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LONG-TIME TAILS IN THE PARABOLIC ANDERSON MODEL WITH BOUNDED POTENTIAL

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We consider the parabolic Anderson problem $\partial_t u = \kappa\Delta u + \xi u$ on $(0,\infty)\times\mathbb{Z}^d$ with random i.i.d. potential $\xi = (\xi(z))_{z\in\mathbb{Z}^d}$ and the initial condition $u(0,\cdot)\equiv 1$. Our main assumption is that $\operatorname{esssup}\xi(0)=0$. Depending on the thickness of the distribution $\operatorname{Prob}(\xi(0)\in\cdot)$ close to its essential supremum, we identify both the asymptotics of the moments of $u(t,0)$ and the almost-sure asymptotics of $u(t,0)$ as $t\to\infty$ in terms of variational problems. As a by-product, we establish Lifshitz tails for the random Schrödinger operator $-\kappa\Delta-\xi$ at the bottom of its spectrum. In our class of ξ distributions, the Lifshitz exponent ranges from $d/2$ to ∞; the power law is typically accompanied by lower-order corrections.

1. Introduction and statement of results.

1.1. *Model and motivation.* In recent years, systems with a priori disorder have become one of the central objects of study in both probability theory and mathematical physics. Two of the pending open problems are the behavior of the simple random walk in random environment on the side of probability theory and understanding of the spectral properties of the so-called Anderson Hamiltonian on the side of (mathematical) solid state physics. The parabolic Anderson model studied in this paper encompasses various features of both aforementioned problems and thus provides a close link between the two seemingly rather remote areas. In particular, long-time tails in the parabolic model are intimately connected with the mass distribution of the spectral measure at the bottom of the spectrum for a class of Anderson Hamiltonians, and with the asymptotic scaling behavior of the random walk in random environment.

The parabolic Anderson model is the Euclidean-time (or diffusion) version of the Schrödinger equation with a random potential. More precisely, the name refers to the initial problem

$$(1.1)\qquad \begin{aligned} \partial_t u(t,z) &= \kappa\Delta^{\mathrm{d}}u(t,z)+\xi(z)u(t,z), && (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,z)&=1, && z\in\mathbb{Z}^d, \end{aligned}$$

where ∂_t is the time derivative, $u\colon[0,\infty)\times\mathbb{Z}^d\to[0,\infty)$ is a function, $\kappa>0$ is a diffusion constant, Δ^{d} is the discrete Laplacian $[\Delta^{\mathrm{d}}f](z)=\sum_{y\sim z}(f(y)-f(z))$ (here $y\sim z$ denotes that y and z are nearest neighbors), and $\xi=(\xi(z))_{z\in\mathbb{Z}^d}$ is a random i.i.d. potential. Let us use $\langle\,\cdot\,\rangle$ to denote the expectation with respect

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to ξ and let $\text{Prob}(\cdot)$ denote the corresponding probability measure. The main subject of our interest concerning (1.1) is the large time behavior of the pth moment $\langle u(t,0)^p\rangle$ for all $p > 0$ and the almost-sure asymptotics of $u(t,0)$.

The quantity $u(t,z)$ can be interpreted as the expected total mass at time t carried by a particle placed at time 0 at site z with a unit mass on it. The particle diffuses on $\mathbb{Z}^d$ like a simple random walk with generator $\kappa\Delta^{\mathrm{d}}$; when present at site x, its mass is increased/decreased by an infinitesimal amount at rate $\pm\xi(x) \vee 0$. Of particular interest is the phenomenon of *intermittency*: The total mass at time t comes mainly from passing through certain small t-dependent regions, the "relevant islands," where the potential ξ is large and of particular preferred shape. Intermittency is reflected (and sometimes defined) by a comparison of the asymptotics of $\langle u(t,0)^p\rangle^{1/p}$ for different p and/or by a comparison of the growths of $\langle u(t,0)\rangle$ and $u(t,0)$, see also Remarks 4 and 5 below. For general aspects of intermittency see Gärtner and Molchanov (1990) and the monograph of Carmona and Molchanov (1994).

1.2. *Assumptions.* Since the time evolution in (1.1) is driven by the operator $\kappa\Delta^{\mathrm{d}}+\xi$, it is clear that both large t asymptotics of $u(t,0)$ are determined by the upper tails of the random variable $\xi(0)$. Our principal assumption is that the support of $\xi(0)$ is bounded from above. As then follows by applying a criterion derived in Gärtner and Molchanov (1990), there is a unique non-negative solution to (1.1) for almost all ξ. Moreover, since $\xi(\cdot) \to \xi(\cdot)+a$ is compensated by $u(t,\cdot) \to e^{at}u(t,\cdot)$ in (1.1), we assume without loss of generality that $\xi(0)$ is a non-degenerate random variable with

$$\operatorname{esssup} \xi(0) = 0. \tag{1.2}$$

Hence, our potential ξ is non-positive throughout $\mathbb{Z}^d$, that is, every lattice site x is either neutral ($\xi(x) = 0$) or a "soft trap" ($-\infty < \xi(x) < 0$) or a "hard trap" ($\xi(x) = -\infty$). Furthermore, $\xi(x)$ exceeds any negative value with positive probability. Note that *a priori* we do not exclude hard traps, but some restrictions to the size of $\text{Prob}(\xi(0) = -\infty)$ have to be imposed in order to have an interesting almost-sure asymptotics (see Theorem 1.5). The important special case of "Bernoulli traps," where the potential attains only the values 0 and $-\infty$, has already extensively been studied by, for example, Donsker and Varadhan (1979), Antal (1995) and in a continuous analogue by Sznitman (1998).

As we have indicated above, our results will prominently depend on the asymptotics of $\text{Prob}(\xi(0) > -x)$ as $x \downarrow 0$. Actually, they turn out to depend on two parameters $A \in (0,\infty)$ and $\gamma \in [0,1)$ only, which appear as follows:

$$\text{Prob}\big(\xi(0) > -x\big) = \exp\Big\{-A\, x^{-\frac{\gamma}{1-\gamma}+o(1)}\Big\}, \qquad x \downarrow 0. \tag{1.3}$$

The reader should keep (1.3) in mind as the main representative of the distributions we are considering. The case $\gamma = 0$ contains the above mentioned special case of "Bernoulli traps."

However, our precise assumption on the thickness of $\text{Prob}(\xi(0) \in \cdot)$ at zero will be more technical. As turns out to be more convenient for our proofs, we describe the upper tail of $\mathrm{Prob}(\xi(0) \in \cdot)$ in terms of scaling properties of the cumulant generating function

$$H(\ell) = \log\langle e^{\ell\xi(0)}\rangle, \qquad \ell \geq 0. \tag{1.4}$$

The reason is that H naturally appears once expectation with respect to ξ is taken on the Feynman-Kac representation of $u(t, 0)$; see, for example, formula (4.8). Note that H is convex and, by (1.2), decreasing and strictly negative on $(0, \infty)$.

SCALING ASSUMPTION. *We assume that there is a non-decreasing function* $t \mapsto \alpha_t \in (0, \infty)$ *and a function* $\widetilde{H}: [0, \infty) \to (-\infty, 0]$, $\widetilde{H} \not\equiv 0$, *such that*

$$\lim_{t\to\infty} \frac{\alpha_t^{d+2}}{t} H\left(\frac{t}{\alpha_t^d}\, y\right) = \widetilde{H}(y), \qquad y \geq 0, \tag{1.5}$$

uniformly on compact sets in $(0, \infty)$.

Informally and intuitively, the scale function α_t admits the interpretation as the asymptotic diameter of the "relevant islands" from which the main contribution to the expected total mass $\langle u(t, 0)\rangle$ comes; see also subsection 2.1. The choice of the scaling ratios α_t^{d+2}/t and t/α_t^d in (1.2) is dictated by matching two large-deviation scales: one (roughly) for the range of the simple random walk, the other for the size of the field ξ; see subsection 2.1.

REMARK 1. The finiteness and non-triviality of $\widetilde{H}$ necessitate that $t/\alpha_t^d \to \infty$ and $\alpha_t = O(t^{1/(d+2)})$. In the asymptotic sense, (1.5) and non-triviality of $\widetilde{H}$ determine the pair $(\alpha_t, \widetilde{H})$ uniquely up to a constant multiple resp. scaling. Indeed, if $(\widehat{\alpha}_t, \widehat{H})$ is another pair satisfying the Scaling Assumption then, necessarily, $\widehat{\alpha}_t/\alpha_t \to c \neq 0, \infty$ and $\widehat{H}(\cdot) = c^{d+2}\widetilde{H}(\cdot/c^d)$. Moreover, if $t \mapsto \widehat{\alpha}_t$ is a positive function with $\widehat{\alpha}_t/\alpha_t \to 0$, then the limit in (1.5) gives $\widehat{H} \equiv 0$. Similarly, if $\widehat{\alpha}_t/\alpha_t \to \infty$, then $\widehat{H} \equiv -\infty$. These assertions follow directly from convexity of H (see also subsection 3.2).

Our Scaling Assumption should be viewed as a more general form of (1.3) that is better adapted to our proofs. Remarkably, it actually constrains the form of possible $\widetilde{H}$ to a two-parameter family and forces the scale function α_t to be regularly varying. The following claim is proved in subsection 3.2.

PROPOSITION 1.1. *Suppose that* (1.2) *and the Scaling Assumption hold. Then*

$$\widetilde{H}(y) = \widetilde{H}(1)y^\gamma, \qquad y > 0, \tag{1.6}$$

for some $\gamma \in [0, 1]$. *Moreover,*

$$\lim_{t\to\infty} \frac{\alpha_{pt}}{\alpha_t} = p^\nu \quad \textit{for all } p > 0 \textit{ and } \quad \lim_{t\to\infty} \frac{\log \alpha_t}{\log t} = \nu, \tag{1.7}$$

where

$$\nu = \frac{1-\gamma}{d+2-d\gamma} \in \left[0, \frac{1}{d+2}\right]. \tag{1.8}$$

REMARK 2. As is seen from (1.3), each value $\gamma \in [0,1)$ can be attained. Note that, despite the simplicity of possible $\widetilde{H}$, the richness of the class of all ξ distributions persists in the scaling behavior of $\alpha_t = t^{\nu+o(1)}$. For instance, the case $\gamma = 0$ includes both distributions with an atom at 0 and those with no atom but with a density ρ (w.r.t. the Lebesgue measure) having the asymptotic behavior $\rho(x) \sim (-x)^\sigma$ $(x \uparrow 0)$ for a $\sigma > -1$. It is easy to find that $\alpha_t = t^{1/(d+2)}$ [and $\widetilde{H}(1) = \log \operatorname{Prob}(\xi(0) = 0)$] in the first case while $\alpha_t = (t/\log t)^{1/(d+2)}$ in the second one. Yet thinner a tail has $\rho(x) \sim \exp(-\log^\tau |x|^{-1})$ with $\tau > 1$, for which we find $\alpha_t = (t/\log^\tau t)^{1/(d+2)}$. Similar examples exist for any $\gamma \in [0,1)$.

Proposition 1.1 leads us to the following useful concept:

DEFINITION. Given a $\gamma \in [0,1]$, we say that H is *in the γ-class*, if (1.2) holds and there is a function $t \mapsto \alpha_t$ such that (H, α_t) satisfies the Scaling Assumption and the limiting $\widetilde{H}$ is homogeneous with exponent γ, as in (1.6).

Throughout the remainder of this paper, we restrict ourselves to the case $\gamma < 1$. The case $\gamma = 1$ is qualitatively different from that of $\gamma < 1$; for more explanation see subsections 2.2 and 2.5.

The rest of this paper is organized as follows. In the remainder of this section we state our results (Theorems 1.2 and 1.5) on the moment and almost-sure asymptotics of $u(t,0)$ and on Lifshitz tails of the Schrödinger operator $-\kappa\Delta^{\mathrm{d}} - \xi$ (Theorem 1.3). The next section contains heuristic explanation of the proofs, discussion of the case $\gamma = 1$ in (1.3), some literature remarks, and a list of open problems. Section 3 contains necessary definitions and proofs of some technical claims (in particular, Proposition 1.1). The proofs of our main results (Theorems 1.2 and 1.5) come in Sections 4 and 5.

1.3. *Main results.*

1.3.1. *Fundamental objects.* First we introduce some objects needed for the definition of the quantity χ which is basic for all our results. An uninterested reader may consider skipping these definitions and passing directly to subsection 1.3.2.

Function spaces. Define

$$\mathcal{F} = \left\{ f \in C_{\mathrm{c}}(\mathbb{R}^d, [0,\infty)) \colon \|f\|_1 = 1 \right\}, \tag{1.9}$$

and for $R > 0$, let $\mathcal{F}_R$ be set of $f \in \mathcal{F}$ with support in $[-R,R]^d$. By $C^+(R)$ (resp. $C^-(R)$) we denote the set of continuous functions $[-R,R]^d \to [0,\infty)$ (resp. $[-R,R]^d \to (-\infty,0]$). Note that functions in $\mathcal{F}_R$ vanish at the boundary of $[-R,R]^d$, while those in $C^\pm(R)$ may not.

Functionals. Let $\mathscr{I}:\mathscr{F}\to[0,\infty]$ be the Donsker-Varadhan rate functional

$$\mathscr{I}(f)=\begin{cases}\kappa\|(-\Delta)^{\frac12}\sqrt{f}\|_2^2, & \text{if } \sqrt{f}\in\mathscr{D}\big((-\Delta)^{\frac12}\big),\\ \infty, & \text{otherwise,}\end{cases} \tag{1.10}$$

where Δ is the Laplace operator on $L^2(\mathbb{R}^d)$ (defined as a self-adjoint extension of $\sum_i(\partial^2/\partial x_i^2)$ from, for example, the Schwarz class on $\mathbb{R}^d$) and $\mathscr{D}((-\Delta)^{1/2})$ denotes the domain of its square root. Note that $\mathscr{I}(f)$ is nothing but the Dirichlet form of the Laplacian evaluated at $f^{1/2}$.

For $R>0$ we define the functional $\mathscr{H}_R: C^+(R)\to(-\infty,0]$ by putting

$$\mathscr{H}_R(f)=\int_{[-R,R]^d}\widetilde{H}\big(f(x)\big)\,dx. \tag{1.11}$$

Note that for H in the γ-class, $\mathscr{H}_R(f)=\widetilde{H}(1)\int f(x)^\gamma dx$, with the interpretation $\mathscr{H}_R(f)=\widetilde{H}(1)|\operatorname{supp} f|$ when $\gamma=0$. Here $|\cdot|$ denotes the Lebesgue measure.

Legendre transforms. Let $\mathscr{L}_R: C^-(R)\to[0,\infty]$ be the Legendre transform of $\mathscr{H}_R$,

$$\mathscr{L}_R(\psi)=\sup\big\{(f,\psi)-\mathscr{H}_R(f): f\in C^+(R),\ \operatorname{supp} f\subset\operatorname{supp}\psi\big\}, \tag{1.12}$$

where we used the shorthand notation $(f,\psi)=\int f(x)\psi(x)\,dx$. If H is in the γ-class, we get $\mathscr{L}_R(\psi)=\text{const.}\int|\psi(x)|^{-\frac{\gamma}{1-\gamma}}\,dx$ for $\gamma\in(0,1)$ and $\mathscr{L}_R(\psi)=-\widetilde{H}(1)\,|\operatorname{supp}\psi|$ for $\gamma=0$.

For any potential $\psi\in C^-(R)$, we also need the principal (i.e., the largest) eigenvalue of the operator $\kappa\Delta+\psi$ on $L^2([-R,R]^d)$ with Dirichlet boundary conditions, expressed either as the Legendre transform of $\mathscr{I}$ or in terms of the Rayleigh-Ritz principle:

$$\begin{aligned}\lambda_R(\psi)&=\sup\big\{(f,\psi)-\mathscr{I}(f): f\in\mathscr{F}_R,\ \operatorname{supp} f\subset\operatorname{supp}\psi\big\}\\ &=\sup\big\{(\psi,g^2)-\kappa\|\nabla g\|_2^2: g\in C_c^\infty(\operatorname{supp}\psi,\mathbb{R}),\ \|g\|_2=1\big\},\end{aligned} \tag{1.13}$$

with the interpretation $\lambda_R(0)=-\infty$.

Variational principles. Here is the main quantity of this subsection:

$$\chi=\inf_{R>0}\inf\big\{\mathscr{I}(f)-\mathscr{H}_R(f): f\in\mathscr{F}_R\big\} \tag{1.14}$$

$$=\inf_{R>0}\inf\big\{\mathscr{L}_R(\psi)-\lambda_R(\psi): \psi\in C^-(R)\big\}. \tag{1.15}$$

where (1.15) is obtained from (1.14) by inserting (1.12) and the second line in (1.13). Note that χ depends on γ and the constant $\widetilde{H}(1)$.

1.3.2. *Moment asymptotics.* We proceed by describing the logarithmic asymptotics of the pth moment of $u(t,0)$; for the proof see Section 4.

THEOREM 1.2. *Suppose that* (1.2) *and the Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$. Then $\chi \in (0,\infty)$ and*

$$\lim_{t\to\infty} \frac{\alpha_{pt}^2}{pt} \log\langle u(t,0)^p\rangle = -\chi, \tag{1.16}$$

for every $p \in (0,\infty)$.

REMARK 3. Both formulas (1.14) and (1.15) arise in well-known large-deviation statements: the former for an exponential functional of Brownian occupation times, the latter for the principal eigenvalue for a scaled version of the field ξ. Our proof pursues the route leading to (1.14); an approach based on the second formula is heuristically explained in subsection 2.1.1.

REMARK 4. Formula (1.16), together with the results of Proposition 1.1, imply that

$$\lim_{t\to\infty} \frac{\alpha_t^2}{t} \log \frac{\langle u(t,0)^p\rangle^{1/p}}{\langle u(t,0)^q\rangle^{1/q}} = \chi\big(q^{-2\nu} - p^{-2\nu}\big), \qquad p,q \in (0,\infty), \tag{1.17}$$

whenever H is in the γ-class, where $\nu > 0$ is as in (1.8). In particular, $\langle u(t,0)^p\rangle$ for $p > 1$ decays much slower than $\langle u(t,0)\rangle^p$. This is one widely used manifestation of intermittency.

1.3.3. *Lifshitz tails.* Based on Theorem 1.2, we can compute the asymptotics of the so-called *integrated density of states* (IDS) of the operator $-\kappa\Delta^{\mathrm{d}} - \xi$ on the right-hand side of (1.1), at the bottom of its spectrum. Below we define the IDS and list some of its basic properties. For a comprehensive treatment and proofs we refer to the book by Carmona and Lacroix (1990).

The IDS is defined as follows: Let $R > 0$ and let us consider the operator $\mathfrak{H}_R = -\kappa\Delta^{\mathrm{d}} - \xi$ in $[-R,R]^d \cap \{x \in \mathbb{Z}^d : \xi(x) > -\infty\}$ with Dirichlet boundary conditions. Clearly, $\mathfrak{H}_R$ has a finite number of eigenvalues that we denote E_k, so it is meaningful to consider the quantity

$$N_R(E) = \#\{k : E_k \le E\}, \qquad E \in \mathbb{R}. \tag{1.18}$$

The integrated density of states is then the limit

$$n(E) = \lim_{R\to\infty} \frac{N_R(E)}{(2R)^d}, \tag{1.19}$$

giving $n(E)$ the interpretation as the number of energy levels below E per unit volume. The limit exists and is almost surely constant, as can be proved using, for example, subadditivity.

It is clear that $E \mapsto n(E)$ is monotone and that $n(E) = 0$ for all $E < 0$, provided (1.2) is assumed. In the 1960's, based on heuristic arguments, Lifshitz postulated that $n(E)$ behaves like $\exp(-\text{const.}\, E^{-\delta})$ as $E \downarrow 0$. This asymptotic form has been established rigorously in the so called "obstacle cases" (see subsection 2.4) treated by Donsker and Varadhan (1979) and Sznitman (1998), with $\delta = d/2$. Here we generalize this result to our class of distributions with $\gamma < 1$; however, in our cases the power-law is typically supplemented with a lower-order correction. The result can concisely be formulated in terms of the inverse function of $t \mapsto \alpha_t$:

THEOREM 1.3. *Suppose that* (1.2) *and the Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$ and let α^{-1} be the inverse to the scaling function $t \mapsto \alpha_t$. Then*

$$\lim_{E\downarrow 0} \frac{\log n(E)}{E\alpha^{-1}(E^{-\frac{1}{2}})} = -\frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}} \tag{1.20}$$

where χ is as in (1.14) *and ν is defined in* (1.8).

Invoking (1.7), $E\alpha^{-1}(E^{-1/2}) = E^{-1/\beta+o(1)}$ as $E \downarrow 0$, where

$$\beta = \frac{2}{d+2\frac{\gamma}{1-\gamma}} = \frac{2\nu}{1-2\nu} \in \left(0, \frac{2}{d}\right]. \tag{1.21}$$

In particular, $1/\beta$ is the Lifshitz exponent. Theorem 1.3 is proved in subsection 4.3.

