SYNOPSIS OF ELEMENTARY MATHEMATICAL STATISTICS[1]

By

B. L. Shook

Section IV. The Graphical Representation of Frequency Distributions

25. The investigation of a frequency distribution is greatly facilitated by presenting the data graphically by means of either a *Frequency Polygon* or a *Histogram*, depending upon the nature of the distribution.

For a distribution of discrete variates the frequencies are represented by ordinates whose lengths are proportional to the various frequencies and whose abscissae correspond to the variates of the distribution. The shape of the distribution is rendered more apparent by either connecting the tops of the ordinates by straight lines, thus forming a *Frequency Polygon*, or drawing a *Frequency Curve* that approximately passes through the vertices of the polygon. Figure I presents the Frequency Polygon derived from the data of Table XI. In addition a curve has been drawn to illustrate the general trend of the distribution.

If the frequency distribution under examination be one of grouped discrete or continuous variates it will be found that the *Histogram* is best suited for graphical representation. A Histogram is a series of rectangles erected on bases that are proportional to the class intervals and with altitudes proportional to the respective class frequencies. Thus, in this case, the frequencies are represented by areas. The shape of the distribution may be emphasized by constructing a continuous fre-

1 A continuationn of an article bearing the same caption appearing in Vol. I, No. 1, of the Annals.

quency curve such that the areas under the curve between the ordinates at the lower and upper boundaries of the various rectangles should equal approximately the areas of the corresponding rectangles. Two examples are presented, both the distributions are composed of continuous variates, one exhibiting positive skewness and the second negative. The numerical data and corresponding Histograms are presented in Tables XII and XIII and Figures II and III respectively.

TABLE XI

Distribution of Frequency of glands in the right fore-leg of 2,000 female swine[1]

V	fV	t	ft
0	15	-2.083	.013
1	209	-1.488	.176
2	365	- .893	.307
3	482	- .298	.405
4	414	.297	.348
5	277	.892	.233
6	134	1.487	.113
7	72	2.082	.061
8	22	2.677	.018
9	8	3.272	.007
10	2	3.867	.002

$M_v = 3.501$ $N = 2000$

$\sigma_v = 1.68077$ $\frac{1}{\sigma} = .594965$

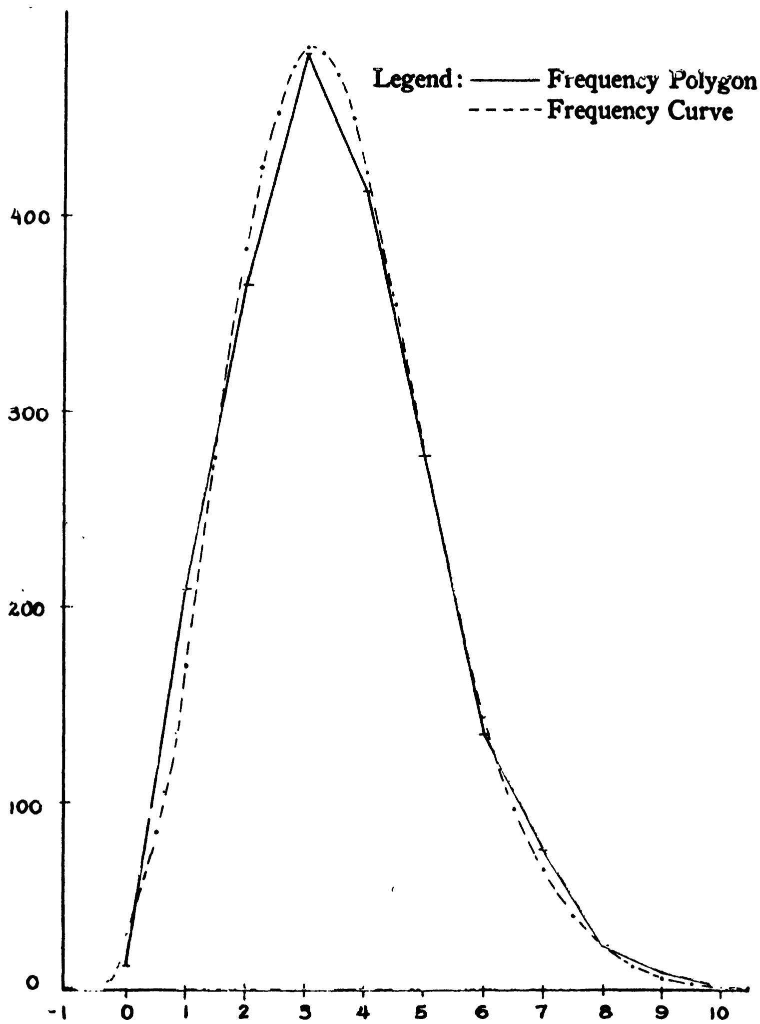
$\alpha_{3 \cdot v} = .508462$ $\frac{\sigma}{N} = .000840385$

26. It has previously been stated that the three fundamental statistical functions are the Mean, Standard Deviation, and Skewness. The Mean has been defined as a convenient average, and the Standard

1 Davenport, "Statistical Methods," page 35.

FIGURE I

Frequency Distribution of glands in the right fore-leg of 2,000 female swine



Deviation measures the concentration of the variates about this average. Skewness has not, however, been so clearly explained. If the variates of a distribution be symmetrically arranged about their mean, then $\mu_{3 \cdot v}$ or the third moment about the mean will be zero. Under these conditions $\alpha_{3 \cdot v}$, or the coefficient of skewness, must also be zero. Thus $\alpha_{3 \cdot v}$ measures the degree to which a frequency distribution is symmetrical. If $\alpha_{3 \cdot v}$ is zero, then from the standpoint

TABLE XII

Weights of White Boys - 30 to 33 months
(Correct to nearest pound)

Class Mark	f	t	ft
21	3	-2.90	.009
22	3	-2.50	.009
23	11	-2.09	.032
24	27	-1.69	.079
25	65	-1.28	.191
26	101	- .88	.297
27	135	- .47	.397
28	136	- .07	.400
29	128	.34	.376
30	105	.75	.309
31	59	1.15	.173
32	30	1.56	.088
33	15	1.96	.044
34	7	2.37	.021
35	5	2.77	.015
36	8	3.18	.024
37	1	3.58	.003
38	1	3.99	.003

$M_v = 28.16190$ $N = 840$

$\sigma_v = 2.46837$ $\frac{1}{\sigma} = .405126$

$\alpha_{3 \cdot v} = .427969$ $\frac{\sigma}{N} = .00293854$

of the present synopsis the distribution may be considered normal, for if such a distribution be graphed in standard units it will follow the locus of the well known Normal Curve of Error. Accordingly it would seem logical to expect that for each value of α_3 there is *one* standard curve which is the locus toward which all distributions with that degree

TABLE XIII

Barometric Heights for Daily Observations During Thirteen Years at Llandudno, England[1]

(Original measurements to nearest millimeter)

Class Mark	t	f	ft
28.35	-4.38	1	.001
28.55	-3.82	2	.001
28.75	-3.26	8	.005
28.95	-2.71	30	.018
29.15	-2.15	74	.045
29.35	-1.59	166	.102
29.55	-1.04	368	.226
29.75	- .48	509	.313
29.95	.08	656	.403
30.15	.63	580	.356
30.35	1.19	353	.217
30.55	1.75	140	.086
30.75	2.31	30	.018
30.95	2.86	5	.003

$M_v = 29.9221$ $\quad N = 2922$

$\sigma_v = .359014$ $\quad \frac{1}{\sigma} = 2.78541$

$\alpha_{3 \cdot v} = -.32919$ $\quad \frac{\sigma}{N} = .000614329$[2]

1 Karl Pearson and A. Lee, "Philosophic Transactions," p. 428 (1897).

2 This formula assumes that the class interval is unity, the proper value of $\frac{\sigma}{N}$ is therefore 5 times the value as ordinarily computed.

