error" is $\sqrt{\frac{2}{\pi}}\lambda_2$. The only reason which may be advanced in defence of the use of this idea is that we are spared some little computations, viz. some squarings and the extraction of a square root, which, however, we rarely need work out with more than three significant figures.

## IX. FREE FUNCTIONS.

§ 36. The foregoing propositions concerning the laws of errors of functions — especially of linear functions — form the basis of the theory of computation with observed values, a theory which in several important things differs from exact mathematics. The result, particularly, is not an exact quantity, but always a law of errors which can be represented by its mean value and its mean error, just like the single observation. Moreover, the computation must be founded on a correct apprehension of what observations we may consider mutually unbound, another thing which is quite foreign to exact mathematics. For it is only upon the supposition that the result $R = r_1 o_1 + \dots r_n o_n = [ro]$ — observe the abbreviated notation — is a linear function of unbound observations only, $o_1 \dots o_n$, that we have demonstrated the rules of computation (35)

$$\lambda_1(R) = r_1\lambda_1(o_1) + \dots + r_n\lambda_1(o_n) = [r\lambda_1(o)] \tag{52}$$

$$\lambda_2(R) = r_1^2\lambda_2(o_1) + \dots + r_n^2\lambda_2(o_n) = [r^2\lambda_2(o)]. \tag{53}$$

While the results of computations with observed quantities, taken singly, have laws of errors in the same way as the observations, they also resemble the observations in the circumstances that there can be bonds between them, and, unfortunately, there can be bonds between "results", even though they are derived from unbound observations. If only some observations have been employed in the computation of both $R' = [r'o]$ and $R'' = [r''o]$, these results will generally be bound to each other. This, however, does not prevent us from computing a law of errors, for instance for $aR' + bR''$. We can, at any rate, represent the function of the results directly as a function of the unbound observations, $o_1, \dots o_n$,

$$aR' + bR'' = [(ar' + br'')o]. \tag{54}$$

This possibility is of some importance for the treatment of those cases in which the single observations are bound. They must be treated then just like results, and we must try to represent them as functions of the circumstances which they have in common, and which must be given instead of them as original observations. This may be difficult to do, but as a principle it must be possible, and functions of bound observations must therefore always have laws of errors as well as others; only, in general, it is not possible to compute these laws of errors correctly simply by means of the laws of errors of the observations only, just as we cannot, in general, compute the law of errors for $aR' + bR''$ by means of the laws of errors for $R'$ and $R''$.

In example 5, § 29, we found the mean error in the determination of a direction $R$ between two points, which were given by bond-free and equally good ($\lambda_2(x) = \lambda_2(y) = 1$) measurements of their rectangular co-ordinates, viz.: $\lambda_2(R) = \frac{2}{\Delta^2}$, and then, in example 6, we determined the angle $V$ in a triangle whose points were determined in the same way. It seems an obvious conclusion then that, as $V = R' - R''$, we must have $\lambda_2(V) = \lambda_2(R') + \lambda_2(R'') = \frac{2}{\Delta'^2} + \frac{2}{\Delta''^2}$. But this is not correct; the solution is $\lambda_2(V) = \frac{\Delta^2 + \Delta'^2 + \Delta''^2}{\Delta'^2 . \Delta''^2}$, where $\Delta$, $\Delta'$, and $\Delta''$ are the sides of the triangle. The cause of this is, of course, that the co-ordinates of the angular point enter into both directions and bind $R'$ and $R''$ together. But it is remarkable then that, when $V$ is a right angle, the solutions are identical.

With equally good unbound observations, $o_0$, $o_1$, $o_2$, and $o_3$, we get

$$\lambda_2(o_2 - 2o_1 + o_0) = 6\lambda_2(o)$$
$$\lambda_2(o_3 - 2o_2 + o_1) = 6\lambda_2(o),$$

but

$$\lambda_2(o_3 - 3o_2 + 3o_1 - o_0) = 20\lambda_2(o),$$

although $o_3 - 3o_2 + 3o_1 - o_0 = (o_3 - 2o_2 + o_1) - (o_2 - 2o_1 + o_0)$, according to which we should expect to find

$$\lambda_2(o_3 - 3o_2 + 3o_1 - o_0) = \lambda_2(o_3 - 2o_2 + o_1) + \lambda_2(o_2 - 2o_1 + o_0) = 12\lambda_2(o).$$

But if, on the other hand, we combine the two functions

$$R' = o_0 + 6o_1 - 4o_2 \quad \text{and} \quad R'' = 2o_1 + 3o_2 - o_3,$$

where $\lambda_2(R') = 53\lambda_2(o)$ and $\lambda_2(R'') = 14\lambda_2(o)$, and from this compute $\lambda_2$ for any function $aR' + bR''$, then, curiously enough, we get as the correct result $\lambda_2(aR' + bR'') = (53a^2 + 14b^2)\lambda_2(o) = a^2\lambda_2(R') + b^2\lambda_2(R'')$.

