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SPECTRAL GAP FOR KAC'S MODEL OF BOLTZMANN EQUATION

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We consider a random walk on S^{n-1}, the standard sphere of dimension $n-1$, generated by random rotations on randomly selected coordinate planes i, j with $1 \le i < j \le n$. This dynamic was used by Marc Kac as a model for the spatially homogeneous Boltzmann equation. We prove that the spectral gap on S^{n-1} is n^{-1} up to a constant independent of n.

1. Introduction. We consider a random walk on the standard sphere S^{n-1} generated by random rotations on randomly selected coordinate planes i, j with $1 \le i < j \le n$. This dynamics was used by Kac ([5], [6]) as a model for the spatially homogeneous Boltzmann equation. Kac's idea was developed later on by, for example, [8], [3], [9] and [1]. The same walk was used by Hastings on the orthogonal group ([4]).

More precisely, let $R^{\theta}_{i,j}$ be a clockwise rotation by $\theta \in [0, 2\pi)$ in the coordinate plane i, j. Explicitly, $R^{\theta}_{i,j}$ is given by

$$R^{\theta}_{i,j} = \begin{array}{c} \begin{matrix} & & i & & j & \end{matrix} \\ \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & c & & s & \\ \vdots & & & \ddots & & \vdots \\ & & -s & & c & \\ & & & & \ddots & 0 \\ 0 & & & \cdots & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{matrix}$$

where all the diagonal entries are 1 except for the (i, i) and (j, j) entries that are $c = \cos(\theta)$, and all the off-diagonal entries are 0 except for the (i, j) and (j, i) entries that are equal to $s = \sin(\theta)$ and $-s$, respectively.

We consider the random walk on $S^{n-1}(1)$, the $(n-1)$-dimensional sphere of radius 1, generated by repeatedly multiplying by $R^{\theta}_{i,j}$ for $1 \le i < j \le n$ chosen uniformly and θ chosen uniformly in $[0, 2\pi)$. Then the generator of the dynamics is defined by

$$\mathscr{L}_n f(x) = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \int_0^{2\pi} \Big(f(R^{\theta}_{i,j} x) - f(x) \Big) \frac{d\theta}{2\pi},$$

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for $x \in S^{n-1}(1)$. Let P_t denote the corresponding semigroup. Let S_{n-1} denote the uniform probability measure on $S^{n-1}(1)$. One can check that S_{n-1} is invariant and reversible with respect to the generator $\mathscr{L}_n$. The Dirichlet form is given by

$$\mathscr{D}_n(f) = -\int f \mathscr{L}_n f \, dS_{n-1} = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} D_{i,j}(f),$$

where

$$D_{i,j}(f) = \frac{1}{2} \int_0^{2\pi} \int \left(f\left(R_{i,j}^{\theta} x\right) - f(x) \right)^2 dS_{n-1}(x) \frac{d\theta}{2\pi}.$$

With the generator $\mathscr{L}_n$, we define a Markov process in the standard way. Let $f_0(x)$ be the initial density on the sphere. Then the density $f_t(x) = P_t f_0(x)$ of the process at time t satisfies Kac's master equation: $\partial_t f_t(x) = \mathscr{L}_n f_t(x)$. If we assume that the initial distribution $f_0(x)$ is of product form, Kac proved that this property is approximately preserved in the limit $n \to \infty$. In modern terminology, Kac proved the "propagation of chaos." Once propagation of chaos is proved, it is straightforward to show that the marginal density of a particle satisfies the following analog of a Boltzmann equation:

$$\frac{\partial g_t(x)}{\partial t} = C \int \int_0^{2\pi} \Big(g_t(x\cos\theta + y\sin\theta) g_t(-x\sin\theta + y\cos\theta) - g_t(x) g_t(y) \Big) \, d\theta dy. \tag{1.1}$$

Clearly, the spectral properties of the collision operator on the right hand side of (1.1) is of critical importance to understand this equation. Since this collision operator is generated by the process with generator $\mathscr{L}_n$, a very basic property is the size of the spectral gap for $\mathscr{L}_n$. It turns out this is very difficult and Kac conjectured it to be of order $1/n$.

Since Kac conjectured it more than four decades ago, the only result is the recent work of Diaconis and Saloff-Coste [2]. Using comparisons techniques, they proved that the spectral gap of $\mathscr{L}_n$ is bounded below by c/n^3 where c is a constant independent of n. Using a test function [e.g., $f(x) = x_1$], one gets an upper bound of the form C/n.

The goal of this paper is to prove Kac's conjecture by supplying a lower bound of the form c/n to the spectral gap of $\mathscr{L}_n$.

The paper is organized as follows. In Section 2, we state our main result (Theorem 2.1) and outline its proof. We use the martingale method developed by Lu and Yau [7] to reduce our problem to estimating the variance of a function conditioned on one coordinate (Proposition 2.1). The rest of the paper is devoted to estimate this term. As in [7], a critical step of this approach is an estimate of a covariance term (cf. Lemma 3.1). We follow [7] by using a substitution lemma (see Section 5, Case 2), which has its origin from the study of interacting particle systems. Our major difficulties in using this approach, however, is in certain region of rare events (Section 5, Case 1). For this region, we need new arguments for performing various cut-offs. Since this part of the arguments is complicated, we recommend the reader to focus on the so-called "normal case" in the first reading, although the estimates of rare events are technically the key part of this paper.

Notice that the standard logarithmic Sobolev inequality does not hold for this model.

For any $\varepsilon > 0$, one can find a set A_ε of probability ε. Choosing $f = 1_{\{A_\varepsilon\}}$, the indicator function of A_ε, we get that the entropy of f is equal to

$$\int f \log \frac{f}{\int f dS_{n-1}} dS_{n-1} = -\varepsilon \log \varepsilon$$

and that $\mathscr{D}_n(\sqrt{f}) \leq \varepsilon$. Hence, the logarithmic Sobolev inequality is wrong for this model.

2. Main result.

THEOREM 2.1 (Spectral gap on the sphere). *There exists a constant $C > 0$ such that for all function $f \in L^2(S_{n-1})$*

$$E_{S_{n-1}}[f; f] \leq C\, n\, \mathscr{D}_n(f).$$

$E_{S^r_{n-1}}[f; g]$ denotes the covariance of f and g with respect to S^r_{n-1}, the uniform measure on the $(n-1)$-dimensional sphere of radius r. When $r = 1$, we omit to mention the radius.

This result means that the spectral gap is bounded below by $1/Cn$. Hence, this identifies the order in n of the spectral gap on S^{n-1}.

This also implies that the rate of convergence to equilibrium of our process in L^2 with respect to the uniform measure on S^{n-1} is of order $\exp(-t/Cn)$.

PROOF OF THEOREM 2.1. For all integer $l \geq 2$, let a_l be the smallest constant such that

$$E_{S_{l-1}}[h; h] \leq a_l\, l\, \mathscr{D}_l(h) \tag{2.1}$$

for all function $h \in L^2(S_{l-1})$. We will use this induction hypothesis to set up a recursive equation for a_n and to prove that a_n is bounded by a constant independent of n.