FIGURE II

Weights of White Boys (30 to 33 months)

Histogram and Frequency Curve

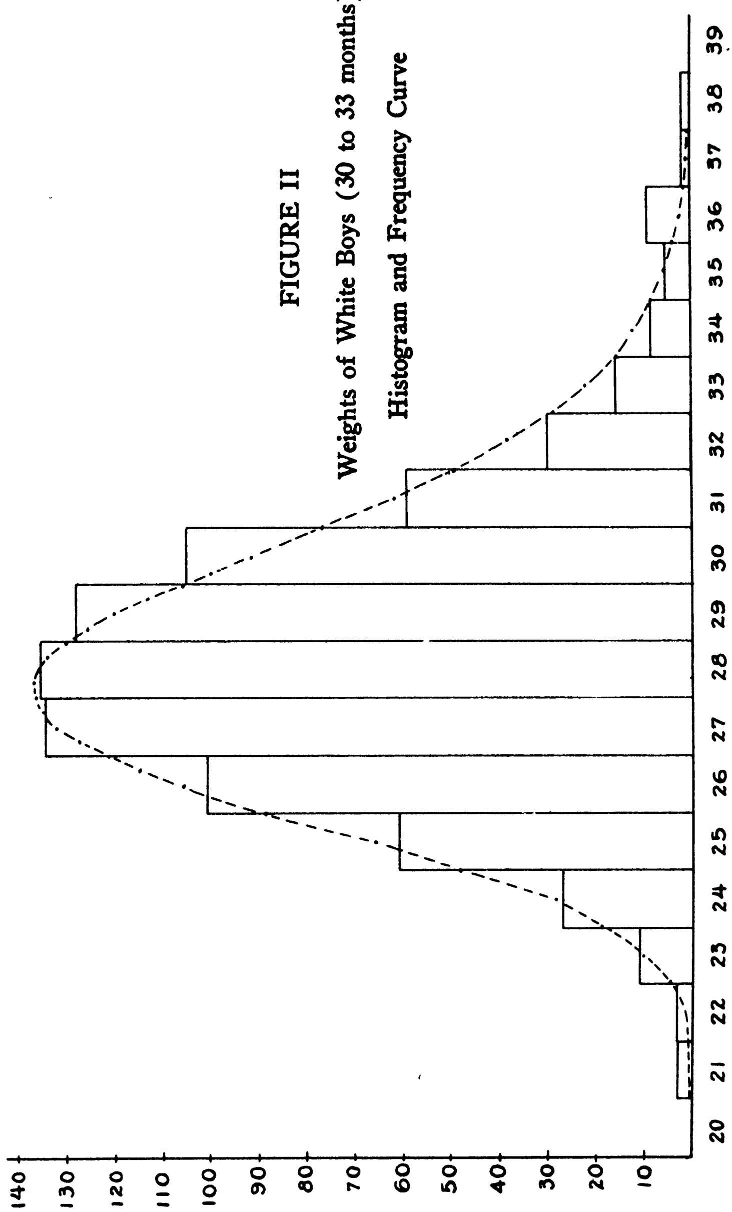
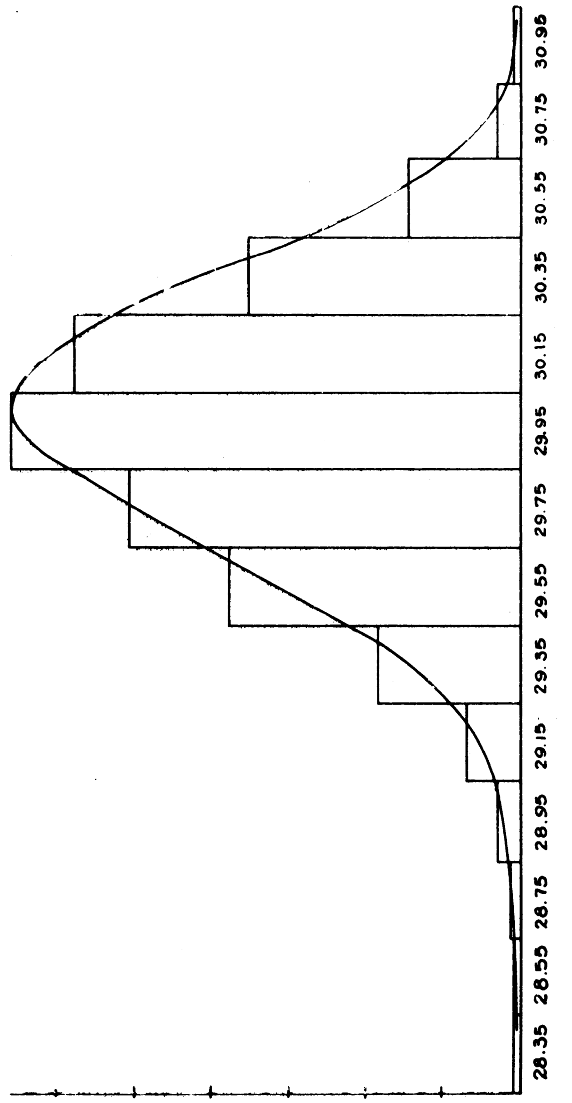


FIGURE III

Barometric Heights Recorded Daily at Llandudno, England



of skewness approach. The one essential is that the unit of measurement must be removed from the data, that is each distribution should be expressed in terms of the standard variates t and the corresponding frequencies f_t As before, the standard variate t_i corresponding to v_i is obtained from the following formula:

$$t_i = \frac{v_i - M_v}{\sigma_v} = \frac{\bar{v}_i}{\sigma_v}$$

Similarly, the frequencies for each of the standard variates is defined as follows:

$$(26) \qquad f_t = \frac{\sigma_v}{N} f_v .$$

These two formulae will enable one to analyze all distributions entirely independent of the unit involved. In Figure IV the three distributions graphically presented in Figures I, II and III are shown contrasted with the Normal Curve. The numerical values of t and f_t for each distribution are given in the corresponding table. The values may be obtained in each case by employing the continuous process described in Section I. It will be noticed that the two distributions with positive skewness of .5 and .4 respectively reach their maximum in advance of the Normal Curve and approach the zero limit more gradually for positive values of the standard variates. Accordingly, for the distribution exhibiting negative skewness, the positions are reversed and the more gradual approach to the zero limit occurs for the negative values of the standard variates. In general, a distribution having skewness within the limits ± .3 will exhibit very little deviation from the normal curve when presented graphically in this manner.

