and $\lambda_2(R) = 4\lambda_2$, the mean error is easily found. Thus the squares of the mean errors of the co-ordinates x and y are

$$\lambda_2(x) = \{2{\cdot}3490(\;\;0{\cdot}4257 + 0{\cdot}0228 \cos V)^2 + 1{\cdot}2943(-0{\cdot}1752 \cos V)^2 \qquad + 4(0{\cdot}25 \cos V)^2\}\lambda_2$$

$$\lambda_2(y) = \{2{\cdot}3490(-0{\cdot}1444 + 0{\cdot}0228 \sin V)^2 + 1{\cdot}2943(\;\;0{\cdot}7726 - 0{\cdot}1752 \sin V)^2 + 4(0{\cdot}25 \sin V)^2\}\lambda_2.$$

Only the value $\lambda_2 = 0{\cdot}00236$, found by the summary criticism, is here very uncertain.

XIII. SPECIAL AUXILIARY METHODS.

§ 57. We have often occasion to use the method of least squares, particularly adjustment by elements; and this sometimes requires so much work that we must try to shorten it as much as possible, even by means which are not quite lawful. Several temptations lie near enough to tempt the many who are soon tired by a somewhat lengthened computation, but not so much by looking for subtleties and short cuts. And as, moreover, the method was formerly considered the best solution — among other more or less good — not the only one that was justified under the given supposition, it is no wonder that it has come to be used in many modifications which must be regarded as unsafe or wrong. After what we have seen of the difference between free and bound functions, it will be understood that the consequences of transgressions against the method of least squares stand out much more clearly in the mean errors of the results than in their adjusted values. And as — to some extent justly — more importance is attached to getting tolerably correct values computed for the elements, than to getting a correct idea of the uncertainty, the lax morals with respect to adjustments have taken the form of an assertion to the effect that we can, within this domain, do almost as we like, without any great harm, especially if we take care that a sum of squares, either the correct one or another, becomes a minimum. This, of course, is wrong. In a text-book we should do more harm than good by stating all the artifices which even experienced computers have allowed themselves to employ, under special circumstances and in face of particularly great difficulties. Only a few auxiliary methods will be mentioned here, which are either quite correct or nearly so, when simple caution is observed.

§ 58. When methodic adjustment was first employed, large numbers of figures were used in the computations (logarithms with 7 decimal places), and people often complained of the great labour this caused; but it was regarded as an unavoidable evil, when the elements were to be determined with tolerable exactness. We can very often manage, however, to get on by means of a much simpler apparatus, if we do not seek something which cannot be determined. During the adjustment properly so called, we ought to be able to work with three figures. But this ideal presupposes that two conditions are satisfied: the elements we seek must be small and free of one another, or nearly so; and in both respects it can be difficult enough to protect oneself in time by appropriate transformation. Often it is only through the adjustment itself that we learn to know the artifices which would have made the work easy. This applies particularly to the mutual freedom of the elements. The condition of their smallness is satisfied, if we everywhere use the same preparatory computation as is necessary when the theory is not of linear form.

By such means as are used in the exact mathematics, or by a provisional, more or less allowable adjustment, we get, corresponding to the several observations $o_1 \dots o_n$, a set of values $v_1 \dots v_n$ which are computed by means of the values $x_0 \dots z_0$ of the several elements $x \dots z$, and which, while they satisfy all the conditions of the theory with perfect or at any rate considerable exactness, nowhere show any great deviation from the corresponding observed value. It is then these deviations $o_i - v_i$ and $x - x_0 \dots$ which are made the object of the adjustment, instead of the observations and elements themselves with which, we know, they have mean error in common. When in a non-linear theory the equations between the adjusted observation and the elements are of the general form

$$u_i = F(x, \dots z),$$

they are changed into

$$u_i - v_i = \left(\frac{dF}{dx}\right)_0 (x - x_0) + \dots + \left(\frac{dF}{dz}\right)_0 (z - z_0) \qquad (109)$$

by means of the terms of the first degree in Taylor's series, or by some other method of approximation. If the equations are linear

$$u_i = p_i x + \dots + r_i z,$$

we have, without any change, for the deviations:

$$u_i - v_i = p_i(x - x_0) + \dots + r_i(z - z_0). \qquad (110)$$

No special luck is necessary to find sets of values, $v_i, \dots x_0, \dots z_0$, whose deviations $o_i - v_i$ show only two significant figures; and then computation by 3 figures is, as far as that goes, sufficient for the needs of the adjustment.

The method certainly requires a considerable extra-work in the preparatory computation, and it must not be overlooked that computations with an exactness of many decimal places will often be necessary in this part; especially v_i ought to be computed with the utmost care as a function of $x_0 \dots z_0$, lest any uncertainty in this computation should increase the mean errors, so that we dare not put $\lambda_2(o-v) = \lambda_2(o)$.

This additional work, however, is not quite wasted, even when the theory is linear. The list of the deviations $o_i - v_i$ will, by easy estimates, graphic construction, or directly by the eye, with tolerable certainty lead to the discovery of gross errors in the series of observations, slips of the pen, etc., which must not be allowed to get into the adjustment. The preliminary rejection of such observations may save a whole adjustment; the ultimate rejection, however, falls under the criticism after the adjustment.

In computing the adjusted values, particularly u_i, after the solution of the normal equations, we ought not to rely too confidently on the transformation of the equations into linear form or into equations of deviations for $o_i - v_i$. Where it is possible, the actual equations $u_i = F(x, \ldots z)$ ought to be employed, and with the same degree of accuracy as in the computation of v_i. In this way only can we see whether the approximate system of elements and values has been so near to the final result as to justify the rejection of the higher terms in Taylor's series. If not, the adjustment may only be regarded as provisional, and must be repeated until the values of u_i, got by direct computation, agree with the values through $u_i - v_i$ in the linear equations of adjustment.

On the whole the adjustment ought to be repeated frequently till we get a sufficient approximation. This, for instance, is the rule where the observations represent probabilities, for which $\lambda_2(o_i)$ is generally known only as functions of the unknown quantities which the adjustment itself is to give us.

§ 59. The form of the theory, and in particular the selection of its system of elements, is as a rule determined by purely mathematical considerations as to the elegance of the formulæ, and only exceptionally by that freedom between the elements which is wanted for the adjustment. On the other hand it will generally be impossible to arrange the adjustment in such a way that the free elements with which it ends, can all be of direct, theoretical interest. A middle course, however, is always desirable, for the reasons mentioned in the foregoing paragraph, and very frequently it is also possible, if only the theory pays so much respect to the adjustments that it avoids setting up, in the same system, elements between which we may expect beforehand that strong bonds will exist. Thus, in systems of elements of the orbits of planets, the length of the nodes and the distance of the perihelion from the node ought not both to be introduced as elements; for a positive change in the former will, in consequence of the frequent, small angles of inclination, nearly always entail an almost equally large negative change in the latter. If a theory says that the observation is a linear function of a single parameter, t, the formula ought not to be written $u = p + qt$, unless all the t's are small, some positive, and others negative, but $u = r + q(t - t_0)$, where t_0 is an average of the parameters corresponding to the observations. If we succeed, in this way, in avoiding all strongly operating bonds, and this can be known by the coefficients of all the normal equations outside the diagonal line becoming numerically small in comparison with the mean proportional between the two corresponding coefficients in the diagonal line, then we have at any rate attained so much that we need not use in the calculations for the adjustment many more decimal places than about the 3, which will always be sufficient when the elements are originally mutually free, and not during the adjustment are first to be transformed into freedom with painful accuracy in the transformation operations.