STEP 1. *Martingale decomposition.* Let $x = (x_1, \ldots, x_n)^t \in S^{n-1}(1)$. For $1 \leq j \leq n$, let $f_{1,\ldots,j}$ be the expectation of f conditioned on $x_1, \ldots, x_j$: $f_{1,\ldots,j}(x_1, \ldots, x_j) = E_{S_{n-1}}[f | x_1, \ldots, x_j]$. For any integer $M < n$ fixed, we have

$$\begin{aligned} E_{S_{n-1}}[f; f] = {} & E_{S_{n-1}}\left[E_{S_{n-1}}[f; f | x_1, \ldots, x_M]\right] \\ & + \sum_{i=0}^{M-1} E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\ldots,i+1}; f_{1,\ldots,i+1} | x_1, \ldots, x_i]\right] \end{aligned} \tag{2.2}$$

where $E_{S_{n-1}}[f; g | \mathscr{F}]$ denotes the covariance of f and g with respect to the measure S_{n-1} conditioned on $\mathscr{F}$:

$$E_{S_{n-1}}[f; g | \mathscr{F}] = E_{S_{n-1}}[fg | \mathscr{F}] - E_{S_{n-1}}[f | \mathscr{F}]\, E_{S_{n-1}}[g | \mathscr{F}].$$

STEP 2. *Induction.* Notice that S_{n-1} conditioned on $x_1, \dots, x_M$ is the uniform measure on $S^{n-M-1}(\sqrt{1-\sum_{j=1}^M x_j^2})$, the $(n-M-1)$-dimensional sphere of radius $\sqrt{1-\sum_{j=1}^M x_j^2}$. Hence, we can use (2.1) to bound the first term on the right-hand side of (2.2) by

$$a_{n-M}(n-M) Av_{j,l=M+1}^n D_{j,l}(f),$$

where $Av_{j,l=M+1}^n$ denotes the average over $j, l \in \{M+1, \dots, n\}$.

STEP 3. *Variance on one coordinate.* All the terms of the sum on the right-hand side of (2.2) are of the same form: $E_{S_{n-1}}[f_{1,\dots,i+1}; f_{1,\dots,i+1} | x_1, \dots, x_i]$ is the variance, with respect to the uniform measure

$$\mu = S_{n-i-1}^{\sqrt{1-\sum_{j=1}^i x_j^2}},$$

of $E_\mu[f | x_{i+1}]$, where $x_1, \dots, x_i$ are fixed.

We will use the following proposition to control these terms.

PROPOSITION 2.1. *For all $k > F > 0$ large enough, there exist a finite constant $C > 0$, $\varepsilon_F > 0$ and $C_F > 0$, two finite constants depending only on F, $\varepsilon_{k,n} > 0$ a finite constant depending only on k and n and $C_{k,F} > 0$, a finite constant depending only on k and F, such that*

$$\begin{aligned} E_{S_{n-1}}[f_1; f_1] \le C_F \frac{\varepsilon_{k,n}}{n} &E_{S_{n-1}}\big[E_{S_{n-1}}[f; f | x_1]\big] + C_{k,F} \mathscr{D}_n(f) \\ &+ C Av_{j=2}^n D_{1,j}(f) + \varepsilon_F \, Av_{j=2}^n \, E_{S_{n-1}}\big[E_{S_{n-1}}[f_{1,j}; f_{1,j} | x_1]\big], \\ &\lim_{F\to\infty} \varepsilon_F = 0 \quad \text{and} \quad \lim_{k\to\infty} \lim_{n\to\infty} \varepsilon_{k,n} = 0, \end{aligned} \tag{2.3}$$

where $Av_{j=2}^n$ denotes the average over $2 \le j \le n$.

STEP 4. *Conclusion.* Suppose we have ε_F / n instead of ε_F in (2.3). By the Schwarz inequality, $E_{S_{n-1}}[f_{1,j}; f_{1,j} | x_1] \le E_{S_{n-1}}[f; f | x_1]$. Choosing $M = 1$ in (2.2) and using the results of Steps 2 and 3, we obtain that

$$\begin{aligned} E_{S_{n-1}}[f; f] \le \left(1 + \frac{C_F \varepsilon_{k,n} + \varepsilon_F}{n}\right) a_{n-1}(n-1) Av_{j,l=2}^n D_{j,l}(f) \\ + C_{k,F} \mathscr{D}_n(f) + C Av_{j=2}^n D_{1,j}(f). \end{aligned}$$

Since we can do the same replacing x_1 by x_i in (2.2) and average over $i \in \{1, \dots, n\}$, we get that

$$E_{S_{n-1}}[f; f] \le \left(\left(1 + \frac{C_F \varepsilon_{k,n} + \varepsilon_F}{n}\right) a_{n-1}(n-1) + C'_{k,F}\right) \mathscr{D}_n(f).$$

From the definition (2.1) of a_n, this proves that a_n is less than $a_{n-1} - \frac{1}{n}[(1 - C_F \varepsilon_{k,n} - \varepsilon_F) a_{n-1} - C'_{k,F}]$, which is enough to conclude that a_n is bounded by a constant independent of n.

In our case, we do not have a factor n^{-1} in front of ε_F. However, since in the last term on the right-hand side of (2.3) we have the variance of $f_{1,j}$ (and *not* f), we will see that it is enough.

By Proposition 2.1, we know that

$$\begin{aligned}
&E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,i+1}; f_{1,\dots,i+1} | x_1, \dots, x_i]\right] \\
&\quad \le C\ Av^n_{j=i+1} D_{i,j}(f) \ + \ C_{k,F}\ Av^n_{j,l=i+1} D_{j,l}(f) \\
&\quad\quad + C_F\ \frac{\varepsilon_{k,n-i}}{n-i}\ E_{S_{n-1}}\left[E_{S_{n-1}}[f; f | x_1, \dots, x_{i+1}]\right] \\
&\quad\quad + \varepsilon_F\ Av^n_{j=i+2}\ E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,i+1,j}; f_{1,\dots,i+1,j} | x_1, \dots, x_{i+1}]\right]
\end{aligned}$$

for any $0 \le i \le M-1$.

Since we could have chosen in (2.2) any other set of size M instead of $x_1, \dots, x_M$ and since we will average over all possible choices, we can simply assume that $j = i+2$ in the last term of the above expression. But this is exactly $E_{S_{n-1}}[E_{S_{n-1}}[f_{1,\dots,i+2}; f_{1,\dots,i+2} | x_1, \dots, x_{i+1}]]$. Thus,

$$\begin{aligned}
&\sum_{i=0}^{M-1} E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,i+1}; f_{1,\dots,i+1} | x_1, \dots, x_i]\right] \\
&\quad \le \sum_{i=0}^{M-1} C_F \frac{\varepsilon_{k,n-i}}{n-i} E_{S_{n-1}}\left[E_{S_{n-1}}[f; f | x_1, \dots, x_{i+1}]\right] \\
&\quad\quad + \sum_{i=0}^{M-1} C\ Av^n_{j=i+1} D_{i,j}(f) \ + \ C_{k,F}\, Av^n_{j,l=i+1} D_{j,l}(f) \\
&\quad\quad + \varepsilon_F \sum_{i=1}^{M} E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,i+1}; f_{1,\dots,i+1} | x_1, \dots, x_i]\right].
\end{aligned}$$