Summary of Section IV—

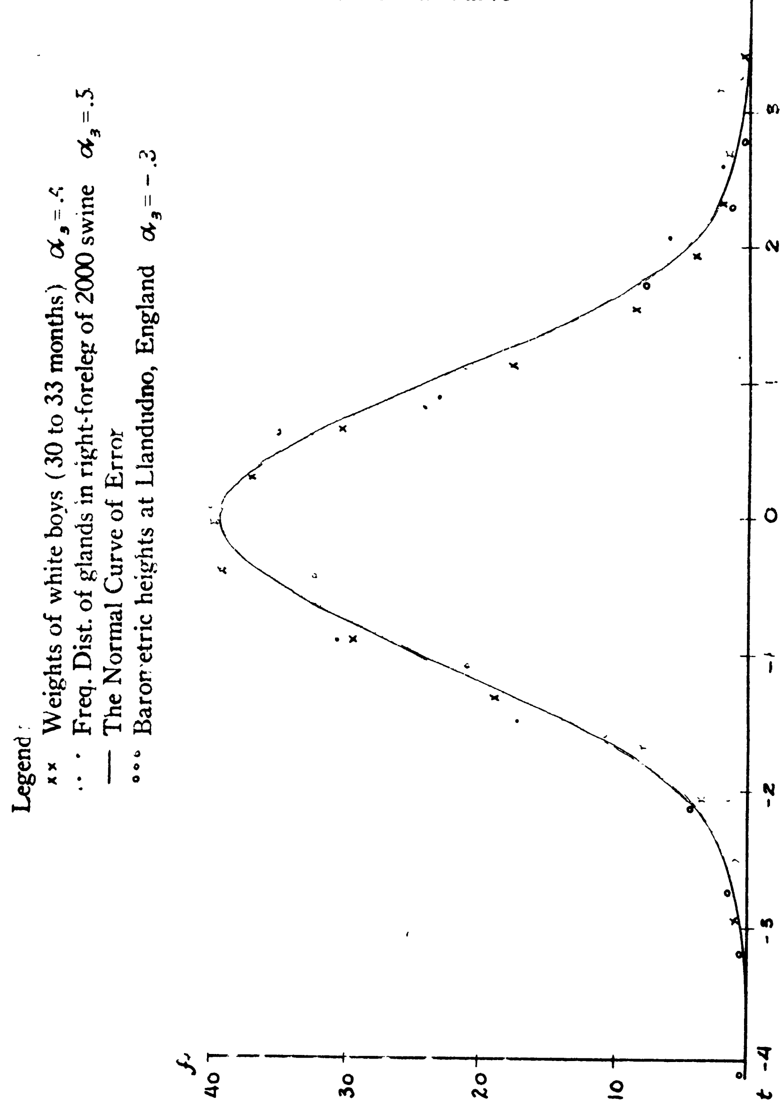
It is usually found very advantageous in the investigation of frequency distributions to present the data graphically. A distribution of discrete variates should be represented by a Frequency Polygon and one of continuous variates by a Histogram. In either case a free hand curve may be drawn indicating the general trend of the distribution and is called the Frequency Curve. The *Standardized* Curve is obtained by plotting the variates and their corresponding frequencies in *standard* form by means of the following formulae:

$$t_i = \frac{v_i - M_v}{\sigma_v} = \frac{\bar{v}_i}{\sigma_v}$$

$$f_t = \frac{\sigma_v}{N} \cdot f_v .$$

FIGURE IV

The Graphical Comparison of Three Distributions with the Normal Curve



SECTION V. THE INVERSE PROBLEM

27. From the standpoint of Elementary Mathematical Statistics we may say that the Mean, Standard Deviation, and Skewness together with its total frequency completely characterize a distribution. If this statement were accurate it would be possible to reproduce any distribution if its three elementary functions and total frequency were known. A tabulation of Pearson's Type III Curves for various degrees of skewness affords, for the purposes of Elementary Statistics, the most satisfactory representation of frequency distributions from the point of view of both effectiveness and facility in using[1]. In order to illustrate the method several numerical examples are included. In Table XIV the illustration is one of discrete variates.

1 L. R. Salvosa, "Tables of Pearson's Type III Function," The Annals of Mathematical Statistics, May, 1930.

TABLE XIV

Frequency Distribution of Number of Glands in the Right Foreleg of 2,000 Female Swine

v (1)	t (2)	f_t (3)	Predicted Frequency (4)	Observed Frequency (5)
0	−2.08	.026952	32	15
1	−1.49	.141661	169	209
2	− .89	.320068	381	365
3	− .30	.409193	487	482
4	.30	.353689	421	414
5	.89	.229770	274[1]	277
6	1.49	.118287	141	134
7	2.08	.051638	62[1]	72
8	2.68	.019220	23	22
9	3.27	.006459	8	8
10	3.87	.001925	2	2
Total			2000	2000

$M = 3.501$ $\quad N = 2000$

$\sigma = 1.68077$ $\quad \frac{1}{\sigma} = .594965$

$\alpha_3 = .508462$ $\quad \frac{N}{\sigma} = 1189.93$

Explanation. In every case the value of α_3 is taken to the nearest tenth and the value of t to the nearest hundredth. In the examples included no interpolation has been made for any value.

Columns (1) and (2) of Table XIV contain the variates and the corresponding values of t obtained by means of the continuous process. Column (3) is obtained directly from the Table of Ordinates of the Pearson Type III Function. All values may be found in the

1 In order to obtain $N = 2000$ it was necessary to increase these frequencies by 1, although the fractional value was slightly less than than the necessary .5.

column with skewness = .5 and opposite the respective value of t. Since these are the Standard Frequencies f_t, the predicted frequencies for each variate may be obtained from the following formula.

$$\because f_t = \frac{\sigma}{N} \cdot f_v$$

(27) $$\therefore f_v = \frac{N}{\sigma} \cdot f_t$$

The predicted frequencies in column (4), therefore, are obtained by multiplying column (3) by the value 1189.93. These values are the *graduated* frequencies. The actual observed frequencies are given in column (5).

TABLE XV

Distribution of Weights of White Boys - 30 to 33 Months

(Measurements correct to nearest pound)

Lower Limit of Class (1)	t at Lower Limit (2)	Accumulative Percent Frequency (3)	Percent Frequency (4)	Predicted Frequency (5)	Observed Frequency (6)
20.5	−3.10	.000021	.000357	0	3
21.5	−2.70	.000378	.002990	3	3
22.5	−2.29	.003368	.013174	11	11
23.5	−1.89	.016542	.039440	33	27
24.5	−1.48	.055982	.079399	67	65
25.5	−1.08	.135381	.127331	107	101
26.5	− .67	.262712	.154908	130	135
27.5	− .27	.417620	.163707	138	136
28.5	.14	.581327	.140357	118	128
29.5	.54	.721684	.110157	93	105
30.5	.95	.831841	.073061	61	59
31.5	1.35	.904902	.045897	39	30
32.5	1.76	.950799	.025019	21	15
33.5	2.16	.975818	.013225	11	7
34.5	2.57	.989043	.006176	5	5
35.5	2.97	.995219	.002845	2	8
36.5	3.38	.998064	.001172	1	1
37.5	3.78	.999236	.000483	0	1
38.5	4.19	.999719	.000218	0	0
Total				840	840

$M = 28.16190$ $\quad N = 840$

$\sigma = 2.46837$ $\quad \frac{1}{\sigma} = .404126$

$\alpha_3 = .427969$

Explanation:

28. Since Table XV is a distribution of continuous variates, it is necessary to use the Table of Areas of the Pearson Type III Curve. The values in this table are the *accumulated percent* of the standard curve *below* a specified value of t. The method of prediction is therefore to estimate the per cent of the distribution lying *between* the consecutive lower limits of each class. In column (1) of Table XV are given the lower limit of each class and in Column (2) the value of t at this lower limit. Column (3) is taken directly from the Table of Areas of the Pearson Type III Function, $\alpha_3 = .4$, and represent the percent of the distribution lying *below* the particular value of t. In order to find the percentage of the distribution in each class, it is merely necessary, therefore, to difference column (3). For example, the first value, .000357, is found by subtracting .000021 from .000378. In order to find the predicted frequencies in column (5), N, or the total frequency, should be multiplied by each value in column (4). The observed frequencies are given in column (6).