Gauss's general prohibition against regarding results of computations — especially those of mean errors — from the same observations as analogous to unbound observations, has long hampered the development of the theory of observations.

To Oppermann and, somewhat later, to Helmert is due the honour of having discovered that the prohibition is not absolute, but that wide exceptions enable us to simplify our calculations. We must therefore study thoroughly the conditions on which actually existing bonds may be harmless.

Let $o_1, \ldots o_n$ be mutually unbound observations with known laws of errors, $\lambda_1(o_i)$, $\lambda_2(o_i)$, of typical form. Let two general, linear functions of them be

$$[po] = p_1 o_1 + \cdots + p_n o_n$$
$$[qo] = q_1 o_1 + \cdots + q_n o_n.$$

For these then we know the laws of errors

$$\left.\begin{array}{l}\lambda_1[po] = [p\lambda_1(o)],\ \ \lambda_2[po] = [p^2\lambda_2(o)],\ \ \lambda_r[po] = 0 \\ \lambda_1[qo] = [q\lambda_1(o)],\ \ \lambda_2[qo] = [q^2\lambda_2(o)],\ \ \lambda_r[qo] = 0\end{array}\right\}\ \text{for } r > 2.$$

For a general function of these, $F = a[po] + b[qo]$, the correct computation of the law of errors by means of $F = [(ap+bq)o]$ will further give

$$\left.\begin{array}{l}\lambda_1(F) = (ap_1+bq_1)\lambda_1(o_1) + \dots + (ap_n+bq_n)\lambda_1(o_n) = \\ \quad = a\lambda_1[po] + b\lambda_1[qo]\end{array}\right\} \tag{55}$$

$$\left.\begin{array}{l}\lambda_2(F) = (ap_1+bq_1)^2\lambda_2(o_1) + \dots + (ap_n+bq_n)^2\lambda_2(o_n) = \\ \quad = a^2\lambda_2[po] + b^2\lambda_2[qo] + 2ab[pq\lambda_2(o)]\end{array}\right\} \tag{56}$$

$$\lambda_r(F) = 0 \text{ for } r > 2.$$

It appears then, both that the mean values can be computed unconditionally, as if $[po]$ and $[qo]$ were unbound observations, and that the law of errors remains typical. Only in the square of the mean error there is a difference, as the term containing the factor $2ab$ in $\lambda_2(F)$ ought not to be found in the formula, if $[po]$ and $[qo]$ were not bound to one another.

When consequently

$$[pq\lambda_2(o)] = p_1q_1\lambda_2(o_1) + \dots + p_nq_n\lambda_2(o_n) = 0 \tag{57}$$

the functions $[po]$ and $[qo]$ can indeed be treated in all respects like unbound observations, for the law of errors for every linear function of them is found correctly determined also upon this supposition. We call such functions mutually "free functions", and for such, consequently, the formula for the mean error

$$\lambda_2([po]a + [qo]b) = a^2[p^2\lambda_2(o)] + b^2[q^2\lambda_2(o)] \tag{58}$$

holds good.

If this formula holds good for one set of finite values of $a$ and $b$, it holds good for all.

If two functions are mutually free, each of them is said to be "free of the other", and inversely.

Example 1. The sum and difference of two equally good, unbound observations are mutually free.

Example 2. When the co-ordinates of a point are observed with equal accuracy and without any bonds, any transformed rectangular co-ordinates for the same will be mutually free.

Example 3. The sum or the mean value of equally good, unbound observations is free of every difference between two of these, and generally also free of every (linear) function of such differences.

Example 4. The differences between one observation and two other arbitrary, unbound observations cannot be mutually free.

Example 5. Linear functions of unbound observations, which are all different, are always free.

Example 6. Functions with a constant proportion cannot be mutually free.

§ 37. In accordance with what we have now seen of free functions, corresponding propositions must hold good also of observations which are influenced by the same circumstances: it is not necessary to respect all connecting bonds; it is possible that actually bound observations may be regarded as free. The conditions on which this may be the case, must be sought, as in (57), by means of the mean errors caused by each circumstance and the coefficients by which the circumstance influences the several observations. — Note particularly:

If two observations are supposed to be connected by one single circumstance which they have in common, such a bond must not be left out of consideration, but is to be respected. Likewise, if there are several bonds, each of which influences both observations in the same direction.

If, on the other hand, some common circumstances influence the observations in the same direction, others in opposite directions, and if, moreover, one class must be supposed to work as forcibly as the other, the observations may possibly be free, and the danger of treating them as unbound is at any rate less than in the other cases.