Notice that the sum in the last term of the above expression is the same than the one we need to estimate except that it is over $i \in \{1, \dots, M\}$ and that it is multiplied by ε_F which is very small. Hence, instead of having one term for each i, we have only one boundary term:

$$\begin{aligned}
&(1-\varepsilon_F) \sum_{i=0}^{M-1} E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,i+1}; f_{1,\dots,i+1} | x_1, \dots, x_i]\right] \\
&\quad \le C_F \sum_{i=0}^{M-1} \frac{\varepsilon_{k,n-i}}{n-i} E_{S_{n-1}}\left[E_{S_{n-1}}[f; f | x_1, \dots, x_{i+1}]\right] \\
&\quad\quad + \sum_{i=0}^{M-1} C\ Av^n_{j=i+1} D_{i,j}(f) \ + \ C_{k,F}\ Av^n_{j,l=i+1}\, D_{j,l}(f) \\
&\quad\quad + \varepsilon_F E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,\dots,M+1}; f_{1,\dots,M+1} | x_1, \dots, x_M]\right].
\end{aligned}$$

Therefore, we "gained" a factor M^{-1}. Using the Schwarz inequality to replace $f_{1,\dots,M+1}$ by f in the above formula and using induction (2.1) for each variance, we get from (2.2) that $E_{S_{n-1}}[f;f]$ is bounded by

$$\frac{1}{1-\varepsilon_F}\sum_{i=0}^{M-1} C\ Av^n_{j=i+1} D_{i,j}(f) + C_{k,F}\ Av^n_{j,l=i+1} D_{j,l}(f)$$
$$+\frac{C_F}{1-\varepsilon_F}\sum_{i=0}^{M-1} \varepsilon_{k,n-i}\ a_{n-i-1}\ Av^n_{j,l=i+2} D_{j,l}(f)$$
$$+\frac{1}{1-\varepsilon_F}\ a_{n-M}\ (n-M)\ Av^n_{j,l=M+1} D_{j,l}(f).$$

As we mentioned before, we could have chosen in (2.2) any other set of size M instead of $x_1,\dots,x_M$. Averaging over all possible choices, we obtain that $E_{S_{n-1}}[f;f]$ is bounded above by

$$\frac{1}{1-\varepsilon_F}\left(\left(C_F\sum_{i=0}^{M-1}\varepsilon_{k,n-i}+(n-M)\right)A_{n-1}+MC'_{k,F}\right)\mathscr{D}_n(f)$$

where A_{n-1} =Sup$(a_1,\dots,a_{n-1})$. Hence, from the definition (2.1) of a_n, choosing $M=\lfloor\frac{n}{2}\rfloor$, the integer part of $\frac{n}{2}$, we get

$$a_n \le \frac{1}{1-\varepsilon_F}\left(\left(\frac{C_F}{n}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\varepsilon_{k,n-i}+\frac{3}{5}\right)A_{n-1}+\frac{1}{2}C'_{k,F}\right).$$

This is enough to conclude the proof of the theorem. □

3. Proof of Proposition 2.1. It is well known that the first marginal ν_{n-1} of S_{n-1} has density

$$d\nu_{n-1}(x_1) = C_{n-1}(1-x_1^2)^{\frac{n-3}{2}}\,dx_1$$

where

$$C_{n-1} = \mathscr{B}\left(\frac{1}{2},\frac{n-1}{2}\right)^{-1} = \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)}$$

is a normalizing constant (notice it is of order $\sqrt{n}$). We have

$$E_{\nu_{n-1}}[f_1;f_1] = \int\int (f_1(x_1)-f_1(y_1))^2\, 1_{\{x_1^2\ge y_1^2\}}\,d\nu_{n-1}(x_1)\,d\nu_{n-1}(y_1).$$

Lemma 3.1 below gives us a formula for $f_1(x_1)-f_1(y_1)$. Instead of using it directly, it is more convenient to use it to compute $f_1(x_1)-f_1(w_1)$ and $f_1(w_1)-f_1(y_1)$ and then average over $w_1\in[|x_1|/2,|x_1|]$.

LEMMA 3.1. *Fix $x_1^2\ge w_1^2$. Let $w^{(j,+)}$ (resp., $w^{(j,-)}$) be obtained from x by changing values at 1 and j: $w_1^{(j,+)}=w_1$ and $w_j^{(j,+)}=\sqrt{x_j^2+x_1^2-w_1^2}$ (resp.,*

$w_1^{(j,-)} = w_1$ *and* $w_j^{(j,-)} = -\sqrt{x_j^2 + x_1^2 - w_1^2}$ *). We have*

$$
\begin{aligned}
f_1(x_1) - f_1(w_1) = {} & \tfrac{1}{2} E_{S_{n-1}} \left[Av_{j=2}^n \left(f(x) - f(w^{(j,+)}) \right) \Big| x_1 \right] \\
& + \tfrac{1}{2} E_{S_{n-1}} \left[Av_{j=2}^n \left(f(x) - f(w^{(j,-)}) \right) \Big| x_1 \right] \qquad (3.1) \\
& + E_{S_{n-1}} \left[f(w); Av_{j=2}^n G_{x_1,w_1}(w_j) \Big| w_1 \right]
\end{aligned}
$$

where

$$
G_{x_1,w_1}(w_j) = 1_{\{w_j^2 \geq x_1^2 - w_1^2\}} \frac{|w_j|}{\sqrt{w_1^2 + w_j^2 - x_1^2}} \left(\frac{1 - w_1^2}{1 - x_1^2} \right)^{\frac{n-3}{2}}. \qquad (3.2)
$$

From now on, all the bounds are made up to a multiplicative constant. $E_{\nu_{n-1}}[f_1; f_1]$ is bounded by

$$
\int\int\int_{|x_1|/2}^{|x_1|} (f_1(x_1) - f_1(w_1))^2 \frac{dw_1}{|x_1|} 1_{\{x_1^2 \geq y_1^2\}} \, d\nu_{n-1}(x_1) d\nu_{n-1}(y_1) \qquad (3.3)
$$

$$
+ \int\int\int_{|x_1|/2}^{|x_1|} (f_1(w_1) - f_1(y_1))^2 \frac{dw_1}{|x_1|} 1_{\{x_1^2 \geq y_1^2\}} \, d\nu_{n-1}(x_1) d\nu_{n-1}(y_1). \qquad (3.4)
$$

Since $x_1^2 \geq w_1^2$, we use Lemma 3.1 to bound (3.3) by

$$
\begin{aligned}
& \int Av_{j=2}^n E_{S_{n-1}} \left[f(x) - f(w^{(j,+)}) \Big| x_1 \right]^2 \frac{dw_1}{|x_1|} \int_0^{|x_1|} d\nu_{n-1}(y_1) d\nu_{n-1}(x_1) \\
& + \int Av_{j=2}^n E_{S_{n-1}} \left[f(x) - f(w^{(j,-)}) \Big| x_1 \right]^2 \frac{dw_1}{|x_1|} \int_0^{|x_1|} d\nu_{n-1}(y_1) d\nu_{n-1}(x_1) \qquad (3.5) \\
& + \int\int E_{S_{n-1}} \left[f(w); Av_{j=2}^n G_{x_1,w_1}(w_j) \Big| w_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1),
\end{aligned}
$$

where the integration in x_1 and w_1 is only for $|x_1| \geq w_1 \geq |x_1|/2$.