TABLE XVI

Barometric Heights for Daily Observations During Thirteen Years at Llandudno, England

(Correct to the nearest millemeter)

Lower Limit of Class (1)	t (2)	Accumulative Percent Frequency (3)	Percent Frequency (4)	Predicted Frequency (5)	Observed Frequency (6)
28.25	-4.66	.999956	.000171	1	1
28.45	-4.10	.999785	.000739	2	2
28.65	-3.54	.999046	.002716	8	8
28.85	-2.99	.996330	.009081	27	30
29.05	-2.43	.987249	.025770	75	74
29.25	-1.87	.961479	.061380	179	166
29.45	-1.31	.900099	.117068	342	368
29.65	- .76	.783031	.185122	541	509
29.85	- .20	.597909	.222074	649	656
30.05	.36	.375835	.192952	564	580
30.25	.91	.182883	.121528	355	353
30.45	1.47	.061355	.048526	142	140
30.65	2.03	.012829	.011285	33	30
30.85	2.58	.001544	.001468	4	5
31.05	3.14	.000076	.000076	0	0
Total				2922	2922

$M = 29.92207$ $N = 2922$

$\sigma = .359014$ $\frac{1}{\sigma} = 2.78541$

$\alpha_3 = -.32919$

Explanation:

29. Although the data of Table XVI is also a distribution of continuous variates, it will be noticed that in this case the coefficient of

skewness is negative. Since the Tables include only positive values of α_3, it seems desirable to explain the procedure for such a distribution. If a frequency curve having pronounced positive skewness be graphed on rather fine paper and then held to the light or in front of a mirror, it will be seen that the distribution will seem to show negative skewness to the same degree in which it formerly displayed positive. This being true, it is possible to use the Tables for all cases of negative skewness by merely changing the sign of t, and if an area is desired it is necessary to reverse the order of differencing. Three examples are given in order to cover as many different cases.

Illustration 1, $\alpha_3 = -.5$, required the percentage of the area of the standardized curve lying between $t = -2.43$ and $t = -1.98$. From the tables under the column for skewness $= .5$.

$t = +2.43$, percent of area $= .983883$

$t = 1.98$, percent of area $= .964416$

The percentage lying between these two values of t is therefore $.983883 - .964416 = .019467$.

Illustration 2, if $\alpha_3 = -.8$, required the percentage of the area lying between $t = -.02$ and $t = .25$. Using the Table of Areas in the column for skewness of .8,

If $t = +.02$, percent of area $= .561064$

$t = -.25$, percent of area $= .450687$

To find the percent of the area merely subtract as before, $.561064 - .450687 = .110377$.

Illustration 3, if $\alpha_3 = -.2$, required the percentage of the area lying between $t = .52$ and $t = 1.63$. Again referring to the Tables of Areas, we find for $\alpha_3 = .2$

If $t = -.52$, percent of area $= .310015$

If $t = -1.63$, percent of area $= .045108$

Accordingly, the required percentage is $.310015 - .045108 = .264907$.

TABLE XVII

Expansion of $(5/6+1/6)^{180}$

v (1)	t (2)	f_t (3)	Pred. Freq. (4)	Obs. Freq. (5)
14	−3.21	.001468	0	0
15	−3.01	.003013	1	1
16	−2.81	.005919	1	1
17	−2.61	.010954	2	2
18	−2.41	.019227	4	4
19	−2.21	.032053	7	6
20	−2.01	.050807	10	10
21	−1.81	.076658	15	16
22	−1.60	.112095	23	23
23	−1.40	.153377	31	31
24	−1.20	.200401	40	41
25	−1.00	.250281	50	51
26	− .80	.299057	60	61
27	− .60	.342196	69	69
28	− .40	.375301	76	75
29	− .20	.394857	80	79
30	.01	.398640	80	80
31	.21	.386166	78	77
32	.41	.359746	72	72
33	.61	.322535	65	64
34	.81	.278510	56	56
35	1.01	.231792	47	46
36	1.21	.186059	38	37
37	1.41	.144144	29	29
38	1.62	.106201	21	22
39	1.82	.076661	15	16
40	2.02	.053513	11	11
41	2.22	.036145	7	8
42	2.42	.024163	5	5
43	2.62	.014975	3	3
44	2.82	.009196	2	2
45	3.02	.005476	1	1
46	3.23	.003077	1	1
47	3.43	.001723	0	0

$M_v = 29.973$ $\qquad N = 1000$

$\sigma_v = 4.96853$ $\qquad t_i = .201267\, v_i - 6.032569$

$\alpha_{3:v} = .108097$ $\qquad \frac{N}{\sigma} = 201.267$

TABLE XVIII

Expansion of $(5/6+1/6)^{180}$

Class (1)	Lower Limit (2)	t (3)	Accumulated Percent Freq. (4)	Percent Freq. (5)	Pred. Freq. (6)	Obs. Freq. (7)
11–	10.5	–3.92	.000014	.000213	0	0
14–	13.5	–3.32	.000227	.002128	2	2
17–	16.5	–2.71	.002355	.012561	13	12
20–	19.5	–2.11	.014916	.049031	49	49
23–	22.5	–1.50	.063947	.120876	121	123
26–	25.5	– .90	.184823	.203066	203	205
29–	28.5	– .30	.387889	.239543	239	236
32–	31.5	.31	.627432	.191988	192	192
35–	34.5	.91	.819420	.112488	112	112
38–	37.5	1.51	.931908	.048675	49	49
41–	40.5	2.12	.980583	.015048	15	16
44–	43.5	2.72	.995631	.003627	4	4
47–	46.5	3.33	.999258	.000742	1	0

$M_v = 29.973$ $\quad N = 1000$

$\sigma_v = 4.96848$ $\quad \frac{1}{\sigma} = .201269$

$\alpha_{3,v} = .105899$

30. As further numerical examples the three illustrated problems used in Section III have been graduated. The complete numerical solution will be found in Tables XVII, XVIII and XIX.

Summary of Section V—

Knowing the three fundamental functions and the total frequency of a distribution, it is possible to obtain predicted or graduated frequencies for that distribution with a surprising degree of accuracy. This is accomplished through the use of tables of the standard ordinates and accumulated percentage areas of the Pearson Type III Curves.

TABLE XIX

Weights of 1000 Female Students

(Original measurements to nearest .1 lb.)

Class (1)	Lower Limit (2)	t (3)	Accumulated Percent Freq. (4)	Percent Freq. (5)	Pred. Freq. (6)	Obs. Freq. (7)
70–	69.95	–2.88	.000000	.000000	0	2
80–	79.95	–2.29	.000000	.003358	4	16
90–	89.95	–1.70	.003358	.102159	102	82
100–	99.95	–1.11	.105517	.238290	238	231
110–	109.95	– .52	.343807	.249585	250	248
120–	119.95	.07	.593392	.183665	184	196
130–	129.95	.66	.777057	.111093	111	122
140–	139.95	1.25	.888150	.059338	59	63
150–	149.95	1.84	.947488	.029412	29	23
160–	159.95	2.44	.976900	.013209	13	5
170–	169.95	3.03	.990109	.005791	6	7
180–	179.95	3.62	.995900	.002445	3	1
190–	189.95	4.21	.998345	.001002	1	2
200–	199.95	4.80	.999347	.000400	0	1
210–	209.95	5.39	.999747	.000157	0	1
220–	219.95	5.98	.999904	.000096	0	0

$M_v = 118.74$ $N = 1000$

$\sigma_v = 16.9175$ $\frac{1}{\sigma} = .0591104$

$\alpha_{3:v} = .976424$

It should be remembered that in advanced statistics moments higher than the third are necessary to characterize a distribution, but from the elementary viewpoint, the Mean, Standard Deviation and Skewness are considered to completely characterize a distribution.