1.3.4. *Almost-sure asymptotics.* The almost-sure behavior of $u(t,0)$ depends strongly on whether the origin belongs to a finite or infinite component of the set $\mathscr{C} = \{z \in \mathbb{Z}^d : \xi(z) > -\infty\}$. Indeed, if 0 is in a finite component of $\mathscr{C}$, then $u(t,0)$ decays exponentially with t. Thus, in order to get a non-trivial almost-sure behavior of $u(t,0)$ as $t \to \infty$, we need that $\mathscr{C}$ contains an infinite component $\mathscr{C}_\infty$ and that $0 \in \mathscr{C}_\infty$ occurs with a non-zero probability. In $d \geq 2$, this is guaranteed by requiring that $\mathrm{Prob}(\xi(0) > -\infty)$ exceed the percolation threshold $p_c(d)$ for site percolation on $\mathbb{Z}^d$. In $d = 1$, $\mathscr{C}$ is percolating if and only if $\mathrm{Prob}(\xi(0) > -\infty) = 1$; sufficient "connectivity" can be ensured only under an extra condition on the *lower* tail of $\xi(0)$.

Suppose, without loss of generality, that $t \mapsto t/\alpha_t^2$ is strictly increasing (recall that $\alpha_t = t^{\nu+o(1)}$ with $\nu \leq 1/3$). Then we can define another scale function $t \mapsto b_t \in (0,\infty)$ by setting

$$\frac{b_t}{\alpha_{b_t}^2} = \log t, \qquad t > 0. \tag{1.22}$$

(In other words, b_t is the inverse function of $t \mapsto t/\alpha_t^2$ evaluated at $\log t$.) Let

$$\widetilde{\chi} = -\sup_{R>0} \sup\Big\{\lambda_R(\psi) : \psi \in C^-(R),\ \mathscr{L}_R(\psi) \leq d\Big\}. \tag{1.23}$$

In our description of the almost sure asymptotics, the pair $(\alpha_{b_t}, \widetilde{\chi})$ will play a role analogous to the pair (α_t, χ) in Theorem 1.2 [in particular, α_{b_t} is the diameter of the "islands" in the "ξ landscape" dominating the a.s. asymptotics of $u(t,0)$]. It is clear from Proposition 1.1 that

$$b_t = (\log t)^{\frac{1}{1-2\nu}+o(1)} \quad \text{and} \quad \alpha_{b_t}^2 = \left(\log t\right)^{\beta+o(1)}, \qquad t\to\infty, \tag{1.24}$$

where β is as in (1.21). It turns out that $\widetilde{\chi}$ can be computed from χ:

PROPOSITION 1.4. *Suppose that* (1.2) *and the Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$. Let ν and β be as in* (1.7) *and* (1.21). *Then $\widetilde{\chi} \in (0,\infty)$ and*

$$\widetilde{\chi} = \chi^{\frac{1}{1-2\nu}}(1-2\nu)\left(\frac{2\nu}{d}\right)^{\beta}, \tag{1.25}$$

where χ and $\widetilde{\chi}$ are as in (1.14) *and* (1.23).

The proof of Proposition 1.4 is given in subsection 3.3. In the special case $\gamma = 0$, the relation (1.25) can independently be verified by inserting the explicit expressions for χ and $\widetilde{\chi}$ derived, for example, in Sznitman (1998).

Our main result on the almost sure asymptotics reads as follows:

THEOREM 1.5. *Suppose that* (1.2) *and the Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$. In $d \geq 2$, let* $\mathrm{Prob}(\xi(0) > -\infty) > p_c(d)$; *in $d=1$, let $\langle \log(-\xi(0)\vee 1)\rangle < \infty$. Then*

$$\lim_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) = -\widetilde{\chi}, \qquad \mathrm{Prob}(\,\cdot\,|0 \in \mathscr{C}_\infty) \text{ -almost surely}. \tag{1.26}$$

Theorem 1.5 is proved in Section 5; for a heuristic derivation see subsection 2.1.2.

REMARK 5. From a comparison of the asymptotics in (1.16) and in (1.26), we obtain another manifestation of intermittency: The moments of $u(t,0)$ decay much slower than the $u(t,0)$ itself.

Assuming that there is no critical site percolation in dimensions $d \geq 2$, Theorem 1.5 and the arguments at the beginning of this subsection give a complete description of possible leading-order almost-sure asymptotics of $u(t,0)$.

REMARK 6. In $d = 1$, there *is* site percolation at $p_c(1) = 1$ which is the reason why an extra condition on the lower tail of $\mathrm{Prob}(\xi(0) \in \cdot)$ needs to be assumed. If the lower tail is too heavy, that is, if $\log(-\xi(0)\vee 1)$ is not integrable, then a *screening effect* occurs: The mass flow over large distances is hampered by regions of large negative field, which cannot be circumvented due to one-dimensional topology. As has recently been shown in Biskup and König (2000), $u(t,0)$ decays faster than in the cases described in Theorem 1.5.

2. Heuristics, literature remarks and open problems.

2.1. *Heuristic derivation.* In our heuristics we use the interpretation of (1.1) in terms of a particle *system* that randomly evolves in a random potential of traps: A particle at z either jumps to its nearest neighbor at rate κ or is killed at rate $-\xi(z)$. Then $u(t,0)$ is the total expected number of particles located at the origin at time t, provided the initial configuration had exactly one particle at each lattice site.

It is clear from (1.2) that, by time t, the origin is not likely to be reached by any particle from regions having distance more than t from the origin. If $u_t(t,0)$ is the expected number of particles at the origin at time t under the constraint that none of the particles has ever been outside of the box $Q_t = [-t,t]^d \cap \mathbb{Z}^d$, then this should imply that

$$u(t,0) \approx u_t(t,0). \tag{2.1}$$

The particle system in the box Q_t is driven by the operator $\kappa\Delta^{\mathrm{d}}+\xi$ on the right-hand side of (1.1) with zero boundary conditions on ∂Q_t and the leading-order behavior of u_t should be governed by its principal (i.e., the largest) eigenvalue $\lambda_t^{\mathrm{d}}(\xi)$ in the sense that

$$u_t(t,0) \approx e^{t\lambda_t^{\mathrm{d}}(\xi)}. \tag{2.2}$$

Based on (2.2), we can give a plausible explanation of our Theorems 1.2 and 1.5.

2.1.1. *Moment asymptotics.* Under the expectation with respect to ξ, there is a possibility that $\langle u(t,0)\rangle$ will be dominated by a set of ξ's with exponentially small probability. But then the decisive contribution to the average particle-number at zero may come from much smaller a box than Q_t. Let $R\alpha_t$ denote the diameter of the purported box. Then we should have

$$\big\langle u_t(t,0)\big\rangle \approx \big\langle \exp\{t\lambda^{\mathrm{d}}_{R\alpha_t}\}\big\rangle. \tag{2.3}$$

The proper choice of the scale function α_t is determined by balancing the gain in $\lambda^{\mathrm{d}}_{R\alpha_t}(\xi)$ and the loss due to taking ξ's with exponentially small probability. Introducing the scaled field

$$\bar{\xi}_t(x) = \alpha_t^2\,\xi\big(\lfloor x\alpha_t\rfloor\big), \tag{2.4}$$

the condition that these scales match for $\bar{\xi}_t \approx \psi \in C^-(R)$ reads

$$\log \operatorname{Prob}(\bar{\xi}_t \approx \psi) \asymp t\lambda^{\mathrm{d}}_{R\alpha_t}\big(\alpha_t^{-2}\psi(\cdot\,\alpha_t^{-1})\big). \tag{2.5}$$

By scaling properties of the continuous Laplace operator, the right-hand side is approximately equal to $(t/\alpha_t^2)\lambda_R(\psi)$, where $\lambda_R(\psi)$ is defined in (1.13). On the other hand, by our Scaling Assumption,

$$\log \operatorname{Prob}(\bar{\xi}_t \approx \psi) \approx -\frac{t}{\alpha_t^2}\mathscr{L}_R(\psi), \tag{2.6}$$

that is, we expect $\bar{\xi}_t$ to satisfy a large-deviation principle with rate t/α_t^2 and rate function $\mathscr{L}_R$. Then the rates on both sides of (2.5) are identical and, comparing also the prefactors, we have

$$\Big\langle \exp\{t\lambda^{\mathrm{d}}_{R\alpha_t}\}\mathbb{1}\{\bar{\xi}_t \approx \psi\}\Big\rangle \approx \exp\left\{\frac{t}{\alpha_t^2}\left[\lambda_R(\psi) - \mathscr{L}_R(\psi)\right]\right\}. \tag{2.7}$$

Now collect (2.1), (2.3) and (2.7) and maximize over $\psi \in C^-(R)$ and over $R > 0$ to obtain formally the statement on the moment asymptotics in Theorem 1.2 for $p = 1$. Note that, by the above heuristic argument, α_t is the spatial scale of the "islands" in the potential landscape that are only relevant for the moments of $u(t, 0)$.

2.1.2. *Almost-sure asymptotics.* Based on the intuition developed for the moment asymptotics, the decisive contribution to (2.2) should come from some quite localized region in Q_t. Suppose this region has size α_{b_t}, where b_t is some new running time scale, and divide Q_t regularly into boxes of diameter $R\alpha_{b_t}$ ("microboxes") with some $R > 0$. According to (2.6) with t replaced by b_t, we have for any $\psi \in C^-(R)$ with $\mathscr{L}_R(\psi) \le d$ that

$$\mathrm{Prob}(\bar{\xi}_{b_t} \approx \psi) \approx \exp\left\{-\frac{b_t}{\alpha_{b_t}^2}\mathscr{L}_R(\psi)\right\} \ge \exp\{-db_t/\alpha_{b_t}^2\}. \tag{2.8}$$

Suppose that b_t obeys (1.22). Then the right-hand side of (2.8) decays as fast as t^{-d}. Since there are of order t^d microboxes in Q_t, a Borel-Cantelli argument implies that for any ψ with $\mathscr{L}_R(\psi) < d$, there will be a microbox in Q_t where $\bar{\xi}_{b_t} \approx \psi$. As before, $t\lambda^{\mathrm{d}}_{R\alpha_{b_t}}(\psi(\cdot/\alpha_{b_t})/\alpha_{b_t}^2) \approx (t/\alpha_{b_t}^2)\lambda_R(\psi)$, and by optimizing over ψ, any value smaller than $\widetilde{\chi}$ can be attained by $\lambda_R(\psi)$ in some microbox in Q_t.

This suggests that $u(t,\cdot)$ in the favorable microbox decays as described by (1.26). It remains to ensure, and this is a non-trivial part of the argument, that the particles that have survived in this microbox by time t can always reach the origin within a negligible portion of time t. This requires, in particular, that sites x with $\xi(x) > -\infty$ form an infinite cluster containing the origin. If the connection between 0 and the microbox can be guaranteed, $u(t, 0)$ should exhibit the same leading-order decay, which is the essence of the claim in Theorem 1.5. Note that, as before, α_{b_t} is the spatial scale of the islands relevant for the random variable $u(t, 0)$.

2.2. *The case* $\gamma = 1$. In the boundary case $\gamma = 1$ the relevant islands grow (presumably) slower than any polynomial as $t \to \infty$ (i.e., $\alpha_t = t^{o(1)}$), and $\widetilde{H}$ is linear. As a consequence, the asymptotic expansions of $\langle u(t,0)^p\rangle$ and $u(t, 0)$ itself start with a *field-driven* term (i.e, a term independent of κ). In particular, no variational problem is involved at the leading order and no information about the "typical" configuration of the fields is gained.

To understand which ξ dominate the moments of $u(t, 0)$ we have to analyze the next-order term. This requires imposing an additional assumption: We suppose the existence of a new scale function $t \mapsto \vartheta_t$, with $\alpha_t = o(\vartheta_t)$, such that

$$\lim_{t\to\infty} \frac{\vartheta_t^{d+2}}{t} \left[H\left(\frac{t}{\vartheta_t^d} y\right) - H\left(\frac{t}{\vartheta_t^d}\right) y \right] = \widehat{H}(y) \tag{2.9}$$

exists (and is not identically zero) locally uniformly in $y \in (0, \infty)$. Analogous heuristic to that we used to explain the main idea of Theorem 1.2 outputs the asymptotic expansion of the first moment

$$\langle u(t, 0) \rangle = \exp\left[\vartheta_t^d H\left(\frac{t}{\vartheta_t^d}\right) - (t/\vartheta_t^2)\,(\widehat{\chi} + o(1)) \right], \tag{2.10}$$

where $\widehat{\chi}$ is defined as in subsection 1.3.1 with $\widetilde{H}$ replaced by $\widehat{H}$.

Similar scenario should occur for the almost-sure asymptotics. Indeed, setting

$$\psi(x) = (\vartheta_t^d/t) H(t/\vartheta_t^d) + \vartheta_t^{-2} \psi_\star(x/\vartheta_t) \tag{2.11}$$

with some $\psi_\star \in C^-(R)$, formula (2.8) should be rewritten as $\operatorname{Prob}(\xi \approx \psi) \approx \exp\{-(t/\vartheta_t^2) \mathscr{L}_R^\star(\psi_\star)\}$, where $\mathscr{L}_R^\star$ is defined by (1.12) with $\widetilde{H}$ replaced by $\widehat{H}$. Let $b_t^\star$ solve for s in $s/\vartheta_s^2 = \log t$. By following the heuristic derivation of Theorem 1.5 (and, in particular, invoking the scaling and additivity of the continuum eigenvalue $\lambda_R(\psi)$, see subsection 2.1.2) we find that

$$u(t, 0) = \exp\left[\left(t \vartheta_{b_t^\star}^d / b_t^\star\right) H\left(b_t^\star/\vartheta_{b_t^\star}^d\right) - \left(t/\vartheta_{b_t^\star}^2\right)(\widehat{\chi}_\star + o(1)) \right] \tag{2.12}$$

should hold $\operatorname{Prob}(\cdot|0 \in \mathscr{C}_\infty)$-almost surely, where and $\widehat{\chi}_\star$ is defined by (1.23) with $\widetilde{H}$ everywhere replaced by $\widehat{H}$. However, we have not made any serious attempt to carry out the details.

Surprisingly, unlike in the cases discussed in Proposition 1.1, $\widehat{H}$ takes a *unique* functional form:

$$\widehat{H}(y) = \sigma y \log y, \tag{2.13}$$

where $\sigma > 0$ is a parameter. This fact is established by arguments similar to those used in the proof of Proposition 1.1. (As a by-product, we also get that $t \mapsto \vartheta_t$ is slowly varying as $t \to \infty$.) An interesting consequence of this is that, unlike in $\gamma < 1$ situations, the variational problems for $\widehat{\chi}$ and $\widehat{\chi}_\star$ factorize to one-dimensional problems; see Gärtner and den Hollander (1999).

2.3. *An application*: *Self-attractive random walks.* One of our original sources of motivation for this work have been self-attractive path measures as models for "squeezed polymers." Consider a polymer $S = (S_0, \ldots, S_n)$ of length n modeled by a path of simple random walk with weight $\exp[\beta \sum_x V(\ell_n(x))]$. Here $V: \mathbb{Z} \to (-\infty, 0]$, and $\ell_n(x) = \#\{k \le n: S_k = x\}$ is the local time at x. Assuming that V is convex and $V(0) = 0$, for example, $V(\ell) = -\ell^\gamma$ with $\gamma \in [0, 1)$, the interaction has an attractive effect. A large class of such functions V (i.e., the completely monotonous ones) are the cumulant generating functions of probability distributions on $[-\infty, 0]$, like H in (1.4). Via the Feynman-Kac representation, this makes the study of the above path measure essentially equivalent to the study of the moments of a parabolic Anderson model. In fact, the only difference is that for polymer models the time of the walk is discrete.

We have no doubt that Theorem 1.2 extends to the discrete-time setting. Hence, the endpoint S_n of the polymer should fluctuate on the scale α_n as in our Scaling Assumption, which is $\alpha_n = n^\nu$ in the $V(\ell) = -\ell^\gamma$ case. Since $\gamma \mapsto \nu$ is decreasing, we are confronted with the counterintuitive fact that the squeezing effect is the more extreme the "closer" is V to the linear function. This is even more surprising if one recalls that for the boundary case $\gamma = 1$, the Hamiltonian $\sum_x V(\ell_n(x))$ is deterministic, and therefore the endpoint runs on scale $n^{1/2}$. Note that, on the other hand, for $\gamma > 1$, which is the self-repellent case, it is known in $d = 1$ (and expected in dimensions $d = 2$ and 3) that the scale of the endpoint is a power larger than 1/2. Hence, at least in low dimensions, there is an intriguing phase transition for the path scale at $\gamma = 1$.

As a nice side remark, the following model of an *annealed randomly-charged polymer* also falls into the class of models considered above. Consider an n-step simple random walk $S = (S_0, \dots, S_n)$ with weight $\exp\{-\beta \mathscr{I}_n(S)\}$, where $\beta > 0$ and

$$\mathscr{I}_n(S) = \sum_{0 \le i < j \le n} \omega_i \omega_j \mathbb{1}\{S_i = S_j\}. \tag{2.14}$$

Here $\omega = (\omega_i)_{i \in \mathbb{N}_0}$ is an i.i.d. sequence with a symmetric distribution on $\mathbb{R}$ having variance one. Think of ω_i as an electric charge at site i of the polymer. [For continuous variants of this model and more motivation see, e.g., Buffet and Pulé (1997).]

If the charges equilibrate faster than the walk, the interaction they effectively induce on the walk is given by the expectation $E(\exp\{-\beta \mathscr{I}_n(S)\})$ and is thus of the above type with

$$V(\ell) = -\log E \exp\big((\omega_0 + \cdots + \omega_\ell)^2\big), \tag{2.15}$$

where E denotes the expectation with respect to ω. By the invariance principle, we have $V(\ell) = -(1/2 + o(1)) \log \ell$ as $\ell \to \infty$, which means that V satisfies our Scaling Assumption with $\alpha_n = (n/\log n)^{1/(d+2)}$. Hence, we can identify the logarithmic asymptotics of the partition function $\mathbb{E}_0 \otimes E(\exp\{-\beta \mathscr{I}_n\})$ and see that the typical end-to-end distance of the annealed charged polymer runs on the scale α_n, that is, the averaging over the charges has a strong self-attractive effect.

2.4. *Relation to earlier work.* General mathematical aspects of the problem 1.1, including the existence and uniqueness of solutions and a criterion for intermittency [see (1.17) and the comments thereafter], were first addressed by Gärtner and Molchanov (1990). In the subsequent paper, Gärtner and Molchanov (1998), the authors focused on the case of *double-exponential* distributions

$$\operatorname{Prob}(\xi(0) > x) \sim \exp\{-e^{x/\varrho}\}, \qquad x \to \infty. \tag{2.16}$$

For $0 < \varrho < \infty$, the main contribution to $\langle u(t,0)^p \rangle$ comes from islands in $\mathbb{Z}^d$ of asymptotically finite size (which corresponds to a constant α_t in our notation). When the upper tails of $\text{Prob}(\xi(0) \in \cdot)$ are yet thicker (i.e., $\varrho = \infty$), for example, when $\xi(0)$ is Gaussian, then the overwhelming contribution to $\langle u(t,0)^p \rangle$ comes from very high peaks of ξ concentrated at single sites. [In a continuous setting the scaling can still be non-trivial; see Gärtner and König (2000) and Gärtner, König and Molchanov (1999).] For thinner tails than double-exponential [i.e., when $\varrho = 0$, called the *almost bounded* case in Gärtner and Molchanov (1998)], the relevant islands grow unboundedly as $t \to \infty$, that is, $\alpha_t \to \infty$ in our notation. The distribution (2.16) thus constitutes a certain critical class for having a non-degenerate but still discrete spatial structure.

The opposite extreme of tail behaviors was addressed in Donsker and Varadhan (1979) (moment asymptotics) and in Antal (1995) (almost-sure asymptotics); see also Antal (1994). The distribution considered by these authors is $\xi(0) = 0$ or $-\infty$ with probability p and $1-p$, respectively. The analysis of the moments can be reduced to a self-interacting polymer problem (see subsection 2.3), which is essentially the route taken by Donsker and Varadhan. In the almost-sure case, the problem is a discrete analogue of the Brownian motion in a Poissonian potential analyzed extensively by Sznitman in the 1990's using his celebrated method of enlargement of obstacles (MEO); see Sznitman (1998).

The MEO bears on the problem (1.1) because of the special form of the ξ distribution: Recall the interpretation of points z with $\xi(z) = -\infty$ as "hard traps" where the simple random walk is strictly killed. If $\mathscr{O} = \{z \in \mathbb{Z}^d : \xi(z) = -\infty\}$ denotes the trap region and $T_{\mathscr{O}} = \inf\{t > 0 : X(t) \in \mathscr{O}\}$ the first entrance time, then

$$u(t,z) = \mathbb{P}_z(T_{\mathscr{O}} > t), \tag{2.17}$$

that is, $u(t,z)$ is the survival probability at time t for a walk started at z. In his thesis, Antal derives a discrete version of the MEO and demonstrates its value in Antal (1994) and Antal (1995) by proving results which are (slight refinements of) our Theorems 1.2 and 1.5 for $\gamma = 0$ and $\alpha_t = t^{1/(d+2)}$.

The primary goal of this paper was to fill in the gap between the two regimes considered in Gärtner and Molchanov (1998) and Donsker and Varadhan (1979), resp., Antal (1995); that is, we wanted to study the general case in which the diameter α_t of the relevant islands grows to infinity. We succeeded in doing that under the restrictions that the field is bounded from above and α_t diverges at least like a power of t. As already noted in subsection 2.2, in the boundary case $\alpha_t = t^{o(1)}$ (i.e., $\gamma = 1$) another phenomenon occurs which cannot be handled in a unified manner; see the discussion of "almost-bounded" cases in the next subsection.