§ 38. Assuming that the functions of which we shall speak in the folloving are linear, or at any rate may be regarded as linear when expanded by Taylor's formula, because the errors are so small that we may reject squares and products of the deviations of the observations from fixed values; and assuming that the observations $o_1, \dots o_n$, on which all the functions depend, are unbound, and that the values of $\lambda_2(o_1) \dots \lambda_2(o_n)$ are given, we can now demonstrate a series of important propositions.

Out of the total system of all functions

$$[po] = p_1 o_1 + \dots + p_n o_n$$

of the given $n$ observations we can arbitrarily select partial systems of functions, each partial system containing all those, which can be represented as functions of a number of $m < n$ mutually independent functions, representative of the system,

$$[ao] = a_1 o_1 + \dots + a_n o_n$$

$$\dots\dots\dots\dots\dots\dots$$

$$[do] = d_1 o_1 + \dots + d_n o_n ,$$

of which no one can be expressed as a function of the others. We can then demonstrate the existence of other functions which are free of every function belonging to the partial system represented by $[ao] \ldots [do]$. It is sufficient to prove that such a function $[go] = g_1 o_1 + \ldots g_n o_n$ is free of $[ao] \ldots [do]$ in consequence of the equations $[ga\lambda_2] = 0 \ldots [gd\lambda_2] = 0$. For if so, $[go]$ must be free of every function of the partial system,

$$[(xa + \ldots + zd) o] = x[ao] + \ldots + z[do],$$

because

$$[g(xa + \ldots + zd)\lambda_2] = x[ga\lambda_2] + \ldots + z[gd\lambda_2] = 0.$$

Any function of the total system $[po]$ can now in one single way be resolved into a sum of two functions of the same observations, one of which is free of the partial system represented by $[ao] \ldots [do]$, while the other belongs to this system.

If we call the free addendum $[p'o]$, this proposition may be written

$$[po] = [p'o] + \{x[ao] + \ldots + z[do]\}. \tag{59}$$

By means of the conditions of freedom, $[p'a\lambda_2] = \ldots = [p'd\lambda_2] = 0$, all that concerns the unknown function $[p'o]$ can be eliminated. We find

$$\left.\begin{array}{l} [pa\lambda_2] = x[aa\lambda_2] + \ldots + z[da\lambda_2] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [pd\lambda_2] = x[ad\lambda_2] + \ldots + z[dd\lambda_2], \end{array}\right\} \tag{60}$$

from which we determine the coefficients $x \ldots z$ unambiguously. The number $m$ of these equations is equal to the number of the unknown quantities, and they must be sufficient for the determination of the latter, because, according to a well known proposition from the theory of determinants, the determinant of the coefficients

$$\begin{vmatrix} [aa\lambda_2], \ldots [da\lambda_2] \\ \ldots\ldots\ldots\ldots \\ [ad\lambda_2], \ldots [dd\lambda_2] \end{vmatrix} = \sum \begin{vmatrix} a_r, \ldots a_s \\ \ldots\ldots \\ d_r, \ldots d_s \end{vmatrix}^2 \lambda_2(o_r) \ldots \lambda_2(o_s)$$

is positive, being a sum of squares, and cannot be $= 0$, unless at least one of the functions $[ao] \ldots [do]$ could, contrary to our supposition, be represented as a function of the others.

From the values of $x \ldots z$ thus found, we find likewise

$$[p'o] = [po] - x[ao] - \ldots - z[do]. \tag{61}$$

If $[po]$ belongs to the partial system represented by $[ao] \ldots [do]$, the determination of $x \ldots z$ expresses its coefficients in that system only, and then we get identically $[p'o] = 0$.

But if we take $[po]$ out of the partial system, then (61) gives us $[p'o]$ as different from zero and free of that partial system. If $[po] - [qo]$ belongs to the partial system of $[ao] \dots [do]$, $[qo]$ must produce in this manner the very same free function as $[po]$.

Let $[po] \dots [ro]$ be $n - m$ functions, independent of one another and of the $m$ functions $[ao] \dots [do]$; if we then find $[p'o]$ out of $[po]$ and $[r'o]$ out of $[ro]$ as the free functional parts in respect to $[ao] \dots [do]$, the $n$ functions $[ao] \dots [do]$ and $[p'o] \dots [r'o]$ may be the representative functions of the total system of the functions of $o_1 \dots o_n$, because no relation $\alpha[p'o] + \dots + \delta[r'o] = 0$ is possible; for by (61) it might result in a relation $\alpha[po] + \dots + \delta[ro] + \pi[ao] + \dots + \varphi[do] = 0$ in contradiction to the presumed representative character of $[po] \dots [ro]$ and $[ao] \dots [do]$.