Section VI. Bernoulli's Theorem

31. *Factorials.* For convenience, the product of the first n consecutive integers is called "factorial n" and is designated by the symbol $\underline{|n}$. Thus

$$\underline{|3} = 1\cdot 2\cdot 3 = 6,\quad \underline{|5} = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120,\quad \frac{\underline{|8}}{\underline{|5}} = 8\cdot 7\cdot 6 = 336.$$

Combinations. The number of combinations, each of r things, that can be formed from n things, is represented by the symbol $_nC_r$. Texts on elementary algebra show that

$$(28)\qquad {}_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = \frac{n\,(n-1)\,(n-2)\cdots(n-\overline{r-1})}{1\cdot 2\cdot 3\cdots r}$$

For example, suppose we desire to find the number of different committees, each of three persons, that can be selected from five individuals. If we designate the five individuals by the letters A, B, C, D and E, we observe that committees of three may be systematically enumerated as follows:

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE

The number of committees, which we just enumerated as 10, agrees with the value found by formula (28), for since here, $n=5$, $r=3$,

$$ {}_nC_r = {}_5C_3 = \frac{\underline{|5}}{\underline{|3}\cdot\underline{|2}} = \frac{5\cdot 4}{1\cdot 2} = 10$$

Another illustration: The number of different committees, each composed of seven individuals, that can be selected from ten candidates is

$$ {}_{10}C_7 = \frac{\underline{|10}}{\underline{|7}\,\underline{|3}} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$$

and the number of combinations, each of three, that can be formed from ten items is

$${}_{10}C_3 = \frac{|\underline{10}}{|\underline{3}\ |\underline{7}} = 120$$

It should be noted that ${}_{10}C_7 = {}_{10}C_3$, and in general that

$$(29) \qquad {}_nC_r = {}_nC_{n-r}$$

This follows from the fact that the number of ways of selecting r items from n is equal to the number of ways of rejecting $(n-r)$ from n. Thus, every time three are selected from ten, seven are rejected. Therefore the number of ways of selecting three from ten, ${}_{10}C_3$, is also equal to ${}_{10}C_7$.

We shall have occasion to refer to the following tabulation of values of ${}_nC_r$.

TABLE XX

Values of ${}_nC_r$

N	r												
	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	1	2											
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

32 *Binomial Theorem.* By repeated multiplication we find that

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

etc.

By mathematical induction it can be shown that for positive integer values of n

$$(30)\quad (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots$$

This equation is known as the binomial theorem and may be written more compactly, if n is an integer, in the following form:

$$(31)\quad (a+b)^n = a^n + {}_nC_1\, a^{n-1}b + {}_nC_2\, a^{n-2}b^2 + {}_nC_3\, a^{n-3}b^3 + \cdots\cdots\cdots$$

Using Table XX, we may write down at once that

$$(a+b)^{12} = a^{12} + 12a^{11}b + 66a^{10}b^2 + 220a^9b^3 + \cdots\cdots + 66a^2b^{10} + 12ab^{11} + b^{12}$$

Bernoulli's Series. If p denote the probability that an event will happen in a single trial, and q the probability that it will not happen in that trial, $p+q = 1$, then the probability that the event will happen exactly 0, 1, 2, . . . x times during r trials is given by the respective terms of the binomial expansion

$$(32)\quad (q+p)^r = q^r + {}_rC_1\, q^{r-1}p + {}_rC_2\, q^{r-2}p^2 + \cdots {}_rC_x\, q^{r-x}p^x + \cdots\cdots\cdots$$

To illustrate. If a coin be tossed, we may assume *a priori* that the probability that heads will turn up is $p = 1/2$ and the probability that heads will not turn up is $q = 1/2$. If an individual tosses the coin twelve times in succession, it is possible that heads may turn up on no occasion, or heads may turn up exactly 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12 times, respectively. By formula (32), these chances are equal respectively to the successive terms of the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$, namely

$$\left(\frac{1}{2}\right)^{12} + {}_{12}C_1\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right) + {}_{12}C_2\,\frac{1}{2}^{10}\left(\frac{1}{2}\right)^2 + \cdots\cdots + {}_{12}C_{11}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{11} + {}_{12}C_{12}\left(\frac{1}{2}\right)^{12}$$

Denoting the probability that heads will turn up on exactly x occasions by P_x , and referring to Table XX for values of ${}_{12}C_x$, we have that

$$P_{12} = \frac{1}{4096},\quad P_{11} = \frac{12}{4096},\quad P_{10} = \frac{66}{4096},\quad P_9 = \frac{220}{4096}$$

TABLE XXI

Values of the Terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$

Number of Successes	$r = 12$, $q = .5$, $p = .5$		Observed Frequencies
	Probability P_x	Expected Freq. $4096\,P_x$	
(1)	(2)	(3)	(4)
0	1/4096	1	0
1	12/4096	12	7
2	66/4096	66	60
3	220/4096	220	198
4	495/4096	495	430
5	792/4096	792	731
6	924/4096	924	948
7	792/4096	792	847
8	495/4096	495	536
9	220/4096	220	257
10	66/4096	66	71
11	12/4096	12	11
12	1/4096	1	0
Total	1	4096	4096

33. *Expectation.* If p denote the probability of success for each of n trials, then pn is defined as the expected number of successes in n trials. For example, we have just shown that the *a priori* probability of throwing heads twelve successive times with a coin is equal to $P_{12} = \frac{1}{4096}$. Therefore if twelve coins be tossed simultaneously on 4096 occasions, we expect that all twelve coins will turn up heads on only one occasion. Likewise, the expected number of times that exactly ten heads and two tails would turn up is equal to $4096 \cdot P_{10} = 66$, and that exactly half of the coins would turn heads only $4096 \cdot P_6 = 924$ times.

It will be seen that the sum of all the probabilities in column (2) is unity. This follows from the fact that these values are the several terms of the expansion of $(q+p)^r$, and since $q+p=1$, therefore $(q+p)^r = 1$.

TABLE XXII

Values of the Terms in the Expansion of $\left(\frac{5}{6}+\frac{1}{6}\right)^{12}$

Number of Successes (1)	$r = 12, \quad q = 5/6 \quad p = 1/6$ Probability P_x (2)	Expected Freq. (3)	Observed Frequencies (4)
0	.11216	459	447
1	.26918	1103	1145
2	.29609	1213	1181
3	.19739	808	796
4	.08883	364	380
5	.02843	116	115
6	.00663	27	24
7	.00114	5	7
8	.00014	1	1
9	.00001	0	0
10	.00000	0	0
11	.00000	0	0
12	.00000	0	0
Total	1.00000	4096	4096

A second illustration: Suppose twelve dice are thrown and that only a throw of 6 is to be considered a success. By formula (32), therefore, the expansion of

$$\left(\frac{5}{6}+\frac{1}{6}\right)^{12}=\left(\frac{5}{6}\right)^{12}+{}_{12}C_1\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)+{}_{12}C_2\left(\frac{5}{6}\right)^{10}\left(\frac{1}{6}\right)^{2}+\cdots\cdots$$

are equal respectively to the probabilities that exactly 0, 1, 2, successes will be obtained in a single throw of the twelve dice, or what is the same thing, in twelve successive throws with a single die.

In this case the probabilities P_x are expressed as decimals, since the expansion contains values of $(6)^{12}$ in the denominators. Therefore 6^{12} is the smallest value of N that will produce integer expected frequencies.

34. We shall now attack a more important problem. Let us consider a hypothetical group of 100,000 individuals, all of the same age and all exposed to the same hazards of life. Moreover, let us assume that the probability that each individual will die within one year is $p = .008$, or that the probability that any specified individual will survive a year is $q = .992$.

