative finite constants c. Then we demonstrate that

$$H_c \to_w \Psi_\alpha, \qquad c \to \infty. \tag{5.11}$$

Finally, we show that suitably rescaled versions of H_c tend to the normal distribution as $c \to 0$.

1. As $\mathbb{E}X_1^m \le 1$ for all $m \ge 0$, the Taylor expansion for the characteristic function $\widehat{F}_{c,n}$ of $X_1^{cp(n)}$ is valid. Lemma 5.1(b) yields

$$\widehat{F}_{c,n}(u) = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!} \mathbb{E}X_1^{kcp(n)} = 1 + \frac{\Gamma(\alpha+1)}{c^\alpha n} \sum_{k=1}^{\infty} \frac{(iu)^k}{k!k^\alpha} + R_n(u), \tag{5.12}$$

where the remainder R_n satisfies

$$\begin{aligned} |nR_n(u)| \le &\sum_{k=1}^{\infty} \frac{|u|^k}{k!k^\alpha} \delta(n) \\ &+ C \sum_{k=1}^{\infty} \frac{|u|^k}{k!k^\alpha} \Big(k^{\varepsilon(n)} \exp\big(|\log c|\varepsilon(n)\big)\big(1+\eta(n)\big) - 1 \Big). \end{aligned} \tag{5.13}$$

The remainder $R_n(u)$ is of order $o(1/n)$ for fixed u, since $k^{-\alpha}(k^{\varepsilon(n)} e^{|\log c|\varepsilon(n)}$ $(1+\eta(n)) - 1)$ is uniformly bounded in k and n for n large enough, and converges to 0 for fixed k as $n \to \infty$. Hence, the characteristic function of $\sum_{i=1}^n X_1^{cp(n)}$ converges pointwise to

$$\widehat{G}_c(u) = \lim_{n\to\infty} \bigg(1 + \frac{\Gamma(\alpha+1)}{c^\alpha n} \sum_{k=1}^{\infty} \frac{(iu)^k}{k!k^\alpha}\bigg)^n = \exp\bigg(c^{-\alpha}\Gamma(\alpha+1) \sum_{k=1}^{\infty} \frac{(iu)^k}{k!k^\alpha}\bigg),$$

which is continuous at the origin; that is, its inverse Fourier transform G_c is a probability distribution. Let $p_k(n) = k^{-1/\alpha} p(n)$; then $(1 - p_k(n)^{-1})^{cp(n)} \to \exp(-ck^{1/\alpha}) > 0$ as $n \to \infty$ and

$$\mathbb{P}\big(X_1 > 1 - p_k(n)^{-1}\big) = kp(n)^{-\alpha} L\big(k^{-1/\alpha} p(n)\big) \sim \frac{k}{n}$$

by (5.6). Let $\xi_n = |\{j \in \{1, \dots, n\} : X_j > 1 - p_k(n)^{-1}\}|$. If $k \ge 4$, then

$$\lim_{n\to\infty} \mathbb{P}\Bigg(\sum_{i=1}^n X_i^{cp(n)} < \frac{\exp(-ck^{1/\alpha})}{2}\Bigg) \le \lim_{n\to\infty} P(\xi_n = 0) \le \frac{2}{k}$$

by Chebyshev's inequality. This shows that $G_c(\{0\}) = 0$. If we choose $I = (0, \infty)$ and $h(x) = c^{-1} \log x$ in Theorem 3.1, we get

$$H_c(x) = \lim_{n\to\infty} \mathbb{P}\big(p(n)(|X_{1n}|_{cp(n)} - 1) < x\big) = G_c(e^{cx}). \tag{5.14}$$