Let us now turn to (3.4). We apply Lemma 3.1 again. Notice that we will not get the same formula whether $w_1^2 \geq y_1^2$ or $w_1^2 \leq y_1^2$. (3.4) is bounded by

$$
\begin{aligned}
& \int Av_{j=2}^n E_{S_{n-1}} \left[f(w) - f(y^{(j,+)}) \Big| w_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1) \\
& + \int Av_{j=2}^n E_{S_{n-1}} \left[f(w) - f(y^{(j,-)}) \Big| w_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1) \qquad (3.6) \\
& + \int E_{S_{n-1}} \left[f(y); Av_{j=2}^n G_{w_1,y_1}(y_j) \Big| y_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1)
\end{aligned}
$$

where the integration is above $2w_1 \geq |x_1| \geq w_1 \geq |y_1|$, plus

$$
\begin{aligned}
&\int Av_{j=2}^{n} E_{S_{n-1}} \left[f(y) - f(w^{(j,+)}) \Big| y_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1) \\
&+ \int Av_{j=2}^{n} E_{S_{n-1}} \left[f(y) - f(w^{(j,-)}) \Big| y_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1) \qquad (3.7) \\
&+ \int E_{S_{n-1}} \left[f(w); Av_{j=2}^{n} G_{y_1, w_1}(w_j) \Big| w_1 \right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1) d\nu_{n-1}(y_1)
\end{aligned}
$$

where the integration is above $|x_1| \geq |y_1| \geq w_1 \geq |x_1|/2$.

CLAIM 1. The first and the second terms of (3.5), (3.6) and (3.7) are bounded by $CAv_{j=2}^{n} D_{1,j}(f)$, where C is a constant independent of n.

CLAIM 2. The third terms of (3.5), (3.6) and (3.7) are bounded by

$$
\begin{aligned}
&C_{k,F}\, Av_{i,j=2}^{n}\, D_{i,j}(f) + \varepsilon_F\, Av_{j=2}^{n}\, E_{S_{n-1}} \left[E_{S_{n-1}}[f_{1,j}; f_{1,j} | x_1] \right] \\
&+ C_F\, \frac{\varepsilon_{k,n}}{n}\, E_{S_{n-1}} \left[E_{S_{n-1}}[f; f | x_1] \right], \qquad (3.8)
\end{aligned}
$$

where the constants $C_{k,F}$, ε_F, C_F and $\varepsilon_{k,n}$ are like in Proposition 2.1.

Claim 1 (resp., Claim 2) is proved in Section 4 (resp., Sections 5 and 6). This concludes the proof of the proposition. □

4. Proof of Claim 1. Let's look at the first term of (3.5).

Since $E_{S_{n-1}} \left[\sqrt{x_1^2 + x_j^2 - w_1^2} \Big| x_1 \right] \leq \sqrt{x_1^2 - w_1^2} + O(1/\sqrt{n})$, by the Schwarz inequality, $E_{S_{n-1}} \left[f(x) - f(w^{(j,+)}) \Big| x_1 \right]^2$ is bounded by

$$
E_{S_{n-1}} \left[\left(f(x) - f(w^{(j,+)}) \right)^2 \frac{\sqrt{x_1^2 - w_1^2} + O\left(1/\sqrt{n}\right)}{\sqrt{x_1^2 + x_j^2 - w_1^2}} \Bigg| x_1 \right].
$$

Moreover, since C_{n-1} is of order $\sqrt{n}$, we have

$$
\frac{\sqrt{x_1^2 - w_1^2} + O\left(1/\sqrt{n}\right)}{|x_1|} \int_0^{|x_1|} d\nu_{n-1}(y_1) = O(1).
$$

Thus, the first term of (3.5) is bounded by

$$
Av_{j=2}^{n} E_{S_{n-1}} \left[\int_{|x_1|/2}^{|x_1|} \left(f(x) - f(w^{(j,+)}) \right)^2 \frac{dw_1}{\sqrt{x_1^2 + x_j^2 - w_1^2}} \right].
$$

Let's make the following change of variable: $w_1 = x_1 \cos\theta + x_j \sin\theta$. Since $dw_1 = (-x_1 \sin\theta + x_j \cos\theta) d\theta = \pm\sqrt{x_1^2 + x_j^2 - w_1^2} d\theta$, we get that the first term of (3.5) is bounded by

$$Av_{j=2}^{n}E_{S_{n-1}}\left[\int_0^{2\pi}\left(f(x)-f(R_{1,j}^{\theta}x)\right)^2 d\theta\right]=4\pi Av_{j=2}^{n}D_{1,j}(f).$$

Using the same argument, we prove that the first term of (3.6) is bounded by

$$Av_{j=2}^{n}E_{S_{n-1}}\left[\int_{-w_1}^{w_1}\left(f(w)-f(y^{(j,+)})\right)^2 A(w_1,y_1)\frac{dy_1}{\sqrt{w_1^2+w_j^2-y_1^2}}\right],$$

where $A(w_1, y_1)$ is equal to

$$\begin{aligned}
&\int_{w_1}^{2w_1}\frac{\sqrt{w_1^2-y_1^2}+O(1/\sqrt{n})}{x_1}d\nu_{n-1}(x_1)\left(\frac{1-y_1^2}{1-w_1^2}\right)^{\frac{n-3}{2}}\\
&\quad\le\left(\frac{1}{1-w_1^2}\right)^{\frac{n-3}{2}}\int_{w_1}^{2w_1}\frac{\sqrt{w_1^2-y_1^2}}{x_1}d\nu_{n-1}(x_1)+O(1)\\
&\quad\le\frac{1}{w_1}\left(1-w_1^2\right)^{-\frac{n-3}{2}}\int_{w_1}^{2w_1}x_1 d\nu_{n-1}(x_1)1_{\{w_1\ge 1/\sqrt{n}\}}\\
&\qquad+C_{n-1}\int_{w_1}^{2w_1}dx_1 1_{\{w_1\le 1/\sqrt{n}\}}+O(1),
\end{aligned}$$

which is of order 1. In the same way, since

$$\int_{|y_1|\le|x_1|\le 2|y_1|}\frac{\sqrt{y_1^2-w_1^2}+O(1/\sqrt{n})}{|x_1|}d\nu_{n-1}(x_1)=O(1),$$

the first term of (3.7) is bounded by

$$Av_{j=2}^{n}E_{S_{n-1}}\left[\int_{|y_1|/2}^{|y_1|}\left(f(y)-f(w^{(j,+)})\right)^2\frac{dw_1}{\sqrt{y_1^2+y_j^2-w_1^2}}\right].$$

Making a change of variable, we conclude that the first term of (3.5), (3.6) and (3.7) are bounded by $CAv_{j=2}^{n}D_{1,j}(f)$.

We deal with the second term of (3.5), (3.6) and (3.7) in the same way.

5. Proof of Claim 2. We first deal with the third term of (3.5). The terms of (3.6) and (3.7) are treated in section 6.

Let us fix $F > 0$. We divide the proof into two cases:

Case 1. $x_1^2 \ge Fn^{-1}$.
Case 2. $x_1^2 \le Fn^{-1}$.