If, by careful selection of the elements, we even get so far that no sum of the products $[pq]$ [1]) in numerical value exceeds about $\frac{1}{30}$ of the mean proportional between the corresponding sums of squares $\sqrt{[pp][qq]}$, or in many cases only $\frac{1}{10}$ of these amounts, then we may consider the bonds between the elements insignificant. The normal equations themselves may then be used to determine the law of error for the elements; we compute provisionally a first approximation by putting all the small sums of products $= 0$, and in the second approximation we correct the $[po]$'s by substituting the sums of the products and the values of the elements as found in the first approximation. For instance:

$$[po] - [pq]\,y_1 - \ldots - [pr]\,z_1 = [pp]\,x_2 \tag{111}$$

while

$$\lambda_2(x_2) = \frac{1}{[pp]^2}\left\{[pp] + \frac{[pq]^2}{[qq]} + \ldots + \frac{[pr]^2}{[rr]}\right\} = \tag{112}$$

$$= 1 : \left\{[pp] - \frac{[pq]^2}{[qq]} - \ldots - \frac{[pr]^2}{[rr]}\right\}. \tag{113}$$

As the errors in these determinations are of the second order, it will not, if the o's themselves are small deviations from a provisional computation, be necessary to make any further approximations.

Even if the bonds between the elements, which are stated in terms of the sums of the products, are stronger, we can sometimes get them untied without any transformation. If we can get new observations, which are just such functions of the elements that the sums of the products will vanish if they are also taken into consideration, we will of course put off the adjustment until, by introducing them into it, we cannot only facilitate the computation but also increase the theoretical value and clearness of the result. And if we can attain freedom of the elements by rejecting from a long series of observations some single ones, we do not hesitate to use this means; especially as such unused observations may very well be employed in the criticism. If, for instance, an arctic expedition has made meteorological observations at some fixed station for a little more than a complete year, we shall not hesitate in the adjustment, by means of periodical functions, to leave out the overlapping observations, or to make use of the means of the double values, giving them the weight of single observations.

[1]) In what follows we write, for the sake of brevity, $[pq]$ for $\left[\frac{pq}{\lambda}\right]$.

§ 60. Though of course the fabrication of observations is, in general, the greatest sin which an applied science can commit, there exists, nevertheless, a rather numerous and important class of cases, in which we both can and ought to use a method which just depends on the fabrication of such observations as might bring about the freedom of the theoretical elements. As a warning, however, against misuse I give it a harsh name: *the method of fabricated observations.*

If, for instance, we consider the problem which has served us as an example in the adjustment, both by correlates and by elements, viz. the determination of the abscissae for 4 points whose 6 mutual distances have been measured by equally good, bondfree observations, we can scarcely after the now given indications look at the normal equations,

$$\begin{aligned} o_{12}+o_{13}+o_{14} &= 3x_1-1x_2-1x_3-1x_4 \\ -o_{12}+o_{23}+o_{24} &= -1x_1+3x_2-1x_3-1x_4 \\ -o_{13}-o_{23}+o_{34} &= -1x_1-1x_2+3x_3-1x_4 \\ -o_{14}-o_{24}-o_{34} &= -1x_1-1x_2-1x_3+3x_4, \end{aligned}$$

without immediately feeling the want of a further observation:

$$O = 1x_1+1x_2+1x_3+1x_4\,,$$

which, if we imagine it to have the same weight = 1 as each of the measurements of distance $\lambda_1(o_{rs}) = x_r - x_s$, will give by addition to the others, but without specifying the value of O,

$$\begin{aligned} O+o_{12}+o_{13}+o_{14} &= 4x_1 \\ O-o_{12}+o_{23}+o_{24} &= 4x_2 \\ O-o_{13}-o_{23}+o_{34} &= 4x_3 \\ O-o_{14}-o_{24}-o_{34} &= 4x_4, \end{aligned}$$

and consequently determine all 4 abscissae as mutually free and with fourfold weight.

What in this and other cases entitles us to fabricate observations is *indeterminateness* in the original problem of adjustment — here, the impossibility of determining any of the abscissae by means of the distances between the points. When we treat such problems in exact mathematics we get simpler, more symmetrical, and easier solutions by introducing values which can only be determined arbitrarily; and so it is also in the theory of observation. But the arbitrariness gets here a greater extent, because not only mean values, but also mean errors must be introduced for greater convenience. And while we can always make use of a fabricated observation in indeterminate problems for the complete or partial liberation of the elements, we must here carefully demonstrate, by criticism in each case, that the fabrication we have used has not changed anything which was really determined without it.

In the above example, this is seen in the first place by O disappearing from all the adjusted values for the distances $x_r - x_s$, and then by O's own adjusted value, determined as the sum $x_1 + x_2 + x_3 + x_4$, and leading only to the identity $O = O$. The adjustment will consequently neither determine O nor let it get any influence on the other determinations. The mean errors show the same and, moreover, in such a way that the criterion becomes independent of whether O has been brought into the computation as an indeterminate number or with an arbitrary value, for, after the adjustment as well as before, we have for O_1 $\lambda_2(O) = 1$. *The scale for O is consequently* = 0, and this is also generally a sufficient proof of our right to use the method of fabricated observations.

§ 61. *The method of partial eliminations.* When the number of elements is large, it becomes a very considerable task to transform the normal equations and eliminate the elements. The difficulty is nearly proportional to the square of that number. Long before the elements would become so numerous that adjustment by correlates could be indicated, a correct adjustment by elements can become practically impossible. The special criticism is quite out of the question, the summary criticism can scarcely be suggested, and the very elimination must be made easier at any price. If it then happens that some of the elements enter into the expressions for some of the observations only, and not at all in the others, then there can be no doubt that the expedient which ought first to be employed is the partial elimination (before we form the normal equations) of such elements from the observations concerning them. These observations will by this means be replaced by certain functions of two observations or more, *which will generally be bound*; and they will be so in a higher and more dangerous degree the fewer elements we have eliminated. By this proceeding we may, consequently, imperil the whole ensuing adjustment, the foundation of which, we know, is *unbound or free* observations as functions of its elements.

If now it must be granted that the difficulties can become so great that we cannot insist on *an absolute prohibition against illegitimate elimination*, we must on the other hand emphatically warn against every elimination which is not performed through free functions, and much the more so, as it is quite possible, in a great many cases in which abuses have taken place, to remain within the strictly legitimate limits of the free functions, by the use of *"the method of partial eliminations"*.

