The Annals of Probability
2001, Vol. 29, No. 2, 683–690

ULAM'S PROBLEM AND HAMMERSLEY'S PROCESS

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Let L_n be the length of the longest increasing subsequence of a random permutation of the numbers $1, \dots, n$, for the uniform distribution on the set of permutations. Hammersley's interacting particle process, implicit in Hammersley (1972), has been used in Aldous and Diaconis (1995) to provide a "soft" hydrodynamical argument for proving that $\lim_{n\to\infty} EL_n/\sqrt{n} = 2$. We show in this note that the latter result is in fact an immediate consequence of properties of a random 2-dimensional signed measure, associated with Hammersley's process.

1. Introduction. Let L_n be the length of the longest increasing subsequence of a random permutation of the numbers $1, \dots, n$, for the uniform distribution on the set of permutations. It was proved in Hammersley (1972) that, as $n \to \infty$,

$$L_n/\sqrt{n} \xrightarrow{p} c,$$

where $\xrightarrow{p}$ denotes convergence in probability and

$$\lim_{n\to\infty} EL_n/\sqrt{n} = c,$$

for some positive constant c, where $\pi/2 \le c \le e$. Subsequently Kingman (1973) showed that

$$1.59 < c < 2.49$$

and later work by Logan and Shepp (1977) and Vershik and Kerov (1977) showed that actually $c = 2$. The problem of proving that the limit exists and finding the value of c has been called "Ulam's problem;" see, for example, Deift (2000), page 633. In Aldous and Diaconis (1995) the hard combinatorial work in Logan and Shepp (1977) and Vershik and Kerov (1977), using Young tableaux, is replaced by what they call a "soft hydrodynamical argument," using Hammersley's interacting particle process that is implicit in Hammersley (1972). Another hydrodynamical argument, also based on Hammersley's interacting particle process is given in Seppäläinen (1996). Both Aldous and Diaconis (1995) and Seppäläinen (1996) use arguments, based on coupling Hammersley's process on $\mathbb{R}_+$ with a stationary version of this process, starting on $\mathbb{R}$ instead of $\mathbb{R}_+$.

The purpose of the present note is to provide an "even softer" argument for $c = 2$, only using almost sure convergence of a random signed measure,

Received July 2000; revised October 2000.
AMS 2000 *subject classifications*. Primary 60C05, 60K35; secondary 60F05.
Key words and phrases. Longest increasing subsequence, Ulam's problem, Hammersley's process.

associated with Hammersley's interacting particle process. To avoid a possible misunderstanding, we note here that we [and no doubt also Aldous and Diaconis (1995)] use the word "soft" in analogy with its use in "soft analysis" versus "hard analysis." We do not intend to say that the argument is non-rigorous! The advantage of using almost sure convergence instead of convergence of expectations is that we do not have to do any work on finding upper bounds of moments, but instead can use purely probabilistic arguments, only using the subadditive ergodic theorem and the continuity theorem for almost sure convergence.

The following gives an intuitive description of Hammersley's process on $\mathbb{R}_+$, developing according to the rules specified in Aldous and Diaconis (1995). Start with a Poisson point process of intensity 1 on $\mathbb{R}_+^2$. Now shift the positive x-axis vertically through (a realization of) this point process and, each time a point is caught, shift to this point the previously caught point that is immediately to the right.

Alternatively, if one finds it hard to imagine that the process gets started, because of the infinitely many jumps it has to make to get away from zero (and continues to make at each positive time), imagine an interval $[0, x]$, moving vertically through the Poisson point process. If this interval catches a point that is to the right of the points caught before, a new extra point is created in $[0, x]$, otherwise we have a shift to this point of the previously caught point that is immediately to the right and belongs to $[0, x]$. The number of points, resulting from this "catch and shift" procedure at time y on the interval $[0, x]$, is denoted in Aldous and Diaconis (1995) by

$$N^+(x, y), \qquad x, y \geq 0.$$

So the process evolves in time according to "Rule 1" in Aldous and Diaconis (1995), which is repeated here for ease of reference.

RULE 1. *At times of a Poisson* (*rate* x) *process in time, a point* U *is chosen uniformly on* $[0, x]$*, independent of the past and the particle nearest to the right of* U *is moved to* U*, with a new particle created at* U *if no such particle exists in* $[0, x]$.