Notice that if x_1^2 is too large (as in Case 1), G_{x_1,w_1} is very large and it may be difficult to control the third term of (3.5). However, since we integrate x_1

with respect to ν_{n-1}, x_1 is typically of order n^{-1} and thus Case 1 is unlikely to happen if F is large. Case 2 is the "normal" case.

Case 1^*. $x_1^2 \geq Fn^{-1}$.

We will prove that in this case the third term of (3.5) is bounded above by

$$\varepsilon_F \, Av_{j=2}^n \, E_{S_{n-1}}\left[E_{S_{n-1}}\left[f_{1,j}; f_{1,j}|w_1\right]\right], \tag{5.1}$$

where $f_{1,j}(w_1, w_j) = E_{S_{n-1}}\left[f(w)|w_1, w_j\right]$ and ε_F goes to 0 as F goes to ∞.

Recall definition (3.2) of G_{x_1,w_1} and the one of ν_{n-1}. We can rewrite the third term of (3.5) as

$$Av_{j=2}^n \int\int_{w_1 \leq |x_1| \leq 2w_1} E_{S_{n-1}}\left[f(w); \frac{|w_j|}{\sqrt{w_1^2 + w_j^2 - x_1^2}} 1_{\{w_j^2 \geq x_1^2 - w_1^2\}}\middle| w_1\right]^2$$
$$\times 1_{\{x_1^2 \geq Fn^{-1}\}} \left(\frac{1 - w_1^2}{1 - x_1^2}\right)^{\frac{n-3}{2}} \frac{dx_1}{|x_1|} d\nu_{n-1}(w_1)$$

and replace f by $f_{1,j}$ in the above covariance. Using the Schwarz inequality, one can bound the covariance term by

$$E_{S_{n-1}}\left[(f_{1,j} - f_1)^2 \frac{1}{|w_j|^\gamma (w_1^2 + w_j^2 - x_1^2)^{3/4}} 1_{\{w_j^2 \geq x_1^2 - w_1^2\}}\middle| w_1\right]$$
$$\times E_{S_{n-1}}\left[\frac{|w_j|^{2+\gamma}}{(w_1^2 + w_j^2 - x_1^2)^{1/4}} 1_{\{w_j^2 \geq x_1^2 - w_1^2\}}\middle| w_1\right]$$

for any γ. Making the change of variable $x_j^2 = w_1^2 + w_j^2 - x_1^2$, we obtain that the second expectation is equal to

$$E_{S_{n-1}}\left[|x_j|^{1/2}(x_1^2 + x_j^2 - w_1^2)^{\frac{1+\gamma}{2}}\middle| x_1\right] \left(\frac{1 - x_1^2}{1 - w_1^2}\right)^{\frac{n-3}{2}}.$$

Choosing $\gamma = 1/2$, we bound the third term of (3.5) by

$$n^{-1/4} \, Av_{j=2}^n \, E_{S_{n-1}}\Bigg[\, E_{S_{n-1}}\Bigg[\, (f_{1,j} - f_1)^2 \, \frac{1}{|w_j|^{1/2}} \int \frac{(x_1^2 - w_1^2)^{3/4} + O(n^{-3/4})}{(w_1^2 + w_j^2 - x_1^2)^{3/4}}$$
$$\times \, 1_{\{w_j^2 + w_1^2 \geq x_1^2 \geq w_1^2\}} 1_{\{x_1^2 \geq Fn^{-1}\}} \frac{dx_1}{|x_1|}\Bigg| w_1\Bigg]\Bigg].$$

We need to estimate

$$\int_{w_1}^{\sqrt{w_1^2 + w_j^2}} \frac{(x_1^2 - w_1^2)^{3/4} + O(n^{-3/4})}{x_1^2} \frac{x_1}{(w_1^2 + w_j^2 - x_1^2)^{3/4}} 1_{\{x_1^2 \geq Fn^{-1}\}} \, dx_1.$$

Since $x_1^2 \geq Fn^{-1}$, the above integral is less than

$$\left(\frac{n}{F}\right)^{1/4} \int_{w_1}^{\sqrt{w_1^2+w_j^2}} \frac{x_1}{(w_1^2+w_j^2-x_1^2)^{3/4}}\, dx_1 = \left(\frac{n}{F}\right)^{1/4} |w_j|^{1/2}.$$

Hence, the third term of (3.5) is bounded above by (5.1) with $\varepsilon_F = O(F^{-1/4})$. $\square$

Case 2* (*"normal" case*) $x_1^2 \leq Fn^{-1}$. Let us introduce some notation. Fix a positive integer k independent of n (k will be chosen larger than F).

Divide $\{2, \dots, n\}$ into L sets A_α of size k. If k does not divide $n-1$, then the set A_L has size $l \in \{1, \dots, k-1\}$.

For $j \in A_\alpha$, we consider the expectation of $G_{x_1,w_1}(w_j)$ conditioned on $(w_i)_{i\notin A_\alpha}$. Notice this depends only on $\bar{w}_\alpha^2 = 1 - \sum_{i\notin A_\alpha} w_i^2 = \sum_{i\in A_\alpha} w_i^2$. Let's denote it by

$$\hat{G}_{x_1,w_1}(\bar{w}_\alpha^2) = E_{S_{n-1}}\left[G_{x_1,w_1}(w_j)\Big|(w_i)_{i\notin A_\alpha}\right].$$

The third term of (3.5) is bounded above by

$$\begin{aligned}
&\int\int_{|x_1|/2}^{|x_1|} E_{S_{n-1}}\left[f; Av_\alpha Av_{j\in A_\alpha} G_{x_1,w_1}(w_j) - \hat{G}_{x_1,w_1}(\bar{w}_\alpha^2)\Big| w_1\right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1)\\
&\quad + \int\int_{|x_1|/2}^{|x_1|} E_{S_{n-1}}\left[f(w); Av_{\alpha=1}^L \hat{G}_{x_1,w_1}(\bar{w}_\alpha^2)\Big| w_1\right]^2 \frac{dw_1}{|x_1|} d\nu_{n-1}(x_1)\\
&\quad = B_1 + B_2.
\end{aligned}$$

LEMMA 5.1.

$$B_1 \leq C_{k,F}\ Av_{\alpha=1}^L Av_{i,j\in A_\alpha} D_{i,j}(f),$$

where $C_{k,F}$ is a constant depending only on k and F.

LEMMA 5.2.

$$B_2 \leq C_F \frac{\varepsilon_{k,n}}{n} E_{S_{n-1}}\left[E_{S_{n-1}}[f; f|w_1]\right],$$

where C_F and $\varepsilon_{k,n}$ are as in Proposition 2.1.