This is connected with the cases, in which some of the observations, for instance $o_1 \dots o_m$, according to the theory, depend on certain elements, for instance $x, \dots y$, which do not occur in the theoretical expression for any other of the observations. Our object is then, by the formation of the normal equations to separate $o_1 \dots o_m$ as a special series of observations. We begin by forming the partial normal equations for this, and then immediately perform the elimination of $x, \dots y$ from them, without taking into consideration whether these equations alone would be sufficient for a determination of the other elements.

As soon as $x \ldots y$ are eliminated, the process of elimination is suspended. The transformed equations containing these elements (which now represent functions that are free of all observations, and functions which depend only on the remaining elements $z, \ldots u$), are put aside till we come back to the determination of $x \ldots y$. The other partially transformed normal equations, originating in the group $o_1 \ldots o_m$, are on the other hand to be added, term by term, to the normal equations for the elements $z, \ldots u$, formed out of the remaining observations, before the process of elimination is continued for these elements.

That this proceeding is quite legitimate becomes evident if we imagine the elements $x \ldots y$ transformed into the elements $x' \ldots y'$, which are free of $z \ldots u$, and then imagine $x' \ldots y'$ inserted instead of $x \ldots y$ in the original equations for the observations. For then all the sums of products with the coefficients of $x' \ldots y'$ will identically become $= 0$, and the sums of squares and sums of products for the separated part of the observations will, as addenda in the coefficients of the normal equations (compare (57)), come out, immediately, with the same values as now the transformed normal equations.

As an example we may treat the following series of measurements of the position of 3 points on a straight line. The mode of observation is as follows. We apply a millimeter scale several times along the straight line, and then each time read off by inspection with the unaided eye either the places of all the points against the scale or the places of two of them. The readings for each point are found in its separate column, and those on the same row belong to the same position of the scale. (Considered as absolute abscissa-observations such observations are bound by the position of the zero by every laying down of the scale; but these bonds are evidently loosened by our taking up the position against the scale of an arbitrarily selected fixed origin y_r as an element beside the abscissae x_1, x_2, x_3 of the three points). All mean errors are supposed to be equal.

Position of the Scale	Point I	Point II	Point III	Eliminated free Elements	
	mm.	mm.	mm.	mm.	
1	6·9	27·54		$17{\cdot}22 = y_1 + \frac{1}{2}(x_1 + x_2)$	Weight = 2
2	8·35		54·95	$31{\cdot}65 = y_2 + \frac{1}{2}(x_1 + x_3)$	
3	7·9		54·5	$31{\cdot}20 = y_3 + \frac{1}{2}(x_1 + x_3)$	
4		21·16	47·2	$34{\cdot}18 = y_4 + \frac{1}{2}(x_2 + x_3)$	
5		10·74	36·7	$23{\cdot}72 = y_5 + \frac{1}{2}(x_2 + x_3)$	
6		4·06	30·1	$17{\cdot}08 = y_6 + \frac{1}{2}(x_2 + x_3)$	
7	31·45	51·98	78·06	$53{\cdot}83 = y_7 + \frac{1}{3}(x_1 + x_2 + x_3)$	Weight = 3
8	32·9	53·5	79·5	$55{\cdot}30 = y_8 + \frac{1}{3}(x_1 + x_2 + x_3)$	
9	9·6	30·3	56·22	$32{\cdot}04 = y_9 + \frac{1}{3}(x_1 + x_2 + x_3)$	
10	20·16	40·78	66·8	$42{\cdot}58 = y_{10} + \frac{1}{3}(x_1 + x_2 + x_3)$	
11	18·9	39·5	65·56	$41.32 = y_{11} + \frac{1}{3}(x_1 + x_2 + x_3)$	

As the theoretical equation for the i^{th} observation in the s^{th} column has the form

$$o_{i,s} = y_i + x_s\,,$$

and every observation, therefore, is a function of only two elements, there is every reason to use the method of partial elimination. If we choose first to eliminate the y's, we have consequently to form normal equations for each of the 11 rows. Where only two points are observed these normal equations get the form

$$\begin{aligned} o_r + o_s &= 2y_i + x_r + x_s \\ o_r &= y_i + x_r \\ o_s &= y_i \qquad + x_s; \end{aligned}$$

for three points the form of the normal equations is

$$\begin{aligned} o_1 + o_2 + o_3 &= 3y_i + x_1 + x_2 + x_3 \\ o_1 &= y_i + x_1 \\ o_2 &= y_i \qquad + x_2 \\ o_3 &= y_i \qquad\qquad + x_3. \end{aligned}$$

Of these equations those referring to the y_i have given the eliminated free elements stated above to the right of the observations after the perpendicular.

By subtracting these equations from the corresponding other equations we get, in the cases where there are 2 points:

$$\begin{aligned} o_r - \tfrac{1}{2}(o_r + o_s) &= \tfrac{1}{2}x_r - \tfrac{1}{2}x_s \\ o_s - \tfrac{1}{2}(o_r + o_s) &= -\tfrac{1}{2}x_r + \tfrac{1}{2}x_s\,, \end{aligned}$$

and in cases where there are 3 points:

$$\begin{aligned} o_1 - \tfrac{1}{3}(o_1 + o_2 + o_3) &= \tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 \\ o_2 - \tfrac{1}{3}(o_1 + o_2 + o_3) &= -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3 \\ o_3 - \tfrac{1}{3}(o_1 + o_2 + o_3) &= -\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3. \end{aligned}$$

By forming the sum of these differences for each column, and counting, on the right side of the equations, how often each element occurs with one other or with two others, we consequently get the ultimate normal equations:

$$\begin{aligned} -\,168\overset{\mathrm{mm}}{\cdot}98 &= \tfrac{29}{6}x_1 - \tfrac{13}{6}x_2 - \tfrac{16}{6}x_3 \\ -\;\;37{\cdot}71 &= -\tfrac{13}{6}x_1 + \tfrac{32}{6}x_2 - \tfrac{19}{6}x_3 \\ +\,206{\cdot}69 &= -\tfrac{16}{6}x_1 - \tfrac{19}{6}x_2 + \tfrac{35}{6}x_3. \end{aligned}$$

The case is here simple enough to be solved by a fabricated observation. How is its most advantageous form found, when its existence is given?