For further details, see Aldous and Diaconis (1995) and for a picture, see Figure 1. Now, in contrast with the approach in Aldous and Diaconis (1995) and Seppäläinen (1996), we consider Hammersley's process as a point process in the plane instead of considering it as a 1-dimensional counting process in one argument, keeping the other argument fixed. By doing this the inherent symmetry of Hammersley's process can be used to our advantage. In fact, $N^+(x, y)$ equals the number of points of the Poisson process in the plane, contained in $[0, x] \times [0, y]$ minus the number of corners of the space-times trajectories of Hammersley's process where we have a turn to the left (corresponding to a shift of the corresponding point to the left). Let us call the points of the Poisson process in the plane "α-points" and the locations of corners of the space-times curves of Hammersley's process, corresponding to a left turn,

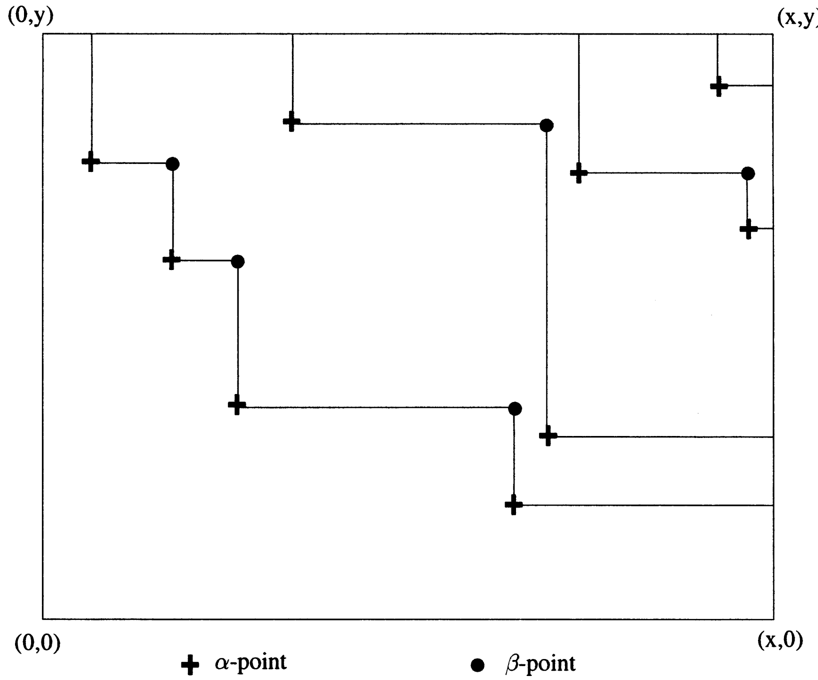

FIG. 1. *Space-time curves of Hammersley's process, contained in* $[0, x] \times [0, y]$.

"β-points." Then we can define a random signed measure ξ on the Borel sets B of $\mathbb{R}_+^2$ by

$$\xi(B) = \text{ number of } \alpha\text{-points in } B \text{ minus number of } \beta\text{-points in } B.$$

We shall write

$$\xi(B) = \int_B dN(x, y), \qquad B \in \mathscr{B}.$$

Note that $N^+(x, y) = N(x, y) \stackrel{\text{def}}{=} \int_{[0,x]\times[0,y]} dN(x, y) \geq 0$, for all $x, y \geq 0$, but that the corresponding measure $dN(x, y)$ can assign negative values to a set B (depending on the difference of the number of α- and β-points, contained in B). The situation is illustrated in Figure 1, where the number of α-points and β-points is 9 and 5, respectively, and $N(x, y) = 4$.

In a similar way we can associate a random measure $d\xi_t = dV_t(x, y)$ with the process

$$(x, y) \mapsto V_t(x, y) = t^{-1}N(tx, ty), \qquad x, y \geq 0. \tag{1.1}$$

We get

$$\begin{aligned} \xi_t(B) &= \int_B dV_t(x, y) \\ &= t^{-1} \times \{ \text{ number of } \alpha\text{-points in } tB \text{ minus number of } \beta\text{-points in } tB\}, \end{aligned}$$

where the set tB is defined by

$$tB = \{(tx, ty) : (x, y) \in B\}.$$

Our rescaling of ξ to ξ_t is similar to the rescaling, used in Seppäläinen (1996). However, he uses a capital N, where we use t, since we want to denote by N a (random) measure.