The technique of our proofs draws heavily on that of Gärtner and König (2000) and Gärtner, König and Molchanov (1999), however, non-trivial adaptations had to be made. An interesting feature of this technique is the handle of the compactification argument: We do not use folding (as Donsker and Varadhan did in their seminal papers from 1975 and 1979) nor do we coarse-grain the field as is done in the MEO; instead, we develop comparison arguments for Dirichlet eigenvalues in large and small boxes. The task is in many places facilitated by switching between the dual languages of Dirichlet eigenvalues *vs* local times of the simple random walk.

After this paper had been submitted, we learned that F. Merkl and M. Wüthrich had independently used rather similar techniques to describe the scaling of the principal eigenvalue of the continuous Dirichlet operator $-\Delta + (\log t)^{-2/d} V_\omega$ in $[-t, t]^d$, where V_ω is the potential generated by convoluting a shape function with the Poissonian cloud. (The scaling of V_ω is chosen such that the eigenvalue is not dominated solely by the potential, as in a certain sense happens in the "obstacle case.") The first part of the results appeared in Merkl and Wüthrich (2000).

2.5. *Discussion and open problems.*

(i) *"Almost-bounded" cases.* As discussed in subsection 2.2, the $\gamma = 1$ case requires analyzing a lower-order scale than considered in this paper. Interestingly, the variational problem driving this scale coincides with that of $\rho = 0$ limit of the double exponential case; see (2.16) and, for example, Gärtner and den Hollander (1999). This makes us believe that the $\gamma = 1$ case actually reflects the *whole* regime of "almost bounded" but unbounded potentials, that is, those interpolating between our cases $\gamma < 1$ and the double exponential distribution. (In all these cases, we expect the following strategy of proof to be universally applicable: identify the maximum of ξ in a box of size t and, subtracting this term away, map the problem to the effectively bounded case; see subsection 2.2 for an example.) For these reasons, we leave its investigation to future work.

(ii) *Generalized MEO.* Despite the fact that our current technique circumvents the use of the MEO, it would be interesting to develop its extension including other fields in our class (in particular, those with $\gamma \neq 0$). The main reason is that this should allow for going beyond the leading order term. However, the so called "confinement property," which is the main result of the MEO we cannot obtain, would require rather detailed knowledge of the *shape* of the field that brings the main contribution to the moments of $u(t, 0)$, resp., to $u(t, 0)$ itself. Thus, while the MEO can help in controlling the "probability part" of the statements (1.16) and (1.26), an analysis of the minimizers in (1.14) and (1.23) is also needed. The latter is expected to be delicate in higher dimensions (in $d = 1$ this task has fully been carried out in Biskup and König (1998)).

(iii) *Adding a drift.* An interesting open problem arises if a homogeneous drift term $\mathbf{h} \cdot \nabla u$ is added on the right-hand side of (1.1). This problem is considered hard (especially in $d \geq 2$), since the associated Anderson Hamiltonian lacks self-adjoinedness with respect to the canonical inner product on $\ell^2(\mathbb{Z}^d)$. Self-adjointness can be restored if the inner product is appropriately modified; however, this case seems to be much more difficult to handle. One expects an interesting phase transition of the decay rate as $|\mathbf{h}|$ increases, but the rigorous understanding is rather poor at the moment.

(iv) *Intermittency.* Our results imply intermittency for our model in the sense of asymptotic properties of positive moments of $u(t,0)$; see Remarks 4 and 5. The picture would round up very nicely if one could identify precisely the set of "islands" (or rather peaks) in the "ξ landscape," where the main contribution to $\langle u(t,0)\rangle$, resp., $u(t,0)$ comes from. At the moment, work of Gärtner, König and Molchanov (2001) for the double-exponential distributions of the potentials is going on in this direction. Some additional complications stemming from $\alpha_t \to \infty$ can be expected in our present cases.

(v) *Correlation structure.* Another open problem concerns the asymptotic correlation structure of the random field $u(t,\cdot)$, as has been analyzed by Gärtner and den Hollander (1999) in the case of the double-exponential distribution. Also for answering this question, quite some control of the minimizers in (1.14) and (1.23) is required. Unfortunately, the compactification technique of Gärtner and den Hollander (1999) cannot be applied without additional work, since it seems to rely on the discreteness of the underlying space in several important places. As already alluded to, extension of this technique to continuous space may also be relevant for the analysis of (1.1) with "almost-bounded" fields.

3. Preliminaries. In this section we first introduce some necessary notation needed in the proof of Theorems 1.2 and 1.5 and then prove Propositions 1.1 and 1.4. In the last subsection, we prove a claim on the convergence of certain approximants to the variational problem (1.14).

3.1. *Feynman-Kac formula and Dirichlet eigenvalues.* Our analysis is based on the link between the random-walk and random-field descriptions provided by the Feynman-Kac formula. Let $(X(s))_{s\in[0,\infty)}$ be the continuous-time simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta^{\mathrm{d}}$. By $\mathbb{P}_z$ and $\mathbb{E}_z$ we denote the probability measure, resp., the expectation with respect to the walk starting at $X(0)=z\in\mathbb{Z}^d$.

3.1.1. *General initial problem.* For any potential $V\colon\mathbb{Z}^d\to[-\infty,0]$, we denote by u^V the unique solution to the initial problem

$$\begin{aligned} &\partial_t u(t,z)=\kappa\Delta^{\mathrm{d}}u(t,z)+V(z)u(t,z), \qquad &&(t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ &u(0,z)=1, &&z\in\mathbb{Z}^d. \end{aligned} \tag{3.1}$$

Note that we have to set $u(t,z)\equiv 0$ whenever $V(z)=-\infty$, in order that (3.1) is well defined. The Feynman-Kac formula allows us to express u^V as

$$u^V(t,z)=\mathbb{E}_z\left[\exp\int_0^t V\big(X(s)\big)\,ds\right], \qquad z\in\mathbb{Z}^d,\ t>0. \tag{3.2}$$

Introduce the local times of the walk

$$\ell_t(z)=\int_0^t \mathbb{1}\{X(s)=z\}\,ds, \qquad z\in\mathbb{Z}^d,\ t>0, \tag{3.3}$$

that is, $\ell_t(z)$ is the amount of time the random walk has spent at $z \in \mathbb{Z}^d$ by time t. Note that $\int_0^t V(X(s))\,ds = (V, \ell_t)$, where $(\cdot,\cdot)$ stands for the inner product on $\ell^2(\mathbb{Z}^d)$.

In view of (2.1), of particular importance will be the finite-volume version of (3.1) with Dirichlet boundary condition. Let $R > 0$ and let $Q_R = -[R, R]^d \cap \mathbb{Z}^d$ be a box in $\mathbb{Z}^d$. The solution of the initial-boundary value problem

$$\begin{aligned} &\partial_t u(t,z) = \kappa\Delta^{\mathrm{d}} u(t,z) + V(z)u(t,z), && (t,z) \in (0,\infty)\times Q_R,\\ &u(0,z) = 1, && z \in Q_R,\\ &u(t,z) = 0, && t > 0,\ z \notin Q_R, \end{aligned} \tag{3.4}$$

will be denoted by $u_R^V: [0,\infty)\times\mathbb{Z}^d \to [0,\infty)$. Similarly to (3.2), we have the representation

$$u_R^V(t,z) = \mathbb{E}_z\left[\exp\left\{\int_0^t V\big(X(s)\big)\,ds\right\} \mathbb{1}\{\tau_R > t\}\right], \qquad z \in \mathbb{Z}^d,\ t > 0, \tag{3.5}$$

where τ_R is the first exit time from the set Q_R, that is,

$$\tau_R = \inf\big\{t > 0: X(t) \notin Q_R\big\}. \tag{3.6}$$

Alternatively,

$$u_R^V(t,z) = \mathbb{E}_z\Big[\exp\{(V,\ell_t)\}\mathbb{1}\big\{\operatorname{supp}(\ell_t) \subset Q_R\big\}\Big], \tag{3.7}$$

where we recalled (3.3). Note that, for $0 < r < R < \infty$,

$$u_r^V \le u_R^V \le u^V \qquad \text{in } [0,\infty)\times\mathbb{Z}^d, \tag{3.8}$$

as follows by (3.5) because $\{\tau_r > t\} \subset \{\tau_R > t\}$.

Apart from u^V, we also need the fundamental solution $p_R^V(t,\cdot,z)$ of (3.4), that is, the solution to (3.4) with $p_R^V(0,\cdot,z) = \delta_z(\cdot)$ instead of the second line. The Feynman-Kac representation is

$$p_R^V(t,y,z) = \mathbb{E}_y\Big[\exp\{(V,\ell_t)\}\mathbb{1}\big\{\operatorname{supp}(\ell_t) \subset Q_R\big\}\mathbb{1}\big\{X(t) = z\big\}\Big], \tag{3.9}$$

for all $y, z \in \mathbb{Z}^d$. Note that $\sum_{z\in Q_R} p_R^V(t,y,z) = u_R^V(t,y)$.

3.1.2. *Eigenvalue representations.* The second crucial tool for our proofs will be the principal (i.e., the largest) eigenvalue $\lambda_R^{\mathrm{d}}(V)$ of the operator $\kappa\Delta^{\mathrm{d}} + V$ in Q_R with Dirichlet boundary condition. The Rayleigh-Ritz formula reads

$$\lambda_R^{\mathrm{d}}(V) = \sup\big\{(V, g^2) - \kappa\|\nabla g\|_2^2 : g \in \ell^2(\mathbb{Z}^d),\ \|g\|_2 = 1,\ \operatorname{supp}(g) \subset Q_R\big\}. \tag{3.10}$$

Here ∇ denotes the discrete gradient.

Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n$, $n = \#Q_R$, be the eigenvalues of the operator $\kappa\Delta^{\mathrm{d}} + V$ in $\ell^2(Q_R)$ with Dirichlet boundary condition (some of them can be $-\infty$). We also write $\lambda_R^{\mathrm{d},k}(V) = \lambda_k$ for the kth eigenvalue to emphasize its dependence on the potential and the box Q_R. Let $(\mathrm{e}_k)_k$ be an orthonormal basis in $\ell^2(Q_R)$

consisting of the corresponding eigenfunctions $\mathrm{e}_k = \mathrm{e}_R^{\mathrm{d},k}(V)$. (Conventionally, e_k vanishes outside Q_R.) Then we have the Fourier expansions

$$p_R^V(t, y, z) = \sum_k \exp\{t\lambda_k\}\mathrm{e}_k(y)\mathrm{e}_k(z) \tag{3.11}$$

and, by summing this over all $y \in Q_R$,

$$u_R^V(t, \cdot) = \sum_k \exp\{t\lambda_k\}(\mathrm{e}_k, \mathsf{1})_R\, \mathrm{e}_k(\cdot), \tag{3.12}$$

where we used $(\cdot, \cdot)_R$ to denote the inner product in $\ell^2(Q_R)$. Here and henceforth "$\mathsf{1}$" is the function taking everywhere value 1.

3.2. *Power-law scaling.*

PROOF OF PROPOSITION 1.1. Let $\widetilde{H}_t$ be the function given by

$$\widetilde{H}_t(\,\cdot\,) = \frac{\alpha_t^{d+2}}{t} H\biggl(\frac{t}{\alpha_t^d}\,\cdot\biggr). \tag{3.13}$$

By our Scaling Assumption, $\lim_{t\to\infty} \widetilde{H}_t = \widetilde{H}$ on $[0, \infty)$. Note that both $\widetilde{H}_t$ and $\widetilde{H}$ are convex, non-positive and not identically vanishing with value 0 at zero. Consequently, $\widetilde{H}_t$ and $\widetilde{H}$ are continuous and strictly negative in $(0, \infty)$. Moreover, by applying Jensen's inequality to the definition of H, we have that $y \mapsto \widetilde{H}_t(y)/y$ and $y \mapsto \widetilde{H}(y)/y$ are both non-decreasing functions.

Next we shall show that α_{pt}/α_t tends to a finite non-zero limit for all p. Let us pick a $y > 0$ and a $p \in (0, \infty)$ and consider the identity

$$\widetilde{H}_t\left(p\left(\frac{\alpha_t}{\alpha_{pt}}\right)^d y\right) = p\left(\frac{\alpha_t}{\alpha_{pt}}\right)^{d+2} \widetilde{H}_{pt}(y), \tag{3.14}$$

which results by comparing (3.13) with the "time" parameter interpreted once as t and next time as pt. Invoking the monotonicity of $y \mapsto \widetilde{H}_t(y)/y$, it follows that

$$p\left(\frac{\alpha_t}{\alpha_{pt}}\right)^2 \widetilde{H}_{pt}(y) \ge \widetilde{H}_t(py) \quad \text{whenever } \alpha_t \ge \alpha_{pt}. \tag{3.15}$$

This implies that α_{pt}/α_t is bounded away from zero, because we have

$$\liminf_{t\to\infty} \left(\frac{\alpha_{pt}}{\alpha_t}\right)^2 \ge \frac{p\widetilde{H}(y)}{\widetilde{H}(py)} \wedge 1 > 0, \tag{3.16}$$

where "$\wedge$" stands for minimum. Since $p \in (0, \infty)$ was arbitrary, α_{pt}/α_t is also uniformly bounded, by replacing t with t/p.

Let $\phi(p)$ be defined for each p as a subsequential limit of α_{pt}/α_t, that is, $\phi(p) = \lim_{n\to\infty} \alpha_{pt_n}/\alpha_{t_n}$ with some (p-dependent) $t_n \to \infty$. By our previous reasoning $\phi(p)^{-1}$ is non-zero, finite and, for all $y > 0$, it solves for z in the equation

$$\widetilde{H}\left(pz^d y\right) = pz^{d+2}\widetilde{H}(y). \tag{3.17}$$

Here we were allowed to pass to the limiting function $\widetilde{H}$ on the left-hand side of (3.14) because $\widetilde{H}$ is continuous and the scaling limit (1.5) is uniform on compact sets in $(0,\infty)$. But $z \mapsto \widetilde{H}(pz^d y)/z^d$ is non-decreasing while $z \mapsto pz^2\widetilde{H}(y)$ is *strictly* decreasing, so the solution to (3.17) is unique. Hence, the limit $\phi(p) = \lim_{t\to\infty} \alpha_{pt}/\alpha_t$ exists in $(0,\infty)$ for all $p \in (0,\infty)$.

It is easily seen that ϕ is multiplicative on $(0,\infty)$, that is, $\phi(pq) = \phi(p)\phi(q)$. Since $\phi(p) \geq 1$ for $p \geq 1$, by the same token we also have that $p \mapsto \phi(p)$ is non-decreasing. These two properties imply that $\phi(2^n) = \phi(2)^n$ and that $\phi(2)^{\frac{n}{m}} \leq \phi(p) \leq \phi(2)^{\frac{n+1}{m}}$ for any $p > 0$, and m, n integer such that $2^n \leq p^m < 2^{n+1}$. Consequently, $\phi(p) = p^\nu$ with $\nu = \log_2 \phi(2)$. By plugging this back into (3.17) and setting $y = 1$ we get that

$$\widetilde{H}\big(p^{1-d\nu}\big) = \widetilde{H}(1)\, p^{1-(d+2)\nu}. \tag{3.18}$$

The claims (1.6) and (1.7) are thus established by putting $\gamma(1-d\nu) = 1-(d+2)\nu$, which is (1.8). Clearly, $\gamma \in [0,1]$, in order to have the correct monotonicity properties of $y \mapsto \widetilde{H}(y)$ and $y \mapsto \widetilde{H}(y)/y$.

To prove also the second statement in (1.7), we first write

$$\alpha_{2^N} = \alpha_1 \prod_{m=0}^{N-1} \frac{\alpha_{2^{m+1}}}{\alpha_{2^m}} \tag{3.19}$$

which, after taking the logarithm, dividing by $\log 2^N$, and noting that $\alpha_{2^{m+1}}/\alpha_{2^m} \to \phi(2)$ as $m \to \infty$, allows us to conclude that

$$\lim_{N\to\infty} \frac{\log \alpha_{2^N}}{\log 2^N} = \log_2 \phi(2) = \nu. \tag{3.20}$$

The limit for general t is then proved again by sandwiching t between 2^{N-1} and 2^N and invoking the monotonicity of $t \mapsto \alpha_t$. $\square$

3.3. *Relation between χ and $\widetilde{\chi}$.*

PROOF OF PROPOSITION 1.4. Suppose H is in the γ-class and define ν as in Proposition 1.1. Suppose $\chi \neq 0, \infty$ (for a proof of this statement, see Proposition 3.1). The argument hinges on particular scaling properties of the functionals $\psi \mapsto \mathscr{L}_R(\psi)$ and $\psi \mapsto \lambda_R(\psi)$, which enable us to convert (1.15) into (1.23). Given $\psi \in C^-(R)$, let us for each $b \in (0,\infty)$ define $\psi_b \in C^-(bR)$ by

$$\psi_b(x) = \frac{1}{b^2}\psi\left(\frac{x}{b}\right). \tag{3.21}$$

Then we have

$$\mathscr{L}_{bR}(\psi_b) = b^{\frac{1}{\nu}-2}\mathscr{L}_R(\psi) \qquad \text{and} \qquad \lambda_{bR}(\psi_b) = b^{-2}\lambda_R(\psi), \tag{3.22}$$

where in the first relation we used that ψ_b can be converted into ψ in (1.12) by substituting $b^{2/(1-\gamma)}f(\cdot/b)$ in the place of $f(\,\cdot\,)$; the second relation is a result of a simple spatial scaling of the first line in (1.13). Note that $\frac{1}{\nu}-2\geq 1>0$.

Let $\psi^{(n)}\in C^-(R_n)$ be a minimizing sequence of the variational problem in (1.15). Suppose, without loss of generality, that $\mathscr{L}_{R_n}(\psi^{(n)})\to\bar{\mathscr{L}}$ and $\lambda_{R_n}(\psi^{(n)})\to\bar{\lambda}$. Then we have

$$\chi=\bar{\mathscr{L}}-\bar{\lambda}. \tag{3.23}$$

Now pick any $b\in(0,\infty)$ and consider instead the sequence $(\psi_b^{(n)})$. Clearly,

$$\chi\leq\lim_{n\to\infty}\left[\mathscr{L}_{bR_n}\left(\psi_b^{(n)}\right)-\lambda_{bR_n}\left(\psi_b^{(n)}\right)\right]=b^{\frac{1}{\nu}-2}\bar{\mathscr{L}}-b^{-2}\bar{\lambda} \tag{3.24}$$

for all b. By (3.23), the derivative of the right-hand side must vanish at $b=1$, that is,

$$\left(\frac{1}{\nu}-2\right)\bar{\mathscr{L}}+2\bar{\lambda}=0. \tag{3.25}$$

By putting (3.23) and (3.25) together, we easily compute that

$$\bar{\mathscr{L}}=2\nu\chi. \tag{3.26}$$

Note that while $b\mapsto\mathscr{L}_{bR}(\psi_b)$ is strictly increasing, $b\mapsto\lambda_{bR}(\psi_b)$ is strictly decreasing. This allows us to recast (1.15) as

$$\chi=\bar{\mathscr{L}}+\inf_{R>0}\inf\left\{-\lambda_R(\psi):\psi\in C^-(R),\ \mathscr{L}_R(\psi)\leq\bar{\mathscr{L}}\right\}. \tag{3.27}$$

Indeed, we begin by observing that "$\leq$" holds in (3.27), as is verified by pulling $\bar{\mathscr{L}}$ inside the bracket, replacing it with $\mathscr{L}_R(\psi)$, and dropping the last condition. To prove the "$\geq$" part, note that the above sequence $(\psi_b^{(n)})$ for $b<1$ eventually fulfills the last condition in (3.27) because $\mathscr{L}_{bR_n}(\psi_b^{(n)})\to b^{\frac{1}{\nu}-2}\bar{\mathscr{L}}<\bar{\mathscr{L}}$. Since $\lambda_{bR_n}(\psi_b^{(n)})\to b^{-2}\bar{\lambda}$, the right-hand side of (3.27) is no more than $\bar{\mathscr{L}}-b^{-2}\bar{\lambda}$ for any $b<1$. Taking $b\uparrow 1$ and recalling (3.23) proves the equality in (3.27).

With (3.27) in the hand we can finally prove (1.25). By using ψ_b instead of ψ in (3.27), the condition $\mathscr{L}_R(\psi)\leq\bar{\mathscr{L}}$ becomes $\mathscr{L}_R(\psi)\leq b^{\frac{1}{\nu}-2}\bar{\mathscr{L}}$ and the factor b^{-2} appears in front of the infimum. Thus, setting $b^{\frac{1}{\nu}-2}\bar{\mathscr{L}}=d$, which by (3.26) requires that

$$b=\left(\frac{2\nu\chi}{d}\right)^{\frac{\nu}{1-2\nu}}, \tag{3.28}$$

(note that $b\neq 0,\infty$) and invoking (3.26), we recover the variational problem (1.23). Therefore,

$$\chi=\bar{\mathscr{L}}+b^{-2}\widetilde{\chi}=2\nu\chi+\left(\frac{2\nu\chi}{d}\right)^{-\frac{2\nu}{1-2\nu}}\widetilde{\chi}. \tag{3.29}$$

From this, (1.25) follows by simple algebraic manipulations. The claim $\widetilde{\chi}\in(0,\infty)$ is a consequence of (1.25) and the fact that $\chi\in(0,\infty)$. $\square$

3.4. *Approximate variational problems.* The proof of Theorem 1.2 will require some technical approximation properties of the variational problem (1.14). These are stated in Proposition 3.1 below. The reader may gain more motivation for digesting the proof by reading first subsection 4.1.