Answer: $\dfrac{759 \cdot o}{23712} = \dfrac{x_1}{114} + \dfrac{x_2}{96} + \dfrac{x_3}{78}$, weight $= 23712$, after which we get the normal equations:

$$\frac{759}{114}\,o - 168{\cdot}98 = \frac{759}{114}\,x_1$$
$$\frac{759}{96}\,o - 37{\cdot}71 = \frac{759}{96}\,x_2$$
$$\frac{759}{78}\,o + 206{\cdot}69 = \frac{759}{78}\,x_3\,,$$

consequently,

$$x_1 = o - \overset{\text{mm.}}{25{\cdot}38}\,,\quad x_2 = o - \overset{\text{mm.}}{4{\cdot}77}\,,\ \text{and}\ x_3 = o + \overset{\text{mm.}}{21{\cdot}24}.$$

From these we now compute the y's:

$$\begin{aligned}
y_1 &= 32{\cdot}295 - o\,, & y_7 &= 56{\cdot}80 - o\,,\\
y_2 &= 33{\cdot}72 - o\,, & y_8 &= 58{\cdot}27 - o\,,\\
y_3 &= 33{\cdot}27 - o\,, & y_9 &= 35{\cdot}01 - o\,,\\
y_4 &= 25{\cdot}945 - o\,, & y_{10} &= 45{\cdot}55 - o\,,\\
y_5 &= 15{\cdot}485 - o\,, & y_{11} &= 44{\cdot}29 - o.\\
y_6 &= 8{\cdot}845 - o\,, & &
\end{aligned}$$

We need not here state the adjusted values for the several observations, nor their differences, of which it is enough to say that their sum vanishes both for each row and for each column; their squares, on the other hand, will be found to be:

	I	II	III	Total:	
	·0002	·0002		·0004	
	1		·0001	2	
	1		1	2	$\Sigma = {\cdot}0028$
		2	2	4	
		6	6	12	
		2	2	4	
	9	25	4	38	
	1	0	1	2	
	9	36	9	54	$\Sigma = {\cdot}0110$
	1	0	1	2	
	1	4	9	14	
Total:	·0025	·0077	·0036	·0138	

For the summary criticism we notice that the number of observations is 27, the number of the elements is $3 + 11 - 1 = 13$, divisor consequently $= 14$ (one element being wholly engaged by the fabricated observation o). The unit of the mean error is therefore determined by $E^2 = 0{\cdot}0010$, and the mean error on single reading $\pm\,\overset{\text{mm.}}{0{\cdot}032}$, which agrees well with what we may expect to attain by practice in estimates of tenth parts.

As to special criticism it is here, where the weights of the eliminated free functions are respectively 2 and 3 times the weight of the single observation, while the weights of x_1, x_2, and x_3 after the adjustment become respectively $\frac{759}{114}$, $\frac{759}{96}$, and $\frac{759}{78}$, very easy to compute the scales

$$1 - \frac{\lambda_2(u)}{\lambda_2(o)} = 1 - \frac{1}{\text{Weight after the adjustment}}.$$

With 759 as common denominator we find for the several scales and the sums of their most natural groups:

	I	II	III		
1	327	327		654	
2	331·5		331·5	663	
3	331·5		331·5	663	$\Sigma = 3996$
4		336	336	672	
5		336	336	672	
6		336	336	672	
7	436	442	448	1326	
8	436	442	448	1326	
9	436	442	448	1326	$\Sigma = 6630$
10	436	442	448	1326	
11	436	442	448	1326	
	3170	3545	3911	10626	

The comparison with the sums of squares in the groups, divided by E^2, shows then for point I 2·5 instead of $\frac{3170}{759} = 4{\cdot}2 \pm \sqrt{8{\cdot}4}$, for point II 7·7 instead of $4{\cdot}7 \pm \sqrt{9{\cdot}4}$, for point III 3·6 instead of $5{\cdot}1 \pm \sqrt{10{\cdot}2}$, for all positions of the scale with two readings 2·8 instead of $5{\cdot}3 \pm \sqrt{10{\cdot}6}$, and for positions with 3 readings 11·0 instead of $8{\cdot}7 \pm \sqrt{17{\cdot}4}$. The limit of the mean error is consequently reached only in the group of point II, where $(7{\cdot}7 - 4{\cdot}7)^2 = 9{\cdot}0 < 9{\cdot}4$, and it is nowhere exceeded. We have a check by summing the scales:

$$\frac{10626}{759} = 14 = 27 - 11 - 3 + 1.$$

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§ 62. In such cases in which the circumstances and weights of the observations are distributed in some regular way, this will often facilitate the treatment of the normal equations. The elimination of the elements and the transformation of the normal equations into such whose left hand sides can be regarded as unbound observations, as they are free functions of the original observations, need not always be so firmly connected with one another as in the ordinary method. If we, in a suitable way, take advantage of regularity in the observations, and thereby are able to find a transformation which sets the normal equations free, then the determination of the several elements will scarcely throw any material obstacles in our way. But in order to find out any special transformations, we must know the general form of the changes of the normal equations resulting from transformation of the original elements into such as are any homogeneous linear funtions of them whatever.

If the equations for the unbound observations in terms of the original elements have been

$$o_i = p_i x + q_i y + r_i z,$$

the normal equations will be:

$$\begin{aligned}[po] &= [pp]x + [pq]y + [pr]z \\ [qo] &= [qp]x + [qq]y + [qr]z \\ [ro] &= [rp]x + [rq]y + [rr]z.\end{aligned}$$

And if we wish to substitute new elements, ξ, η, and ζ, for the old ones, we make use of substitutions in which *the original elements are represented as functions of the new ones*, therefore

$$\left.\begin{aligned}x &= h_1\xi + k_1\eta + l_1\zeta \\ y &= h_2\xi + k_2\eta + l_2\zeta \\ z &= h_3\xi + k_3\eta + l_3\zeta.\end{aligned}\right\} \quad (114)$$

The equations for the observations then have the form

$$o_i = (p_i h_1 + q_i h_2 + r_i h_3)\xi + (p_i k_1 + q_i k_2 + r_i k_3)\eta + (p_i l_1 + q_i l_2 + r_i l_3)\zeta. \quad (115)$$

The new normal equations may be formed from these, but the form becomes very cumbrous, the equation which specially refers to ξ being

$$[(ph_1+qh_2+rh_3)o] = [(ph_1+qh_2+rh_3)^2]\xi + [(ph_1+qh_2+rh_3)(pk_1+qk_2+rk_3)]\eta + \\ + [(ph_1+qh_2+rh_3)(pl_1+ql_2+rl_3)]\zeta.$$

The computation ought not to be performed according to the expressions for the coefficients which come out when we get rid of the round brackets under the signs of summation [ ]. But it is easy to give the rule of the computation with full clearness. The old normal equations are first treated exactly as if they were equations for unbound observations, for x, y, and z, respectively; expressed by the new elements, consequently by multiplication, by *columns*, by h_1, h_2, and h_3 and addition; by multiplication by k_1, k_2, and k_3 and addition; and by multiplication by l_1, l_2, and l_3 and succeeding addition. Thereby, certainly, we get the new normal equations, but still with preservation of the old elements:

$$\left.\begin{array}{l}[(ph_1+qh_2+rh_3)o] = [(ph_1+qh_2+rh_3)p]x+[(ph_1+qh_2+rh_3)q]y+[(ph_1+qh_2+rh_3)r]z\\ [(pk_1+qk_2+rk_3)o] = [(pk_1+qk_2+rk_3)p]x+[(pk_1+qk_2+rk_3)q]y+[(pk_1+qk_2+rk_3)r]z\\ [(pl_1+ql_2+rl_3)o] = [(pl_1+ql_2+rl_3)p]x+[(pl_1+ql_2+rl_3)q]y+[(pl_1+ql_2+rl_3)r]z\end{array}\right\}(116)$$

The second part of the operation must therefore consist in the substitution of the new elements for the original ones in the right hand sides of these equations. In order to find the coefficients of ξ, η, and ζ, we must therefore here again multiply the sums of the products, *now by rows*, by

$$\begin{array}{l}h_1,\ h_2,\ h_3\\ k_1,\ k_2,\ k_3\\ l_1,\ l_2,\ l_3\end{array}$$

and add them up.