Therefore, assuming Lemma 5.1 and 5.2 and recalling the result (5.1) obtained in Case 1, we proved that the third term of (3.5) is bounded above by

$$\begin{aligned}
&C_{k,F}\ Av_{\alpha=1}^L Av_{i,j\in A_\alpha} D_{i,j}(f) + C_F \frac{\varepsilon_{k,n}}{n} E_{S_{n-1}}\left[E_{S_{n-1}}[f; f|w_1]\right]\\
&\quad + \varepsilon_F\ Av_{j=2}^n E_{S_{n-1}}\left[E_{S_{n-1}}[f_{1,j}; f_{1,j}|x_1]\right].
\end{aligned}$$

Since the choice of the sets A_α is arbitrary, we average over all possible choices and we get (3.8). $\square$

PROOF OF LEMMA 5.1. We can bound the square of the covariance in B_1 by

$$Av_{\alpha=1}^{L} E_{S_{n-1}}\left[f(w); Av_{j\in A_\alpha} G_{x_1,w_1}(w_j) - \hat{G}_{x_1,w_1}(\bar{w}_\alpha^2) \middle| w_1 \right]^2.$$

By definition of $\hat{G}_{x_1,w_1}$, the expectation of $G_{x_1,w_1}(w_j) - \hat{G}_{x_1,w_1}(\bar{w}_\alpha^2)$ conditioned on $(w_\ell)_{\ell\notin A_\alpha}$ is equal to 0. Hence, by the Schwarz inequality, the above expression is bounded above by

$$Av_{\alpha=1}^{L} E_{S_{n-1}}\left[E_{S_{n-1}}\left[f(w); Av_{j\in A_\alpha} G_{x_1,w_1}(w_j) \middle| (w_\ell)_{\ell\notin A_\alpha} \right]^2 \middle| w_1 \right].$$

Using the Schwarz inequality again, the above covariance term is less than

$$\begin{aligned} &E_{S_{n-1}}\left[\left(f(w) - E_{S_{n-1}}\left[f \middle| (w_\ell)_{\ell\notin A_\alpha} \right] \right)^2 Av_{j\in A_\alpha} G_{x_1,w_1}(w_j) \middle| (w_\ell)_{\ell\notin A_\alpha} \right] \\ &\quad \times E_{S_{n-1}}\left[G_{x_1,w_1}(w_j) \middle| (w_\ell)_{\ell\notin A_\alpha} \right]. \end{aligned}$$

The expectation on the right-hand side is $\hat{G}_{x_1,w_1}(\bar{w}_\alpha^2)$. Let's compute it. We need $\bar{w}_\alpha^2 \geq x_1^2 - w_1^2$. (Otherwise, it is equal to 0.) $\hat{G}_{x_1,w_1}(\bar{w}_\alpha^2)$ is equal to

$$\begin{aligned} &\int G_{x_1,w_1}(w_j)\, dS_{k-1}^{\bar{w}_\alpha}(w_j) \\ &= \int G_{x_1,w_1}(\bar{w}_\alpha z)\, d\nu_{k-1}(z) \\ &= \left(\frac{1-w_1^2}{1-x_1^2} \right)^{\frac{n-3}{2}} \int \frac{|\bar{w}_\alpha z|}{\sqrt{\bar{w}_\alpha^2 z^2 + w_1^2 - x_1^2}} 1_{\{\bar{w}_\alpha^2 z^2 \geq x_1^2 - w_1^2\}}\, d\nu_{k-1}(z). \end{aligned}$$

Making the change of variable $\bar{w}_\alpha^2 z^2 + w_1^2 - x_1^2 = \left(\bar{w}_\alpha^2 + w_1^2 - x_1^2\right) a^2$, we obtain

$$\hat{G}_{x_1,w_1}(\bar{w}_\alpha^2) = \left(\frac{1-w_1^2}{1-x_1^2} \right)^{\frac{n-3}{2}} 1_{\{\bar{w}_\alpha^2 \geq x_1^2 - w_1^2\}} \left(\frac{\bar{w}_\alpha^2 - (x_1^2 - w_1^2)}{\bar{w}_\alpha^2} \right)^{\frac{k-2}{2}}. \tag{5.2}$$

Since $x_1^2 \leq Fn^{-1}$, this term is bounded by a constant C_F depending only on F. Hence, B_1 is bounded above by C_F times

$$\begin{aligned} &Av_\alpha Av_{j\in A_\alpha} \int\int_{|x_1|/2}^{|x_1|} E_{S_{n-1}}\left[E_{S_{n-1}}\left[\left(f(w) - E_{S_{n-1}}\left[f(w) \middle| (w_\ell)_{\ell\notin A_\alpha} \right] \right)^2 \right.\right. \\ &\qquad\qquad \left.\left. \times G_{x_1,w_1}(w_j) \middle| (w_k)_{k\notin A_\alpha} \right] \middle| w_1 \right] \frac{dw_1}{|x_1|}\, d\nu_{n-1}(x_1). \end{aligned}$$

Let's first integrate in x_1. Recall the definition of $G_{x_1,w_1}(w_j)$. Since

$$\int_{w_1}^{2w_1}\left(\frac{1-w_1^2}{1-x_1^2}\right)^{\frac{n-3}{2}}\left(1+\frac{x_1^2-w_1^2}{w_1^2+w_j^2-x_1^2}\right)^{1/2} 1_{\{w_j^2\geq x_1^2-w_1^2\}}\frac{d\nu_{n-1}(x_1)}{x_1}$$
$$\leq C_{n-1}(1-w_1^2)^{\frac{n-3}{2}}\left(\int_{w_1}^{2w_1}\frac{dx_1}{x_1}+\int_{w_1}^{\sqrt{w_1^2+w_j^2}}\frac{dx_1}{\sqrt{w_1^2+w_j^2-x_1^2}}\right)$$
$$\leq C_{n-1}(1-w_1^2)^{\frac{n-3}{2}}\left(\log 2+2\frac{w_1}{|w_j|}\left(\sqrt{1+\frac{w_j^2}{w_1^2}}-1\right)\right)$$

is $O(1)C_{n-1}(1-w_1^2)^{\frac{n-3}{2}}$, we proved that B_1 is bounded above by

$$C_F Av_{\alpha=1}^L\int E_{S_{n-1}}\left[E_{S_{n-1}}\left[f;f\middle|(w_\ell)_{\ell\notin A_\alpha}\right]\middle|w_1\right]d\nu_{n-1}(w_1)$$
$$=C_F Av_{\alpha=1}^L E_{S_{n-1}}\left[E_{S_{n-1}}\left[f;f\middle|(w_\ell)_{\ell\notin A_\alpha}\right]\right].$$

Using induction (2.1) in order to bound the variance of f, we obtain

$$B_1\leq C_{k,F}\,Av_{\alpha=1}^L Av_{i,j\in A_\alpha}E_{S_{n-1}}\left[D_{i,j}\left(f\middle|(w_\ell)_{\ell\notin A_\alpha}\right)\right]$$
$$\leq C_{k,F}\,Av_{\alpha=1}^L Av_{i,j\in A_\alpha}D_{i,j}(f),$$

where $C_{k,F}$ is a constant depending only on k and F. □

PROOF OF LEMMA 5.2. Recalling (5.2), we can rewrite B_2 as

$$\int\int_{w_1}^{2w_1}E_{S_{n-1}}\left[f;Av_{\alpha=1}^L h(\bar{w}_\alpha^2)\middle|w_1\right]^2\left(\frac{1-w_1^2}{1-x_1^2}\right)^{\frac{n-3}{2}}\frac{dx_1}{x_1}d\nu_{n-1}(w_1)\tag{5.3}$$
$$\text{where }\ h(a)=\left(\frac{a-(x_1^2-w_1^2)}{a}\right)^{\frac{k-2}{2}}1_{\{a\geq x_1^2-w_1^2\}}.$$

Let $r_\alpha^2=E_{S_{n-1}}\left[\bar{w}_\alpha^2|w_1\right]=\frac{|A_\alpha|}{n-1}(1-w_1^2)$. Since this is a constant with respect to the uniform measure conditioned on w_1, we can replace $h(\bar{w}_\alpha^2)$ by $h(\bar{w}_\alpha^2)-h(r_\alpha^2)$ in (5.3).