If we employ $[p'o] \dots [r'o]$ or other $n - m$ mutually independent functions

$$[go] \dots [ko],$$

all free of the partial set $[ao] \dots [do]$, as representative functions of another partial system of $o_1 \dots o_n$, then every function of this system must be free of every function of the partial system $[ao] \dots [do]$ (Compare the introduction to this §). No other function of $o_1 \dots o_n$ can be free of $[ao] \dots [do]$ than those belonging to the system $[go] \dots [ko]$; otherwise we should have more than $n$ independent functions of the $n$ variables $o_1 \dots o_n$.

Thus selecting arbitrarily a partial system of functions of the observations $o_1 \dots o_n$ we can — with reference to given squares of mean errors $\lambda_2(o_1) \dots \lambda_2(o_n)$ — distribute the linear homogeneous functions of these observations into three divisions:

1) the given partial system $[ao] \dots [do]$,
2) the partial system of functions $[go] \dots [ko]$, which are free of the former, and
3) all the rest, of which it is proved that every such function is always in only one way compounded by addition of one function of the first partial system to one of the second.

The freedom of functions is a reciprocal property. If the second partial system $[go] \dots [ko]$ were selected arbitrarily instead of the first $[ao] \dots [do]$, then only this latter would be found as the free functions in 2); the composition of every function in 3) would remain the same.

Example. Determine the parts of $o_1 + o_2$, $o_3 + o_4$, $o_1 + o_4$, and $o_2 + o_3$, which are free of $o_1 + o_3$ and $o_2 + o_4$, on the supposition that all 4 observations are equally exact and unbound.

Answer: $\frac{1}{2}(o_1 + o_2 - o_3 - o_4)$, etc.

§ 39. Like all other functions of the observations $o_1 \ldots o_n$, each of these observed values, for instance $o_i$, is the sum of two quantities, one $o_i''$ belonging to the system of $[ao] \ldots [do]$, the other $o_i'$ to the partial system of $[go] \ldots [ko]$, which is free of this. But from $o_i = o_i'' + o_i'$ follows, generally, that $[po] = [po''] + [po']$, and $[po'']$ evidently belongs to the system of $[ao] \ldots [do]$, $[po']$ to the system which is free of this. Accordingly there must between the $n$ functions $o_1' \ldots o_n'$ exist $m$ relations $[ao'] = \ldots [do'] = 0$; likewise $n - m$ relations $[go''] = \ldots [ko''] = 0$, between $o_1'' \ldots o_n''$.

§ 40. That the functions of observations can be split up, in an analogous way, into three or more free quantities, is of no consequence for the following, *except when we imagine this operation to be carried through to the utmost.* It is easy enough to see, however, that also the partial systems of functions can be split up. We could, for instance, among the representatives $[ao] \ldots [do]$ of one partial system select a smaller number $[ao] \ldots [bo]$, and from the others $[co] \ldots [do]$, according to (37), separate the functions $[c'o] \ldots [d'o]$ which were free of $[ao] \ldots [bo]$. $[c'o] \ldots [d'o]$ would then represent the sub-system of functions, free of $[ao] \ldots [bo]$, within the partial system $[ao] \ldots [do]$; and in this way we may continue till all representative functions are mutually free, every single one of all the rest. Such a collection of representative functions we call a *complete set of free functions.* Their number is sufficient to enable us to express all the observations, and all functions of these observations, as functions of them; and their mutual freedom has the effect that they can be treated, by all computations of laws of errors, quite like unbound observations, and thus wholly replace the original observations.

§ 41. The mathematical theory of the transformations of observations into free functions is analogous to the theory of the transformation of rectangular co-ordinates (comp. § 36, example 2), and is treated in several text-books of the higher algebra and determinants under the name of the theory of the orthogonal substitutions. I shall here enter into those propositions only, which we are to use in what follows.

When we have transformed the unbound observations $o_1 \ldots o_n$ into the complete set of free functions $[ao]$, $[b'o] \ldots [d^{\nu}o]$, it is often important to be able to undertake the opposite transformation back to the observations. This is very easily done, for we have

$$o_i = \left\{\frac{a_i}{[aa\lambda_2]}[ao] + \ldots + \frac{d_i^{\nu}}{[d^{\nu}d^{\nu}\lambda_2]}[d^{\nu}o]\right\}\lambda_2(o_i), \tag{62}$$

which is demonstrated by substitution in the equations for the direct transformation

$$[ao] = a_1 o_1 + \ldots + a_n o_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[d^{\nu}o] = d_1^{\nu} o_1 + \ldots + d_n^{\nu} o_n,$$

because $[ab'\lambda_2] = [ad^{\nu}\lambda_2] = \ldots [b'd^{\nu}\lambda_2] = 0$.