Example. It happens pretty often, for instance in investigations of scales for linear measures, that there is symmetry between the elements, two and two, x_r and x_{m-r}, so that for instance the normal equation which specially refers to x_r, has the same coefficients, only in inverted order, as the normal equation corresponding to x_{m-r}; of course, irrespective of the two observed terms $[po]$ on the left hand sides of the equations. Already P. A. Hansen pointed out that this indicates a transformation of the elements into the mean values $s_r = \frac{1}{2}(x_r+x_{m-r})$ and their half differences $d_r = \frac{1}{2}(x_r-x_{m-r})$. In this case therefore the equations for the old elements by the new ones have the form

$$\begin{array}{l}x_r = s_r + d_r\\ x_{m-r} = s_r - d_r\,,\end{array}$$

and the transformation of the normal equations is, consequently, performed just by forming sums and differences of the original coefficients. If the normal equations are

$$\begin{array}{l}[ao] = 4x+3y+2z+1u\\ [bo] = 3x+6y+4z+2u\\ [co] = 2x+4y+6z+3u\\ [do] = 1x+2y+3z+4u\,,\end{array}$$

the procedure is as follows:

$$[ao]+[do] = 5x+5y+5z+5u = 10\frac{x+u}{2}+10\frac{y+z}{2}$$

$$[bo]+[co] = 5x+10y+10z+5u = 10\frac{x+u}{2}+20\frac{y+z}{2}$$

$$[ao]-[do] = 3x+1y-1z-3u = 6\frac{x-u}{2}+2\frac{y-z}{2}$$

$$[bo]-[co] = 1x+2y-2z-1u = 2\frac{x-u}{2}+4\frac{y-z}{2}$$

As in this example, we always succeed in separating the mean values from the half differences, as two mutually free systems of functions of the observations.

§ 63. The great simplification that results when the observations are mere repetitions, in contradistinction to the general case when there are varying circumstances in the observations, is owing to the fact that the whole adjustment is then reduced to the determination of the mean values and the mean errors of the observations. Before an adjustment, therefore, we not only take the means of any observations, which are strictly speaking repetitions, but we also save a good deal of work in the cases which only approximate to repetitions, viz. those where the variations of circumstances have been small enough to allow us to neglect their products and squares. It has not been necessary to await the systematic development of the theory of observations to know how to act in such cases.

When astronomers have observed the place of a planet or a comet several times in the same night, they form a mean time of observation t, a mean right ascension α, and a mean declination δ, and consider α and δ the spherical co-ordinates of the star at the time t.

With the obvious extensions this is what is called the *normal place* method, the most important device in practical adjustment. Such observations whose essential circumstances have "small" variations, are, before the adjustment, brought into a normal place, by forming mean values both for the observed values themselves and for each of their essential circumstances, and on the supposition that the law which connects the observations and circumstances, holds good also, without any change, with respect to their mean values.

Much trouble may be spared by employing the normal place method. The question is, whether we lose thereby in exactness, and then how much.

We shall first consider the case where the unbound observations o are linear functions of the varying essential circumstances $x, \ldots. z$, the equation for the observations being:

$$\lambda_1(o) = a + bx + \ldots. + dz.$$

With the weights v we form the normal equations:

$$[vo\,] = a[v] + b[vx] + \ldots + d[vz] \tag{117}$$

$$\left.\begin{aligned} [vxo] &= a[vx] + b[vx^2] + \ldots + d[vxz] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [vzo] &= a[vz] + b[vzx] + \ldots + d[vz^2]. \end{aligned}\right\} \tag{118}$$

If the whole series of observations is gathered into a single normal place, O, corresponding to the circumstances X [illegible], and with the weight V, we shall have:

$$V = [v]$$
$$VO = [vo]$$
$$VX = [vx]$$
$$\ldots\ldots\ldots\ldots$$
$$VZ = [vz],$$

and as

$$O = a + bX + \ldots + dZ, \tag{117a}$$

this normal place will exhaust the normal equation (117) corresponding to the constant term, both with respect to mean value and mean error. But if we make the other normal equations free of (117), we get, by the correct method of least squares:

$$\left.\begin{array}{l} [v(o-O)(x-X)] = b[v(x-X)^2] + \ldots + d[v(x-X)(z-Z)] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [v(o-O)(z-Z)] = b[v(x-X)(z-Z)] + \ldots + d[v(z-Z)^2] \end{array}\right\} \tag{118a}$$

for the determination of the elements $b \ldots d$, and these determinations are lost completely if the whole series is gathered into a single normal place. Certainly, the coefficients of these equations (118a) are small quantities of the second order, if the $x - X$ and $z - Z$ are small of the first order.

If, on the other hand, we split up the series, forming for each part a normal place, and adjusting these normal places instead of the observations according to the method of the least squares, then the normal equation corresponding to the constant term is still exhausted by the normal place method; and besides this determination of $a + bX + \ldots + dZ$ the normal place method now also affords a determination of the other elements $b \ldots d$, in such a way, however, that we suffer a loss of the weights for their determination. This loss can become great, nay total, if the normal places are selected in a way that does not suit the purpose; but it can be made rather insignificant by a suitable selection of normal places in not too small a number.

Let us suppose, in order to simplify matters, that the observations have only one variable essential circumstance x, of which their mean values are linear functions, consequently

$$\lambda_1(o) = a + bx,$$

and that the x's are uniformly distributed within the utmost limits, x_0 and x_1; we then let each normal place encompass an equally large part of this interval, and we shall find then, this being the most favourable case, with n normal places, that the weight on the adjusted value of the element b becomes $1 - \left(\frac{1}{n}\right)^2$, if by a correct adjustment by elements the corresponding weight is taken as unity. The loss is thus, at any rate, not very great. And it can be made still smaller, if the distribution of the essential circumstance of the observations is uneven, and if we can get a normal place everywhere where the observations become particularly frequent, while empty spaces separate the normal places from each other.