Assume that $\frac{n-1}{k}\in\mathbb{N}$. Thus, $r_\alpha^2=\frac{k}{n-1}(1-w_1^2)$ does not depend on α. Let's denote it by r^2. We will use the Taylor expansion to estimate $h(\bar{w}_\alpha^2)-h(r^2)$ when $|\bar{w}_\alpha^2-r^2|\leq\frac{k^{1-\varepsilon}}{n-1}$. Moreover, since $\sum_{\alpha=1}^L(\bar{w}_\alpha^2-r^2)=0$,

$$Av_{\alpha=1}^L\left(h(\bar{w}_\alpha^2)-h(r^2)\right)$$
$$=\frac{1}{L}\sum_{\alpha=1}^L\left(h(\bar{w}_\alpha^2)-h(r^2)-h'(r^2)(\bar{w}_\alpha^2-r^2)\right)1_{\{|\bar{w}_\alpha^2-r^2|>\frac{k^{1-\varepsilon}}{n-1}\}}\tag{5.4}$$
$$+\frac{1}{L}\sum_{\alpha=1}^L h''(c_\alpha)(\bar{w}_\alpha^2-r^2)^2 1_{\{|\bar{w}_\alpha^2-r^2|\leq\frac{k^{1-\varepsilon}}{n-1}\}},$$

where $c_\alpha\in[r^2,\bar{w}_\alpha^2]$.

We split the covariance appearing in (5.3) into two parts, considering each of the above terms separately.

Let's look at the part corresponding to the first term of (5.4):

$$E_{S_{n-1}}\left[f; Av_\alpha\left(h(\bar{w}_\alpha^2)-h(r^2)-h'(r^2)(\bar{w}_\alpha^2-r^2)\right)1_{\{|\bar{w}_\alpha^2-r^2|>\frac{k^{1-\varepsilon}}{n-1}\}}\middle| w_1\right]^2.$$

From the Schwarz inequality, it is bounded by

$$E_{S_{n-1}}\left[\left(Av_\alpha\left(h(\bar{w}_\alpha^2)-h(r^2)-h'(r^2)(\bar{w}_\alpha^2-r^2)\right)1_{\{|\bar{w}_\alpha^2-r^2|>\frac{k^{1-\varepsilon}}{n-1}\}}\right)^2\middle| w_1\right]$$
$$\times E_{S_{n-1}}\left[f;f\middle|w_1\right].$$

We need to estimate the expectation on the left-hand side. Developing the square and using Chebyshev's inequality, we get that it is bounded by

$$\frac{1}{L}\left(\|h\|_\infty^2\left(\frac{n-1}{k^{1-\varepsilon}}\right)^2+h'(r^2)^2\right)\left(\frac{n-1}{k^{1-\varepsilon}}\right)^2 E_{S_{n-1}}\left[|\bar{w}_\alpha^2-r^2|^4\middle|w_1\right]$$
$$+\left(\|h\|_\infty^2\left(\frac{n-1}{k^{1-\varepsilon}}\right)^2+\|h\|_\infty|h'(r^2)|\left(\frac{n-1}{k^{1-\varepsilon}}\right)+h'(r^2)^2\right)\left(\frac{n-1}{k^{1-\varepsilon}}\right)^2$$
$$\times E_{S_{n-1}}\left[(\bar{w}_\alpha^2-r^2)^2(\bar{w}_\beta^2-r^2)^2\middle|w_1\right].$$

An easy but wearisome computation shows that

$$E_{S_{n-1}}\left[(\bar{w}_\alpha^2-r^2)^2(\bar{w}_\beta^2-r^2)^2\middle|w_1\right]=\begin{cases}O\left(k^2/n^5\right), & \text{if } \alpha\neq\beta,\\ O\left(k^2/n^4\right), & \text{otherwise.}\end{cases}\tag{5.5}$$

Since $\|h\|_\infty\le 1$ and $h'(r^2)\le O\left(\frac{Fn}{k}\right)$ (because $x_1^2\le F/n$), we obtain that the above formula is of order $F^2k^{4\varepsilon-1}/n$. We now have to multiply by

$$\left(\frac{1-w_1^2}{1-x_1^2}\right)^{\frac{n-3}{2}}\frac{1}{|x_1|}\le\frac{C_F}{|x_1|}$$

and to integrate in x_1 and w_1 for $w_1\le|x_1|\le 2w_1$. Thus, the part of (5.3) corresponding to the first term of (5.4) is bounded by

$$\frac{C_Fk^{4\varepsilon-1}}{n}E_{S_{n-1}}\left[E_{S_{n-1}}\left[f;f\middle|w_1\right]\right].\tag{5.6}$$

Let's turn to the part corresponding to the second term of (5.4): By the Schwarz inequality,

$$E_{S_{n-1}}\left[f(w);\frac{1}{L}\sum_{\alpha=1}^{L}h''(c_\alpha)(\bar{w}_\alpha^2-r^2)^2 1_{\{|\bar{w}_\alpha^2-r^2|\le\frac{k^{1-\varepsilon}}{n-1}\}}\middle|w_1\right]^2$$

is bounded above by

$$E_{S_{n-1}}\left[f;f\middle|w_1\right]E_{S_{n-1}}\left[\left(Av_\alpha h''(c_\alpha)(\bar{w}_\alpha^2-r^2)^2 1_{\{|\bar{w}_\alpha^2-r^2|\le\frac{k^{1-\varepsilon}}{n-1}\}}\right)^2\middle|w_1\right].$$

Since $|\bar{w}_\alpha^2 - r^2| \le \frac{k^{1-\varepsilon}}{n-1}$, we have $c_\alpha \ge O(k/n)$. Hence, we get that $h''(c_\alpha)$ is of order $F^2 n^2/k^2$ and thus, using (5.5),

$$E_{S_{n-1}}\left[\left(Av_{\alpha=1}^L h''(c_\alpha)(\bar{w}_\alpha^2 - r^2)^2 1_{\{|\bar{w}_\alpha^2 - r^2| \le \frac{k^{1-\varepsilon}}{n-1}\}}\right)^2 \middle| w_1\right]$$
$$\le C_F \frac{n^4}{k^4} E_{S_{n-1}}\left[\left(Av_{\alpha=1}^L (\bar{w}_\alpha^2 - r^2)^2\right)^2 \middle| w_1\right] = O\left(\frac{C_F}{kn}\right)$$

where C_F depends only on F.

Thus, we proved that the part of (5.3) corresponding to the last term of (5.4) is bounded by

$$\frac{C_F}{kn} E_{S_{n-1}}\left[E_{S_{n-1}}\left[f; f \middle| w_1\right]\right]. \tag{5.7}$$

Recall the bounds (5.6) and (5.7) we obtained. This proves Lemma 5.2 when $\frac{n-1}{k} \in \mathbb{N}$ with $\varepsilon_{k,n} = k^{-1} + k^{4\varepsilon - 1}$, which go to 0 as k goes to ∞ for $\varepsilon < 1/4$.