Let χ_R be the finite-volume counterpart of χ:

$$\chi_R = \inf\{\mathscr{I}(f) - \mathscr{H}_R(f): f \in \mathscr{F}_R\}, \qquad R > 0. \tag{3.30}$$

Suppose H is in the γ-class and introduce the following quantities: In the case $\gamma \in (0, 1)$, let

$$\chi_R^\star(M) = \inf\{\mathscr{I}(f) - \mathscr{H}_R(f \wedge M): f \in \mathscr{F}_R\}, \qquad M > 0, \tag{3.31}$$

for any $R > 0$. For $\gamma = 0$ and any $R > 0$, let

$$\chi_R^\#(\varepsilon) = \inf\{\mathscr{I}(f) - \widetilde{H}(1)|\{f > \varepsilon\}|: f \in \mathscr{F}_R\}, \qquad 0 < \varepsilon \ll R. \tag{3.32}$$

The needed relations between χ, χ_R, $\chi_R^\star(M)$ and $\chi_R^\#(\varepsilon)$ are summarized as follows

PROPOSITION 3.1. *Let H be in the γ-class and let χ be as in* (1.14). *Then*:

(1) $\chi \in (0, \infty)$.
(2) *For $\gamma \in (0, 1)$ and any $R > 0$*, $\lim_{M\to\infty} \chi_R^\star(M) = \chi_R$.
(3) *For $\gamma = 0$ and any $R > 0$*, $\lim_{\varepsilon\downarrow 0} \chi_R^\#(\varepsilon) = \chi_R$.

PROOF OF (1) AND (2). Assertion (1) for $\gamma = 0$ is well-known. Assume that $\gamma \in (0, 1)$ and observe that, due to the perfect scaling properties of both $f \mapsto \mathscr{I}(f)$ and $f \mapsto \mathscr{H}_R(f)$, (3.30) can alternatively be written as

$$\chi_R = \inf\{R^{-2}\mathscr{I}(f) - R^{d(1-\gamma)}\mathscr{H}_1(f): f \in \mathscr{F}_1\}. \tag{3.33}$$

Let $(\lambda_1, \widehat{g})$ be the principal eigenvalue, resp., an associated eigenvector of $-\Delta$ in $[-1, 1]^d$ with Dirichlet boundary condition. Then $\mathscr{I}(\widehat{g}^2) = \kappa\lambda_1 \neq 0, \infty$, which means that

$$\chi_R \le R^{-2}\kappa\lambda_1 - R^{d(1-\gamma)}\widetilde{H}(1)\int |\widehat{g}|^{2\gamma} =: \bar{\chi}_R. \tag{3.34}$$

Since $\widehat{g}$ is continuous and bounded, the integral is finite, whereby $\chi \le \inf_{R>0} \bar{\chi}_R < \infty$.

Claim (2) and the remainder of (1) are then simple consequences of the following observation, whose justification we defer to the end of this proof:

$$\begin{aligned} &\inf\left\{\mathscr{I}(f): f \in \mathscr{F}_R,\ \left\|f\mathbb{1}_{\{f\ge M\}}\right\|_1 \ge \varepsilon\right\} \\ &\qquad \ge \kappa\,\frac{\varepsilon}{2}\left(\frac{M}{8\pi_d}\right)^{2/d}, \quad R, \varepsilon > 0,\ M \ge 8\pi_d d^d/R^d, \end{aligned} \tag{3.35}$$

where π_d is the volume of the unit sphere in $\mathbb{R}^d$. Indeed, to get that χ is non-vanishing, set $\varepsilon = 1/2$ and choose M such that the infimum in (3.35) is strictly larger than $-\widetilde{H}(1)M^{\gamma-1}/2$ for all $R \ge 1$. Clearly, M is finite, so

$C := -\widetilde{H}(1)M^{\gamma-1}/2 > 0$. Then for any $f \in \mathscr{F}_R$ either $\|f\mathsf{1}_{\{f\geq M\}}\|_1 \geq 1/2$, which implies $\mathscr{I}(f) \geq C$, or $\|f\mathsf{1}_{\{f\geq M\}}\|_1 < 1/2$ which implies

$$\begin{aligned} -\mathscr{H}_R(f) &\geq -\widetilde{H}(1)\int f^\gamma \mathsf{1}_{\{f<M\}} \\ &\geq -\widetilde{H}(1)M^{\gamma-1}\int f\,\mathsf{1}_{\{f<M\}} \geq -\widetilde{H}(1)M^{\gamma-1}/2 = C. \end{aligned} \tag{3.36}$$

Thus, in both cases, $\mathscr{I}(f) - \mathscr{H}_R(f) \geq C > 0$ independent of R. Since $R \mapsto \chi_R$ is decreasing, the restriction to $R \geq 1$ is irrelevant which finishes part 1.

To prove also part (2), note first that $\chi_R^\star(M) \leq \chi_R$ for all $M > 0$. Given $\varepsilon > 0$, let $M \geq 1$ be such that the infimum in (3.35) is larger than $\bar{\chi}_R$ in (3.34). Consider (3.31) restricted to $f \in \mathscr{F}_R$ with $\|f\mathsf{1}_{\{f\geq M\}}\|_1 < \varepsilon$. Since for any such f

$$\begin{aligned} -\mathscr{H}_R(f\wedge M) &\geq -\widetilde{H}(1)\int f^\gamma \mathsf{1}_{\{f<M\}} \geq -\mathscr{H}_R(f) + \widetilde{H}(1)\int f^\gamma \mathsf{1}_{\{f\geq M\}} \\ &\geq -\mathscr{H}_R(f) + \widetilde{H}(1)\int f\,\mathsf{1}_{\{f\geq M\}} \geq -\mathscr{H}_R(f) + \widetilde{H}(1)\varepsilon, \end{aligned} \tag{3.37}$$

the restricted infimum is no less than $\chi_R + \widetilde{H}(1)\varepsilon$. Therefore, $\chi_R^\star(M) \geq \bar{\chi}_R \wedge (\chi_R + \widetilde{H}(1)\varepsilon)$, which by $\varepsilon \downarrow 0$ and (3.34) proves part (2) of the claim.

It remains to prove (3.35). To that end, denote the infimum by $\Psi_R(\varepsilon, M)$ and note that

$$\Psi_R(\varepsilon, M) = R^{-2}\Psi_1(\varepsilon, MR^d). \tag{3.38}$$

Indeed, denoting $f^*(\,\cdot\,) = R^d f(\cdot R)$ for any $f \in \mathscr{F}_R$, we have $f^* \in \mathscr{F}_1$, $\mathscr{I}(f^*) = R^2\mathscr{I}(f)$, and $\|f^*\mathsf{1}_{\{f^*\geq MR^d\}}\|_1 = \|f\mathsf{1}_{\{f\geq M\}}\|_1$, whereby (3.38) immediately follows. Since $R^{-2}(MR^d)^{2/d} = M^{2/d}$, it suffices to prove (3.35) just for $R = 1$.

Recall that the operator $-\Delta$ on $[-1,1]^d$ with Dirichlet boundary condition has a compact resolvent, so its spectrum $\sigma(-\Delta)$ is a discrete set of finitely-degenerate eigenvalues. For each $k \in \mathbb{N}$, define the function

$$\varphi_k(x) = \begin{cases} \cos\left(\dfrac{\pi}{2}kx\right), & \text{if } k \text{ is odd,} \\ \sin\left(\dfrac{\pi}{2}kx\right), & \text{if } k \text{ is even.} \end{cases} \tag{3.39}$$

Then $\sigma(-\Delta) = \{\pi^2|k|_2^2/4\colon k \in \mathbb{N}^d\}$, with $|k|_2^2 = k_1^2+\ldots+k_d^2$ and the eigenvectors given as $\omega_k = \varphi_{k_1} \otimes \ldots \otimes \varphi_{k_d}$. Note that the latter form a (Fourier) basis in $L^2([-1,1]^d)$.

Let $\varepsilon > 0$ and $M > 0$ be fixed. Let r be such that $8\pi_d r^d = M$. Note that $r \geq d$. Pick a function $f \in \mathscr{F}_1$ such that $\|f\mathsf{1}_{\{f\geq M\}}\|_1 \geq \varepsilon$ and let $g = \sqrt{f}$. Let g_1, resp., g_2, be the normalized projections of g onto the Hilbert spaces generated by (ω_k) with $|k|_2 \leq r$, resp., $|k|_2 > r$. Then $g = a_1 g_1 + a_2 g_2$ with $|a_1|^2 + |a_2|^2 = 1$. We claim that $\|g_1\|_\infty \leq \sqrt{M}/2$. Indeed, $g_1 = \sum_k c_k\omega_k$ where $(c_k) \in \ell^2(\mathbb{N}^d)$ is such that $c_k = 0$ for all $k \in \mathbb{N}^d$ with $|k|_2 > r$ and

$$\|g_1\|_\infty \leq \sum_k |c_k|\,\|\omega_k\|_\infty \leq \sqrt{\#\{k\colon c_k \neq 0\}} \leq \sqrt{2\pi_d r^d} = \sqrt{M}/2. \tag{3.40}$$

Here we used that $\|\omega_k\|_\infty \leq 1$, then we applied Cauchy-Schwarz inequality and noted that (c_k) is normalized to one in $\ell^2(\mathbb{N}^d)$, because $\|\omega_k\|_2 = 1$ for all $k \in \mathbb{N}^d$. The third inequality follows by the observation $\#\{k: c_k \neq 0\} \leq \pi_d(r+1)^d/2d \leq 2\pi_d r^d$ implied by $r \geq d$.

Let x be such that $g(x) \geq \sqrt{M}$. Then we have $\sqrt{M} \leq g(x) \leq |g_1(x)| + |a_2||g_2(x)|$. Using (3.40), we derive that $|a_2||g_2(x)| \geq \sqrt{M}/2$, whereby we have that $g(x) \leq 2|a_2||g_2(x)|$. This gives us the bound

$$\varepsilon \leq \|f \mathbb{1}_{\{f\geq M\}}\|_1 = \left\|g\mathbb{1}_{\{g\geq\sqrt{M}\}}\right\|_2^2 \leq 4|a_2|^2\|g_2\|_2^2 = 4|a_2|^2, \tag{3.41}$$

that is, $|a_2|^2 \geq \varepsilon/4$. On the other hand,

$$\mathscr{I}(f) = \kappa\|\nabla g\|_2^2 \geq \kappa|a_2|^2\|\nabla g_2\|_2^2 \geq \kappa|a_2|^2 \frac{\pi^2}{4} r^2. \tag{3.42}$$

where we used that $g_1 \perp g_2$ and that g_2 has no overlap with ω_k such that $|k|_2 \leq r$. By putting (3.41) and (3.42) together and noting that $\pi^2/16 \geq 1/2$, (3.35) for $R = 1$ follows. □

PROOF OF (3). Let $\varepsilon \ll (2R)^d$ and consider $f \in \mathscr{F}_R$. Let $g = \sqrt{f}$ and define $g_\varepsilon = (g - \sqrt{\varepsilon})\mathbb{1}\{g \geq \sqrt{\varepsilon}\}$. By a straightforward calculation, $\|g_\varepsilon\|_2^2 \geq 1 - 2\varepsilon(2R)^d - 2\sqrt{\varepsilon(2R)^d}$. Let $f_\varepsilon = (g_\varepsilon/\|g_\varepsilon\|_2)^2$. Then $\mathscr{I}(f) \geq \|g_\varepsilon\|_2^2 \mathscr{I}(f_\varepsilon)$, while $|\{f > \varepsilon\}| = |\{f_\varepsilon > 0\}|$. This implies that $\chi_R^\#(\varepsilon) \geq \chi_R(1 - O(\sqrt{\varepsilon}))$. Since $\chi_R^\#(\varepsilon) \leq \chi_R$, the proof is complete. □

4. Proof of Theorems 1.2 and 1.3 .

We begin by deriving the logarithmic asymptotics for the moments of $u(t,0)$ as stated in Theorem 1.2. The proof is divided into two parts: we separately prove the lower bound and the upper bound. Whenever convenient, we write $\alpha(t)$ instead of α_t.

4.1. *The lower bound.*

We translate the corresponding proof of Gärtner and König (2000) into the discrete setting. Let u denote the solution to (1.1), denoted by u^ξ in Section 3. Similarly, let u_R stand for u_R^ξ for any $R > 0$. Fix $p \in (0, \infty)$, $R > 0$, and consider the box $Q_{R\alpha(pt)} = [-R\alpha(pt), R\alpha(pt)]^d \cap \mathbb{Z}^d$. Note that $\#Q_{R\alpha(pt)} = e^{o(t\alpha_{pt}^{-2})}$ as $t \to \infty$. Recall that $u_{R\alpha(pt)}(t,\cdot) = 0$ outside $Q_{R\alpha(pt)}$ and that $(\cdot,\cdot)$ denotes the inner product in $\ell^2(\mathbb{Z}^d)$. Our first observation is the following.

LEMMA 4.1. *As* $t \to \infty$,

$$\langle u(t,0)^p\rangle \geq \exp\{o(t\alpha_{pt}^{-2})\}\langle (u_{R\alpha(pt)}(t,\cdot), \mathbb{1})^p\rangle. \tag{4.1}$$

PROOF. In the case $p \geq 1$, use the shift-invariance of $z \mapsto u(t,z)$, Jensen's inequality, and the monotonicity assertion (3.8) to obtain

$$\begin{aligned} \langle u(t,0)^p\rangle &= \left\langle \frac{1}{\#Q_{R\alpha(pt)}} \sum_{z\in Q_{R\alpha(pt)}} u(t,z)^p \right\rangle \\ &\geq \left\langle \left(\frac{1}{\#Q_{R\alpha(pt)}} \sum_{z\in Q_{R\alpha(pt)}} u(t,z)\right)^p \right\rangle \\ &\geq \exp\left\{o(t\alpha_{pt}^{-2})\right\}\left\langle (u_{R\alpha(pt)}(t,\cdot), \mathbb{1})^p\right\rangle. \end{aligned} \tag{4.2}$$

In the case $p < 1$, instead of Jensen's inequality we apply

$$\sum_{i=1}^{n} x_i^p \geq \left(\sum_{i=1}^{n} x_i\right)^p, \qquad x_1, \dots, x_n \geq 0,\ n \in \mathbb{N}, \tag{4.3}$$

to deduce similarly as in (4.2) that

$$\begin{aligned} \langle u(t,0)^p \rangle &= \exp\{o(t\alpha_{pt}^{-2})\}\Big\langle \sum_{z \in Q_{R\alpha(pt)}} u(t,z)^p \Big\rangle \\ &\geq \exp\{o(t\alpha_{pt}^{-2})\}\Big\langle \Big(\sum_{z \in Q_{R\alpha(pt)}} u(t,z)\Big)^p \Big\rangle \\ &\geq \exp\{o(t\alpha_{pt}^{-2})\}\Big\langle (u_{R\alpha(pt)}(t,\cdot), 1)^p \Big\rangle. \end{aligned} \tag{4.4}$$

□

The following lemma carries out the necessary large-deviation arguments for the case $p = 1$. Lemma 4.3 then reduces the proof of arbitrary p to the case $p = 1$. Recall the "finite-R" version χ_R of (1.14) defined in (3.30).

LEMMA 4.2. *Let $R > 0$. Then for $t \to \infty$,*

$$-\chi_R + o(1) \leq \frac{\alpha_t^2}{t} \log \langle (u_{R\alpha(t)}(t,\cdot), 1) \rangle \leq -\chi_{3R} + o(1), \tag{4.5}$$

$$\frac{\alpha_t^2}{t} \log \left\langle \sum_k \exp\left\{ t\lambda_{R\alpha(t)}^{\mathrm{d},k}(\xi) \right\} \right\rangle \leq -\chi_{3R} + o(1). \tag{4.6}$$

LEMMA 4.3. *Let $R > 0$. Then for $t \to \infty$,*

$$\langle (u_{R\alpha(pt)}(t,\cdot), 1)^p \rangle \geq \exp\left\{ o(t\alpha_{pt}^{-2}) \right\} \langle (u_{R\alpha(pt)}(pt,\cdot), 1) \rangle. \tag{4.7}$$

Lemmas 4.1, 4.2 and 4.3 make the proof of the lower bound immediate:

PROOF OF THEOREM 1.2, LOWER BOUND. By combining (4.1), (4.7) and the left inequality in (4.5) for pt instead of t, we see that $(\alpha_{pt}^2/pt)\log\langle u(t,0)^p\rangle \geq -\chi_R + o(1)$. Since $\lim_{R\to\infty} \chi_R = \chi$, the left-hand side of (1.16), with "$\liminf$" instead of "lim," is bounded below by $-\chi$. By Proposition 3.1(1), χ positive, finite and non-zero. □

The remainder of this subsection is devoted to the proof of the two lemmas.

PROOF OF LEMMA 4.2. Recall the notation of subsection 3.1. By taking the expectation over ξ (and using that ξ is an i.i.d. field) and recalling (3.7), we have for any $z \in Q_{R\alpha(t)}$ that

$$
\begin{aligned}
\langle u_{R\alpha(t)}(t,z)\rangle &= \Big\langle \mathbb{E}_z\big[\exp\{(\xi,\ell_t)\}\,1\{\tau_{R\alpha(t)} > t\}\big]\Big\rangle \\
&= \mathbb{E}_z\Bigg[\prod_{y\in\mathbb{Z}^d}\langle \exp\{\ell_t(y)\xi(y)\}\rangle 1\{\tau_{R\alpha(t)} > t\}\Bigg] \\
&= \mathbb{E}_z\Bigg[\exp\Big\{\sum_{y\in\mathbb{Z}^d} H\big(\ell_t(y)\big)\Big\}1\big\{\operatorname{supp}(\ell_t)\subset Q_{R\alpha(t)}\big\}\Bigg],
\end{aligned}
\tag{4.8}
$$

Consider the scaled version $\bar{\ell}_t\colon \mathbb{R}^d \to [0,\infty)$ of the local times

$$
\bar{\ell}_t(x) = \frac{\alpha_t^d}{t}\ell_t\big(\lfloor x\alpha_t\rfloor\big), \qquad x\in\mathbb{R}^d.
\tag{4.9}
$$

Let $\widetilde{\mathscr{F}}$ be the space of all non-negative Lebesgue almost everywhere continuous functions in $L^1(\mathbb{R}^d)$ with a bounded support. Clearly, $\mathscr{F} \subset \widetilde{\mathscr{F}}$ and $\bar{\ell}_t \in \widetilde{\mathscr{F}}$. Introduce the functional $\mathscr{H}^{(t)}\colon \widetilde{\mathscr{F}} \to [-\infty, 0]$, assigning each $f\in\widetilde{\mathscr{F}}$ the value

$$
\mathscr{H}^{(t)}(f) = \int_{\mathbb{R}^d} \widetilde{H}_t\big(f(x)\big)\,dx,
\tag{4.10}
$$

where we recalled (3.13). Substituting $\bar{\ell}_t$ and $\mathscr{H}^{(t)}$ into (4.8), we obtain

$$
\begin{aligned}
&\langle (u_{R\alpha(t)}(t,\cdot),1)\rangle \\
&\quad = \sum_{z\in Q_{R\alpha(t)}} \mathbb{E}_z\Bigg[\exp\Big\{\frac{t}{\alpha_t^2}\mathscr{H}^{(t)}\big(\bar{\ell}_t\big)\Big\}1\{\operatorname{supp}(\bar{\ell}_t)\subset[-R,R+\alpha_t^{-1}]^d\}\Bigg].
\end{aligned}
\tag{4.11}
$$

Using shift-invariance and the fact that $\mathscr{H}^{(t)}(f)\le \mathscr{H}^{(t)}(f\wedge M)$ for any $M>0$, we have

$$
\begin{aligned}
&\mathbb{E}_0\Bigg[\exp\Big\{\frac{t}{\alpha_t^2}\mathscr{H}^{(t)}\big(\bar{\ell}_t\big)\Big\}1\Big\{\operatorname{supp}(\bar{\ell}_t)\subset[-R,R]^d\Big\}1\{\bar{\ell}_t\le M\}\Bigg] \\
&\quad\le \langle (u_{R\alpha(t)}(t,\cdot),1)\rangle \\
&\quad\le \exp\big\{o(t\alpha_t^{-2})\big\} \\
&\qquad\times \mathbb{E}_0\Bigg[\exp\Big\{\frac{t}{\alpha_t^2}\mathscr{H}^{(t)}\big(\bar{\ell}_t\wedge M\big)\Big\}1\Big\{\operatorname{supp}(\bar{\ell}_t)\subset[-3R,3R]^d\Big\}\Bigg].
\end{aligned}
\tag{4.12}
$$

It is well known that the family of scaled local times $(\bar{\ell}_t)_{t>0}$ satisfies a weak large-deviation principle on $L^1(\mathbb{R}^d)$ with rate $t\alpha_t^{-2}$ and rate function $\mathscr{I}$ defined in (1.10). This fact has been first derived by Donsker and Varadhan (1979) for the discrete-time random walk; for the changes of the proof in the continuous time case we refer to Chapter 4 of the monograph by Deuschel and Stroock (1989). The large-deviation principle allows us to use Varadhan's integral lemma to convert both bounds in (4.12) into corresponding variational formulas. Note that, if both $\mathscr{I}$ and $\mathscr{H}$ are appropriately extended to $L^1([-R,R]^d)$, all infima (3.30), (3.31) and (3.32) can be taken over $f\in L^1([-R,R]^d)$ with the same result. In the sequel, we have to make a distinction between the cases $\gamma\in(0,1)$ and $\gamma=0$.