As the original observations, considered as functions of the transformed observations $[ao] \dots [d^\nu o]$, must be mutually free, just as well as the latter are free functions of the former, we find by computing the squares of the mean errors $\lambda_2(o_i)$ and the equation that expresses the formal condition that $o_i$ is free of $o_k$, two of the most remarkable properties of the orthogonal substitutions:

$$\frac{1}{\lambda_2(o_i)} = \frac{a_i^2}{[aa\lambda_2]} + \dots + \frac{d_i^{\nu 2}}{[d^\nu d^\nu \lambda_2]} \tag{63}$$

and

$$0 = \frac{a_i a_k}{[aa\lambda_2]} + \dots + \frac{d_i^\nu d_k^\nu}{[d^\nu d^\nu \lambda_2]}. \tag{64}$$

If all observations and functions are stated with their respective mean error as unity, or are divided by their mean error, a reduction which gives also a more elegant form to all the preceding equations, the sum of the squares of the thus reduced observations is not changed by any (orthogonal) transformation into a complete set of free functions.

We have

$$\frac{[ao]^2}{\lambda_2[ao]} + \dots + \frac{[d^\nu o]^2}{\lambda_2[d^\nu o]} = \frac{o_1^2}{\lambda_2(o_1)} + \dots + \frac{o_n^2}{\lambda_2(o_n)}, \tag{65}$$

which, pursuant to the equations (63) and (64), is easily demonstrated by working out the sums of the squares in the numerators on the left side of the equation. As this equation is identical, the same proposition holds good also, for instance, of the differences between $o_1 \dots o_n$ and $n$ arbitrarily selected variables corresponding to them $v_1 \dots v_n$, and of the corresponding differences between the values of the functions. Also here is

$$\left.\begin{aligned} \frac{([ao]-[av])^2}{\lambda_2[ao]} + \dots + \frac{([d^\nu o]-[d^\nu v])^2}{\lambda_2[d^\nu o]} &= \frac{[a(o-v)]^2}{[aa\lambda_2]} + \dots + \frac{[d^\nu(o-v)]^2}{[d^\nu d^\nu \lambda_2]} = \\ &= \frac{(o_1-v_1)^2}{\lambda_2(o_1)} + \dots + \frac{(o_n-v_n)^2}{\lambda_2(o_n)}. \end{aligned}\right\} \tag{66}$$

§ 42. For the practical computation of a complete set of free functions it will be the easiest way to bring forward the functions of such a set one by one. In this case we must select a sufficient number of functions and fix the order in which these are to be taken into consideration. For a moment we can imagine this order to be arbitrary.

The function $[ao]$, which is the first in this list, is now, unchanged, taken into the transformed set. By multiplying the selected function by suitable constants of the form $\frac{[ba\lambda_2]}{[aa\lambda_2]}$, and subtracting the products from the remaining functions $[bo]$ in the list, we can, according to § 38, from each of these separate the addendum which is free of the selected function. Of these then the one which is founded on function Nr. 2 on the list is taken into the transformed set. This function is multiplied in the same way and subtracted from the still remaining functions, so that they give up the addenda which are free of both the selected functions, and so on. The following schedule shows the course of the operation, for the case $n = 4$.