By formula (32), the terms of the expansion of $(q+p)^r$ $(.992+.008)^{100,000}$, namely, $(.992)^{100,000}$ ${}_{100,000}C_1\,(.992)^{99,999}$ $(.008)^1 +$ ${}_{100,000}C_2\,(.992)^{99,998}\,(.008)^2$ . . . . ${}_{100,000}C_x$ $(.992)^{100,000-x}\,(.008)^x$ . . . . represent the probabilities that exactly 0, 1, 2, . . .,x,. . . individuals will die within the year.

The value of $(.992)^{100,000}$ is very small. Thus $(.992)^{100,000} = \left(\frac{992}{1000}\right)^{100,000}$

$\log 992 = 2.9965117$	$\log 992^{100,000} = 299651.17$
$\log 1000 = 3$	$\log 1000^{100,000} = 300000.00$
	$\log (.992)^{100,000} = \overline{349}.17$

Therefore $.992^{100,000} = .000,000,000\ .\ .\ .\ 15$, where 15 is preceded by 348 zeros. The probability that all would die $(.008)^{100,000}$ is far less than this value.

The values of P_x in Table XXIII are given to the nearest fourth decimal place. Thus to six decimal places $P_{690} = .000005$ and $P_0 + P_1 + P_2 + \ldots + P_{690} = \sum_0^{690} P_x = .000035$. These values appear in Table XXIII, therefore, as .0000. An inspection of Table XXIII shows that for our hypothetical population

(a) The chance that exactly 800 will die within a year is .0142
(b) The chance that 800 or less will die within a year is .5094.
(c) The chance that 850 or less will die within a year is .9625.
(d) The chance that at least 750 will die within a year is

$$P_{750} + P_{751} + P_{752} + \ldots + P_{100,000} = 1 - \sum_0^{749} P_x = 1 - .0355 = .9645$$

Obviously the sum of all terms from P_0 to $P_{100,000}$ is equal to unity. It is interesting to note that although q is relatively much greater than p, nevertheless the values of P_x are very symetrically arranged about their mean. For example, the first significant term of P_x is $P_{707} = .0001$, and the last significant term is $P_{896} = .0001$. Thus there are 93 significant terms above and 96 terms below P_{800}. However, there are 707 insignificant terms before P_{707} and 99,104 insignificant terms after P_{896}. We have arbitrarily rejected as insignificant any value less than .0001. Had we taken .0000001 as the limit of significance, we would have found that the limiting significant values of P_x are $P_{668} = P_{944} = .0000001$. Here again the significant ranges above and below the expected P_{800} are almost the same.

In general it may be said that unequal values of q and p when associated with large values of r are reflected in an unequal number of insignificant terms in the upper and lower ranges. The significant terms form a distribution which, to the eye, is rather symmetrical.

35. Let us now retrace a few steps. Theoretically, formula (32) enables one to compute the probability that exactly x individuals out of any population of r will die within a year, provided, of course, q and p are known. Actually, however, such computation is very laborious. Thus, it is not easy to show that

$$P_{850} = {}_{100,000}C_{850}\,(.992)^{99150}(.008)^{850} = .0029354$$

TABLE XXIII

Values of P_x and $\sum_{x=0}^{x} P_x$ $\quad P_x = {}_rC_x\, q^{r-x} p^x$ and $r = 100{,}000$,

$q = .999, \quad p = .008$

x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$
690	.0000	.0000	730	.0006	.0062	770	.0081	.1474
691	.0000	.0000	731	.0007	.0069	771	.0084	.1558
692	.0000	.0000	732	.0007	.0077	772	.0087	.1645
693	.0000	.0001	733	.0008	.0085	773	.0091	.1736
694	.0000	.0001	734	.0009	.0093	774	.0094	.1830
695	.0000	.0001	735	.0010	.0103	775	.0097	.1926
696	.0000	.0001	736	.0010	.0113	776	.0100	.2026
697	.0000	.0001	737	.0011	.0125	777	.0103	.2128
698	.0000	.0001	738	.0012	.0137	778	.0106	.2234
699	.0000	.0001	739	.0013	.0150	779	.0108	.2342
700	.0000	.0002	740	.0014	.0165	780	.0111	.2454
701	.0000	.0002	741	.0016	.0180	781	.0114	.2568
702	.0000	.0002	742	.0017	.0197	782	.0117	.2684
703	.0000	.0002	743	.0018	.0215	783	.0119	.2803
704	.0000	.0003	744	.0019	.0234	784	.0122	.2925
705	.0000	.0003	745	.0021	.0255	785	.0124	.3049
706	.0000	.0004	746	.0022	.0278	786	.0126	.3175
707	.0001	.0004	747	.0024	.0302	787	.0128	.3303
708	.0001	.0005	748	.0026	.0327	788	.0130	.3433
709	.0001	.0005	749	.0027	.0355	789	.0132	.3565
710	.0001	.0006	750	.0029	.0384	790	.0134	.3699
711	.0001	.0007	751	.0031	.0415	791	.0135	.3835
712	.0001	.0008	752	.0033	.0448	792	.0137	.3971
713	.0001	.0009	753	.0035	.0484	793	.0138	.4109
714	.0001	.0010	754	.0037	.0521	794	.0139	.4248
715	.0001	.0012	755	.0040	.0561	795	.0140	.4388
716	.0001	.0013	756	.0042	.0603	796	.0141	.4528
717	.0002	.0015	757	.0044	.0647	797	.0141	.4669
718	.0002	.0017	758	.0047	.0694	798	.0141	.4811
719	.0002	.0019	759	.0049	.0744	799	.0142	.4952
720	.0002	.0021	760	.0052	.0796	800	.0142	.5094
721	.0003	.0023	761	.0055	.0850	801	.0141	.5235
722	.0003	.0026	762	.0058	.0908	802	.0141	.5377
723	.0003	.0029	763	.0060	.0968	803	.0141	.5517
724	.0003	.0033	764	.0063	.1032	804	.0140	.5657
725	.0004	.0037	765	.0066	.1098	805	.0139	.5796
726	.0004	.0041	766	.0069	.1167	806	.0138	.5934
727	.0005	.0046	767	.0072	.1239	807	.0137	.6070
728	.0005	.0051	768	.0075	.1314	808	.0135	.6206
729	.0006	.0056	769	.0078	.1392	809	.0134	.6340

TABLE XXIII (Continued)

x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$
810	.0132	.6472	850	.0029	.9625	890	.0001	.9992
811	.0130	.6602	851	.0028	.9652	891	.0001	.9993
812	.0128	.6731	852	.0026	.9678	892	.0001	.9994
813	.0126	.6857	853	.0024	.9702	893	.0001	.9994
814	.0124	.6981	854	.0023	.9725	894	.0001	.9995
815	.0122	.7103	855	.0021	.9746	895	.0001	.9996
816	.0119	.7222	856	.0020	.9766	896	.0001	.9996
817	.0117	.7339	857	.0019	.9785	897	.0000	.9997
818	.0114	.7454	858	.0017	.9802	898	.0000	.9997
819	.0112	.7565	859	.0016	.9818	899	.0000	.9997
820	.0109	.7674	860	.0015	.9833	900	.0000	.9998
821	.0106	.7781	861	.0014	.9847	901	.0000	.9998
822	.0103	.7884	862	.0013	.9860	902	.0000	.9998
823	.0100	.7984	863	.0012	.9872	903	.0000	.9998
824	.0097	.8082	864	.0011	.9883	904	.0000	.9999
825	.0094	.8176	865	.0010	.9893	905	.0000	.9999
826	.0091	.8267	866	.0009	.9902	906	.0000	.9999
827	.0088	.8356	867	.0009	.9911	907	.0000	.9999
828	.0085	.8441	868	.0008	.9919	908	.0000	.9999
829	.0082	.8524	869	.0007	.9926	909	.0000	.9999
830	.0079	.8603	870	.0007	.9933	910	.0000	.9999
831	.0076	.8680	871	.0006	.9939	911	.0000	.9999
832	.0073	.8753	872	.0006	.9945	912		
833	.0071	.8824	873	.0005	.9950	913		
834	.0068	.8891	874	.0005	.9955	914		
835	.0065	.8956	875	.0004	.9959	915		
836	.0062	.9018	876	.0004	.9963	916		
837	.0059	.9077	877	.0004	.9967	917		
838	.0056	.9134	878	.0003	.9970	918		
839	.0054	.9188	879	.0003	.9973	919		
840	.0051	.9239	880	.0003	.9976	920		
841	.0049	.9288	881	.0002	.9978	921		
842	.0046	.9334	882	.0002	.9981	922		
843	.0044	.9378	883	.0002	.9983	923		
844	.0042	.9419	884	.0002	.9984	924		
845	.0039	.9459	885	.0002	.9986	925		
846	.0037	.9496	886	.0001	.9988	926		
847	.0035	.9531	887	.0001	.9989	927		
848	.0033	.9564	888	.0001	.9990	928		
849	.0031	.9595	889	.0001	.9991	929		