The case is analogous also when the observations are still functions of a single or a few essential circumstances, but the function is of a higher degree, or transcendental. For it is possible also to form normal places in these cases; and we can do so not only when the variations of the circumstances can be directly treated as infinitely small within each normal place, which case by Taylor's theorem falls within the given rule. For if we have at our disposal a provisional approximate formula, $y = f(x)$, and have calculated the deviation from this, $o - y$, of every observation (considering the deviations as observations with the essential circumstances and mean errors of the original observations), then we can use mean numbers of deviations for reciprocally adjacent circumstances as corrections which, added to the corresponding values from the approximate formula, give the normal values. Further, it is required here only that no normal place is made so comprehensive that the deviations within its limits do not remain linear functions of the essential circumstances.

Also here part of the correctness is lost, and it is difficult to say how much. The loss is, under equal circumstances, smaller, the more normal places we form. With twice (or three times) as many normal places as the number of the unknown elements of the problem, it will rarely become perceptible. With due regard to the essential circumstances and the distribution of the weights we can reduce it, using empty spaces as boundaries between the normal places.

A suitable distribution of the normal places also depends on what function the observations are of their essential circumstances. As to this, however, it is, as a rule, sufficient to know the behaviour of the integral algebraic functions, as we generally, when we have to do with functions which are essentially different from these, will try through transformations of the variables to get back to them and to certain functions which resemble them in this respect.

We need only consider the cases in which we have only one variable essential circumstance, of which the mean value of the observation is an algebraic function of the r^{th} degree. We are able then, on any supposition as to the distribution of the observations, o, and their essential circumstances, x, and weights, v, to determine $r+1$ substitutive observations, O, together with the essential circumstances, X, and weights, V, belonging to them, in such a way that they treated according to the method of the least squares will give the same results as the larger number of actual observations. The conditions are:

$$\left.\begin{array}{l} [ov] = O_0 V_0 + \ldots + O_r V_r \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [ox^r v] = X_0^r O_0 V_0 + \ldots + X_r^r O_r V_r \end{array}\right\} \qquad (119)$$

and

$$\left.\begin{array}{l}[v] = V_0 + \dots + V_r \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ [x^{2r}v] = X_0^{2r} V_0 + \dots + X_r^{2r} V_r.\end{array}\right\} \quad (120)$$

These $3r+2$ equations are not quite sufficient for the determination of the $3r+3$ unknowns. We remove the difficulty in the best way by adding the equation:

$$[x^{2r+1}v] = X_0^{2r+1} V_0 + \dots + X_r^{2r+1} V_r.$$

The elimination of the V's (and O's) then leads to an equation of the $r+1$ degree, whose roots $X_0, \dots X_r$ are all real quantities, if the given x's have been real and the v's positive. When the roots are found, we can compute, first $V_0, \dots V_r$ and afterwards $O_0, \dots O_r$, by means of two systems of $r+1$ linear equations with $r+1$ unknowns.

If, for instance, the essential circumstances of the actual observations are contained in the interval from -1 to $+1$, and if the observations are so numerous and so equally distributed that they may be looked upon as continuous with constant mean error everywhere in this interval; if, further, the sum of the weights $= 2$; then the distribution of the substitutive observations will be symmetrical around 0, and, for functions of the lowest degrees, be

$$r = 0\begin{cases} X = \cdot 000 \\ V = 2\cdot 000\,; \end{cases}$$

$$r = 1\begin{cases} X = -\cdot 577, & +\cdot 577 \\ V = 1\cdot 000, & 1\cdot 000\,; \end{cases}$$

$$r = 2\begin{cases} X = -\cdot 775, & \cdot 000, & +\cdot 775 \\ V = \cdot 556, & \cdot 889, & \cdot 556\,; \end{cases}$$

$$r = 3\begin{cases} X = -\cdot 861, & -\cdot 340, & +\cdot 340, & +\cdot 861 \\ V = \cdot 348, & \cdot 652, & \cdot 652, & \cdot 348\,; \end{cases}$$

$$r = 4\begin{cases} X = -\cdot 906, & -\cdot 538, & \cdot 000, & +\cdot 538, & +\cdot 906 \\ V = \cdot 237, & \cdot 479, & \cdot 569, & \cdot 479, & \cdot 237\,; \end{cases}$$

$$r = 5\begin{cases} X = -\cdot 932, & -\cdot 661, & -\cdot 239, & +\cdot 239, & +\cdot 661, & -\cdot 932 \\ V = \cdot 171, & \cdot 361, & \cdot 468, & \cdot 468, & \cdot 361, & \cdot 171\,; \end{cases}$$

$$r = 6\begin{cases} X = -\cdot 949, & -\cdot 742, & -\cdot 406, & \cdot 000, & +\cdot 406, & +\cdot 742, & +\cdot 949 \\ V = \cdot 129, & \cdot 280, & \cdot 382, & \cdot 418, & \cdot 382, & \cdot 280, & \cdot 129 \end{cases}$$

If, in another example, the distribution of the observations is, likewise, continuous, but the weights within the element dx proportional to $e^{-\frac{1}{2}x^2}$, consequently symmetrical with maximum by $x = 0$, then the distribution for the lowest degrees, the only ones of any practical interest, will be

r								
$r = 0$	$X =$	·000						
	$V =$	2·000 ;						
$r = 1$	$X =$	−1·000,	+1·000					
	$V =$	1·000,	1·000 ;					
$r = 2$	$X =$	−1·732,	·000,	+1·732				
	$V =$	·333,	1·333,	·333 ;				
$r = 3$	$X =$	−2·334,	− ·742,	+ ·742,	+2·334			
	$V =$	·092,	·908,	·908,	·092 ;			
$r = 4$	$X =$	−2·857,	−1·356,	·000,	+1·356,	+2·857		
	$V =$	·023,	·444,	1·067,	·444,	·023 ;		
$r = 5$	$X =$	−3·324,	−1·889,	− ·617,	+ ·617,	+1·889,	+3·324	
	$V =$	·005,	·177,	·818,	·818,	·177,	·005 ;	
$r = 6$	$X =$	−3·750,	−2·367,	−1·154,	·000,	+1·154,	+2·367,	3·750
	$V =$	·001,	·062,	·480,	·914,	·480,	·062,	001

If we were able now to represent these substitutive observations as normal places, then we should be able also, by the use of such tables in analogous cases, to prevent any loss of exactness. It would be possible entirely to evade the application of the method of the least squares; we had but to form such qualified normal places in just the same number as the adjustment formula contains elements that are to be determined. This, however, is not possible. Certainly, we can obtain normal places corresponding to the required values of the essential circumstance, but we cannot by a simple formation of mean numbers give them the weight which each of them ought to have, without employing some of the observations twice, others not at all. By taking into consideration how much the extreme normal places from this reason must lose in weight, compared to the substitutive observations, we can estimate how many per cent the loss, in the worst case, can amount to. In the first of our examples we find the loss to be 0, for $r = 0$ and $r = 1$; but for $r = 2$ we lose 15, for $r = 3$ we lose 19, for $r = 4$ we lose 20, and for greater values of r 21 p. c.