| Functions | Coefficients $\lambda_2(o_1)$ | $\lambda_2(o_2)$ | $\lambda_2(o_3)$ | $\lambda_2(o_4)$ | Sums of the Products | | | | Rule of Computation |
|---|---|---|---|---|---|---|---|---|---|
| $[ao]$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $[aa\lambda]$ | $[ab\lambda]$ | $[ac\lambda]$ | $[ad\lambda]$ | $[ao]$ is selected. |
| $[bo]$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $[ba\lambda]$ | $[bb\lambda]$ | $[bc\lambda]$ | $[bd\lambda]$ | $[bo]-[ao]\cdot[ba\lambda]:[aa\lambda]=[b'o]$ |
| $[co]$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $[ca\lambda]$ | $[cb\lambda]$ | $[cc\lambda]$ | $[cd\lambda]$ | $[co]-[ao]\cdot[ca\lambda]:[aa\lambda]=[c'o]$ |
| $[do]$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $[da\lambda]$ | $[db\lambda]$ | $[dc\lambda]$ | $[dd\lambda]$ | $[do]-[ao]\cdot[da\lambda]:[aa\lambda]=[d'o]$ |
| $[b'o]$ | $b'_1$ | $b'_2$ | $b'_3$ | $b'_4$ | | $[b'b'\lambda]$ | $[b'c'\lambda]$ | $[b'd'\lambda]$ | $[b'o]$ is selected, is free of $[ao]$ |
| $[c'o]$ | $c'_1$ | $c'_2$ | $c'_3$ | $c'_4$ | | $[c'b'\lambda]$ | $[c'c'\lambda]$ | $[c'd'\lambda]$ | $[c'o]-[b'o]\cdot[c'b'\lambda]:[b'b'\lambda]=[c''o]$ |
| $[d'o]$ | $d'_1$ | $d'_2$ | $d'_3$ | $d'_4$ | | $[d'b'\lambda]$ | $[d'c'\lambda]$ | $[d'd'\lambda]$ | $[d'o]-[b'o]\cdot[d'b'\lambda]:[b'b'\lambda]=[d''o]$ |
| $[c''o]$ | $c''_1$ | $c''_2$ | $c''_3$ | $c''_4$ | | | $[c''c''\lambda]$ | $[c''d''\lambda]$ | $[c''o]$ is selected, is free of $[b'o]$ and $[a'o]$ |
| $[d''o]$ | $d''_1$ | $d''_2$ | $d''_3$ | $d''_4$ | | | $[d''c''\lambda]$ | $[d''d''\lambda]$ | $[d''o]-[c''o]\cdot[d''c''\lambda]:[c''c''\lambda]=[d'''o]$ |
| $[d'''o]$ | $d'''_1$ | $d'''_2$ | $d'''_3$ | $d'''_4$ | | | | $[d'''d'''\lambda]$ | $[d'''o]$ is free of $[c''o]$, $[b'o]$, and $[ao]$. |

The computations of the sums of the products (in which for the sake of brevity we have written $\lambda$ for $\lambda_2(o)$) could be made all through by means of the single coefficients in the transformed functions, as it must be done in the beginning by means of the coefficients in the original functions. It is much easier, however, (particularly if for some reason or other we might otherwise do without the computation of the coefficients of the transformed functions), to make use, for this purpose, of the following remarkable property of these sums of the products. We have, for instance,

$$\left.\begin{aligned} [b'c'\lambda] &= \left[\left(b-a\frac{[ba\lambda]}{[aa\lambda]}\right)\left(c-a\frac{[ca\lambda]}{[aa\lambda]}\right)\lambda\right] = \\ &= [bc\lambda]-[ac\lambda]\frac{[ba\lambda]}{[aa\lambda]}-[ba\lambda]\frac{[ca\lambda]}{[aa\lambda]}+[aa\lambda]\frac{[ba\lambda]}{[aa\lambda]}\frac{[ca\lambda]}{[aa\lambda]} = \\ &= [bc\lambda]-[ac\lambda]\cdot[ba\lambda]:[aa\lambda]\,. \end{aligned}\right\} \tag{67}$$

Consequently, the same general rule of computation as, according to the schedule, holds good of the functions and their coefficients, holds good also of the sums of the products and of the squares. The schedule gets the following appendix:

$$[bb\lambda]-[ab\lambda]\cdot[ba\lambda]:[aa\lambda]=[b'b'\lambda],\quad [bc\lambda]-[ac\lambda]\cdot[ba\lambda]:[aa\lambda]=[b'c'\lambda],\quad [bd\lambda]-[ad\lambda]\cdot[ba\lambda]:[aa\lambda]=[b'd'\lambda]$$
$$[cb\lambda]-[ab\lambda]\cdot[ca\lambda]:[aa\lambda]=[c'b'\lambda],\quad [cc\lambda]-[ac\lambda]\cdot[ca\lambda]:[aa\lambda]=[c'c'\lambda],\quad [cd\lambda]-[ad\lambda]\cdot[ca\lambda]:[aa\lambda]=[c'd'\lambda]$$
$$[db\lambda]-[ab\lambda]\cdot[da\lambda]:[aa\lambda]=[d'b'\lambda],\quad [dc\lambda]-[ac\lambda]\cdot[da\lambda]:[aa\lambda]=[d'c'\lambda],\quad [dd\lambda]-[ad\lambda]\cdot[da\lambda]:[aa\lambda]=[d'd'\lambda]$$

$$[c'c'\lambda]-[b'c'\lambda]\cdot[c'b'\lambda]:[b'b'\lambda]=[c''c''\lambda],\quad [c'd'\lambda]-[b'd'\lambda]\cdot[c'b'\lambda]:[b'b'\lambda]=[c''d''\lambda]$$
$$[d'c'\lambda]-[b'c'\lambda]\cdot[d'b'\lambda]:[b'b'\lambda]=[d''c''\lambda],\quad [d'd'\lambda]-[b'd'\lambda]\cdot[d'b'\lambda]:[b'b'\lambda]=[d''d''\lambda]$$

$$[d''d''\lambda]-[c''d''\lambda]\cdot[d''c''\lambda]:[c''c''\lambda]=[d'''d'''\lambda]$$

As will be seen, there is a check by means of double computation for each of the sums of the products properly so called. The sums of the squares are of special importance as they are the squares of the mean errors of the transformed functions, $\lambda_2[ao]=[aa\lambda]$, $\lambda_2[b'o]=[b'b'\lambda]$, $\lambda_2[c''o]=[c''c''\lambda]$, and $\lambda_2[d'''o]=[d'''d'''\lambda]$.