If $\frac{n-1}{k} \notin \mathbb{N}$, we treat the terms corresponding to the set A_L separately. Since we have a factor $L^{-2} = O\left(k^2/n^2\right)$ in front of them, we prove Lemma 5.2 with $\varepsilon_{k,n} = k^{-1} + k^{4\varepsilon-1} + k^2/n$. $\square$

6. Proof of the third terms of (3.6) and (3.7). Let's turn to the third term of (3.6). It is equal to twice

$$\int E_{S_{n-1}}\left[f(y); Av_{j=2}^n G_{w_1,y_1}(y_j) \middle| y_1\right]^2 \int_{w_1}^{2w_1} \frac{d\nu_{n-1}(x_1)}{x_1} dw_1 d\nu_{n-1}(y_1)$$

where the integration in w_1 is over $w_1 \ge |y_1|$. Since

$$\int_{w_1}^{2w_1} \frac{d\nu_{n-1}(x_1)}{x_1} \le C \frac{(1-w_1^2)^{\frac{n-3}{2}}}{w_1},$$

we can bound the third term of (3.6) by

$$\int\int_{w_1 \ge |y_1|} E_{S_{n-1}}\left[f(y); Av_{j=2}^n G_{w_1,y_1}(y_j) \middle| y_1\right]^2 \frac{dy_1}{w_1} d\nu_{n-1}(w_1)$$

in Case 1. Notice that in this case, we did not use the fact that $|x_1| \le 2w_1$ when treating the third term of (3.5). Hence, we can conclude in the same way.

In Case 2, we used that $|x_1| \le 2w_1$, so we have to be careful. But it works in the same way because we can use in Lemma 5.1 that

$$\int_{|y_1|}^{\sqrt{y_1^2+y_j^2}} \int_{w_1}^{2w_1} \frac{d\nu_{n-1}(x_1)}{x_1}\left(1 + \frac{w_1^2 - y_1^2}{y_1^2 + y_j^2 - w_1^2}\right)^{1/2} dw_1$$
$$\le \int\int_{x_1/2}^{x_1} dw_1 \frac{d\nu_{n-1}(x_1)}{x_1} + \int_{|y_1|}^{\sqrt{y_j^2+y_1^2}} \frac{dw_1}{\sqrt{y_1^2 + y_j^2 - w_1^2}}$$

is of order 1, and in Lemma 5.2 that

$$\int\int_{w_1}^{2w_1} \frac{d\nu_{n-1}(x_1)}{x_1} dw_1 = O(1).$$

We claim that the third term of (3.7) is bounded by the third term of (3.5). Notice that it can be rewritten as

$$2\int E_{S_{n-1}}\left[f(w); Av^n_{j=2}G_{y_1,w_1}(w_j)\Big|w_1\right]^2 \int_{|y_1|}^1 \frac{d\nu_{n-1}(x_1)}{x_1} dw_1 d\nu_{n-1}(y_1)$$

where the integration in w_1 is over $|y_1| \geq w_1 \geq |y_1|/2$. The above expression is bounded by

$$\int\int_{|y_1|/2}^{|y_1|} E_{S_{n-1}}\left[f(w); Av^n_{j=2}G_{y_1,w_1}(w_j)\Big|w_1\right]^2 \frac{dw_1}{|y_1|} d\nu_{n-1}(y_1),$$

which is equal to the third term of (3.5). □

7. Proof of Lemma 3.1. We can rewrite $f_1(x_1) - f_1(w_1)$ as

$$\begin{aligned}(7.1)\qquad &\tfrac{1}{2}E_{S_{n-1}}\left[\left(f(x) - f(w^{(j,+)})\right)\Big|x_1\right] + \tfrac{1}{2}E_{S_{n-1}}\left[\left(f(x) - f(w^{(j,-)})\right)\Big|x_1\right] \\ &+\tfrac{1}{2}E_{S_{n-1}}\left[f(w^{(j,+)})\Big|x_1\right] + \tfrac{1}{2}E_{S_{n-1}}\left[f(w^{(j,-)})\Big|x_1\right] - f_1(w_1).\end{aligned}$$

Let's compute $E_{S_{n-1}}[f(w^{(j,+)})|x_1]$. It is equal to

$$\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} E_{S_{n-3}^{\sqrt{1-x_1^2-x_j^2}}}\left[f(w^{(j,+)})\right] C_{n-2}\frac{(1-x_1^2-x_j^2)^{\frac{n-4}{2}}}{(1-x_1^2)^{\frac{n-3}{2}}}\, dx_j.$$

In the above formula, we first integrate in $(x_i)_{i\neq 1,j}$ and then in x_j. This means we integrate $(x_i)_{i\neq 1,j}$ with respect to the uniform measure on the $(n-3)$-dimensional sphere of radius $\sqrt{1-x_1^2-x_j^2}$ and then integrate in x_j. Making the change of variable $w_j = \sqrt{x_1^2 + x_j^2 - w_1^2}$, we obtain that the above integral is equal to

$$2\int E_{S_{n-3}^{\sqrt{1-w_1^2-w_j^2}}}[f(w)]\, \frac{w_j}{\sqrt{w_1^2+w_j^2-x_1^2}}\, C_{n-2}\frac{(1-w_1^2-w_j^2)^{\frac{n-4}{2}}}{(1-x_1^2)^{\frac{n-3}{2}}}\, dw_j,$$

where the integration in w_j is from $\sqrt{x_1^2-w_1^2}$ to $\sqrt{1-w_1^2}$. Hence, it is equal to

$$2\, E_{S_{n-1}}\left[f(w)G_{x_1,w_1}(w_j)1_{\{w_j\geq 0\}}\Big|w_1\right],$$

where G_{x_1,w_1} is given by (3.2). Let's turn to $E_{S_{n-1}}\left[f(w^{(j,-)})\Big|x_1\right]$. Making the change of variable $w_j = -\sqrt{x_1^2+x_j^2-w_1^2}$, we easily prove that

$$E_{S_{n-1}}\left[f(w^{(j,-)})\Big|x_1\right] = 2E_{S_{n-1}}\left[f(w)G_{x_1,w_1}(w_j)1_{\{w_j\leq 0\}}\Big|w_1\right].$$

Eventually, we obtain that

$$\begin{aligned}&\tfrac{1}{2}E_{S_{n-1}}\left[f(w^{(j,+)})\Big|x_1\right] + \tfrac{1}{2}E_{S_{n-1}}\left[f(w^{(j,-)})\Big|x_1\right] \\ &\quad= E_{S_{n-1}}\left[f(w)G_{x_1,w_1}(w_j)\Big|w_1\right].\end{aligned}$$

Choosing $f = 1$, we deduce $E_{S_{n-1}}[G_{x_1,w_1}(w_j)|w_1] = 1$, and thus the above expression is equal to

$$E_{S_{n-1}}\left[f(w); G_{x_1,w_1}(w_j)\middle|w_1\right] + f_1(w_1).$$

Recalling (7.1) and averaging over $2 \le j \le n$, we get (3.1). □

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