2. In order to show (5.11), it suffices to apply Lemma 3.2 for $v_{t,n} = \mathbb{P}(p(n)(|X_{1n}|_{tp(n)} - 1) \le x) = \mathbb{P}(Y_{t,n} \le x)$, $w_t = H_t(x)$, $z = \Psi_\alpha(x)$ and a specific distribution function $F \in D_\infty(\Psi_\alpha)$. For simplicity, we choose $F(x) = 1 - (1 - x)^\alpha$ for $x \in [0, 1]$, $p(n) = n^{1/\alpha}$ and $g(n) = \log n$; see Lemma 3.2. If $\beta(n) = c(n)p(n)$, then Lemma 3.2 states that (5.11) is implied by

$$\lim_{n\to\infty} \mathbb{P}\big(p(n)(|X_{1n}|_{\beta(n)} - 1) \le x\big) = \Psi_\alpha(x) \tag{5.15}$$

for all $x \in \mathbb{R}$ and all sequences $c(n) \le \log(n)$ that tend to ∞ as $n \to \infty$. Subsequently, we show that (5.15) holds. Let $M(n) = \max\{X_1, \dots, X_n\}$. As

$$\begin{aligned}
&\mathbb{P}\big(p(n)(M(n) - 1) \le x\big)\\
&\quad \ge \mathbb{P}\big(p(n)(|X_{1n}|_{\beta(n)} - 1) \le x\big)\\
&\quad = \mathbb{P}\Big(p(n)\big[M(n) - 1\big] + M(n)p(n)\big[|X_{1n}|_{\beta(n)}/M(n) - 1\big] \le x\Big)\\
&\quad \ge \mathbb{P}\Big(p(n)\big[M(n) - 1\big] + p(n)\big[|X_{1n}|_{\beta(n)}/M(n) - 1\big] \le x\Big)
\end{aligned}$$

and

$$1 \le |X_{1n}|_{\beta(n)}/M(n) \le n^{1/\beta(n)} \to 1, \qquad n \to \infty,$$

equality (5.15) is equivalent to [Theorem 25.4 in Billingsley (1995)]

$$\begin{aligned}
1 &= \lim_{n\to\infty} \mathbb{P}\big(p(n)(|X_{1n}|_{\beta(n)}/M(n) - 1) < \varepsilon\big) \quad \text{for all } \varepsilon > 0\\
&= \lim_{n\to\infty} \mathbb{P}\left(c(n)^{-1} \log \sum_{i=1}^n (X_i/M(n))^{\beta(n)} < \varepsilon\right)\\
&= \lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^n X_i^{\beta(n)} < M(n)^{\beta(n)} e^{\varepsilon c(n)}\right).
\end{aligned}$$

Let $\varepsilon \in (0, 1)$, $\xi \ge (-\log(\varepsilon/3))^{1/\alpha}$,

$$R_n = \sum_{\substack{i=1\\ \xi+\log n \le p(n)(1-X_i)}}^n X_i^{\beta(n)}, \quad S_n = \sum_{\substack{i=1\\ p(n)(1-X_i)\le\xi}}^n X_i^{\beta(n)}, \quad T_n = \sum_{\substack{i=1\\ \xi< p(n)(1-X_i)<\xi+\log n}}^n X_i^{\beta(n)}.$$

Let N_S and N_T be the random number of summands of S_n and T_n, respectively. Then, by (5.6),

$$\operatorname{Var} N_S \sim \mathbb{E} N_S \sim \xi^\alpha$$

and

$$\operatorname{Var} N_T \sim \mathbb{E} N_T \sim (\xi + \log n)^\alpha.$$

Assume in the following that n is large enough; then $R_n \le n(1 - (\xi + \log n)/p(n))^{\beta(n)} \sim n^{1-c(n)} e^{-\xi c(n)}$,

$$\mathbb{E}\frac{S_n}{M(n)^{\beta(n)}} \le 2\xi^\alpha$$

and

$$\operatorname{Var} \frac{S_n}{M(n)^{2\beta(n)}} \le \mathbb{E} N_S + \operatorname{Var} N_s \le 3\xi^\alpha$$