A by now standard application of the subadditive ergodic theorem shows

$$V_t(x, y) \xrightarrow{\text{a.s.}} V(x, y) \stackrel{\text{def}}{=} c\sqrt{xy},$$

for all $x, y \geq 0$; see, for example, Durrett [(1991), Example 7.2, Chapter 6], where c is the constant, discussed above. Hence, using a continuity argument, we also have

$$\begin{aligned}\int \phi(x, y)\, dV_t(x, y) &= \int \frac{\partial^2}{\partial x \partial y}\phi(x, y)\; V_t(x, y)\, dx\, dy \\ &\xrightarrow{\text{a.s.}} \int \frac{\partial^2}{\partial x \partial y}\phi(x, y)\; V(x, y)\, dx\, dy \\ &= \int \phi(x, y) \frac{c}{4\sqrt{xy}}\, dx\, dy, \qquad t \to \infty,\end{aligned}$$

for all smooth "test functions" $\phi : \mathbb{R}_+^2 \to \mathbb{R}$ with compact support.

We can also associate a random measure $dV_t(x, y)^2$ with the process

$$V_t(x, y)^2 = t^{-2} N(tx, ty)^2, \qquad x, y \geq 0. \tag{1.2}$$

It is shown in the next section that

$$dV_t(x, y)^2 \sim 2V_t(x, y)\, dV_t(x, y) + 2\, dx\, dy, \qquad t \to \infty,$$

in a sense to be specified there. This will yield the conclusion that $c = 2$.

2. The proof. In the following we disregard events happening with probability zero, like the event that two points of the Poisson point process in a rectangle $[0, a] \times [0, b]$ have the same $x-$ (or $y-$) coordinate (in fact, for events of this type, the space-time curves of Hammersley's process are not even well-defined).

We now first define

$$V_t^+(x, y) = t^{-1} \times \{\text{number of } \alpha\text{-points in } [0, tx] \times [0, ty]\}$$

and

$$V_t^-(x, y) = t^{-1} \times \{\text{number of } \beta\text{-points in } [0, tx] \times [0, ty]\}.$$

With these definitions we clearly have

$$V_t(x, y) = V_t^+(x, y) - V_t^-(x, y).$$

Moreover, we define $\widetilde{V}_t(x, y)$ by

$$\widetilde{V}_t(x, y) = \int_{[0,x]\times[0,y)} dV_t(u, v),$$

so we omit the upper edge of the rectangle $[0, x] \times [0, y]$ (we can also omit the right edge of this rectangle, but not both edges, as will be clear from the sequel!). Finally we define the measure $dV_t(x, y)^2$ by

$$\int_B dV_t(x, y)^2 = t^{-1} \int_B d\left(V_t^+ + V_t^-\right)(x, y) + 2 \int_B \widetilde{V}_t(x, y)\, dV_t(x, y), \tag{2.1}$$

for Borel sets $B \subset \mathbb{R}_+^2$. With these definitions we have the following lemma.

LEMMA 2.1. *For each rectangle* $B = [0, x] \times [0, y]$ *we have*

$$V_t(x, y)^2 = t^{-1}\left\{V_t^+(x, y) + V_t^-(x, y)\right\} + 2 \int_B \widetilde{V}_t(u, v)\, dV_t(u, v). \tag{2.2}$$

PROOF. Suppose (as we may) that the boundary of the rectangle $[0, tx] \times [0, ty]$ does not contain α- or β-points. Further suppose that there are m space-time curves, going through the rectangle $[0, tx] \times [0, ty]$ [meaning that $N(tx, ty) = m$].