In the case $\gamma \in (0,1)$, our Scaling Assumption implies that, for every $M > 0$, $f \mapsto \mathscr{H}(f)$ is continuous and $\mathscr{H}^{(t)}$ converges to $\mathscr{H}$ uniformly on the space of all measurable functions $[-R,R]^d \to [0,M]$ with L^∞ topology. Indeed, for any such function f and any $\varepsilon > 0$, the integral (4.10) can be split into $\mathscr{H}^{(t)}(f\mathbb{1}_{\{f>\varepsilon\}})$ and $\mathscr{H}^{(t)}(f\mathbb{1}_{\{0<f\le\varepsilon\}})$. The former then converges uniformly to $\mathscr{H}(f\mathbb{1}_{\{f>\varepsilon\}})$, while the latter can be bounded as

$$0 \ge \mathscr{H}^{(t)}\big(f\mathbb{1}_{\{0<f\le\varepsilon\}}\big) \ge \widetilde{H}_t(\varepsilon)\big|\{0<f\le\varepsilon\}\big| \ge (2R)^d\widetilde{H}_t(\varepsilon), \tag{4.13}$$

where we invoked the monotonicity of $y \mapsto \widetilde{H}_t(y)$. Taking $\varepsilon \downarrow 0$ proves that this part is negligible for $\mathscr{H}^{(t)}(f)$ and, if $t \to \infty$ is invoked before $\varepsilon \downarrow 0$, it also shows that $\mathscr{H}(f\mathbb{1}_{\{f>\varepsilon\}}) \to \mathscr{H}(f)$ uniformly in f as $\varepsilon \downarrow 0$. Having verified continuity, Varadhan's lemma (and $M \to \infty$) readily outputs the left inequality in 4.5, while on the right-hand side it yields a bound in terms of the quantity $\chi^\star_{3R}(M)$ defined in (3.31). By Proposition 3.1(2), $\chi^\star_{3R}(M)$ tends to χ_{3R} as $M \to \infty$, which proves the inequality on the right of (4.5).

In the case $\gamma = 0$, the lower bound goes along the same line, but we have to be more careful with (4.13), since $\lim_{\varepsilon\downarrow 0}\lim_{t\to\infty}\widetilde{H}_t(\varepsilon) \ne 0$ in this case. Let us estimate

$$\begin{aligned}\mathscr{H}^{(t)}(f) &= \mathscr{H}^{(t)}\big(f\mathbb{1}_{\{0<f\le\varepsilon\}}\big) + \mathscr{H}^{(t)}\big(f\mathbb{1}_{\{f>\varepsilon\}}\big)\\ &\ge \widetilde{H}_t(\varepsilon)\big|\{0<f\le\varepsilon\}\big| + \mathscr{H}^{(t)}\big(f\mathbb{1}_{\{f>\varepsilon\}}\big)\\ &\ge \mathscr{H}(f) - \big|\mathscr{H}^{(t)}(f\mathbb{1}_{\{f>\varepsilon\}}) - \mathscr{H}(f\mathbb{1}_{\{f>\varepsilon\}})\big| - (2R)^d\big|\widetilde{H}_t(\varepsilon) - \widetilde{H}(\varepsilon)\big|,\end{aligned} \tag{4.14}$$

where we invoked the explicit form of $f \mapsto \mathscr{H}(f)$. Since both absolute values on the right-hand side tend to 0 as $t \to \infty$ uniformly in $f \le M$, the lower bound in (4.5) follows again by Varadhan's lemma and limit $M \to \infty$. For the upper bound, the estimate and uniform limit $\mathscr{H}^{(t)}(f) \le \mathscr{H}^{(t)}(f\mathbb{1}_{\{f>\varepsilon\}}) \to \mathscr{H}(f\mathbb{1}_{\{f>\varepsilon\}})$ give us a bound in terms of the quantity $\chi^\#_{3R}(\varepsilon)$ defined in (3.32). By then M is irrelevant, so by invoking Proposition 3.1(3), the claim is proved by taking $\varepsilon \downarrow 0$.

It remains to prove (4.6). Recall the shorthand $\lambda_k = \lambda^{\mathrm{d},k}_{R\alpha(t)}(\xi)$. By (3.11), (3.9) and analogously to (4.8), we have

$$\begin{aligned}\Big\langle \sum_k \exp\{t\lambda_k\}\Big\rangle &= \sum_{z\in Q_{R\alpha(t)}} \langle p_{R\alpha(t)}(t,z,z)\rangle\\ &= \Big\langle \sum_{z\in Q_{R\alpha(t)}} \mathbb{E}_z\big[\exp\{(\xi,\ell_t)\}\mathbb{1}\{\tau_{R\alpha_t} > t\}\mathbb{1}\{X(t)=z\}\big]\Big\rangle.\end{aligned} \tag{4.15}$$

Noting that $\mathbb{1}\{X(t)=z\} \le 1$, we thus have $\langle \sum_k \exp\{t\lambda_k\}\rangle \le \langle (u_{R\alpha(t)}(t,\cdot),\mathbb{1})\rangle$. With this in the hand, (4.6) directly follows by the right inequality in (4.5). □

Proof of Lemma (4.3). In the course of the proof, we use abbreviations $r = R\alpha(pt)$ and $\lambda_k = \lambda^{\mathrm{d},k}_r(\xi)$. Recall that $(\mathrm{e}_k)_k$ denotes an orthonormal basis in $\ell^2(Q_r)$ (with inner product $(\cdot,\cdot)_r$) consisting of the eigenfunctions of $\kappa\Delta^{\rm d}+\xi$ with Dirichlet boundary condition.

We first turn to the case $p \geq 1$. Use the Fourier expansion (3.12) and the inequality

$$\left(\sum_{i=1}^n x_i\right)^p \geq \sum_{i=1}^n x_i^p, \qquad x_1,\dots,x_n \geq 0,\ n\in\mathbb{N}, \tag{4.16}$$

to obtain

$$\langle (u_r(t,\cdot),\mathbb{1})^p\rangle = \left\langle \left(\sum_k \exp\{t\lambda_k\}\,(\mathrm{e}_k,\mathbb{1})_r^2\right)^p\right\rangle \geq \left\langle \sum_k \exp\{pt\lambda_k\}\,(\mathrm{e}_k,\mathbb{1})_r^{2p}\right\rangle. \tag{4.17}$$

By Jensen's inequality for the probability measure

$$(l, d\xi)\mapsto \left\langle \sum_k \exp\{pt\lambda_k\}\right\rangle^{-1} \exp\{pt\lambda_l\}\mathrm{Prob}(d\xi), \tag{4.18}$$

we have that the r.h.s. of (4.17) is greater than or equal to

$$\begin{aligned} &\left(\frac{\langle \sum_k \exp\{pt\lambda_k\}(\mathrm{e}_k,\mathbb{1})_r^2\rangle}{\langle \sum_k \exp\{pt\lambda_k\}\rangle}\right)^p \left\langle \sum_k \exp\{pt\lambda_k\}\right\rangle \\ &\quad\geq \exp\{o(t\alpha_{pt}^{-2})\}\left\langle \sum_k \exp\{pt\lambda_k\}\,(\mathrm{e}_k,\mathbb{1})_r^2\right\rangle \\ &\quad= \exp\{o(t\alpha_{pt}^{-2})\}\,\langle (u_r(pt,\cdot),\mathbb{1})\rangle, \end{aligned} \tag{4.19}$$

where we recalled from the end of the proof of Lemma 4.2 that $\langle \sum_k \exp\{pt\lambda_k\}\rangle \leq \langle (u_r(pt,\cdot),\mathbb{1})\rangle = \langle \sum_k \exp\{pt\lambda_k\}(\mathrm{e}_k,\mathbb{1})_r^2\rangle$, inserted $1 \geq \exp\{o(t\alpha_{pt}^{-2})\}(\mathrm{e}_k,\mathbb{1})_r^2$ and applied (3.12).

In the case $p\in(0,1)$, we apply Jensen's inequality as follows:

$$\begin{aligned} \langle (u_r(t,\cdot),\mathbb{1})^p\rangle &= (\mathbb{1},\mathbb{1})_r^p \left\langle \left(\sum_k \exp\{t\lambda_k\}\frac{(\mathrm{e}_k,\mathbb{1})_r^2}{(\mathbb{1},\mathbb{1})_r}\right)^p\right\rangle \\ &\geq (\mathbb{1},\mathbb{1})_r^p \left\langle \sum_k \exp\{pt\lambda_k\}\frac{(\mathrm{e}_k,\mathbb{1})_r^2}{(\mathbb{1},\mathbb{1})_r}\right\rangle. \end{aligned} \tag{4.20}$$

Invoking that $(\mathbb{1},\mathbb{1})_r = \exp\{o(t\alpha_{pt}^{-2})\}$, the proof is complete by recalling (3.12) once again. □

4.2. *The upper bound.* Recall that Q_R denotes the discrete box $[-R,R]^d\cap\mathbb{Z}^d$. We abbreviate $r(t)=t\log t$ for $t>0$. For $z\in\mathbb{Z}^d$ and $R>0$, we denote by $\lambda_{z;R}^{\rm d}(V)$ the principal eigenvalue of the operator $\kappa\Delta^{\rm d}+V$ with Dirichlet boundary conditions in the *shifted* box $z+Q_R$. The main ingredient in the proof of the upper bound in Theorem 1.2 is (the following) Proposition 4.4, which provides an estimate of $u(t,0)$ in terms of the maximal principal eigenvalue of $\kappa\Delta^{\rm d}+V$ in small subboxes ("microboxes") of the "macrobox" $Q_{r(t)}$.

PROPOSITION 4.4. *Let $B_R(t) = Q_{r(t)+2\lfloor R\rfloor}$. Then there is a constant $C = C(d,\kappa) > 0$ such that, for any $R, t > C$ and any potential $V\colon \mathbb{Z}^d \to [-\infty, 0]$,*

$$u^V(t,0) \le \exp\{-t\} + \exp\{Ct/R^2\}\big(3r(t)\big)^d \exp\left\{t \max_{z\in B_R(t)} \lambda^{\mathrm{d}}_{z;2R}(V)\right\}. \tag{4.21}$$

By Proposition 4.4 and inequality (4.6), the upper bound in Theorem 1.2 is now easy:

PROOF OF THEOREM 1.2, UPPER BOUND. Let $p \in (0,\infty)$. First, notice that the second term in (4.21) can be estimated in terms of a sum:

$$\exp\left\{t \max_{z\in B_R(t)} \lambda^{\mathrm{d}}_{z;2R}(V)\right\} \le \sum_{z\in B_R(t)} \exp\{t\lambda^{\mathrm{d}}_{z;2R}(V)\}. \tag{4.22}$$

Thus, applying (4.21) to $u(t,0)$ (i.e., for $V = \xi$) with R replaced by $R\alpha(pt)$ for some fixed $R > 0$, raising both sides to the pth power and using (4.22) we get

$$\begin{aligned} u(t,0)^p \le 2^p \max\Bigg\{ &\exp\{-pt\},\ \exp\{Cpt/(R^2\alpha(pt)^2)\}\big(3r(t)\big)^{pd} \\ &\times \sum_{z\in B_{R\alpha(pt)}(t)} \exp\left\{pt\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi)\right\}\Bigg\}. \end{aligned} \tag{4.23}$$

Next we take the expectation w.r.t. ξ and note that, by the shift-invariance of ξ, the distribution of $\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi)$ does not depend on $z \in \mathbb{Z}^d$. Take logarithm, multiply by $\alpha_{pt}^2/(pt)$ and let $t \to \infty$. Then we have that

$$\limsup_{t\to\infty} \frac{\alpha_{pt}^2}{pt} \log\langle u(t,0)^p\rangle \le \frac{C}{R^2} + \limsup_{t\to\infty} \frac{\alpha_{pt}^2}{pt} \log\left\langle \exp\{pt\lambda^{\mathrm{d}}_{2R\alpha(pt)}(\xi)\}\right\rangle, \tag{4.24}$$

where we also used that $\exp\{-pt\}$, $r(t)^{pd}$ and $\#B_{R\alpha(pt)}(t)$ are all $\exp\{o(t\alpha_{pt}^{-2})\}$ as $t \to \infty$. Since

$$\exp\left\{pt\lambda^{\mathrm{d}}_{R\alpha(pt)}(\xi)\right\} \le \sum_k \exp\left\{pt\lambda^{\mathrm{d},k}_{R\alpha(pt)}(\xi)\right\}, \tag{4.25}$$

(4.6) for pt instead of t implies that the second term on the right-hand side of (4.24) is bounded by $-\chi_{6R}$. The upper bound in Theorem 1.2 then follows by letting $R \to \infty$. $\square$

Now we can turn to the proof of Proposition 4.4. We begin by showing that $u^V(t,0)$ is very close to the solution $u^V_{r(t)}(t,0)$ of the initial-boundary problem (3.4), whenever the size $r(t) = t\log t$ of the "macrobox" $Q_{r(t)}$ is large enough.

LEMMA 4.5. *For sufficiently large $t > 0$,*

$$u^V(t,0) \le e^{-t} + u^V_{r(t)}(t,0). \tag{4.26}$$

PROOF. It is immediate from (3.2) and (3.5) with $r = r(t)$ that

$$u^V(t,0) - u^V_{r(t)}(t,0) = \mathbb{E}_0\left[\exp\left\{\int_0^t V\big(X(s)\big)\,ds\right\} \mathbb{1}\{\tau_{r(t)} \le t\}\right]. \tag{4.27}$$

According to Lemma 2.5(a) in Gärtner and Molchanov (1998), we have, for every $r > 0$,

$$\mathbb{P}_0(\tau_r \le t) \le 2^{d+1} \exp\left\{-r\left(\log \frac{r}{d\kappa t} - 1\right)\right\}. \tag{4.28}$$

Using this for $r = r(t) = t\log t$ in (4.27), we see that, for sufficiently large t (depending only on d and κ), the right-hand side of (4.27) is no more than e^{-t}. □

The crux of our proof of Proposition 4.4 is that the principal eigenvalue in a box Q_r of size r can be bounded by the maximal principal eigenvalue in "microboxes" $z + Q_R$ contained in Q_r, at the cost of changing the potential slightly. This will later allow us to move the t-dependence of the principal eigenvalue from the *size* of $Q_{r(t)}$ to the *number* of "microboxes." The following lemma is a discrete version of Proposition 1 of Gärtner and König (2000) and is based on ideas from Gärtner and Molchanov (2000). However, for the sake of completeness, no familiarity with Gärtner and König (2000) is assumed.

LEMMA 4.6. *There is a number $C > 0$ such that for every integer R, there is a function $\Phi_R: \mathbb{Z}^d \to [0,\infty)$ with the following properties*:

1. *Φ_R is $2R$-periodic in every component.*
2. $\|\Phi_R\|_\infty \le C/R^2$.
3. *For any potential $V: \mathbb{Z}^d \to [-\infty, 0]$ and any $r > R$,*

$$\lambda^{\mathrm{d}}_r(V - \Phi_R) \le \max_{z \in Q_{r+2R}} \lambda^{\mathrm{d}}_{z;2R}(V). \tag{4.29}$$

PROOF. The idea is to construct a partition of unity:

$$\sum_{k\in\mathbb{Z}^d} \eta_k^2(z) = 1, \qquad z \in \mathbb{Z}^d, \tag{4.30}$$

where $\eta_k(z) = \eta(z - 2Rk)$ with

$$\eta: \mathbb{Z}^d \to [0,1] \quad \text{such that } \eta \equiv 1 \text{ on } Q_{R/2},\ \operatorname{supp}(\eta) \subset Q_{3R/2}. \tag{4.31}$$

Then we put

$$\Phi_R(z) = \kappa \sum_{k\in\mathbb{Z}^d} \big|\nabla \eta_k(z)\big|^2, \qquad z \in \mathbb{Z}^d, \tag{4.32}$$

where ∇ is the discrete gradient. Obviously, Φ_R is $2R$-periodic in every component. The construction of η such that Φ_R satisfies claim 2 is given at the end of this proof.

Assuming the existence of the above partition of unity, we turn to the proof of (4.29). Recall the Rayleigh-Ritz formula (3.10), which can be shortened as $\lambda_r^{\mathrm{d}}(V) = \sup G^V(g)$, where

$$G^V(g) = \sum_{z\in\mathbb{Z}^d} \bigl(-\kappa|\nabla g(z)|^2 + V(z)g^2(z)\bigr), \tag{4.33}$$

and where the supremum is over normalized $g \in \ell^2(\mathbb{Z}^d)$ with support in Q_r. Let g be such a function, and define $g_k(z) = g(z)\eta_k(z)$ for $k, z \in \mathbb{Z}^d$. Note that, according to (4.30) and (4.31), we have $\sum_k \|g_k\|_2^2 = 1$ and $\operatorname{supp}(g_k) \subset 2kR + Q_{3R/2}$.

The pivotal point of the proof is the bound

$$G^{V-\Phi_R}(g) \le \sum_{k\in\mathbb{Z}^d} \|g_k\|_2^2\, G^V\left(\frac{g_k}{\|g_k\|_2}\right). \tag{4.34}$$

In order to prove this inequality, we invoke the rewrite

$$\begin{aligned} &g(y)\eta_k(y) - g(x)\eta_k(x) \\ &\quad = g(x)\bigl(\eta_k(y) - \eta_k(x)\bigr) + \eta_k(y)\bigl(g(y) - g(x)\bigr), \end{aligned} \tag{4.35}$$

recall (4.30) and (4.32), and then perform a couple of symmetrizations to derive

$$\kappa \sum_{k\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d} \bigl|\nabla g_k(x)\bigr|^2 = \sum_{x\in\mathbb{Z}^d}\Bigl[\kappa\bigl|\nabla g(x)\bigr|^2 + \Phi_R(x)g(x)^2\Bigr] + \kappa\Theta, \tag{4.36}$$

where Θ is given by the formula

$$\Theta = -\tfrac12 \sum_{k\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d}\sum_{y:y\sim x} \bigl[g(y) - g(x)\bigr]^2\bigl[\eta_k(y) - \eta_k(x)\bigr]^2 \le 0. \tag{4.37}$$

Using this bound on the right-hand side of (4.36), we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} \|g_k\|_2^2\, G^V\left(\frac{g_k}{\|g_k\|_2}\right) &= \sum_{k\in\mathbb{Z}^d} G^V(g_k) \\ &= \sum_{z\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\Bigl[-\kappa\bigl|\nabla g_k(z)\bigr|^2 + V(z)g_k^2(z)\Bigr] \\ &\ge \sum_{z\in\mathbb{Z}^d}\Bigl[-\kappa\bigl|\nabla g(z)\bigr|^2 + \bigl(V(z) - \Phi_R(z)\bigr)g^2(z)\Bigr] \\ &= G^{V-\Phi_R}(g), \end{aligned} \tag{4.38}$$

which is exactly inequality (4.34).

Since the support of g_k is contained in $2kR + Q_{3R/2}$, the Rayleigh-Ritz formula yields that

$$G^V\left(\frac{g_k}{\|g_k\|_2}\right) \le \lambda_{2kR;3R/2}^{\mathrm{d}}(V) \le \lambda_{2kR;2R}^{\mathrm{d}}(V) \tag{4.39}$$

whenever $\|g_k\|_2 \ne 0$ (which requires, in particular, that $2R|k| - 3R/2 \le r$). Estimating these eigenvalues by their maximum and taking into account that

$\sum_{k\in\mathbb{Z}^d}\|g_k\|_2^2=\|g\|_2^2=1$, we find that the right-hand side of (4.34) does not exceed the right-hand side of (4.29). The claim (4.29) is finished by passing to the supremum over g on the left-hand side of (4.34).