Formula (8.357) in Gradshteyn and Ryzhik (2000) for the incomplete gamma function yields

$$\begin{aligned}\mathbb{E} T_n &= n\alpha \int_{1-n^{-1/\alpha}(\xi+\log n)}^{1-n^{-1/\alpha}\xi} x^{\beta(n)}(1-x)^{\alpha-1}\,dx \\ &\le 2\alpha \int_{\xi}^{\xi+\log n} e^{-c(n)y} y^{\alpha-1}\,dy \le \frac{3\alpha\xi^{\alpha-1}}{c(n)} e^{-c(n)\xi}.\end{aligned}$$

Similarly, $\operatorname{Var} T_n \le Cc(n)^{-1}e^{-2c(n)\xi}$ for a suitable finite constant $C = C(\xi, \alpha)$. Furthermore, $M(n) \ge 1 - p(n)^{-1}\xi$. Thus $M(n)^{\beta(n)} \ge e^{-\xi c(n)}/2$, with probability at least $1 - \varepsilon/2$, and

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i^{\beta(n)} < M(n)^{\beta(n)} e^{\varepsilon c(n)}\right) \\ &\quad \ge \lim_{n\to\infty} \mathbb{P}\left(e^{-\xi c(n)} + S_n + T_n \le M(n)^{\beta(n)} e^{\varepsilon c(n)} \mid M(n) > 1 - \frac{\xi}{p(n)}\right)\left(\frac{1-\varepsilon}{2}\right) \\ &\quad \ge 1-\varepsilon\end{aligned}$$

by Chebyshev's inequality and the asymptotic independence of S_n and T_n.

3. In the following we show that a rescaled version of H_c converges to the normal distribution Φ. To this end, Lemma 3.2 is applied for $v_{1/t,n} = \mathbb{P}(p(n)(|X_{1n}|_{tp(n)} - 1) \le x)$, $w_{1/t} = H_t(x)$, $z = \Phi(x)$, $t \to 0$ and $F(x) = 1 - (1-x)^\alpha$ for $x \in [0, 1]$. Let $\beta(n)$, $n \in \mathbb{N}$, be an arbitrary sequence converging to ∞ and let τ be an arbitrary positive number. The inequality $x^{2\beta(n)} > \tau^2 n \mathbb{E} X_1^{2\beta(n)}/2$ is equivalent to

$$\left(x^{\beta(n)} - \mathbb{E} X_1^{\beta(n)}\right)^2 > \tau^2 n \operatorname{Var}\left(X_1^{\beta(n)}\right) + \tau^2 n \mathbb{E} X_1^{2\beta(n)}\left(-1/2 + f_n(x)\right),$$

where $f_n(x) = (-2x^{\beta(n)}\mathbb{E} X_1^{\beta(n)} + (\tau^2 n + 1)(\mathbb{E} X_1^{\beta(n)})^2)/(\tau^2 n \mathbb{E} X_1^{2\beta(n)})$. Since $|f_n(x)| \to 0$ as $n \to \infty$ for $x \in [0, 1]$, by Lemma 5.1(a) we have

$$\begin{aligned}&\left\{x \in [0,1] : \left(x^{\beta(n)} - \mathbb{E} X_1^{\beta(n)}\right)^2 > \tau^2 n \operatorname{Var} X_1^{\beta(n)}\right\} \\ &\quad \subset \left\{x \in [0,1] : x^{2\beta(n)} > \tau^2 n \mathbb{E} X_1^{2\beta(n)}/2\right\}\end{aligned} \tag{5.16}$$

for n large enough. If $c(n)$ converges to 0 and $\beta(n) = c(n)p(n) = c(n)n^{1/\alpha}$ tends to ∞ as $n \to \infty$, then Lemma 5.1(c) yields that the right-hand side of (5.16) is empty for n large enough, and the central limit theorem holds for $\sum_{i=1}^n X_i^{\beta(n)}$ [see, e.g., Theorem 49.2 in Gnedenko (1963)]. Lemma 5.1(b) yields that, for any fixed positive constant $\zeta_0 < 1$,

$$\sup_{\zeta_0 \le c \le 1} \log\left(\frac{n\mathbb{E}X_1^{cp(n)}}{\Gamma(\alpha+1)c^{-\alpha}}\right) \to 0, \qquad n \to \infty.$$