Example. Eighteen unbound observations, equally good, $\lambda_2(o) = \frac{1}{12}$, correspond to an essential circumstance whose values are distributed as the prime numbers p from 1049 to 1171. Taking $(p - 1105) : 100 = x$ as the essential circumstance of the observation o, we have:

x	o	x	o	x	o
−·56	−·41	−·14	−·15	+·18	−·24
−·54	+·50	−·12	−·32	+·24	+·09
−·44	−·03	−·08	+·33	+·46	+·39
−·42	−·15	−·02	−·21	+·48	+·12
−·36	+·48	+·04	+·21	+·58	−·24
−·18	+·18	+·12	+·40	+·66	−·39

Dividing these observations into groups indicated by the horizontal lines, we get the 6 normal places:

x	o	weight
−·550	+·045	2
−·407	+·100	3
−·108	−·034	5
+·145	+·115	4
+·470	+·255	2
+·620	−·315	2

If we suppose the mean values of the observations to be a function of the third, eventually second, degree of x, $\lambda_1(o) = a + bx + cx^2 + dx^3$, we have by ordinary application of the adjustment by elements the normal equations:

$$\begin{aligned} 6{\cdot}72 &= 216{\cdot}00a - 1{\cdot}20b + 29{\cdot}98c + 1{\cdot}94d \\ -3{\cdot}07 &= -1{\cdot}20a + 29{\cdot}98b + 1{\cdot}94c + 8{\cdot}11d \\ -1{\cdot}08 &= 29{\cdot}98a + 1{\cdot}94b + 8{\cdot}11c + 1{\cdot}21d \\ -1{\cdot}44 &= 1{\cdot}94a + 8{\cdot}11b + 1{\cdot}21c + 2{\cdot}56d. \end{aligned}$$

By the free equations:

$$\begin{aligned} 6{\cdot}72 &= 216{\cdot}00a - 1{\cdot}20b + 29{\cdot}98c + 1{\cdot}94d \\ -3{\cdot}03 &= 29{\cdot}97b + 2{\cdot}11c + 8{\cdot}12d \\ -1{\cdot}79 &= 3{\cdot}90c + {\cdot}37d \\ -{\cdot}54 &= {\cdot}305d \end{aligned}$$

we get:

$$\begin{aligned} a &= +{\cdot}09, & a' &= +{\cdot}10, \\ b &= +{\cdot}40, & b' &= -{\cdot}07, \\ c &= -{\cdot}30, & c' &= -{\cdot}47, \\ d &= -1{\cdot}77, \end{aligned}$$

where a', b', c' are the coefficients in the functions of second degree, obtained by presupposing $d = 0$.

Now, by application of the normal places instead of the original observations, we obtain on the same suppositions the normal equations:

$$
\begin{aligned}
6{\cdot}72 &= 216{\cdot}00\,a - 1{\cdot}20\,b + 29{\cdot}45\,c + 1{\cdot}87\,d \\
-2{\cdot}84 &= -1.20\,a + 29{\cdot}45\,b + 1{\cdot}87\,c + 7{\cdot}93\,d \\
-\,{\cdot}54 &= 29{\cdot}45\,a + 1{\cdot}87\,b + 7{\cdot}93\,c + 1{\cdot}14\,d \\
-1{\cdot}57 &= 1{\cdot}87\,a + 7{\cdot}93\,b + 1{\cdot}14\,c + 2{\cdot}45\,d.
\end{aligned}
$$

By the free equations:

$$
\begin{aligned}
6{\cdot}72 &= 216{\cdot}00\,a - 1{\cdot}20\,b + 29{\cdot}45\,c + 1{\cdot}87\,d \\
-2{\cdot}80 &= 29{\cdot}44\,b + 2{\cdot}03\,c + 7{\cdot}94\,d \\
-1{\cdot}26 &= 3{\cdot}77\,c + {\cdot}34\,d \\
-\,{\cdot}76 &= {\cdot}263\,d,
\end{aligned}
$$

we get:

$$
\begin{aligned}
a &= +\,{\cdot}07, & a' &= +\,{\cdot}08, \\
b &= +\,{\cdot}69, & b' &= -\,{\cdot}07, \\
c &= -\,{\cdot}07, & c' &= -\,{\cdot}33, \\
d &= -2{\cdot}88.
\end{aligned}
$$

A comparison between these two calculations, particularly between the leading coefficients in the free equations, shows that the loss of weight amounts to $1 - \frac{263}{306}$, or 14 per cent. But it is only in the equation for d that the loss is so great; in the equations for b and c, respectively, it is only two and one per cent.

Our normal places are very good if the function is only of the first or second degree; for the function of third degree they can be admitted even though the values of the elements a, b, c, d have changed considerably. For functions of 4th or higher degrees these normal places would prove insufficient.

§ 64. That *graphical adjustment* is a means which can carry us through great difficulties, we have shown already in practice by applying it to the drawing of curves of errors. The remarkable powers of the eye and the hand must, like a deus ex machina, help us where all other means fail:

Adjustment by drawing is restricted only by one single condition: if we are to represent a relation between quantities by a plane curve, there must be only two quantities; one of these, represented by the ordinate, is, or is considered to be, the observed value; and the other, represented by the abscissa, is considered the only essential circumstance on which the observed value depends.

Examples of graphical adjustment with two essential circumstances do occur, however, for instance in weather-charts. In periodic phenomena polar co-ordinates are preferred. But otherwise each observation is represented by a point whose ordinate and abscissa are, respectively, the observed value and its essential circumstance; and the adjustment is performed by free-hand drawing of a curve which satisfies the two conditions of being free from irregularities and going as near as possible to the several points of observation. The smoothness of the curve in this process plays the part of the theory, and it is a matter of course that we succeed relatively best when the theory is unknown or extremely intricate; when, for instance, we must confine ourselves to requiring that the phenomenon must be continuous within the observed region, or be a single valued function. But also such a theoretical condition as, for instance, the one that the law of dependence must be of an integral, rational form, may be successfully represented by graphical adjustment, if the operator has had practice in the drawing of parabolas of higher degrees. And we have seen that also such functional forms as have the rapid approximation to an asymptote which the curves of error demand, lie within the province of the graphical adjustment.

As for the approximation to the several observed points, the idea of the adjustment implies that a perfect identity is not necessary; only, the curve must intersect the ordinates so near the points as is required by the several mean errors or laws of errors. If, after all, we know anything as to the exactness of the several observations before we make the adjustment, this ought to be indicated visibly on the drawing-paper and used in the graphical adjustment. We cannot pay much regard, of course, to the presupposed typical form and other properties of the law of errors, but something may be attained, particularly with regard to the number of similar deviations.

If we know nothing whatever as to the exactness of the several observations, or only that they are all to be considered equally good, there can be only a single point in our figure for each observation. In a graphical adjustment, however, we can and ought to take care that the curve we draw has the same number of observed points on each side of it, not only in its whole extent, but also as far as possible for arbitrary divisions. If we know the weights of the observations, they may be indicated on the drawing, and observations with the weight n count n-fold.