For the proof to be complete, it remains to construct the functions η and Φ_R with the properties (4.30) and (4.31) and such that $\|\Phi_R\|_\infty \le C/R^2$ for some $C>0$. First, the ansatz

$$\eta(z)=\prod_{i=1}^{d}\zeta(z_i),\qquad z=(z_1,\dots,z_d)\in\mathbb{Z}^d, \tag{4.40}$$

reduces the construction of η to the case $d=1$ (with η replaced by ζ). In order to define $z\mapsto\zeta(z)$, let $\varphi:\mathbb{R}\to[0,1]$ be such that both $\sqrt{\varphi}$ and $\sqrt{1-\varphi}$ are smooth, $\varphi\equiv 0$ on $(-\infty,-1]$ and $\varphi\equiv 1$ on $[0,\infty)$ and $\varphi(-x)=1-\varphi(x)$ for all $x\in\mathbb{R}$. Then we put

$$\zeta(z)=\sqrt{\varphi\left(\frac{1}{2}+\frac{z}{R}\right)\left[1-\varphi\left(-\frac{3}{2}+\frac{z}{R}\right)\right]},\qquad z\in\mathbb{Z}. \tag{4.41}$$

In order to verify that the functions $\zeta_k^2(z)=\zeta^2(z+2Rk)$ with $k\in\mathbb{Z}$ form a partition of unity on $\mathbb{R}$, we first note that $\zeta(z)\equiv 1$ on $[-R/2,R/2]$ while $\zeta(z)+\zeta(z-2R)=1-\varphi(-3/2+z/R)+\varphi(-3/2+z/R)=1$ for $z\in[R/2,3R/2]$. Moreover, as follows by a direct computation, $\sup_{z\in\mathbb{Z}}\sum_k|\nabla\zeta_k(z)|^2\le 4\|(\sqrt{\varphi})'\|_\infty^2R^{-2}$, which means that claim 2 is satisfied with $C=4d\|(\sqrt{\varphi})'\|_\infty^2$. This completes the construction and also the proof. □

PROOF OF PROPOSITION 4.4. Having all the prerequisites, the proof is easily completed. First,

$$\int_0^t V\big(X(s)\big)\,ds\le t\frac{C}{R^2}+\int_0^t(V-\Phi_R)\big(X(s)\big)\,ds,\qquad t>0. \tag{4.42}$$

by Lemma 4.6(2). Therefore, combining (3.2) with Lemma 4.5, we have that

$$u^V(t,0)\le\exp\{-t\}+\exp\{tC/R^2\}\,u_{r(t)}^{V-\Phi_R}(t,0) \tag{4.43}$$

whenever t is large enough. Invoking also the Fourier expansion (3.12) w.r.t. the eigenfunctions of $\kappa\Delta^{\mathrm{d}}+V-\Phi_R$ in $\ell^2(Q_{r(t)})$ and the fact that $(1,1)_{r(t)}=\#Q_{r(t)}$, we find that

$$u_{r(t)}^{V-\Phi_R}(t,0)\le\sum_{z\in Q_{r(t)}}u_{r(t)}^{V-\Phi_R}(t,z)\le\#Q_{r(t)}\exp\big\{t\lambda_{r(t)}^{\mathrm{d}}(V-\Phi_R)\big\}. \tag{4.44}$$

Now apply Lemma 4.6 for $r=r(t)=t\log t$ to complete the proof. □

4.3. *Proof of Lifshitz tails.* Let ν_R denote the empirical measure on the spectrum of $\mathfrak{H}_R$, that is,

$$\nu_R=\frac{1}{\#Q_R}\sum_k\delta_{\{-\lambda_k\}}, \tag{4.45}$$

where $\lambda_k = \lambda_R^{\mathrm{d},k}(\xi) = -E_k$ denotes the eigenvalues of $-\mathfrak{H}_R$. Note that ν_R has total mass at most 1, because the dimension of the underlying Hilbert space is bounded by $\#Q_R$. Due to (1.2), ν_R is supported on $[0,\infty)$. Moreover, $N_R(E)$ in (1.18) is precisely $\#Q_R\,\nu_R([0,E])$, for any $E\in[0,\infty)$. Let $\mathscr{L}(\nu_R,t)$ be the Laplace transform of ν_R evaluated at $t\ge 0$,

$$\mathscr{L}(\nu_R,t)=\int \nu_R(d\lambda)\,\exp\{-\lambda t\}=\frac{1}{\#Q_R}\sum_k \exp\{t\lambda_k\}. \tag{4.46}$$

Adapting Theorem VI.1.1. in Carmona and Lacroix (1990) to our discrete setting, the existence of the limit (1.19) is proved by establishing the a.s. convergence of ν_R to some non-random ν, which in turn is done by proving that $\mathscr{L}(\nu_R,\cdot)$ has a.s. a non-random limit. In our case, the argument is so short that we find it convenient to reproduce it here.

Invoking (3.11) and (3.9) for $V=\xi$, we have from (4.46) that

$$\mathscr{L}(\nu_R,t)=\frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{E}_z\left\{\exp\left[\int_0^t \xi\big(X(s)\big)ds\right]\mathbb{1}\{\tau_R>t\}\mathbb{1}\big\{X(t)=z\big\}\right\}. \tag{4.47}$$

Next, writing $\mathbb{1}\{\tau_R>t\}=1-\mathbb{1}\{\tau_R\le t\}$ we arrive at two terms, the second of which tends to zero as $R\to\infty$ for any fixed t by the estimate

$$\begin{aligned}0&\le\frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{E}_z\left\{\exp\left\{\int_0^t \xi(X(s))ds\right\}\mathbb{1}\{\tau_R\le t\}\mathbb{1}\{X(t)=z\}\right\}\\&\le\frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{P}_z(\tau_R\le t),\end{aligned} \tag{4.48}$$

where we used that $\xi\le 0$. Indeed, $\mathbb{P}_z(\tau_R\le t)\le\mathbb{P}_0(\tau_{R(z)}\le t)$ with $R(z)=\operatorname{dist}(z,Q_R^{\mathrm{c}})$, which by (4.28) means that $\mathbb{P}_z(\tau_R\le t)$ decays exponentially with $\operatorname{dist}(z,Q_R^{\mathrm{c}})$. Thus, $\mathscr{L}(\nu_R,t)$ is asymptotically given by the right-hand side of (4.47) with $\mathbb{1}\{\tau_R>t\}$ omitted. But then the right-hand side is the average of an L^1 function over the translates in the box Q_R, so by the ergodic theorem,

$$\lim_{R\to\infty}\mathscr{L}(\nu_R,t)=\left\langle\mathbb{E}_0\left\{\exp\left[\int_0^t\xi\big(X(s)\big)ds\right]\mathbb{1}\big\{X(t)=0\big\}\right\}\right\rangle \tag{4.49}$$

ξ-almost surely for every fixed $t\ge 0$ (the exceptional null set is a priori t-dependent). Both the right-hand side of (4.49) and $\mathscr{L}(\nu_R,t)$ for every R are continuous and decreasing in t. Consequently, with probability one, (4.49) holds for all $t\ge 0$.

The right-hand side of (4.49) inherits the complete monotonicity property from $\mathscr{L}(\nu_R,t)$; it thus equals $\mathscr{L}(\nu,t)$ where ν is some measure supported in $[0,\infty)$. Moreover, this also implies that $\nu_R\to\nu$ weakly as $R\to\infty$. In particular, we have $n(E)=\nu([0,E])$ for any $E\ge 0$.

PROOF OF THEOREM 1.3. From (4.49) we immediately have

$$\exp\big\{o(t/\alpha_t^2)\big\}\left\langle\exp\left\{t\lambda^{\mathrm{d}}_{R\alpha(t)}\right\}\right\rangle\le\mathscr{L}(\nu,t)\le\langle u(t,0)\rangle,\qquad R\ge 0, \tag{4.50}$$

where $\lambda^{\mathrm{d}}_{R\alpha(t)}$ is as in (3.10). Here, for the upper bound we simply neglected $\mathbb{1}\{X(t)=0\}$ in (4.49), whereas for the lower bound we first wrote (4.49) as a normalized sum of the right-hand side of (4.49) with the walk starting and ending at all possible $z \in Q_{R\alpha_t}$, and then inserted $\mathbb{1}\{\operatorname{supp}(\ell_t) \subset Q_{R\alpha(t)}\}$, applied (3.9) and (3.11), and then recalled (4.25). The factor $\exp\{o(t/\alpha_t^2)\}$ comes from the normalization by $\#Q_{R\alpha(t)}$ in the first step. Using subsequently (4.24) for $p=1$, the left-hand side of (4.50) is further bounded from below by $\exp\{(t/\alpha_t^2)(-4C/R^2+o(1))\}\langle u(t,0)\rangle$. Then Theorem 1.2 and the limit $R\to\infty$ enable us to conclude that

$$\lim_{t\to\infty}\frac{\alpha_t^2}{t}\log\mathscr{L}(\nu,t)=-\chi. \tag{4.51}$$

In the remainder of the proof, we have to convert this statement into the appropriate limit for the IDS. This is a standard problem in the theory of Laplace transforms and, indeed, there are theorems that can after some work be applied [e.g., de Bruijn's Tauberian Theorem; see Bingham, Goldie and Teugels (1987)]. However, for the sake of both completeness and convenience we provide an independent proof below.

Suppose that H is the γ-class. We begin with an upper bound. Clearly,

$$\mathscr{L}(\nu,t)\ge e^{-tE}n(E)\quad\text{for any } t, E\ge 0. \tag{4.52}$$

Let $t_E=\alpha^{-1}(\sqrt{(1-2\nu)\chi\, E^{-1}})$ and insert this for t in the previous expression. The result is

$$\log n(E)\le t_E E+\log\mathscr{L}(\nu,t_E)=-t_E E\frac{2\nu}{1-2\nu}\big(1+o(1)\big),\qquad E\downarrow 0, \tag{4.53}$$

where we applied (4.51) and the definition of t_E. In order to finish the upper bound, we first remark that from the first assertion in (1.7) it can be deduced that

$$\lim_{E\downarrow 0}\frac{t_E}{\alpha^{-1}(E^{-\frac{1}{2}})}=\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}. \tag{4.54}$$

Indeed, define $t'_E=\alpha^{-1}(E^{-1/2})$ and consider the quantity $p_E=t_E/t'_E$. Clearly,

$$\alpha(p_E t'_E)=\alpha(t'_E)\sqrt{(1-2\nu)\chi}. \tag{4.55}$$

Let $\widetilde{p}=[(1-2\nu)\chi]^{-1/(2\nu)}$. Since $t'_E\to\infty$ as $E\downarrow 0$, there is no $\varepsilon>0$ such that $p_E\ge\widetilde{p}+\varepsilon$ for infinitely many E with an accumulation point at zero, because otherwise the left-hand side (4.55) would, by (1.7), eventually exceed the right-hand side. Similarly we prove that $\liminf_{E\downarrow 0} p_E$ cannot be smaller than $\widetilde{p}-\varepsilon$. Therefore, $p_E\to\widetilde{p}$ as $E\downarrow 0$, which is (4.54).

Using (4.54), we have from (4.53) that

$$\limsup_{E\downarrow 0}\frac{\log n(E)}{E\alpha^{-1}(E^{-\frac{1}{2}})}\le-\frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}. \tag{4.56}$$

The lower bound is slightly harder, but quite standard. First, introduce the probability measure on $[0, \infty)$ defined by

$$\mu_E(d\lambda) = \frac{e^{-t_E\lambda}}{\mathscr{L}(\nu, t_E)}\nu(d\lambda), \quad E \geq 0. \tag{4.57}$$

We claim that, for any $\varepsilon > 0$, all mass of μ_E gets eventually concentrated inside the interval $[E - \varepsilon E, E + \varepsilon E]$ as $E \downarrow 0$. Indeed, for any $0 \leq t < t_E$ we have

$$\begin{aligned} &\mu_E\big((E + \varepsilon E, \infty)\big) \\ &\quad \leq \mathscr{L}(\nu, t_E)^{-1} \int_{E+\varepsilon E}^{\infty} \nu(d\lambda) \exp\big\{-t_E\lambda + t(\lambda - E - \varepsilon E)\big\} \\ &\quad \leq \exp\{-t\varepsilon E\} \frac{\mathscr{L}(\nu, t_E - t)}{\mathscr{L}(\nu, t_E)} \exp\{-tE\}. \end{aligned} \tag{4.58}$$

Pick $0 < \delta < 1$ and set $t = \delta t_E$. Then we have

$$\begin{aligned} &\mu_E\big((E + \varepsilon E, \infty)\big) \\ &\leq \exp\left\{-\delta\varepsilon t_E E - \delta t_E E - \chi \frac{t_E}{\alpha(t_E)^2}\big[(1-\delta)^{1-2\nu} - 1 + o(1)\big]\right\}, \end{aligned} \tag{4.59}$$

where we again used (4.51) and (1.7). Applying that $(1 - \delta)^{1-2\nu} - 1 = -\delta(1 - 2\nu) + o(\delta)$, using

$$t_E E - \chi(1 - 2\nu)\frac{t_E}{\alpha(t_E)^2} = 0, \tag{4.60}$$

and noting that $\alpha(t_E)^{-2} = O(E)$, we have

$$\mu_E\big((E + \varepsilon E, \infty)\big) \leq \exp\big[-t_E E\big(\delta\varepsilon + o(\delta)\big)\big]. \tag{4.61}$$

Choosing δ small enough, the right-hand side vanishes as $E \downarrow 0$. Similarly we proceed in the case $[0, E - \varepsilon E)$.

Now we can finish the lower bound on Lifshitz tails. Indeed, using Jensen's inequality

$$\begin{aligned} &\nu\big([0, E + \varepsilon E]\big) \\ &= \mathscr{L}(\nu, t_E) \int_0^{E+\varepsilon E} \mu_E(d\lambda) \exp\{t_E\lambda\} \\ &\geq \mathscr{L}(\nu, t_E)\mu_E\big([0, E + \varepsilon E]\big) \exp\left\{\frac{t_E}{\mu_E([0, E + \varepsilon E])} \int_0^{E+\varepsilon E} \mu_E(d\lambda)\,\lambda\right\}. \end{aligned} \tag{4.62}$$

But $\int_0^\infty \mu_E(d\lambda)\lambda$ tends to E, by what we have proved about the concentration of the mass of μ_E (note that (4.61) and the similar bound for $[0, E - \varepsilon E)$ are both exponential in ε) and, by the same token, so does $\int_0^{E+\varepsilon E} \mu_E(d\lambda)\lambda$. By putting all this together, dividing both sides of (4.62) by $E'\alpha^{-1}((E')^{-1/2})$ with

$E' = E + \varepsilon E$, interpreting E' as a new variable tending to 0 as $E \downarrow 0$, and invoking (4.53) and the subsequent computation, we get

$$\liminf_{E\downarrow 0} \frac{\log n(E)}{E\alpha^{-1}(E^{-\frac{1}{2}})} \geq -(1+\varepsilon)^{\frac{1-2\nu}{2\nu}} \frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}, \tag{4.63}$$

where we also used that $t_E/t_{E+\varepsilon E} \to (1+\varepsilon)^{1/(2\nu)}$. Since ε was arbitrary, the claim is finished by taking $\varepsilon \to 0$. □

5. Proof of Theorem 1.5.

Again, we divide the proof in two parts: the upper bound and the lower bound. While the former is a simple application of our results on the moment asymptotics (and the exponential Chebyshev inequality), the latter requires two ingredients: a Borel-Cantelli argument for size of the field and a rather tedious percolation argument. These combine in Proposition 5.1, whose proof is deferred to subsection 5.3.

5.1. *The upper bound.*

PROOF OF THEOREM 1.5, UPPER BOUND. Let $r(t) = t\log t$ and let $L \in (0,\infty)$. We want to apply Proposition 4.4 with the random potential $V = \xi$ and with R replaced by $R\alpha(Lb_t)$ for some fixed $R, L > 0$. (Later we shall let $R \to \infty$ and pick L appropriately.)

Recall the definition of $B_R(t)$ in Proposition 4.4 and abbreviate $B(t) = B_{R\alpha(Lb_t)}(t)$. Take logarithms in (4.21), multiply by $\alpha_{b_t}^2/t$ and use (1.7) to obtain

$$\limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) \leq \frac{C}{L^{2\nu}R^2} + \limsup_{t\to\infty}\Big[\alpha_{b_t}^2 \max_{z\in B(t)} \lambda^{\rm d}_{z;2R\alpha(Lb_t)}(\xi)\Big], \tag{5.1}$$

almost surely w.r.t. the field ξ. Thus, we just need to evaluate the almost sure behavior of the maximum of the random variables on the right-hand side. This will be done by showing that

$$\limsup_{R\to\infty}\limsup_{t\to\infty}\Big[\alpha_{b_t}^2 \max_{z\in B(t)} \lambda^{\rm d}_{z;2R\alpha(Lb_t)}(\xi)\Big] \leq -\widetilde{\chi} \tag{5.2}$$

almost surely w.r.t. the field ξ, provided $L > 0$ is chosen appropriately.

For any $t > 0$, let $(\lambda_i(t))_{i=1,\dots,N(t)}$ be an enumeration of the random variables $\lambda^{\rm d}_{z;2R\alpha(Lb_t)}(\xi)$ with $z \in B(t)$. Note that $N(t) \leq 3^d t^d (\log t)^d$ for t large. Clearly, $(\lambda_i(t))$ are identically distributed but not independent. By (4.6), the tail of their distribution is bounded by

$$\limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{b_t} \log\big\langle \exp\{Lb_t \lambda^{\rm d}_{2R\alpha(Lb_t)}(\xi)\}\big\rangle \leq -L^{1-2\nu}\chi_{6R}, \qquad L, R > 0, \tag{5.3}$$

where χ_R is defined in (3.30).

Assertion (5.2) will be proved if we can verify that, with probability one,

$$\max_{i=1,\dots,N(t)} \lambda_i(t) \leq -\frac{\widetilde{\chi}-\varepsilon}{\alpha^2(b_t)}\big(1+o(1)\big), \qquad t\to\infty, \tag{5.4}$$

for any $\varepsilon > 0$ and sufficiently large $R > 0$, as $t \to \infty$. To that end, note first that the left-hand side of (5.4) is increasing in t since the maps $t \mapsto \alpha(Lb_t)$, $R \mapsto \lambda_R^{\rm d}(\xi)$ and $t \mapsto r(t)$ are all increasing. As a consequence, it suffices to prove the assertion (5.4) only for $t \in \{e^n : n \in \mathbb{N}\}$, because also $\alpha(b_s)^{-2} - \alpha(b_{e^n})^{-2} = o(\alpha(b_{e^n})^{-2})$ as $n \to \infty$ for any $e^{n-1} \le s < e^n$. Let

$$p_n = \text{Prob}\left(\max_{i=1,\dots,N(e^n)} \lambda_i(e^n) \ge -\frac{\widetilde{\chi} - \varepsilon}{\alpha^2(b_{e^n})}\right). \tag{5.5}$$

Abbreviating $t = e^n$ and recalling $b_t \alpha_{b_t}^{-2} = \log t = n$, the exponential Chebyshev inequality and (5.3) allow us to write for any $L > 0$ and n large that

$$\begin{aligned} p_n &\le N(\exp\{n\})\,\text{Prob}\Big(\exp\{Lb_t\lambda_1(\exp\{n\})\} \ge \exp\{-Lb_t\alpha^{-2}(b_t)(\widetilde{\chi} - \varepsilon)\}\Big) \\ &\le 3^d n^d \exp\{nd\} \exp\big\{Lb_t\alpha^{-2}(b_t)(\widetilde{\chi} - \varepsilon)\big\}\Big\langle \exp\{Lb_t\lambda^{\rm d}_{2R\alpha(Lb_t)}(\xi)\}\Big\rangle \\ &= \exp\Big\{n\big[-\varepsilon L + d + L\widetilde{\chi} - L^{1-2\nu}\chi_{6R} + o(1)\big]\Big\}. \end{aligned} \tag{5.6}$$

Now let L minimize the function $L \mapsto d + L\widetilde{\chi} - L^{1-2\nu}\chi$ on $[0, \infty]$. An easy calculation reveals that $L = [(1-2\nu)\chi/\widetilde{\chi}\,]^{1/(2\nu)}$. By invoking Proposition 1.4, we also find that $d + L\widetilde{\chi} - L^{1-2\nu}\chi = 0$ for this value of L, and, substituting this into (5.6), we obtain

$$p_n \le \exp\big\{-n\big[\varepsilon L - L^{1-2\nu}(\chi - \chi_{6R}) + o(1)\big]\big\}, \tag{5.7}$$

which is clearly summable on n provided R is sufficiently large. The Borel-Cantelli lemma then guarantees the validity of (5.4), which in turn proves (5.2). The limit $R \to \infty$ then yields the upper bound in Theorem 1.5. □

5.2. *The lower bound.* Recall the notation of subsection 3.1. Let $Q_{\gamma_t} = [-\gamma_t, \gamma_t]^d \cap \mathbb{Z}^d$ denote the "macrobox," where γ_t is the time scale defined by

$$\gamma_t = \frac{t}{\alpha_{b_t}^3}, \qquad t > 0. \tag{5.8}$$

We assume without loss of generality that $t \mapsto \gamma_t$ is strictly increasing. Since we assumed $\text{Prob}(\xi(0) > -\infty) > p_{\rm c}(d)$ for $d \ge 2$, there is a $K \in (0, \infty)$ such that $\text{Prob}(\xi(0) \ge -K) > p_{\rm c}(d)$. Consequently, $\{z \in \mathbb{Z}^d : \xi(z) \ge -K\}$ contains almost-surely a unique infinite cluster $\mathscr{C}_\infty^*$.

Given a $\psi \in C^-([-R, R]^d)$, let $\psi_t : \mathbb{Z}^d \to (-\infty, 0]$ be the function $\psi_t(\cdot) = \psi(\cdot/\alpha(b_t))/\alpha(b_t)^2$. Suppose H is in the γ-class. Abbreviate

$$Q^{(t)} = \begin{cases} Q_{R\alpha(b_t)}, & \text{if } \gamma \ne 0, \\ Q_{R\alpha(b_t)} \cap \operatorname{supp} \psi_t, & \text{if } \gamma = 0. \end{cases} \tag{5.9}$$

The main point of the proof of the lower bound in Theorem 1.5 is the existence of a microbox of diameter of order α_{b_t} in Q_{γ_t} (which is contained in $\mathscr{C}_\infty^*$ for

$d \geq 2$) where the field is bounded from below by ψ_t:

PROPOSITION 5.1. *Let $R > 0$ and fix a function $\psi \in C^-(R)$ satisfying $\mathscr{L}_R(\psi) < d$. Let $\varepsilon > 0$ and let H be in the γ-class with $\gamma \in [0, 1)$. Then the following holds almost surely: There is a $t_0 = t_0(\xi, \psi, \varepsilon, R) < \infty$ such that for each $t \geq t_0$, there exists a $y_t \in Q_{\gamma_t}$ such that*

$$\xi(z + y_t) \geq \frac{1}{\alpha_{b_t}^2}\psi\left(\frac{z}{\alpha_{b_t}}\right) - \frac{\varepsilon}{\alpha_{b_t}^2} \qquad \forall z \in Q^{(t)}. \tag{5.10}$$

In addition, whenever $d \geq 2$, y_t can be chosen such that $y_t \in \mathscr{C}_\infty^*$.