Crossing the space-time curves, going from $(0, 0)$ in a North-East direction, we can number these paths as $P_1, P_2, \dots, P_m$, where P_1 is the path, closest to the origin. Then, for an α-point (u, v) on P_i, we get $\widetilde{V}_t(u, v) = (i-1)/t$, and for a β-point (u, v) on P_i, we get $\widetilde{V}_t(u, v) = i/t$ (here the fact that we omit the upper edge of the rectangle $[0, u] \times [0, v]$ becomes important!). Let A_1 be the set of α-points and A_2 be the set of β-points, contained in $[0, tx] \times [0, ty]$, respectively. Then we get:

$$\begin{aligned}
&\int_{B(x,y)} \widetilde{V}_t(u, v)\, dV_t(u, v) \\
&\qquad = t^{-2} \sum_{i=1}^{m} \{(i-1)\#\{\alpha\text{-points on } P_i\} - i\#\{\beta\text{-points on } P_i\}\} \\
&\qquad = t^{-2} \sum_{i=1}^{m} (i-1)\{\#\{\alpha\text{-points on } P_i\} - \#\{\beta\text{-points on } P_i\}\} - t^{-2}\#A_2.
\end{aligned}$$

But for each space-time curve P_i, contained in $[0, tx] \times [0, ty]$, we have

$$\#\{\alpha\text{-points on } P_i\} - \#\{\beta\text{-points on } P_i\} = 1.$$

So we get

$$\begin{aligned}
\int_{B(x,y)} \widetilde{V}_t(u, v)\, dV_t(u, v) &= t^{-2}\left\{\sum_{i=1}^{m}(i-1) - \#A_2\right\} \\
&= t^{-2}\{m(m-1)/2 - \#A_2\} \\
&= \tfrac{1}{2} V_t(x, y)\left\{V_t(x, y) - t^{-1}\right\} - t^{-2}\#A_2 \\
&= \tfrac{1}{2} V_t(x, y)^2 - \tfrac{1}{2} t^{-1} V_t(x, y) - t^{-2}\#A_2 \\
&= \tfrac{1}{2} V_t(x, y)^2 - \tfrac{1}{2} t^{-2}\{\#A_1 + \#A_2\} \\
&= \tfrac{1}{2} V_t(x, y)^2 - \tfrac{1}{2} t^{-1} \int_{[0,x]\times[0,y]} d\left(V_t^+ + V_t^-\right)(u, v). \qquad \square
\end{aligned}$$

We shall show that, for $B(x, y) = [0, x] \times [0, y]$, the three terms in Lemma 2.1 converge almost surely to the three terms in the relation

$$c^2 xy = 2xy + \tfrac{1}{2} c^2 xy, \tag{2.3}$$

and this equation identifies c as 2.

By the continuity theorem for almost sure convergence we get

$$V_t(x, y)^2 \xrightarrow{\text{a.s.}} c^2 xy,$$

and, again taking $B(x, y) = [0, x] \times [0, y]$, we get:

$$\begin{aligned} t^{-1} \int_{B(x,y)} & d\left(V_t^+ + V_t^-\right)(u, v) \\ &= 2t^{-1} V_t^+(x, y) - t^{-1}\left\{V_t^+(x, y) - V_t^-(x, y)\right\} \\ &= 2t^{-1} V_t^+(x, y) - t^{-1} V_t(x, y) \\ &\xrightarrow{\text{a.s.}} 2xy, \qquad t \to \infty, \end{aligned}$$

since $t^{-1} V_t(x, y) \xrightarrow{\text{a.s.}} 0$, because $V_t(x, y) \xrightarrow{\text{a.s.}} c\sqrt{xy}$, $t \to \infty$, and since

$$\begin{aligned} 2t^{-1} V_t^+(x, y) &= 2t^{-2} \#\{\text{points of the Poisson point process in}[0, tx] \times [0, ty]\} \\ &\xrightarrow{\text{a.s.}} 2xy, \end{aligned}$$

because $V_t^+(x, y)$ only counts the α-points in $[0, tx] \times [0, ty]$, which are just the points of the Poisson point process in $[0, tx] \times [0, ty]$.

We can write

$$\int_{B(x,y)} \widetilde{V}_t(u, v)\, dV_t(u, v) = \int_{u \le u' \le x,\, v < v' \le y} dV_t(u, v)\, dV_t(u', v').$$