*Example.* Five equally good, unbound observations $o_1$, $o_2$, $o_3$, $o_4$, and $o_5$ represent values of a table with equidistant arguments. The function tabulated is known to be an integral algebraic one, not exceeding the 3[rd] degree. The transformation into free functions is to be carried out, in such a way that the higher differences are selected before the lower ones. (Because $\Delta^4$, certainly, $\Delta^3$ etc., possibly, represent equations of condition). With symbols for the differences, and with $\lambda_2(o_i)=1$, we have then:

| Function | Coefficients | | | | | Sums of the Products | | | | | Factors |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $o_3$ | $0o_1$ | $+0o_2$ | $+1o_3$ | $+0o_4$ | $+0o_5$ | 1 | −1 | −2 | 3 | 6 | $-\frac{3}{35}$ |
| $V\Delta o_3$ | 0 | 0 | −1 | 1 | 0 | | 2 | 3 | −6 | −10 | $\frac{1}{7}$ |
| $\Delta^2 o_3$ | 0 | 1 | −2 | 1 | 0 | | 3 | 6 | −10 | −20 | $\frac{2}{7}$ |
| $V\Delta^3 o_3$ | 0 | −1 | 3 | −3 | 1 | 3 | −6 | −10 | 20 | 35 | $-\frac{1}{2}$ |
| $\Delta^4 o_3$ | 1 | −4 | 6 | −4 | 1 | 6 | −10 | −20 | 35 | 70 | is selected |
| $o_3-\frac{3}{35}\Delta^4 o_3$ | $-\frac{3}{35}$ | $\frac{12}{35}$ | $\frac{17}{35}$ | $\frac{12}{35}$ | $-\frac{3}{35}$ | $\frac{17}{35}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | 0 | | 0 |
| $V\Delta o_3+\frac{1}{7}\Delta^4 o_3$ | $\frac{1}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{4}{7}$ | $\frac{1}{7}$ | −1 | | $\frac{2}{5}$ |
| $\Delta^2 o_3+\frac{2}{7}\Delta^4 o_3$ | $\frac{2}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | 0 | | 0 |
| $V\Delta^3 o_3-\frac{1}{2}\Delta^4 o_3$ | $-\frac{1}{2}$ | 1 | 0 | −1 | $\frac{1}{2}$ | 0 | −1 | 0 | $\frac{1}{2}$ | | is selected |
| $o_3-\frac{3}{35}\Delta^4 o_3$ | $-\frac{3}{35}$ | $\frac{12}{35}$ | $\frac{17}{35}$ | $\frac{12}{35}$ | $-\frac{3}{35}$ | $\frac{17}{35}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | | | 1 |
| $V\Delta o_3+\frac{2}{5}V\Delta^3 o_3-\frac{2}{35}\Delta^4 o_3$ | $-\frac{2}{35}$ | $-\frac{6}{35}$ | $-\frac{1}{7}$ | $\frac{1}{35}$ | $\frac{12}{35}$ | $-\frac{1}{7}$ | $\frac{6}{35}$ | $\frac{1}{7}$ | | | $-\frac{1}{2}$ |
| $\Delta^2 o_3+\frac{2}{7}\Delta^4 o_3$ | $\frac{2}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | | | is selected |
| $o_3+\Delta^2 o_3+\frac{1}{5}\Delta^4 o_3$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | 0 | | | | are free |
| $V\Delta o_3-\frac{1}{2}\Delta^2 o_3+\frac{2}{5}V\Delta^3 o_3-\frac{1}{5}\Delta^4 o_3$ | $-\frac{1}{5}$ | $-\frac{1}{10}$ | 0 | $\frac{1}{10}$ | $\frac{1}{5}$ | 0 | $\frac{1}{10}$ | | | | are both selec |