Let $n_0(1) = 1$ and $n_0(k)$ be such that $n_0(k) > n_0(k-1)$ and

$$k^{\alpha/2} \sup_{k^{-1} \le c \le 1} \log\left(\frac{n\mathbb{E}X_1^{cp(n)}}{\Gamma(\alpha+1)c^{-\alpha}}\right) < k^{-1} \quad \text{for all } n \ge n_0(k) \text{ and } k \ge 2.$$

Define $\zeta_1(n) = k^{-1}$ for all n with $n_0(k) \le n < n_0(k+1)$ and $k \ge 1$. Then,

$$c(n)^{-\alpha/2} \log\left(\frac{n\mathbb{E}X_1^{c(n)p(n)}}{\Gamma(\alpha+1)c(n)^{-\alpha}}\right) \to 0, \qquad n \to \infty, \tag{5.17}$$

for all sequences $c(n) \to 0$ such that $\zeta_1(n) \le c(n)$ for all $n \in \mathbb{N}$. The sequence $\zeta(n)$,

$$\zeta(n) = \max\left\{\zeta_1(n),\, p(n)^{-1/(2+\alpha)}\right\}, \qquad n \in \mathbb{N}, \tag{5.18}$$

tends to 0 as $n \to \infty$, and $\zeta(n)p(n) \to \infty$ as $n \to \infty$. In the following let $c(n)$ be any sequence tending to 0 as $n \to \infty$ such that $c(n) \ge \zeta(n)$ for all $n \in \mathbb{N}$. Let

$$a_c(n) = 2^{-\alpha/2} c^{\alpha/2-1} \Gamma(\alpha+1)^{-1/2} p(n)^{-1}$$

and

$$b_c(n) = 1 + c^{-1} p(n)^{-1} \log(\Gamma(\alpha+1)c^{-\alpha}).$$

Then, by Lemma 5.1(a),

$$a_c(n) \sim c^{-1} p(n)^{-1} \sqrt{n \operatorname{Var} X_1^{cp(n)}} \left(n\mathbb{E}X_1^{cp(n)}\right)^{c^{-1}p(n)^{-1}-1}.$$

The Taylor expansion for the exponential function and relation (5.17) yield

$$\begin{aligned} &a_{c(n)}(n)^{-1}\left(n\mathbb{E}X_1^{\beta(n)}\right)^{1/\beta(n)} \\ &\quad = a_{c(n)}(n)^{-1} + a_{c(n)}(n)^{-1}\left[\exp\left(\beta(n)^{-1}\log\left(n\mathbb{E}X_1^{\beta(n)}\right)\right) - 1\right] \\ &\quad = a_{c(n)}(n)^{-1} + 2^{\alpha/2}\Gamma(\alpha+1)^{1/2} c(n)^{-\alpha/2} \log\left(\Gamma(\alpha+1)c(n)^{-\alpha}\right) + o(1) \\ &\quad = b_{c(n)}(n)/a_{c(n)}(n) + o(1). \end{aligned}$$

Since

$$\sqrt{n \operatorname{Var} X_1^{\beta(n)}} \Big/ \left(n\mathbb{E}X_1^{\beta(n)}\right) \sim c(n)^{\alpha/2}\Gamma(\alpha+1)^{-1/2} 2^{-\alpha/2} \to 0, \qquad n \to \infty,$$

by (5.8) and $\sum_{i=1}^{n} X_i^{\beta(n)}$ obeys the c.l.t. as shown previously, Lemma 3.1 and Theorem 3.1 yield

$$\mathbb{P}\left(\frac{|X_{1n}|_{\beta(n)} - b_{c(n)}(n)}{a_{c(n)}(n)} < x\right) \to \Phi(x), \qquad n \to \infty, \quad \text{for all } x \in \mathbb{R}.$$