The proof of Proposition 5.1 is deferred to subsection 5.3. In order to make use of it, we establish that the walk can get to $y_t + Q^{(t)}$ in a reasonable time. In $d \geq 2$, this will be possible whenever the above microbox can be reached from any point in $\mathscr{C}_\infty^* \cap Q_{\gamma_t}$ by a path in $\mathscr{C}_\infty^*$ whose length is comparable to the lattice distance between the path's end-points. Given $x, z \in \mathscr{C}_\infty^*$, let $\mathrm{d}_*(x, z)$ denote the length of the shortest path in $\mathscr{C}_\infty^*$ connecting x and z. Let $|x - z|_1$ be the lattice distance of x and z. The following lemma is the site-percolation version of Lemma 2.4 in Antal's thesis [Antal (1994), page 72]. While the proof is given there in the bond-percolation setting, its inspection shows that it carries over to our case. Therefore, we omit it.

LEMMA 5.2. *Suppose $d \geq 2$. Then, with probability one,*

$$\varrho(x) := \sup_{z \in \mathscr{C}_\infty^* \setminus \{x\}} \frac{\mathrm{d}_*(x, z)}{|x - z|_1} < \infty \quad \textit{for all } x \in \mathscr{C}_\infty^*. \tag{5.11}$$

We proceed with the proof of Theorem 1.5 in the case $d \geq 2$. In $d = 1$, Lemma 5.2 will be substituted by a different argument.

PROOF OF THEOREM 1.5 ($d \geq 2$), LOWER BOUND. Let $R, \varepsilon > 0$ and let $\psi \in C^-(R)$ be twice continuously differentiable with $\mathscr{L}_R(\psi) < d$. If $\gamma = 0$, let $\operatorname{supp}\psi$ be a non-degenerate ball in Q_R centered at 0. Suppose that $\xi = (\xi(z))_{z \in \mathbb{Z}^d}$ does not belong to the exceptional null sets of the preceding assertions. In particular, there are unique infinite clusters $\mathscr{C}_\infty$ in $\{z \in \mathbb{Z}^d : \xi(z) > -\infty\}$ and $\mathscr{C}_\infty^*$ in $\{z \in \mathbb{Z}^d : \xi(z) \geq -K\}$, and ξ satisfies the claims in Proposition 5.1 and Lemma 5.2. Clearly, $\mathscr{C}_\infty^* \subset \mathscr{C}_\infty$. Assume $0 \in \mathscr{C}_\infty$ and pick a $z^* \in \mathscr{C}_\infty^*$. For each $t \geq t_0$ choose a $y_t \in Q_{\gamma_t} \cap \mathscr{C}_\infty^*$ such that (5.10) holds. We assume that t is so large that $z^* \in Q_{\gamma_t}$.

The lower bound on $u(t, 0)$ will be obtained by restricting the random walk $(X(s))_{s \geq 0}$ (which starts at 0) to be at z^* at time 1, at y_t at time γ_t (staying within $\mathscr{C}_\infty^*$ in the meantime) and to remain in $y_t + Q^{(t)}$ until time t. Introduce the exit times from $\mathscr{C}_\infty^*$ and $y_t + Q^{(t)}$, respectively,

$$\tau_\infty^* = \inf\{s > 0 : X(s) \notin \mathscr{C}_\infty^*\} \quad \text{and} \quad \tau_{y_t, t} = \inf\{s > 0 : X(s) \notin y_t + Q^{(t)}\}. \tag{5.12}$$

Let $t \ge t_0(\xi)$. Inserting the indicator on the event described above and using the Markov property twice at times 1 and γ_t, we get

$$u(t,0) \ge \mathrm{I} \times \mathrm{II} \times \mathrm{III}, \tag{5.13}$$

where the three factors are given by

$$\begin{aligned} \mathrm{I} &= \mathbb{E}_0\left[\exp\left\{\int_0^1 \xi\big(X(s)\big)\,ds\right\} 1\{X(1)=z^*\}\right],\\ \mathrm{II} &= \mathbb{E}_{z^*}\left[\exp\left\{\int_0^{\gamma_t-1} \xi\big(X(s)\big)\,ds\right\} 1\{\tau^*_\infty > \gamma_t-1,\, X(\gamma_t-1)=y_t\}\right],\\ \mathrm{III} &= \mathbb{E}_{y_t}\left[\exp\left\{\int_0^{t-\gamma_t} \xi\big(X(s)\big)\,ds\right\} 1\{\tau_{y_t,t} > t-\gamma_t\}\right]. \end{aligned} \tag{5.14}$$

Clearly, the quantity I is independent of t and is non-vanishing because $0, z^* \in \mathscr{C}_\infty$. Our next claim is that $\mathrm{II} \ge \exp\{o(t\alpha_{b_t}^{-2})\}$ as $t\to\infty$. Indeed,

$$\mathrm{II} \ge \exp\{-K\gamma_t\}\mathbb{P}_{z^*}\big(\tau^*_\infty > \gamma_t-1,\, X(\gamma_t-1)=y_t\big), \tag{5.15}$$

since there is at least one path connecting z^* to y_t within $\mathscr{C}^*_\infty$ (recall that the field ξ is bounded from below by $-K$ on $\mathscr{C}^*_\infty$). Denote by $\mathrm{d}_t = \mathrm{d}_*(z^*, y_t)$ the minimal length of such a path and abbreviate $\varrho(z^*) = \varrho$, where $\varrho(z^*)$ is as in (5.11). Then, for $t \ge t_0$,

$$\mathrm{d}_t \le \varrho|z^*-y_t|_1 \le 2d\varrho\gamma_t \le 3d\varrho(\gamma_t-1), \tag{5.16}$$

by Lemma 5.2 and the fact that the both $z^*, y_t \in Q_{\gamma_t}$. Hence, using also that $\mathrm{d}_t! \le \mathrm{d}_t^{\mathrm{d}_t}$,

$$\begin{aligned} &\mathbb{P}_{z^*}\big(\tau^*_\infty > \gamma_t-1,\, X(\gamma_t-1)=y_t\big)\\ &\quad\ge \exp\{-(\gamma_t-1)\}\frac{(\gamma_t-1)^{\mathrm{d}_t}}{\mathrm{d}_t!}(2d)^{-\mathrm{d}_t}\\ &\quad\ge \exp\{-\gamma_t\}\exp\big[-\mathrm{d}_t\log(2d\mathrm{d}_t/(\gamma_t-1))\big]\\ &\quad\ge \exp\big[-\gamma_t\big(1+3d\varrho\log(6d^2\varrho)\big)\big]. \end{aligned} \tag{5.17}$$

In order to see that $\mathrm{II} \ge \exp\{o(t\alpha_{b_t}^{-2})\}$, recall that $\gamma_t = o(t\alpha_{b_t}^{-2})$ as $t\to\infty$ by (5.8) and that z^* does not depend on t.

We turn to the estimate of III. By spatial homogeniety of the random walk, we have

$$\mathrm{III} = \mathbb{E}_0\left[\exp\left\{\int_0^{t-\gamma_t} \xi\big(y_t+X(s)\big)\,ds\right\} 1\{\tau_{0,t} > t-\gamma_t\}\right], \tag{5.18}$$

where $\tau_{0,t}$ is the first exit time from $Q^{(t)}$. Using (5.10), we obtain the estimate

$$\begin{aligned} \mathrm{III} \ge\ & \exp\left\{-\varepsilon(t-\gamma_t)\alpha_{b_t}^{-2}\right\}\\ &\times\mathbb{E}_0\left[\exp\left\{\int_0^{t-\gamma_t} \psi_t\big(X(s)\big)\,ds\right\} 1\{\tau_{0,t} > t-\gamma_t\}\right], \end{aligned} \tag{5.19}$$

By invoking (3.5) and (3.12), the expectation on the right-hand side is bounded from below by

$$\exp\left\{(t-\gamma_t)\lambda^{\mathrm{d}}(t)\right\}\mathrm{e}_t(0)^2, \tag{5.20}$$

where $\lambda^{\mathrm{d}}(t)$, resp., e_t, denote the principal Dirichlet eigenvalue, resp. the ℓ^2-normalized principal eigenfunction of $\kappa\Delta^{\mathrm{d}}+\psi_t$ in $Q^{(t)}$. For $e_t(0)$ and $\lambda^{\mathrm{d}}(t)$ we have the following bounds, whose proofs will be given subsequently.

LEMMA 5.3. *We have*

$$\liminf_{t\to\infty}\frac{\alpha_{b_t}^2}{t}\log \mathrm{e}_t(0)^2 \ge 0, \tag{5.21}$$

$$\liminf_{t\to\infty}\alpha_{b_t}^2\lambda^{\mathrm{d}}(t)\ge \lambda_R(\psi). \tag{5.22}$$

Summarizing all the preceding estimates and applying (5.21) and (5.22), we obtain

$$\liminf_{t\to\infty}\frac{\alpha_{b_t}^2}{t}\log u(t,0)\ge \lambda_R(\psi)-\varepsilon, \tag{5.23}$$

where we also noted that $t-\gamma_t=t(1+o(1))$. In the case $\gamma>0$, let $\varepsilon\downarrow 0$, optimize over $\psi\in C^-(R)$ with $\mathscr{L}_R(\psi)<d$ [clearly, the supremum in (1.23) may be restricted to the set of twice continuously differentiable functions $\psi\in C^-(R)$ such that $\mathscr{L}_R(\psi)<d$] and let $R\to\infty$ to get the lower bound in Theorem 1.5. In the case $\gamma=0$, recall that $\mathscr{L}_R(\psi)=\text{const.}\,|\{\psi<0\}|$. It is classical [see, e.g., Donsker and Varadhan (1975), Lemma 3.13, or argue directly by Faber-Krahn's inequality] that the supremum (1.23) can be restricted to ψ whose support is a ball. The proof is therefore finished by letting $\varepsilon\downarrow 0$, optimizing over such ψ and letting $R\to\infty$. □

PROOF OF LEMMA 5.3. We begin with (5.21). Recall that e_t is also an eigenfunction for the transition densities of the random walk in $Q^{(t)}$ with potential $\psi_t-\lambda^{\mathrm{d}}(t)$. Using this observation at time 1, we can write

$$\mathrm{e}_t(0)=\mathbb{E}_0\left[\exp\left\{\int_0^1\left[\psi_t(X(s))-\lambda^{\mathrm{d}}(t)\right]ds\right\}\mathbb{1}\{\tau_{0,t}>1\}\mathrm{e}_t(X(1))\right], \tag{5.24}$$

Since $\lambda^{\mathrm{d}}(t)$ is nonpositive and ψ is bounded from below, we have

$$\mathrm{e}_t(0)\ge \exp\left[\alpha(b_t)^{-2}\inf\psi\right]\sum_{z\in Q^{(t)}}\mathbb{P}_0(\tau_{0,t}>1, X(1)=z)\mathrm{e}_t(z). \tag{5.25}$$

Using the same strategy as in (5.17), we have $\mathbb{P}_0(\tau_{0,t}>1, X(1)=z)\ge \exp\{-O(\alpha(b_t)\log\alpha(b_t))\}$. Since e_t is nonnegative and satisfies $\|\mathrm{e}_t\|_2=1$, we have $\sum_z \mathrm{e}_t(z)\ge\|\mathrm{e}_t\|_2^2=1$. From these estimates, (5.21) is proved by noting that $\alpha(b_t)\log\alpha(b_t))=o(t/\alpha(b_t)^2)$.

In order to establish (5.22), we shall restrict the supremum in (3.10) to a particular choice of g. Let $Q_R(\psi) = [-R, R]^d$ if $\gamma \neq 0$ and $Q_R(\psi) = \operatorname{supp}\psi$ if $\gamma = 0$. Let $\widehat{g}: [-R, R]^d \to [0, \infty)$ be the L^2-normalized principal eigenfunction of the (continuous) operator $\kappa\Delta + \psi$ on $Q_R(\psi)$ with Dirichlet boundary conditions. Let us insert $\widehat{g}_t(z) = \widehat{g}(z/\alpha(b_t))/\alpha(b_t)^{d/2}$ into (3.10) in the place of g. Thus we get

$$\begin{aligned}\alpha(b_t)^2\lambda^{\mathrm{d}}(t)(\psi_t) & \\ \geq \alpha(b_t)^{-d}\sum_{z\in Q^{(t)}}&\Bigg[(\psi\widehat{g}^2)\left(\frac{z}{\alpha(b_t)}\right) \\ &-\kappa\alpha(b_t)^2\sum_{y:y\sim z}\left(\widehat{g}\left(\frac{z}{\alpha(b_t)}\right)-\widehat{g}\left(\frac{y}{\alpha(b_t)}\right)\right)^2\Bigg],\end{aligned}\tag{5.26}$$

where $y \sim z$ denotes that y and z are nearest neighbors.

Since ψ is smooth, standard theorems guarantee that $\widehat{g}$ is continuously differentiable on $Q_R(\psi)$ and, hence, $\|\nabla\widehat{g}\|_\infty < \infty$. [This fact is derived using regularity properties of Green's function of the Poisson equation, see, e.g., Theorem 10.3 in Lieb and Loss (1997).] Then

$$\widehat{g}\big(z/\alpha(b_t)\big)-\widehat{g}\big(y/\alpha(b_t)\big)=\alpha(b_t)^{-1}(z-y)\cdot\nabla\widehat{g}\big(z_\eta/\alpha(b_t)\big), \qquad z, y \in Q^{(t)},\tag{5.27}$$

where $z_\eta = \eta z + (1-\eta)y$ for some $\eta \in [0, 1]$. For the pairs $z \sim y$ with $y \notin Q^{(t)}$ we only get a bound $|\widehat{g}(z/\alpha(b_t)) - \widehat{g}(y/\alpha(b_t))| \leq (1 + \|\nabla\widehat{g}\|_\infty)/\alpha(b_t)$ (note that $\widehat{g}(y/\alpha(b_t)) = 0$ in this case). Since the total contribution of these boundary terms to (5.26) is clearly bounded by $(1+\|\nabla\widehat{g}\|_\infty)/\alpha(b_t)$, we see that the right-hand side of (5.26) converges to $(\psi, \widehat{g}^2) - \kappa\|\nabla\widehat{g}\|_2$ as $t \to \infty$. By our choice of $\widehat{g}$, this limit is equal to the eigenvalue $\lambda_R(\psi)$, which proves (5.22). □

PROOF OF THEOREM (1.5) ($d = 1$), LOWER BOUND. Suppose that $\langle\log(-\xi(0) \vee 1)\rangle > -\infty$. This implies that $\mathscr{C}_\infty = \mathbb{Z}$ almost surely and, by the law of large numbers,

$$K_\xi := \sup_{y\in\mathbb{Z}\setminus\{0\}} \frac{1}{|y|}\sum_{x=0}^{|y|}\log\big(-\xi(x)\vee 1\big) < \infty \quad \text{almost surely.}\tag{5.28}$$

Suppose that $\xi = (\xi(z))_{z\in\mathbb{Z}}$ does not belong to the exceptional sets of (5.28) and Proposition (5.1). For sufficiently large t, let $y_t \in Q_{\gamma_t}$ be such that (5.10) holds.

Let $r_x = (-1/\xi(x))\wedge 1$. The strategy for the lower bound on $u(t, 0)$ is that the random walk performs $|y_t|$ steps toward y_t, resting at most time r_x at each site x between 0 and y_t, so that y_t is reached before time γ_t. Afterwards the walk stays at y_t until γ_t. Use $E^{(t)}$ to denote the latter event. Then $u(t, 0) \geq \mathrm{II}\times\mathrm{III}$, where III is as in (5.14) and $\mathrm{II} = \mathbb{E}_0[\exp\{\int_0^{\gamma_t}\xi(X(s))\,ds\}1_{E^{(t)}}]$.

The lower bound on III is identical to the case $d \geq 2$. To estimate the term II, suppose that $y_t > 0$ (clearly, if $y_t = 0$ no estimate on II is needed; $y_t < 0$

is handled by symmetry) and abbreviate $|y_t| = n+1$. Using the shorthand $[s]_n = s_0 + \ldots + s_n$, we have

$$\text{II} = \int_0^{r_0} ds_0 \ldots \int_0^{r_n} ds_n \int_0^{\gamma_t - [s]_n} ds_{n+1} \exp\left\{-\sum_{x=0}^{n+1} s_x\big(\kappa - \xi(x)\big)\right\}$$

$$\geq \exp\{O(\gamma_t)\} \prod_{x=0}^{n} \Big[r_x \exp\big(r_x \xi(x)\big)\Big] \tag{5.29}$$

$$\geq \exp\{O(\gamma_t)\} \exp\left\{-\sum_{x=0}^{n} \log\big(-\xi(x) \vee 1\big)\right\}.$$

Indeed, in the first line we noted that $[s]_n \leq \gamma_t$ because $r_x \leq 1$. Then we took out the terms $\exp(-\kappa s_x)$ as well as $\exp(s_{n+1}\xi(y_t))$, recalling that $\xi(y_t) \geq \inf \psi_t = \inf \psi/\alpha(b_t)^2 = O(1)$ and that $|y_t| = O(\gamma_t)$. The last inequality follows by the fact that $r_x \exp(r_x \xi_x) \geq r_x/e$. Invoking (5.28), the sum in the exponent is bounded above by $K_\xi |y_t| = O(\gamma_t)$, whereby we finally get that $\text{II} \geq e^{-O(\gamma_t)}$. □

5.3. *Technical claims.* In this final subsection, we prove Proposition 5.1. First, we need to introduce some notation and prove two auxiliary lemmas. For $\varepsilon > 0$ and $y \in \mathbb{Z}^d$, define the event

$$A_y^{(t)} = \{y \in \mathscr{C}_\infty^*\} \cap \bigcap_{z \in Q^{(t)}} \left\{\xi(y+z) \geq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2}\right\}. \tag{5.30}$$

Note that the distribution of $A_y^{(t)}$ does not depend on y. By $\partial(Q)$ we denote the outer boundary of a set $Q \subset \mathbb{Z}^d$. To estimate $\text{Prob}(A_y^{(t)})$, it is convenient to begin with the first event on the right-hand side of (5.30). Since $\{y \in \mathscr{C}_\infty^*\} \subset \partial(y + Q^{(t)}) \cap \mathscr{C}_\infty^*$ it suffices to know an estimate on $\text{Prob}(\partial Q^{(t)} \cap \mathscr{C}_\infty^*)$:

LEMMA 5.4. *Let $d \geq 2$ and let $\psi \in C^-(R)$ be such that $\psi \not\equiv 0$. Then there is a $c \in (0, \infty)$ such that, for t large enough,*

$$\text{Prob}\big(\partial Q^{(t)} \cap \mathscr{C}_\infty^* = \varnothing\big) \leq \exp\{-c\alpha(b_t)\}. \tag{5.31}$$

PROOF. Since $\psi \not\equiv 0$ is continuous, there is a ball $B_{\alpha(b_t)}$ of radius of order $\alpha(b_t)$ such that $B_{\alpha(b_t)} \subset Q^{(t)}$. If t is so large that $\psi_t \geq \inf \psi/\alpha(b_t)^2 \geq -K$, then $B_{\alpha(b_t)} \subset \{z: \xi(z) \geq -K\}$ and the left-hand side of (5.31) is bounded from above by $\text{Prob}(\partial B_{\alpha(b_t)} \cap \mathscr{C}_\infty^* = \varnothing)$. The proof now proceeds in a different way depending whether $d \geq 3$ or $d = 2$. In the following, the words "percolation," "infinite cluster," etc., refer to site-percolation on $\mathbb{Z}^d$ with parameter $p = \text{Prob}(\xi(0) > -K)$. Recall that $p > p_c(d)$ by our choice of K.

Let $d \geq 3$. Then, by equality of $p_c(d)$ and the limit of slab-percolation thresholds, there is a width k such that the slab $S_k = \mathbb{Z}^{d-1} \times \{1, \ldots, k\}$ contains almost surely an infinite cluster. Pick a lattice direction and decompose $\mathbb{Z}^d$ into a disjoint union of translates of S_k. There is $c' > 0$ such that, for t large, at least $\lfloor c'\alpha(b_t)/k \rfloor$ slabs are intersected by $\partial B_{\alpha(b_t)}$. Then $\{\partial B_{\alpha(b_t)} \cap \mathscr{C}_\infty^* = \varnothing\}$ is contained in the event that in none of the slabs intersecting $\partial B_{\alpha(b_t)}$ the respective infinite cluster reaches $\partial B_{\alpha(b_t)}$. Let $P_\infty(k)$ be minimum probability that a site in S_k belongs to an infinite cluster. Combining the preceding inclusions, we have

$$\operatorname{Prob}(\partial B_{\alpha(b_t)} \cap \mathscr{C}_\infty^* = \varnothing) \leq P_\infty(k)^{c'\alpha(b_t)/k}. \tag{5.32}$$

Now the claim follows by putting $c = -c' k^{-1} \log P_\infty(k)$.