It can be done, provided an extensive table of logarithms are available, by using the so-called Stirling's formula

$$\underline{|n} \sqrt{2\pi}\ n^{n+1/2}\ e^{-n+1/12n-1/360n^3+\cdots}$$

where $\pi = 3.14159\ 26535\ 89793\ldots$
$e = 2.71828\ 18284\ 59045$

We shall now proceed to develop a method which will enable us to find approximately the value of any term of the expansion of $(q+p)^r$ and the sum of any number of consecutive terms of this series.

In Section V we made use of the fact that the mean, standard deviation, and skewness may be regarded as satisfactorily describing any distribution. We shall now show that for any distribution whose frequencies are proportional to the terms of the expansion of $(q+p)^r$,

$$(33)\qquad \begin{aligned} M &= rp \\ \sigma &= \sqrt{rp(1-p)} \\ \alpha_3 &= \frac{1-2p}{\sigma} \end{aligned}$$

Thus, for the expected distribution of Table XXI, column (3), since $r = 12$, $p = 1/2$, $q = 1/2$,

$$\begin{aligned} M &= rp = \frac{12}{2} = 6 \\ \sigma &= \sqrt{12 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}} = \sqrt{3} = 1.732 \\ \alpha_3 &= 0 \end{aligned}$$

Similarly, for the expected distribution of Table XXII, column (3), since $r = 12$, $q = 5/6$, $p = 1/6$,

$$\begin{aligned} M &= \frac{12}{6} = 2 \\ \sigma &= \sqrt{12 \cdot \tfrac{1}{6} \cdot \tfrac{5}{6}} = \sqrt{\tfrac{5}{3}} = 1.291 \\ \alpha_3 &= \frac{1-1/3}{1.291} = .516 \end{aligned}$$

Values for these expected distributions may be calculated from the frequencies ${}_{4096}P_x$ in the usual manner. The results will then be found to agree with those obtained as above by means of formulae (33). Since the P_x column in Table XXI is composed of integers they will agree exactly, but since in Table XXII both the probabilities and expected frequencies are approximations, the values of these functions obtained by the two methods may differ slightly. Theoretically those obtained by employing formulae (33) are the more correct.

If, as before, q denote the probability that each individual will die within a year, and P_x the probability that exactly x out of r individuals will die within one year, then the values of P_0, P_1, P_2, . . . are equal to the terms of the expansion of $(q+p)^r$ which are shown in frequency distribution form in Table XXIV.

The total of column (2) is obviously equal to N since the values of f_x are merely the expansion of $N(q+p)^r$. Since $q+p=1$, therefore $(q+p)^r=1$, and hence $\Sigma f_x = N$.

If one takes the common factor Nrp out of every term in column (3) of the previous table, it is noted that the sum of this column may be written

$$N\Sigma xf_x = Nrp\left[q^{r-1} + (r-1)q^{r-2}p + \frac{(r-1)(r-2)}{1\cdot 2}q^{r-3}p^2 + \cdots\cdots\right]$$

But the expression within the bracket is merely the expansion of the binomial $(q+p)^{r-1}$. Hence $\Sigma xf_x = Nrp\,[1] = Nrp$ Likewise the sum of the terms in columns (4) and (5) may be factored as follows:

$$\Sigma x(x-1)f_x = Nr(r-1)p^2\left[q^{r-2} + (r-2)q^{r-3}p + \frac{(r-2)(r-3)}{1\cdot 2}q^{r-4}p^2 + \cdots\right]$$

$$= Nr(r-1)\,p^2\,(q+p)^{r-2} = Nr(r-1)p^2$$

$$\Sigma x(x-1)(x-2)f_x = Nr(r-1)(r-2)p^3\left[q^{r-3} + (r-3)q^{r-4}p + \frac{(r-3)(r-4)}{1\cdot 2}q^{r-5}p^2 + \cdots\right]$$

$$= Nr(r-1)(r-2)p^3(q+p)^{r-3} = Nr(r-1)(r-2)p^3$$

TABLE XXIV

x (1)	f_x (2)	$x \cdot f_x$ (3)	$x(x-1)f_x$ (4)	$x(x-1)(x-2)f_x$ (5)
0	Nq^r	0	0	0
1	$Nrq^{r-1}p$	$Nrq^{r-1}p$	0	0
2	$N\frac{r(r-1)}{1\cdot 2}q^{r-2}p^2$	$N\frac{r(r-1)}{1}q^{r-2}p^2$	$Nr(r-1)q^{r-2}p^2$	0
3	$N\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}q^{r-3}p^3$	$N\frac{r(r-1)(r-2)}{1\cdot 2}q^{r-3}p^3$	$N\frac{r(r-1)(r-2)}{1}q^{r-3}p^3$	$Nr(r-1)(r-2)q^{r-3}p^3$
4	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2\cdot 3\cdot 4}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2\cdot 3}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1}q^{r-4}p^4$
	etc.	etc.	etc.	etc.
Total	$N(q+p)^r = N$	$Nrp(q+p)^{r-1} = Nrp$	$Nr(r-1)p^2(q+p)^{r-2} =$ $Nr(r-1)p^2$	$Nr(r-1)(r-2)p^3(q+p)^{r-3} =$ $Nr(r-1)(r-2)p^3$

But we may write

$$x(x-1)f_x = x^2 f_x - x f_x,$$

$$\therefore \Sigma x(x-1)f_x = \Sigma x^2 f_x - \Sigma x f_x$$

$$x(x-1)(x-2)f_x = x^3 f_x - 3x^2 f_x + 2x f_x,$$

$$\therefore \Sigma x(x-1)(x-2)f_x = \Sigma x^3 f_x - 3\Sigma x^2 f_x + 2\Sigma x f_x$$