In contradistinction to this it is worth while to remark that, with the exception only of bonds between observations, represented by different points, it is possible to lay down on the paper of adjustment almost all desirable information about the several laws of errors. Around each point whose co-ordinates represent the mean values of an observation and of its essential circumstance, a curve, the curve of mean errors, may be drawn in such a way that a real intersection of it with any curve of adjustment indicates a deviation less than the mean error resulting from the combination of the mean errors of the observed value and that of its essential circumstance, if this is also found by observation, while a passing over or under indicates a deviation exceeding the mean error. Evidently, drawings furnished with such indications enable us to make very good adjustments.

If the laws of errors both for the observation and for its circumstance are typical, then the curve of mean errors is an ellipse with the observed points in its centre.

If, further, there are no bonds between the observation and its circumstance, then the ellipse of mean errors has its axes parallel to the ordinate and the abscissa, and their lengths are double the respective mean errors.

If the essential circumstance of the observation, the abscissa, is known to be free of errors, the ellipse of the mean errors is reduced to the two points on the ordinate, distant by the mean error of the observation from the central point of observation. In special cases other means of illustrating the laws of errors may be used. If, for instance, he mean errors as well as the mean values are continuous functions of the essential circumstance of the observation, continuous curves for the mean errors may be drawn on the adjustment paper.

The principal advantages of the graphical adjustment are its indication of gross errors and its independence of a definitely formulated theory. By measuring the ordinates of the adjusted curve we can get improved observations corresponding to as many values of the circumstance or abscissa as we wish, and we can select them as we please within the limits of the drawing. But these adjusted observations are strongly bound together, and we have no indication whatever of their mean errors. Consequently, no other adjustment can be based immediately upon the results of a graphical adjustment.

On the other hand, graphical adjustment can be very advantageously combined with interpolations, both preceding and following, and we shall see later on that by this means we can remedy its defects, particularly its limited accu[illegible]y and its tendency to place too much confidence in the observations, and too little in the theory, i. e. to give an under-adjustment.

By drawing we attain an exactness of only 3 or 4 significant figures, and that is frequently insufficient. The scale of the drawing must be chosen in such a way that the errors of observations are visible; but then the dimensions may easily become so large that no paper can contain the drawing. In order to give the eye a full grasp of the figure, the latter must in its whole course show only small deviations from the straight line, which is taken as the axis of abscissae. This is a practical hint, founded upon experience. The eye can judge of the smoothness of other curves also, but not by far so well as of that of a straight line. And if the line forms a large angle with the axis of the abscissae, then the exactness is lost by the flat intersections with the ordinates. Therefore, as a rule, it is not the original observations that are marked on the paper when we make a graphical adjustment, but only their differences from values found by a preceding interpolation.

In order to avoid an under-adjustment, we must allow $\frac{1}{3}$ of the deviations of the curve from the observation-points to surpass the mean errors. It is further essential that the said interpolation is based on a minimum number of observed data; and after the graphical adjustment has been made, it is safe to try another interpolation using a smaller number of the adjusted values as the base of a new interpolation and a repeated graphical adjustment.

If the results of a graphical adjustment are required only in the form of a table representing the adjusted observations as a function of the circumstance as argument, this table also ought to be based on an interpolation between relatively few measured values, the interpolated values being checked by comparison with the corresponding measured values. A table of exclusively measured values will show too irregular differences.

When we have corrected these values by measuring the ordinates in a curve of graphical adjustment, they may be employed instead of the observations as a sort of normal places. It has been said, however, and it deserves to be repeated, that they must not be adjusted by means of the method of the least squares, like the normal places properly so called. But we can very well use both sorts of normal places, in a ***just sufficient number***, for the computation of the unknown elements of the problem, according to the rules of exact mathematics.

That we do not know their weights, and that there are bonds between them, will not here injure the graphically determined normal places. The very circumstance that even distant observations by the construction of the curve are made to influence each normal place, is an advantage. It is not necessary here to suffer any loss of exactness, as by the other normal places, which, as they are to be represented as mean numbers, cannot at the same time be put in the most advantageous places and obtain the due weight. As to the rest, however, what has been said p. 108—110 about the necessity of putting the substitutive observations in the right place, holds good also, without any alteration, of the graphical normal places.

The method of the graphical adjustment enables us to execute the drawing with absolute correctness, and it leaves us full liberty to put the normal places where we like, consequently also in the places required for absolute correctness; but in both these respects it leaves everything to our tact and practice, and gives no formal help to it.

As to the criticism, the graphical adjustment gives no information about the mean errors of its results. But, if we can state the mean error of each observation, we are able, nevertheless, to subject the graphical adjustments to a summary criticism, according to the rule

$$\sum \frac{(o-u)^2}{\lambda_2} = n - m.$$

And with respect to the more special criticism on systematical deviations, the graphical method even takes a very high rank. Through graphical representations of the finally remaining deviations, $o-u$, particularly if we can also lay down the mean errors on the same drawing, we get the sharpest check on the objective correctness of any adjustment.

From this reason, and owing to the proportionally slight difficulties attached to it, the graphical adjustment becomes particularly suitable where we are to lay down new empirical laws. In such cases we have to work through, to check, and to reject series of hypotheses as to the functional interdependency of observations and their essential circumstances. We save much labour, and illustrate our results, if we work by graphical adjustment.

Of course, we are not obliged to subject observations to adjustment. In the preliminary stages, or as long as it is doubtful whether a greater number of essential circumstances ought not to be taken into consideration, it may even be the best thing to give the observations just as they are.

But if we use the graphical form in order to illustrate such statements by the drawing of a line which connects the several observed points, then we ought to give this line the form of a continuous curve and not, according to a fashion which unfortunately is widely spread, the form of a rectilinear polygon which is broken in every observed point. Discontinuity in the curve is such a marked geometrical peculiarity that it ought, even more than cusps, double-points, and asymptotes, to be reserved for those cases in which the author expressly wants to give his opinion on its occurrence in reality.

XIV. THE THEORY OF PROBABILITY.

§ 65. We have already, in § 9, defined "*probability*" as the limit to which — the law of the large numbers taken for granted — the relative frequency of an event approaches, when the number of repetitions is increasing indefinitely; or in other words, as the limit of the ratio of the number of favourable events to the total number of trials.

The theory of probabilities treats especially of such observations whose events cannot be naturally or immediately expressed in numbers. But there is no compulsion in this limitation. When an observation can result in different numerical values, then for each of these events we may very well speak of its probability, imagining as the opposite event all the other possible ones. In this way the theory of probabilities has served as the constant foundation of the theory of observation as a whole.

But, on the other hand, it is important to notice that the determination of the law of errors by symmetrical functions may also be employed in the non-numerical cases without the intervention of the notion of probability. For as we can always indicate the mutually complementary opposite events as the "fortunate" or "unfortunate" one, or as "Yes" and "No", we may also use the numbers 0 and 1 as such a formal indication. If