On the other hand, the distribution of

$$\frac{|X_{1n}|_{cp(n)} - b_c(n)}{a_c(n)} = \frac{Y_{c,n} - c^{-1}d_c}{c^{-1}D_c},$$

where

$$d_c = \log(\Gamma(\alpha+1)c^{-\alpha}) \quad \text{and} \quad D_c = 2^{-\alpha/2}c^{\alpha/2}\Gamma(\alpha+1)^{-1/2},$$

tends to $G(\exp(D_c x + d_c))$ as $n \to \infty$. Since the distribution of $Y_{c,n}$ tends to H_c for any distribution function $F \in D_\infty(\Psi_\alpha)$, the coefficients d_c and D_c can be chosen independently of F. Then the coefficients a_c and b_c are given by (2.5) and (2.6) in the general case. □

6. Proof of Theorem 2.3. If $c \le \alpha/2$. Theorem 2.1 can be applied and yields directly the statement for $c < \alpha/2$. If $c = \alpha/2$, let $\widetilde{A}(n) = A(n) + \mathbb{E}X_1^c$ and assume that the variance of X_1 is not finite. Partial integration of (2.1) and changing to the integration measure $d(1-F)$ give

$$\frac{n}{A(n)^2}\left(\left(\mathbb{E}X_1^c\right)^2 - \tfrac{A(n)^2}{\widetilde{A}(n)^2}L\left(\widetilde{A}(n)^{1/c}\right) + 2\int_0^{\widetilde{A}(n)} \frac{t - \mathbb{E}X_1^c}{t^2} L\left(t^{1/c}\right)dt\right) \to 1, \qquad n \to \infty. \tag{6.1}$$

Since $L(\widetilde{A}(n)^{1/c})/\int_0^{\widetilde{A}(n)} t^{-1}L(t^{1/c})\,dt \to 0$ [Bingham, Goldie and Teugels (1987), relation (1.5.8)] and $\int_0^\infty t^{-2}L(t^{1/c})\,dt = \int_0^\infty \overline{F}(t^{1/c})\,dt$ is finite, condition (6.1) is satisfied if and only if

$$\frac{2n}{A(n)^2}\int_0^{\widetilde{A}(n)} \frac{L(t^{1/c})}{t}\,dt \to 1, \qquad n \to \infty. \tag{6.2}$$

Since $t^{-1}L(t^{1/c}) \to 0$ as $t \to \infty$, the upper bound of the integral in (6.2) can be replaced by $A(n)$.

In the following we consider the case $c > \alpha/2$. Theorem 1.8.1 in Samorodnitsky and Taqqu (1994) states that $(|X_{1n}^c|_1 - \tilde{b}_c(n))/\tilde{a}_c(n)$ converges to a stable variable with characteristic function

$$\Psi(\zeta) = \begin{cases} \exp\left(-|\zeta|^{\alpha/c}\exp\left(-(\operatorname{sign}\zeta)i\pi\left(\frac{\alpha}{2c} - 1\right)\right)\right), & \text{if } \frac{\alpha}{2} < c < \alpha, \\ \exp\left(-|\zeta|\left[\frac{\pi}{2} + (\operatorname{sign}\zeta)i\log|\zeta|\right]\right), & \text{if } c = \alpha, \\ \exp\left(-|\zeta|^{\alpha/c}\left(-(\operatorname{sign}\zeta)i\pi\frac{\alpha}{2c}\right)\right), & \text{if } c > \alpha, \end{cases}$$

where

$$\tilde a_c(n)=\left[F^{\leftarrow}\left(1-\frac{1}{n}\right)\right]^c\left|\Gamma\left(1-\frac{\alpha}{c}\right)\right|^{c/\alpha},\quad \tilde b_c(n)=n\mathbb{E}X_1^c,\qquad \text{if } \frac{\alpha}{2}<c<\alpha,$$

$$\tilde a_c(n)=\left[F^{\leftarrow}\left(1-\frac{1}{n}\right)\right]^c,\quad \tilde b_c(n)=n\tilde a_c(n)\times\int_0^\infty \sin\left(\frac{x}{\tilde a_c(n)}\right)dF(x^{1/c}),\quad \text{if } c=\alpha,$$

$$\tilde a_c(n)=\left[F^{\leftarrow}\left(1-\frac{1}{n}\right)\right]^c\Gamma\left(1-\frac{\alpha}{c}\right)^{c/\alpha},\quad \tilde b_c(n)=0,\qquad \text{if } c>\alpha.$$

