On the other hand, if, in a table arranged according to the magnitude of the values, we select a single middle value, preceded and followed by nearly equal numbers of values, we shall get a quantity which is very well fitted to represent the whole series of repetitions.

If, while we are thus counting the results arranged according to their magnitude, we also take note of these two values with which we respectively (*a*) leave the first sixth part of the total number, and (*b*) enter upon the last sixth part (more exactly we ought to say 16 per ct.), we may consider these two as indicating the limits between great and small deviations. If we state these two values along with the middle one above referred to, we give a serviceable expression for the law of errors, in a way which is very convenient, and although rough, is not to be despised. Why we ought to select just the middle value and the two sixth-part values for this purpose, will appear from the following chapters.

IV. CURVES OF ERRORS.

§ 12. Curves of actual errors of repeated observations, each of which we must be able to express by one real number, are generally constructed as follows. On a straight line as the axis of abscissae, we mark off points corresponding to the observed numerical quantities, and at each of these points we draw an ordinate, proportional to the number of the repetitions which gave the result indicated by the abscissa. We then with a free hand draw the curve of errors through the ends of the ordinates, making it as smooth and regular as possible. For quantities and their corresponding abscissae which, from the nature of the case, *might* have appeared, but do not really appear, among the repetitions, the ordinate will be = 0, or the point of the curve falls on the axis of abscissae. Where this case occurs very frequently, the form of the curves of errors becomes very tortuous, almost discontinuous. If the observation is essentially bound to discontinuous numbers, for instance to integers, this cannot be helped.

§ 13. If the observation is either of necessity or arbitrarily, in spite of some inevitable loss of accuracy, made in round numbers, so that it gives a lower and a higher limit for each observation, a somewhat different construction of the curve of errors ought to be applied, viz. such a one, that the area included between the curve of error, the axis of abscissae, and the ordinates of the limits, is proportional to the frequency of repetitions within these limits. But in this way the curve of errors may depend very much on the degree of accuracy involved in the use of round numbers. This construction of areas can be made by laying down rectangles between the bounding ordinates, or still better, trapezoids with their free sides approximately parallel to the tangents of the curve. If the limiting round numbers are equidistant, the mean heights of the trapezoids or rectangles are directly proportional to the frequencies of repetition. In this case a preliminary construction of curve-points can be made as in § 12, and may often be used as sufficient.

It is a very common custom, but one not to be recommended, to draw a broken line between the observed points instead of a curve.

§ 14. There can be no doubt that the curve of errors, as a form for the law of errors, has the advantage of perspicuity, and were not the said uncertainty in so many cases a critical drawback, this would perhaps be sufficient. Moreover, it is in practice quite possible, and not very difficult, to pass from the curve of actual errors to one which may hold good for presumptive errors; though, certainly, this transition cannot be founded upon any positive theory, but depends on skill, which may be acquired by working at good examples, but must be practised judiciously.

According to the law of large numbers we must expect that, when we draw curves of actual errors according to relative frequency, for a numerous series of repetitions, first based upon small numbers, afterwards redrawn every time as we get more and more repetitions, the curves, which at first constantly changed their forms and were plentifully furnished with peaks and valleys, will gradually become more like each other, as also simpler and more smooth, so that at last, when we have a very large but finite number of observations, we cannot distinguish the successive figures we have drawn from one another. We may thus directly construct curves of errors, which may be approved as pictures of curves of presumptive errors, but in order to do so millions of repetitions, rather than thousands, are certainly required.

If from curves of actual errors for small numbers we are to draw conclusions as to the curve of presumptive errors, we must guess, but at the same time support our guess, partly by an estimate of how great irregularities we may expect in a curve of actual errors for the given number, partly by developing our feeling for the form of regular curves of that sort, as we must suppose that the curves of presumptive errors will be very regular. In both respects we must get some practice, but this is easy and interesting.

Without feeling tied down to the particular points that determined the curve of actual errors, we shall nevertheless try to approach them, and especially not allow many large deviations on the same side to come together. We can generally regard as large deviations (the reason why will be mentioned in the chapter on the Theory of Probabilities) those that cause greater errors, as compared with the absolute frequency of the result in question, than the square root of that number (more exactly $\sqrt{h\frac{n-h}{n}}$, where h is the frequency of the result, n the number of all repetitions). But even deviations two or three times as great as this ought not always to be avoided, and we may be satisfied, if only one third of the deviations of the determining points must be called large. We may use the word "adjustment" (graphical) to express the operation by which a curve of presumptive errors is determined. (Comp. § 64). The adjustment is called an over-adjustment, if we have approached too near to some imaginary ideal, but if we have kept too close to the curve of actual errors, then the curve is said to be under-adjusted.