In $d = 2$, suppose without loss of generality that $B_{\alpha(b_t)}$ is centered at the origin. Recall that x and y are $*$-connected if their Euclidean distance is not more than $\sqrt{2}$. On the event $\{\partial B_{\alpha(b_t)} \cap \mathscr{C}_\infty^* = \varnothing\}$, the origin is encircled by a $*$-connected circuit of size at least $c\alpha(b_t)$ for some $c > 0$, not depending on t. Denote by x the nearest point of this circuit in the first coordinate direction. Call sites z with $\xi(z) \geq -K$ "occupied," the other sites are "vacant."

Note that percolation of occupied sites rules out percolation of vacant sites, for example, by the result of Gandolfi, Keane and Russo (1988). Moreover, using the site-perolation version of the famous "$p_c = \pi_c$" result [see, e.g., Grimmett (1989)], the probability that a given site is contained in a vacant $*$-cluster of size n is bounded by $e^{-\sigma(p)n}$, where $\sigma(p) > 0$ since $p > p_c(d)$. If the ball $B_{\alpha(b_t)}$ has diameter at least $r\alpha(b_t)$, then by taking the above circuit for such a cluster we can estimate the probability of its occurrence:

$$\operatorname{Prob}\big(\partial Q^{(t)} \cap \mathscr{C}_\infty^* = \varnothing\big) \leq \sum_{n=\lfloor r\alpha(b_t)\rfloor}^{\infty} n e^{-\sigma(p)n} \leq e^{-\sigma(p) r\alpha(b_t)/2}, \tag{5.33}$$

for t large enough. Here "n" in the sum accounts for the position of the circuit's intersection with the positive part of the first coordinate axis. The minimal size of the circuit is at least $\lfloor r\alpha(b_t)\rfloor$, since it has to stay all outside $B_{\alpha(b_t)}$. The claim follows by putting $c = r\sigma(p)/2$. □

LEMMA 5.5. *For any* $\varepsilon > 0$,

$$\operatorname{Prob}(A_0^{(t)}) \geq t^{-\mathscr{L}_R(\psi)+o(1)}, \qquad t \to \infty. \tag{5.34}$$

Let H be in the γ-class and let $\psi \not\equiv 0$ (otherwise there is nothing to prove because $\mathscr{L}_R(0) = \infty$). Consider the event

$$\widetilde{A}^{(t)} = \bigcap_{z \in Q^{(t)}} \left\{ \xi(z) \geq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2} \right\}. \tag{5.35}$$

Note that both events on the right-hand side of (5.30) are increasing in the partial order $\xi \succeq \xi' \Leftrightarrow \xi(x) \geq \xi'(x)$ for all x. Therefore, by the FKG-inequality,

$$\operatorname{Prob}(A_0^{(t)}) \geq \operatorname{Prob}(0 \in \mathscr{C}_\infty^*) \operatorname{Prob}(\widetilde{A}^{(t)}). \tag{5.36}$$

Since $\operatorname{Prob}(0 \in \mathscr{C}_\infty^*) > 0$, we only need to prove the assertion for $A_0^{(t)}$ replaced by $\widetilde{A}^{(t)}$. The proof proceeds in three steps, depending on γ and on whether there is an atom at 0.

PROOF OF LEMMA 5.5 FOR $\gamma \in (0,1)$. Let $f \in C^+(R)$ be the solution to $\psi - \frac{3}{8}\varepsilon = \widetilde{H}' \circ f$ and let $f_t: \mathbb{Z}^d \to (0,\infty)$ be its scaled version: $f_t(z) = (b_t/\alpha(b_t)^d) f(z/\alpha(b_t))$. Define the tilted probability measure

$$\text{Prob}_{t,z}(\,\cdot\,) = \big\langle \exp\{f_t(z)\xi(z)\} \mathbb{1}\{\xi(z) \in \cdot\,\}\big\rangle \exp\{-H(f_t(z))\}. \tag{5.37}$$

We denote expectation with respect to $\text{Prob}_{t,z}$ by $\langle\,\cdot\,\rangle_{t,z}$. Consider the event

$$D_t(z) = \left\{ -\frac{\varepsilon}{4\alpha(b_t)^2} \ge \xi(z) - \psi_t(z) \ge -\frac{\varepsilon}{2\alpha(b_t)^2} \right\}. \tag{5.38}$$

Then $\text{Prob}(\widetilde{A}^{(t)})$ can be bounded as

$$\text{Prob}\big(\widetilde{A}^{(t)}\big) \ge \prod_{z \in Q^{(t)}} \Big[\exp\{H(f_t(z))\} \big\langle \exp\{-f_t(z)\xi(z)\} \mathbb{1}\{D_t(z)\}\big\rangle_{t,z} \Big]. \tag{5.39}$$

Applying the left inequality in (5.38), we obtain

$$\begin{aligned} \text{Prob}\big(\widetilde{A}^{(t)}\big) \ge \exp&\left\{ \sum_{z\in Q^{(t)}} \left[H(f_t(z)) - f_t(z)\left(\psi_t(z) - \frac{\varepsilon}{4\alpha(b_t)^2}\right)\right]\right\} \\ &\times \prod_{z \in Q^{(t)}} \text{Prob}_{t,z}\big(D_t(z)\big). \end{aligned} \tag{5.40}$$

Since $\gamma > 0$ and f is continuous and bounded, we can use our Scaling Assumption and the fact that $b_t\alpha(b_t)^{-2} = \log t$ to turn the sum over $z \in Q^{(t)}$ into a Riemann integral over $[-R,R]^d$:

$$\text{Prob}(\widetilde{A}^{(t)}) \ge t^{-\int[f\psi - \widetilde{H}\circ f] + \frac{\varepsilon}{4}\int f + o(1)} \prod_{z\in Q^{(t)}} \text{Prob}_{t,z}\big(D_t(z)\big). \tag{5.41}$$

where we also used that $Q^{(t)} = Q_{R\alpha(b_t)}$ in this case. In order to complete the proof of the lower bound in (5.34), we thus need to show that

$$\int \big[f\psi - \widetilde{H}\circ f\big] \le \mathscr{L}_R(\psi) \tag{5.42}$$

and that

$$\prod_{z\in Q^{(t)}} \text{Prob}_{t,z}\big(D_t(z)\big) \ge t^{o(1)}, \qquad t \to \infty. \tag{5.43}$$

Let us begin with (5.42). For simplicity, we restrict ourselves to the case when $\widetilde{H}(1) = -1$. Then $\mathscr{L}_R(\psi) = \gamma^{1/(1-\gamma)}(\gamma^{-1} - 1)\int |\psi|^{-\gamma/(1-\gamma)}$ and $f = \gamma^{1/(1-\gamma)} |\psi - \frac{3}{8}\varepsilon|^{-1/(1-\gamma)}$. Hence,

$$\int \big[f\psi - \widetilde{H}\circ f\big] - \mathscr{L}_R(\psi) = \gamma^{\frac{1}{1-\gamma}} \int |\psi|^{-\frac{\gamma}{1-\gamma}} \zeta_\gamma\left(\left|\frac{\psi}{\psi - \frac{3}{8}\varepsilon}\right|^{\frac{1}{1-\gamma}}\right), \tag{5.44}$$

where $\zeta_\gamma(x) = 1 - x - \frac{1}{\gamma}(1 - x^\gamma)$. Since $\zeta_\gamma(x) \le 0$ for any $x \ge 0$, (5.42) is proved.

In order to prove (5.43), note that

$$
\begin{aligned}
\mathrm{Prob}_{t,z}\big(D_t(z)\big) \geq \; & 1 - \mathrm{Prob}_{t,z}\left(\xi(z) \geq \psi_t(z) - \frac{\varepsilon}{4\alpha(b_t)^2}\right) \\
& -\mathrm{Prob}_{t,z}\left(\xi(z) \leq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2}\right).
\end{aligned} \tag{5.45}
$$

We concentrate on estimating the second term; the first term is handled analogously. By the exponential Chebyshev inequality, we have for any $g_t(z) \in (0, f_t(z))$ that

$$
\begin{aligned}
& \mathrm{Prob}_{t,z}\Big(\xi(z) \leq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2}\Big) \\
& \leq \exp\big\{-H(f_t(z))\big\}\Big\langle \exp\Big\{f_t(z)\xi(z) - g_t(z)\Big[\xi(z) - \psi_t(z) + \frac{\varepsilon}{2\alpha(b_t)^2}\Big]\Big\}\Big\rangle \\
& = \exp\Big\{H\big(f_t(z) - g_t(z)\big) - H\big(f_t(z)\big) + g_t(z)\psi_t(z) - g_t(z)\frac{\varepsilon}{2\alpha(b_t)^2}\Big\}.
\end{aligned} \tag{5.46}
$$

Note that $\widetilde{H}'_t \to \widetilde{H}'$ (recall (3.13)) as $t \to \infty$ uniformly on compact sets in $(0, \infty)$. Also note that f is bounded away from 0. Choose $g_t(z) = \delta_t f_t(z)$, where $\delta_t \downarrow 0$ is still to be chosen appropriately. Then the exponent in the third line of (5.46) can be bounded from above by

$$
\begin{aligned}
& -\delta_t \frac{b_t}{\alpha(b_t)^{d+2}} f\left(\frac{z}{\alpha(b_t)}\right)\left\{\widetilde{H}'_t\left[f\left(\frac{z}{\alpha(b_t)}\right)(1-\delta_t)\right] - \psi\left(\frac{z}{\alpha(b_t)}\right) + \frac{\varepsilon}{2}\right\} \\
& \quad = -\delta_t \frac{b_t}{\alpha(b_t)^{d+2}} f\left(\frac{z}{\alpha(b_t)}\right)\left[\frac{\varepsilon}{8} + o(1)\right],
\end{aligned} \tag{5.47}
$$

where we replaced $\widetilde{H}'_t$ by $\widetilde{H}' + o(1)$ and used the definition relation for f. Pick $\delta_t = (\alpha_{b_t}^{d+2}/b_t)^{1/2}$ for definiteness. Taking the product over $z \in Q^{(t)}$ in (5.45) and using that $[\frac{\varepsilon}{8} + o(1)]f \geq C > 0$, we obtain for t large that

$$
\begin{aligned}
\prod_{z \in Q^{(t)}} \mathrm{Prob}_{t,z}\big(D_t(z)\big) & \geq \Big[1 - 2\exp\Big\{-C\delta_t \frac{b_t}{\alpha(b_t)^{d+2}}\Big\}\Big]^{\# Q^{(t)}} \\
& \geq \exp\Big\{-4\# Q^{(t)} \exp\Big\{-C\delta_t \frac{b_t}{\alpha(b_t)^{d+2}}\Big\}\Big\} \\
& = t^{-C'(\alpha_{b_t}^{d+2}/b_t)\exp\left(-C\delta_t \frac{b_t}{\alpha(b_t)^{d+2}}\right)},
\end{aligned} \tag{5.48}
$$

where also used that $b_t\alpha(b_t)^{-2} = \log t$ and $\# Q^{(t)} \leq \alpha(b_t)^d C'/4$ for some C' as $t \to \infty$. By our choice of δ_t, (5.43) is clearly satisfied, which completes the proof in the case $\gamma \in (0, 1)$. □

PROOF OF LEMMA 5.5 FOR $\gamma = 0$, ATOM AT 0. Suppose $\mathrm{Prob}(\xi(0) \in \cdot)$ has an atom at 0 with mass $p > 0$. Then, noting that $Q^{(t)}$ are only the sites with $\psi_t < 0$, we have

$$\begin{aligned}\mathrm{Prob}(\widetilde{A}^{(t)}) &\geq \mathrm{Prob}\big(\xi(0) = 0\big)^{\#Q^{(t)}}\\ &= \exp\big\{\alpha(b_t)^d(|\mathrm{supp}\,\psi| + o(1))\log p\big\}, \quad t \to \infty.\end{aligned} \tag{5.49}$$

Since $\alpha_t = t^{1/(d+2)}$ and $\widetilde{H}(1) = \log p$, we have $\mathscr{L}_R(\psi) = -\widetilde{H}(1)|\mathrm{supp}\,\psi|$ and $\alpha(b_t)^d = \log t$, whereby (5.34) immediately follows. □

PROOF OF LEMMA 5.5 FOR $\gamma = 0$, NO ATOM AT 0. Suppose that $\gamma = 0$ and $\mathrm{Prob}(\xi(0) = 0) = 0$. Set $f_t = b_t\alpha(b_t)^{-d}$ and consider the probability measure $\mathrm{Prob}_t(\xi(0) \in \cdot)$ with density $\exp[f_t\xi(0) - H(f_t)]$ with respect to $\mathrm{Prob}(\xi(0) \in \cdot)$. Invoking that $\xi(0) \leq 0$, we obtain

$$\begin{aligned}\mathrm{Prob}(\widetilde{A}^{(t)}) &\geq \mathrm{Prob}\Big(\xi(0) \geq -\frac{\varepsilon}{2\alpha(b_t)^2}\Big)^{\#Q^{(t)}}\\ &\geq \exp\{\#Q^{(t)}H(f_t)\}\,\mathrm{Prob}_t\Big(\xi(0) \geq -\frac{\varepsilon}{2\alpha(b_t)^2}\Big)^{\#Q^{(t)}}.\end{aligned} \tag{5.50}$$

Now use the Scaling Assumption and the fact that $\#Q^{(t)} = \alpha(b_t)^d(|\mathrm{supp}\,\psi| + o(1))$ as $t \to \infty$ to extract the term $t^{-\mathscr{L}_R(\psi)}$ from the exponential on the right-hand side [here we recalled that $\mathscr{L}_R(\psi) = -\widetilde{H}(1)|\mathrm{supp}\,\psi|$]. Moreover, by an argument similar to (5.46), the last term on the right-hand side is no smaller than $t^{o(1)}$ as $t \to \infty$. To that end we noted that our choice of f_t corresponds to $f \equiv 1$ and then we used again that $\lim_{t\to\infty} b_t\alpha(b_t)^{-(d+2)} = \infty$, which follows from the fact that $\xi(0)$ has no atom at zero. This finally completes the proof of Lemma 5.5. □

Now we can complete the proof of Proposition 5.1.

PROOF OF PROPOSITION 5.1. Fix $R > 0$ and $\psi \in C^-(R)$ with $\mathscr{L}_R(\psi) < d$. Recall the notation (5.9) and (5.30). Let $t_1 = t_1(\psi, \varepsilon, R)$ be such that for all $t \geq t_1$ and for all $s \in [0, e)$

$$\psi_{et}(z) - \frac{\varepsilon}{2\alpha(b_{et})^2} \geq \psi_{st}(z) - \frac{\varepsilon}{\alpha(b_{st})^2}, \qquad z \in Q^{(st)}. \tag{5.51}$$

Such a $t_1 < \infty$ indeed exists, since $\alpha(b_{st})/\alpha(b_{et}) \to 1$ as $t \to \infty$ and since ψ is uniformly continuous on $[-R, R]^d$. This implies that to prove Proposition 5.1 it suffices to find an almost-surely finite $n_0 = n_0(\xi, \psi, \varepsilon, R)$ such that for each $n \geq n_0$ there is a $y_n \in Q_{\gamma_{e^n}}$ for which the event $A_{y_n}^{(e^{n+1})}$ occurs. Indeed, for any $t = se^n$ with $n \geq n_0$ and $s \in [0, e)$ we have that $Q_{\gamma_{e^n}} \subset Q_{\gamma_t}$ and $y_n + Q_{R\alpha(b_t)} \subset y_n + Q_{R\alpha(b_{e^{n+1}})}$, as follows by monotonicity of the maps $t \mapsto \gamma_t$

and $t \mapsto \alpha(b_t)$ and, consequently,

$$\bigcap_{z\in Q^{(t)}}\left\{\xi(y_n+z)\ge\psi_t(z)-\frac{\varepsilon}{\alpha(b_t)^2}\right\}\supset A_{y_n}^{(e^{n+1})}, \tag{5.52}$$

by invoking (5.51). Then Proposition 5.1 would follow with the choice $t_0 = t_1 \vee e^{n_0}$.

Based on the preceding reduction argument, let $t \in \{e^n : n \in \mathbb{N}\}$ for the remainder of the proof. Let $M_t = Q_{\gamma_t} \cap \lfloor 3R\alpha(b_{et})\rfloor\mathbb{Z}^d$. We claim that, to prove Proposition 5.1 for $t \in \{e^n : n \in \mathbb{N}\}$, it suffices to show the summability of

$$p_t = \mathrm{Prob}\left(\sum_{y\in M_t} 1_{A_y^{(et)}} \le \tfrac{1}{2}\#M_t\,\mathrm{Prob}\left(A_0^{(et)}\right)\right), \qquad t\in\{e^n : n\in\mathbb{N}\}. \tag{5.53}$$

Indeed, since $\#M_t \ge t^{d+o(1)}$ we have by Lemma 5.5

$$\#M_t\,\mathrm{Prob}(A_0^{(et)}) \ge t^{d-\mathscr{L}_R(\psi)+o(1)}, \qquad t\to\infty. \tag{5.54}$$

Since we assumed $\mathscr{L}_R(\psi) < d$, summability of p_t would imply the existence of at least one site $y \in Q_{\gamma_t}$ (in fact, at least $t^{d-\mathscr{L}_R(\psi)+o(1)}$ sites) with $A_y^{(et)}$ satisfied.

To prove a suitable bound on p_t we invoke Chebyshev's inequality to find that

$$p_t \le \frac{4}{\#M_t\,\mathrm{Prob}(A_0^{(et)})} + \frac{4\max_{y\ne y'}\mathrm{cov}(A_y^{(et)}, A_{y'}^{(et)})}{\mathrm{Prob}(A_0^{(et)})^2}. \tag{5.55}$$

As follows from (5.54), the first term on the right-hand side is summable on $t \in \{e^n : n \in \mathbb{N}\}$. In order to estimate $\mathrm{cov}(A_y^{(et)}, A_{y'}^{(et)})$ for $y \ne y'$, let $\mathbb{H}$ and $\mathbb{H}'$ be two disjoint half spaces in $\mathbb{R}^d$ which contain $y+Q^{(et)}$ and $y'+Q^{(et)}$, respectively, including the outer boundaries. By our choice of M_t, $\mathbb{H}$ can be chosen such that $\mathrm{dist}(y+Q^{(et)}, \mathbb{H}^c) \ge R\alpha(b_t)/3$, and similarly for $\mathbb{H}'$. We introduce the event F_y that the outer boundary of $y+Q^{(et)}$ is connected to infinity by a path in $\mathscr{C}_\infty^* \cap \mathbb{H}$, and the analogous event $F_{y'}$ with y' and $\mathbb{H}'$ instead of y and $\mathbb{H}$. By splitting $A_y^{(et)}$ into $A_y^{(et)} \cap F_y$ and $A_y^{(et)} \cap F_y^c$ (and analogously for y') and invoking the independence of $A_y^{(et)} \cap F_y$ and $A_{y'}^{(et)} \cap F_{y'}$ we see that

$$\begin{aligned}\mathrm{cov}\big(A_y^{(et)}, A_{y'}^{(et)}\big) &= \mathrm{cov}\big(A_y^{(et)}\cap F_y^c, A_{y'}^{(et)}\big) + \mathrm{cov}\big(A_y^{(et)}\cap F_y, A_{y'}^{(et)}\cap F_{y'}^c\big)\\ &\le \mathrm{Prob}\big(\widetilde{A}^{(et)}\big)^2\big[\mathrm{Prob}(F_y^c)+\mathrm{Prob}(F_{y'}^c)\big],\end{aligned} \tag{5.56}$$

where we recalled (5.35) for the definition of $\widetilde{A}^{(et)}$.

In order to estimate the last expression, let us observe that

$$F_y^c \subset \big\{\partial(y+Q^{(et)})\cap\mathscr{C}_\infty^* = \varnothing\big\} \cup \bigcup_{x\in\partial(y+Q^{(et)})} G_x \tag{5.57}$$

where G_x is the event that x is in a finite component of $\{z : \xi(z) \ge -K\} \cap \mathbb{H}$ which reaches up to $\mathbb{H}^c$. By Lemma 5.4, the probability of the first event is bounded by $e^{-c\alpha(b_t)/2}$ and, as is well known [see, e.g., Grimmett (1989), proof of Theorem 6.51], $\mathrm{Prob}(G_x)$ is exponentially small in $\mathrm{dist}(x, \mathbb{H}^c)$, which is at least $R\alpha(b_t)/3$. Since $\#\partial(y + Q^{(et)}) = O(\alpha(b_t)^{d-1})$, we have

$$\mathrm{Prob}(F_y^c) \le \exp\{-c_* \alpha(b_t)\} \tag{5.58}$$

for some $c_* > 0$. Since $\alpha(b_t) = n^{\nu/(1-2\nu)+o(1)}$ for $t = e^n$, also the second term is thus summable on $t \in \{e^n : n \in \mathbb{N}\}$, because by (5.36), $\mathrm{Prob}(\widetilde{A}^{(et)}) \le \mathrm{Prob}(A^{(et)})/\mathrm{Prob}(0 \in \mathscr{C}_\infty^*)$. Combining all the preceding reasoning, the proof of Proposition 5.1 is complete. □

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