If $c\le\alpha$, we apply Theorem 3.1 and Lemma 3.1 with $e_n=\tilde b_c(n)$, $\sigma_n=\tilde a_c(n)$ and $p_n=c$. Namely, $\sigma_n/e_n\to 0$ by Potter's theorem if $c<\alpha$. If $c=\alpha$, we have $\tilde a_c(n)=nL(\tilde a_c(n)^{1/c})$ and

$$\begin{aligned}\frac{\tilde b_c(n)}{\tilde a_c(n)}&\ge n\left(\int_0^{\tilde a_c(n)}\sin\frac{x}{\tilde a_c(n)}\,dF(x^{1/c})-\overline{F}\big(\tilde a_c(n)^{1/c}\big)\right)\\&\ge -1-\sin(1)+\frac{1}{L\big(\tilde a_c(n)^{1/c}\big)}\int_0^{\tilde a_c(n)}\frac{L(x^{1/c})}{x}\cos\frac{x}{\tilde a_c(n)}\,dx\\&\ge -2+\frac{0.5}{L\big(\tilde a_c(n)^{1/c}\big)}\int_0^{\tilde a_c(n)}\frac{L(x^{1/c})}{x}\,dx.\end{aligned}$$

Thus, $\tilde a_c(n)/\tilde b_c(n)\to 0$ by relation (1.5.8) in Bingham, Goldie and Teugels (1987). The density functions (2.11) and (2.12) follow from (XVII.6.1) and Lemma XVII.6.1 in Feller (1971) and from equality (14.32) in Sato (1999), respectively. If $c>\alpha$, then

$$\big(|X_{1n}^c|_1-\tilde b_c(n)\big)/\tilde a_c(n)=\big(|X_{1n}|_c/\tilde a_c(n)^{1/c}\big)^c$$

and suitable norming and centering constants for $|X_{1n}|_c$ are $a_c(n)=\tilde a_c(n)^{1/c}$ and $b_c(n)=0$. As the limit density function of $(|X_{1n}^c|_1-\tilde b_c(n))/\tilde a_c(n)$ equals

$$g_c(x)=\frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{\Gamma(c^{-1}k\alpha+1)}{k!}(-1)^{k+1}x^{-\alpha k/c}\sin\left(\frac{k\pi\alpha}{c}\right)\mathbf{1}_{x\ge 0}$$

by Lemma XVII.6.1 in Feller (1971), the coordinate transformation $x'=x^{1/c}$ yields equality (2.13). Integrating both sides of (2.13) gives

$$F_c(x)=1+\sum_{k=1}^{\infty}\frac{\Gamma(c^{-1}k\alpha+1)}{k!}(-x^{-\alpha})^k\frac{c\sin(k\pi\alpha/c)}{k\pi\alpha}$$

and the right-hand side tends to Φ_α as $c\to\infty$. This proves (2.10). □

7. Proof of Theorem 2.4. Stirling's formula [formula 6.1.37 in Abramowitz and Stegun (1984)] yields

$$\Gamma(1+c\log n)\sim\sqrt{2\pi}(1+c\log n)^{c\log(n)+1/2}e^{-c\log(n)-1}. \tag{7.1}$$

It follows that

$$\mathbb{E}X_1^{c\log n}=\Gamma(1+c\log n)\sim\sqrt{2c\pi}\sqrt{\log n}\,n^{c(\log(c)-1+\log\log n)},\quad n\to\infty \tag{7.2}$$

and

$$\operatorname{Var}X_1^{c\log n}\sim\mathbb{E}X_1^{2c\log n}\sim 2\sqrt{c\pi}\sqrt{\log n}\,n^{2c(\log(2c)-1+\log\log n)},\quad n\to\infty. \tag{7.3}$$