So we have

$$\Sigma f_x = N$$

$$\Sigma x f_x = Nrp$$

$$\Sigma x(x-1)f_x = \Sigma x^2 f_x - \Sigma x f_x = Nr(r-1)p^2$$

$$\Sigma x(x-1)(x-2)f_x = \Sigma x^3 f_x - 3\Sigma x^2 f_x + 2\Sigma x f_x$$

$$= Nr(r-1)(r-2)p^3$$

Therefore

$$\Sigma x^2 f_x = \Sigma x f_x + Nr(r-1)p^2 = Nrp + Nr(r-1)p^2.$$

$$= Nrp + Nr^2p^2 - Nrp^2$$

$$\Sigma x^3 f_x = 3\Sigma x^2 f_x - 2\Sigma x f_x + Nr(r-1)(r-2)p^3$$

$$= 3N(rp + r^2p^2 - rp^2) - 2Nrp + Nr(r-1)(r-2)p^3$$

$$= Nrp + 3Nr^2p^2 - 3Nrp^2 + Nr^3p^3 - 3Nr^2p^3 + 2Nrp^3$$

Hence

$$M_x = \frac{\Sigma x f_x}{\Sigma f_x} = \frac{Nrp}{N} = rp$$

$$\mu_2' = \frac{\Sigma x^2 f_x}{\Sigma f_x} = rp + r^2p^2 - rp^2$$

$$\mu_3' = \frac{\Sigma x^3 f_x}{\Sigma f_x} = rp + 3r^2p^2 - 3rp^2 + r^3p^3 - 3r^2p^3 + 2rp^3$$

$$\mu_2 = \mu_2' - M_x^2 = rp - rp^2 = rp(1-p)$$

$$\mu_3 = \mu_3' - 3M_x\mu_2' + 2M_x^3 = rp - 3rp^2 + 2rp^3$$

$$= rp(1-3p+2p^2) = rp(1-p)(1-2p)$$

The reductions follow since $(q+p) = 1$.

We have finally, that

$$M = rp$$

$$\sigma = \sqrt{\mu_2} = \sqrt{rp(1-p)}$$

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{rp(1-p)(1-2p)}{(\sqrt{rp(1-p)})^3} = \frac{1-2p}{\sigma}$$

Formulae (33) are therefore established.

The equation $M = rp$ shows that for a Bernoulli series the "mean" value is also the "expected" value, since, from our definition of expectation, the expected number of deaths from a group of r individuals is rp.

36. For the distribution of the values of P_x shown in Table XXIII, since $r = 100{,}000$, $q = .992$, $p = .008$

$$M_x = rp = 800$$

$$\sigma = \sqrt{rp(1-p)} = \sqrt{793.6} = 28.1709$$

$$\frac{1}{\sigma} = .0354976$$

$$\alpha_3 = \frac{1-2p}{\sigma} = \frac{.992-.008}{\sigma} = .984(.0354976) = .03493$$

If as before we let $t = \frac{x-M}{\sigma}$, and designate the ordinates of the standard frequency curves by f_t , we can compute any value of P_x with a reasonable degree of accuracy by the formula

$$(34) \qquad P_x = \frac{1}{\sigma} f_t$$

For example: Required the probability that exactly 762 individuals will die within one year in a population of 100,000 for which $p = .008$.

As before, we must first express the number of deaths under consideration in standard units, that is since $M_x = rp = 800$,

$$t = \frac{x-M}{\sigma} = \frac{762-800}{\sigma} = -38(.0354976) = -1.3489$$

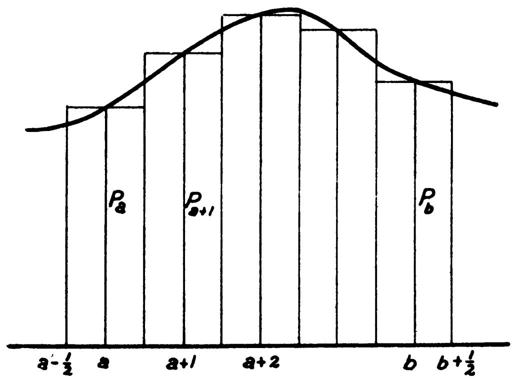
That is, 762 deaths is 38 less than the mean, or $\frac{-38}{\sigma} = -1.3489$ standard units less than the mean.

With $\alpha_3 = 0$ and using the Table of Ordinates of the Pearson Type III Curve, the value of f_t corresponding to $t = -1.35$ is found to be .160383.

$$f_t = .160383$$

$$\therefore \quad P_{762} = \frac{1}{\sigma} f_t = .0354976(.160383) = .005693$$

We shall now consider the following problem: Required the probability that not more than 780 individuals will die within one year, where as before $r = 100{,}000$, $p = .008$. This means that we must obtain the sum of the 781 terms. $P_0 + P_1 + P_2 + \ldots + P_{780}$



Suppose we represent the sum of the probabilities $P_a + P_{a+1} + P_{a+2} + \ldots + P_b$ by a series of ordinates erected at unit intervals along the x axis, and then construct a series of rectangles having these ordinates as altitudes which bisect the bases of the respective unit bases. Then the area of the first rectangle is $P_a = 1 \cdot P_a$, etc. Thus the sum of the series $P_a + P_{a+1} + \ldots + P_b$ is equal to the total area of all the rectangles and is therefore approximately equal to the area under the frequency curve from $a - \frac{1}{2}$ to $b + \frac{1}{2}$. Therefore the sum of all the probabilities, $P_0 + P_1 + \ldots + P_{780}$ can be computed readily by calculating by means of the Tables of Areas of the Standard Curves, the per cent of the area of the standard curve lying below $x = 780.5$, that is below $\bar{x} = -19.5$, or $t = \frac{-19.5}{\sigma} = -.6922$. For $\alpha_3 = 0$, the per cent of the area of the frequency curve lying below $t = -.69$ is 24.5097. Since the sum of all probabilities from P_0 to $P_{100,000}$ inclusive is 1, and $P_0 + P_1 + \ldots + P_{780}$ represents approximately 24.5097 per cent of the total area under the frequency curve, therefore we estimate that $P_0 + P_1 + P_2 + \ldots + P_{780} = .245097$.

By Table XXIII the correct value is .2454, or the error of our approximation is .0003. Using the values $\alpha_3 = 0$, the per cent of the area lying below $t = -.70$ is found to be 24.1964, using straight line interpolation the per cent below $t = -.6922$ is found to be 24.4408. In the same manner, only using $\alpha_3 = .1$, the per cent of the curve lying below $t = -.6922$ is found to be 24.7105. By using straight line interpolation again for the value of α_3, it is found that the per cent of the distribution lying below $t = -.6922$, skewness $= .035$, is 24.5352. The error of our approximation is now zero. In general, however, a sufficient degree of accuracy may be obtained without interpolating for either the value of t or α_3.

Next let it be required to find the probability that less than 840 but more than 780 will die within the year, that is, required the value of $P_{781}+P_{782}+\ldots P_{839}$.

We require therefore the per cent of the area of a standard frequency curve lying between $x=780.5$ and $x=839.5$, that is between $\bar{x}=-19.5$ and $\bar{x}=39.5$ or $t=-.69$ to $t=1.40$.

As has just been shown, 24.5097 per cent of the area of the curve lies below $t=-.69$. Likewise for $\alpha_3=0$ the per cent lying below $t=1.40$ is 91.9243. Consequently 91.9243% - 24.5097%, or 67.4146%, of the area lies between $t=-.69$ and $t=1.40$. Therefore the probability that less than 840 but more than 780 will die within the year is .674146.

By Table XXIII, the correct value is .9188 - .2454 = .6734.

Summary of Section VI.

If p represent the probability that an event will happen in a single trial, then the probability that the event will happen either 0, 1, 2, ... times during r trials are given by the respective terms of the expansionn of $(q+p)^r$. The distribution of these probabilities or the corresponding expected frequencies is adequately described by the three fundamental functions as follows:

$$M=rp$$

$$\sigma=\sqrt{rp(1-p)}$$

$$\alpha_3=\frac{1-2p}{\sigma}$$

The probabilities or expected frequencies may be regarded as a distribution that can be reproduced at will by utilizing the Tables of Pearson's Type III Curves, with the fundamental functions computed from the above formulae. In this way the values of isolated probabilities or the sum of any number of consecutive probabilities may be obtained.