But the measure $dV_t(u, v)\, dV_t(u', v')$ converges almost surely, in the vague topology for measures on the Borel sets of $\mathbb{R}_+^4$, to the measure $dV(u, v) dV(u', v')$ on $\mathbb{R}_+^4$, where $dV(u, v) dV(u', v')$ is defined by

$$\int \phi(u, v, u', v')\, dV(u, v)\, dV(u', v') = \int \phi(u, v, u', v') \frac{c^2}{16\sqrt{uvu'v'}}\, du\, dv\, du'\, dv',$$

for smooth test functions $\phi(u, v, u', v')$ with compact support. For let ϕ be such a smooth test function, then

$$\begin{aligned} \int & \phi(u, v, u', v')\, dV_t(u, v)\, dV_t(u', v') \\ &= \int \frac{\partial^4}{\partial u \partial v \partial u' \partial v'} \phi(u, v, u', v')\, V_t(u, v)\, V_t(u', v')\, du\, dv\, du'\, dv' \\ &\xrightarrow{\text{a.s.}} \int \frac{\partial^4}{\partial u \partial v \partial u' \partial v'} \phi(u, v, u', v')\, c^2 \sqrt{uv}\sqrt{u'v'}\, du\, dv\, du'\, dv' \\ &= \int \phi(u, v, u', v')\, \frac{c^2}{16\sqrt{uvu'v'}}\, du\, dv\, du'\, dv'. \end{aligned}$$

Since the limiting measure does not give mass to the boundary of the set $A(x, y) \stackrel{\text{def}}{=} \{(u, v, u', v') \in \mathbb{R}_+^4 : u \le u' \le x,\ v < v' \le y\}$, we now also have

$$\begin{aligned}\int_{B(x,y)} \widetilde{V}_t(u, v)\, dV_t(u, v) &= \int_{A(x,y)} dV_t(u, v)\, dV_t(u', v') \\ &\stackrel{\text{a.s.}}{\longrightarrow} \int_{A(x,y)} dV(u, v)\, dV(u', v') \\ &= \int_{A(x,y)} \frac{c^2}{16\sqrt{uvu'v'}}\, du\, dv\, du'\, dv' \\ &= \int_{B(x,y)} V(u, v)\, dV(u, v) \\ &= \frac{1}{4} c^2 xy.\end{aligned}$$

So we get (2.3), since the left side of (2.2) converges almost surely to $c^2 xy$, and the right side converges almost surely to $2xy + \frac{1}{2}c^2 xy$. Thus we obtain $c = 2$.

Since the almost sure convergence of $V_t(x, y)$ to $c\sqrt{xy}$ implies the convergence in probability of $L_n/\sqrt{n}$ to c, where L_n is the length of the longest increasing subsequence of a (uniform) random permutation of the numbers $1, \dots, n$ [this connection was the motivation of the results in Hammersley (1972)], we now also have

$$L_n/\sqrt{n} \stackrel{p}{\longrightarrow} 2,$$

and since the constant c is the same for the result on the expectations, it also follows that

$$\lim_{n\to\infty} EL_n/\sqrt{n} = 2.$$

3. Concluding remarks. In Section 2 it was proved that $c = 2$, by using almost sure convergence of certain random signed measures, thereby providing a purely probabilistic argument for this fact, essentially only using the subadditive ergodic theorem. Recent work of Baik, Deift and Johansson (1999) has shown that (in our notation), for fixed $x, y > 0$,

$$t^{2/3}\{V_t(x, y) - 2\sqrt{xy}\} \stackrel{\mathscr{D}}{\longrightarrow} Z(x, y), \qquad t \to \infty,$$

where $\stackrel{\mathscr{D}}{\longrightarrow}$ denotes convergence in distribution, and where the random variable $Z(x, y)$ is distributed as a (rescaled) Tracy-Widom distribution, which can be characterized in terms of the solution of a Painlevé II equation and Airy functions, or, alternatively, as the limit distribution of the largest eigenvalue of certain (Gaussian) random Hermitian matrices. Nice overviews of this and other recent work in the area are given in Aldous and Diaconis (1999) and Deift (2000).

These distribution results have been obtained by analytic tools, and rely in particular on the analysis of a Riemann-Hilbert problem. It is tempting to think that a more purely probabilistic approach to these results is possible, of which the preceding proof might give a hint. In any case, the transition from the (positive) counting process N^+ of Aldous and Diaconis (1995) to the corresponding 2-dimensional signed measure dN, with the property $\int_{[0,x]\times[0,y]} dN(x, y) = N^+(x, y)$, might be a first step in this direction.

Acknowledgments. I want to thank Eric Cator, Ronald Pyke and a referee for their comments.

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