Then

$$\frac{\sigma_n}{e_n}=\frac{\sqrt{n\operatorname{Var}X_1^{c\log n}}}{n\mathbb{E}X_1^{c\log n}}\sim\frac{n^{c\log 2-1/2}}{(c\pi\log n)^{1/4}},\qquad n\to\infty,$$

and Lemma 3.1 holds if $c\le 1/(2\log 2)$. It remains to show that $\sum_{i=1}^n X_i^{c\log n}$ obeys the c.l.t. for $c<1/2$ but not for $c>1/2$. Lindeberg's condition [see Theorem 49.2 in Gnedenko (1963)] requires that, for any positive constant τ,

$$\begin{aligned}K(n)=\operatorname{Var}(X^{c\log n})^{-1}\int_{|x^{c\log n}-\mathbb{E}X_1^{c\log n}|>\tau\sqrt{n\operatorname{Var}X_1^{c\log n}}}^{\infty}&\left(x^{c\log n}-\mathbb{E}X_1^{c\log n}\right)^2\\ &\times e^{-x}\,dx\to 0,\qquad n\to\infty.\end{aligned} \tag{7.4}$$

Equalities (7.2) and (7.3) imply that there exist functions $\eta(n)=\eta(n,\tau)$ and $\xi(n)$ which tend to 1 and $(4c\pi)^{-1/2}$, respectively, as $n\to\infty$, such that

$$K(n)=\xi(n)(\log n)^{-1/2}n^{-2c(\log(2c)-1+\log\log n)}\int_{T\log n}^{\infty}x^{2c\log n}e^{-x}\,dx,$$

where

$$T=T(n)=2ce^{1/(2c)-1}\eta(n).$$

Here it has been assumed that

$$\begin{aligned}\left(n\mathbb{E}X_1^{2c\log n}\right)^{1/(2c\log n)}&=\left(n\Gamma(1+2c\log n)\right)^{1/(2c\log n)}\\ &\sim 2ce^{1/(2c)-1}\log n,\qquad n\to\infty,\end{aligned}$$

and that both $x_n^{c\log n}/\mathbb{E}X_1^{c\log n}$ for any sequence $x_n\ge T(n)\log n$ and $\sqrt{n\operatorname{Var}X^{c\log n}}/\mathbb{E}X_1^{c\log n}$ tend to ∞ as $n\to\infty$.

If $c<1/2$, then $e^{1/(2c)-1}>1$ and $T(n)$ is finally greater than $2c+2/\log n$ as $n\to\infty$. As the function $x\mapsto x^{2+2c\log n}e^{-x}$ is decreasing for $x>2+2c\log n$, it follows that, for large n,

$$\begin{aligned}K(x)\le{}&\xi(n)(\log n)^{-1/2}n^{-2c(\log(2c)-1+\log\log n)}\\ &\times(T\log n)^{2+2c\log n}e^{-T\log n}\int_{T\log n}^{\infty}x^{-2}\,dx.\end{aligned}$$

The prefactor of the integral equals

$$\xi(n)T^2(\log n)^{3/2}n^{2c\log T-T-2c(\log(2c)-1)}.$$

As the function $t\mapsto 2c\log t-t-2c(\log(2c)-1)$ takes its unique maximum at $t=2c$ that values 0 and as T is finally greater than $2c$, the prefactor converges to 0 as $n\to\infty$; that is, Lindeberg's condition (7.4) is satisfied.

If $c>1/2$, then $e^{1/(2c)-1}<1$ and T is finally smaller than $2c$. As the function $x\mapsto x^{2c\log n}e^{-x}$ takes its minimum within the interval $[2c\log n, 2c\log n+\sqrt{\log n}]$ at the upper endpoint, it follows that, for n large enough,

$$\begin{aligned}K(n)&\geq \xi(n)(\log n)^{-1/2}n^{-2c(\log(2c)-1+\log\log n)}\int_{2c\log n}^{2c\log n+\sqrt{\log n}}x^{2c\log n}e^{-x}\,dx\\&\geq \xi(n)n^{-2c(\log(2c)-1+\log\log n)}\Big(2c\log n+\sqrt{\log n}\Big)^{2c\log n}e^{-2c\log n-\sqrt{\log n}}\\&=\xi(n)\left(1+\frac{\sqrt{\log n}}{2c\log n}\right)^{2c\log n}e^{-\sqrt{\log n}}.\end{aligned}$$

The right-hand side converges to $(4c\pi)^{-1/2}e^{-1/(4c)}$ as $n\to\infty$ and Lindeberg's condition does not hold.