The complete set of free observations and the squares of their mean errors are thus:

$$
\begin{array}{llll}
(0) = & o_3 + \Delta^2 o_3 + \frac{1}{5}\Delta^4 o_3 & = \frac{1}{5}(o_1 + o_2 + o_3 + o_4 + o_5), & \lambda_2(0) = \frac{1}{5} \\
(1) = & V\Delta o_3 - \frac{1}{2}\Delta^2 o_3 + \frac{2}{5}(V\Delta^3 o_3 - \frac{1}{2}\Delta^4 o_3) & = \frac{1}{10}(-2o_1 - o_2 + o_4 + 2o_5), & \lambda_2(1) = \frac{1}{10} \\
(2) = & \Delta^2 o_3 + \frac{2}{7}\Delta^4 o_3 & = \frac{1}{7}(2o_1 - o_2 - 2o_3 - o_4 + 2o_5), & \lambda_2(2) = \frac{2}{7} \\
(3) = & V\Delta^3 o_3 - \frac{1}{2}\Delta^4 o_3 & = \frac{1}{2}(-o_1 + 2o_2 - 2o_4 + o_5), & \lambda_2(3) = \frac{5}{2} \\
(4) = & \Delta^4 o_3 & = o_1 - 4o_2 + 6o_3 - 4o_4 + o_5, & \lambda_2(4) = 70
\end{array}
$$

Through this and the preceding chapter we have got a basis which will generally be sufficient for computations with observations and, in a wider sense, for computations with numerical values which are not given in exact form, but only by their laws of errors. We can, in the first place, compute the law of errors for a given, linear function of reciprocally free observations whose laws of presumptive errors we know. By this we can solve all problems in which there is not given a greater number of observations, and other more or less exact data, than of the reciprocally independent unknown values of the problem. When we, in such cases, by the means of the exact mathematics, have expressed each of the unknown numbers as a function of the given observations, and when we have succeeded in bringing these functions into a linear form, then we can, by (35), compute the laws of errors for each of the unknown numbers.

Such a solution of a problem may be looked upon as a transformation, by which $n$ observed or in other ways given values are transformed into $n$ functions, each corresponding to its particular value among the independent, unknown values of the problem. It lies often near thus to look upon the solution of a problem as a transformation, when the solution of the problem is not the end but only the means of determining other unknown quantities, perhaps many other, which are all explicit functions of the independent unknowns of the problem. Thus, for instance, we compute the 6 elements of the orbit of a planet by the rectascensions and declinations corresponding to 3 times, not precisely as our end, but in order thereby to be able to compute ephemerides of the future places of the planet. But while the validity of this view is absolute in exact mathematics, it is only limited when we want to determine the presumptive laws of errors of sought functions by the given laws of errors for the observations. Only the mean values, sought as well as given, can be treated just as exact quantities, and with these the general linear transformation of $n$ given into $n$ sought numbers, with altogether $n^2$ arbitrary constants, remains valid, as also the employment of the found mean numbers as independent variables in the mean value of the explicit functions.

If we want also correctly to determine the mean errors, we may employ no other transformation than that into free functions. And if, to some extent, we may choose the independent unknowns of the problem as we please, we may often succeed in carrying through the treatment of a problem by transformation into free functions; for an unknown number may be chosen quite arbitrarily in all its $n$ coefficients, and each of the following unknowns looses, as a function of the observations, only an arbitrary coefficient in comparison to the preceding one; even the $n^{th}$ unknown can still get an arbitrary factor. Altogether are $\frac{1}{2}n(n+1)$ of the $n^2$ coefficients of these transformations arbitrary.

But if the problem does not admit of any solution through a transformation into free functions, the mean errors for the several unknowns, no matter how many there may be, can be computed only in such a way that each of the sought numbers are directly expressed as a linear function of the observations. The same holds good also when the laws of errors of the observations are not typical, and we are to examine how it is with $\lambda_3$ and the higher half-invariants in the laws of errors of the sought functions.

Still greater importance, nay a privileged position as the only legitimate proceeding, gets the transformation into a complete set of free functions in the over-determined problems, which are rejected as self-contradictory in exact mathematics. When we have a collection of observations whose number is greater than the number of the independent unknowns of the problem, then the question will be to determine laws of actual errors from the standpoint of the observations. We must mediate between the observations that contradict one another, in order to determine their mean numbers, and the discrepancies themselves must be employed to determine their mean deviations, etc. But as we have not to do with repetitions, the discrepancies conceal themselves behind the changes of the circumstances and require transformations for their detection. All the functions of the observations which, as the problem is over-determined, have theoretically necessary values, as, for instance, the sum of the angles of a plane triangle, must be selected for special use. Besides, those of the unknowns of the problem, to the determination of which the theory does not contribute, must come forth by the transformation by which the problem is to be solved.

As we shall see in the following chapters on Adjustment, it becomes of essential moment here that we transform into a system of free functions. The transformation begins with mutually free observations, and must not itself introduce any bond, because the transformed functions in various ways must come forth as observations which determine laws of actual errors.

## X. ADJUSTMENT.

§ 48. Pursuing the plan indicated in § 5 we now proceed to treat the determination of laws of errors in some of the cases of observations made under varying or different