8. Open problems. The paper leaves some questions open. Clearly the behavior of the $l_{cp(n)}$-norms in the Gumbel case remains to be investigated. Discrete positive random variables with upper endpoint obey the central limit theorem if they are not deterministic. However, they lead to trivial limit distributions with respect to l_∞-norms. Their behavior with regard to $l_{cp(n)}$-norms is not known yet. Furthermore, it is unclear how to generalize our approach to multivariate or nonpositive random variables.

Acknowledgment. The author is very grateful to Jonathan Tawn for many hints and discussions.

REFERENCES

ABRAMOWITZ, M. and STEGUN, I. A. (1984). *Pocketbook of Mathematical Functions*. Harri Deutsch, Frankfurt am Main.

ANDERSON, C. W. and TURKMAN, K. F. (1995). Sums and maxima of stationary sequences with heavy tailed distributions. *Sankhyā Ser. A* **57** 1–10.

BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. Wiley, New York.

BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. Cambridge Univ. Press.

CHOW, T. L. and TEUGELS, J. (1979). The sum and the maximum of i.i.d. random variables. In *Proceedings of the Second Prague Symposium on Asymptotic Statistics* (P. Mandl and M. Huskova, eds.) 81–92. North-Holland, Amsterdam.

EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events*. Springer, Berlin.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

GNEDENKO, B. W. (1963). *The Theory of Probability*, 2nd ed. Chelsea, New York.

GRADSHTEYN, I. S. and RYZHIK, I. M. (2000). *Table of Integrals, Series, and Products*, 6th ed. Academic, London.

GREENWOOD, P. E. and HOOGHIEMSTRA, G. (1991). On the domain of attraction of an operator between supremum and sum. *Probab. Theory Related Fields* **89** 201–210.

GRIFFIN, P. and KUELBS, J. (1991). Some extensions of the LIL via self-normalisations. *Ann. Probab.* **19** 380–395.

HAHN, M. G. and WEINER, D. C. (1992). Asymptotic behaviour of self-normalized trimmed sums: nonnormal limits. *Ann. Probab.* **20** 455–482.

HO, H.-C. and HSING, T. (1996). On the asymptotic joint distribution of the sum and maximum of stationary normal random variables. *J. Appl. Probab.* **33** 138–145.

HOOGHIEMSTRA, G. and GREENWOOD, P. E. (1997). The domain of attraction of the α-sun operator for type II and type III distributions. *Bernoulli* **3** 479–489.

HORVÁTH, L. and SHAO, Q.-M. (1996). Large deviations and law of the iterated logarithm for partial sums normalized by the largest absolute observation. *Ann. Probab.* **24** 1368–1387.

HSING, T. (1995). A note on the asymptotic independence of the sum and the maximum of strongly mixing stationary random variables. *Ann. Probab.* **23** 938–947.

KALLENBERG, O. (1997). *Foundations of Modern Probability*. Springer, New York.

LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, Berlin.

LOGAN, B. F., MALLOWS, C. L., RICE, S. O. and SHEPP, L. A. (1973). Limit distributions of self-normalized sums. *Ann. Probab.* **1** 788–809.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes. Appl. Probab.* **4**. Springer, New York.

SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman and Hall, Boca Raton, FL.

SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press.

SHAO, Q.-M. (1997). Self-normalized large deviations. *Ann. Probab.* **25** 285–327.

DEPARTMENT OF MATHEMATICS AND STATISTICS
LANCASTER UNIVERSITY
LANCASTER LA1 4YF
UNITED KINGDOM
E-MAIL: martin.schlather@uni-bayreuth.de