Our second guide, the regularity of the curve of errors, is as an æsthetical notion of a somewhat vague kind. The continuity of the curve is an essential condition, but it is not sufficient. The regularity here is of a somewhat different kind from that seen in the examples of simple, continuous curves with which students more especially become acquainted. The curves of errors get a peculiar stamp, because we would never select the essential circumstances of the observation so absurdly that the deviations could become indefinitely large. Nor would we without necessity retain a form of observation which might bring about discontinuity. It follows that to the abscissae which indicate very large deviations, must correspond rapidly decreasing ordinates. The curve of errors must have the axis of abscissae as an asymptote, both to the right and the left. All frequency being positive, where the curve of errors deviates from the axis of abscissae, it must exclusively keep on the positive side of the latter. It must therefore more or less get the appearance of a bow, with the axis of abscissae for the string. In order to train the eye for the apprehension of this sort of regularity, we recommend the study of figs. 2 & 3, which represent curves of errors of typical forms, exponential and binomial (comp. the next chapter, p. 16, seqq.), and a comparison of them with figures which, like Nr. 1, are drawn from actual observations without any adjustment.

The best way to acquire practice in drawing curves of errors, which is so important that no student ought to neglect it, may be to select a series of observations, for which the law of presumptive errors may be considered as known, and which is before us in tabular form.

We commence by drawing curves of actual errors for the whole series of observations; then for tolerably large groups of the same, and lastly for small groups taken at random and each containing only a few observations. On each drawing we draw also, besides the curve of actual errors, another one of the presumptive errors, on the same scale, so that the abscissae are common, and the ordinates indicate relative frequencies in proportion to the same unit of length for the total number. The proportions ought to be chosen so that the whole part of the axis of abscissae which deviates sensibly from the curve, is between 2 and 5 times as long as the largest ordinate of the curve.

Prepared by the study of the differences between the curves, we pass on at last to the construction of curves of presumptive errors immediately from the scattered points of the curve which correspond to the observed frequencies. In this construction we must not consider ourselves obliged to reproduce the curve of presumptive errors which we may know beforehand; our task is to represent the observations as nearly as possible by means of a curve which is as smooth and regular as that curve.

The following table of 500 results, got by a game of patience, may be treated in this way as an exercise.

Result	Actual frequency for groups of:																									For all 500	The law of errors of the method, interpolated	Result
	25 repetitions																				100 repetitions							
	I				II				III				IV				V				I	II	III	IV	V			
7	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	1	0	1	3	0·0003	7
																											0·0019	
8	0	0	0	1	0	2	2	0	0	1	1	0	0	0	0	0	0	0	0	0	1	4	2	0	0	7	0·0071	8
																											0·0192	
9	1	3	1	1	5	3	1	1	3	2	2	0	3	1	0	3	1	1	2	1	6	10	7	7	5	35	0·0392	9
																											0·0636	
10	9	2	9	5	6	6	6	4	5	4	8	3	3	3	5	6	4	6	3	4	25	22	20	17	17	101	0·0859	10
																											0·1005	
11	3	6	3	3	3	6	4	4	5	5	3	5	3	7	2	5	5	6	3	8	15	17	18	17	22	89	0.1054	11
																											0·1021	
12	8	5	3	4	3	3	2	8	3	7	4	6	5	4	6	5	3	3	5	7	20	16	20	20	18	94	0·0934	12
																											0·0823	
13	2	4	4	3	6	3	3	1	4	1	1	3	5	4	3	6	7	3	6	1	13	13	9	18	17	70	0·0705	13
																											0·0591	
14	1	2	2	4	1	0	2	3	2	1	2	4	3	5	4	0	4	0	2	4	9	6	9	12	10	46	0·0485	14
																											0·0387	
15	0	1	2	2	1	1	3	2	3	2	2	3	1	1	3	0	0	2	1	0	5	7	10	5	3	30	0·0298	15
																											0·0216	
16	1	2	0	1	0	1	1	0	0	1	0	1	2	0	0	0	0	3	2	0	4	2	2	2	5	15	0·0145	16
																											0·0088	
17	0	0	1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	1	0	0	1	0	0	2	1	4	0·0046	17
																											0·0020	
18	0	0	0	1	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	1	1	2	0	1	5	0·0007	18
																											0·0002	
19	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0·0000	19
Total	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	100	100	100	100	100	500	0·9999	

The law of presumptive errors here given is not the direct result of free-hand construction; but the curve so got has been improved by interpolation of the logarithms of its statements of the relative frequencies, together with the formation of mean numbers for the deviations, a proceeding which very often will give good results, but which is not strictly necessary. By this we can also determine the functional law of errors (Comp. the next chapter). The equation of the curve is:

$$\text{Log}\, y = 2{\cdot}0228 + 0{\cdot}0030\,(x-11) - 0{\cdot}6885\,(x-11)^2 + 0{\cdot}01515\,(x-11)^3 - 0{\cdot}001675\,(x-11)^4$$

§ 15. By the study of many curves of presumptive errors, and especially such as represent ideal functional laws of errors, we cannot fail to get the impression that there exists a typical form of curves of errors, which is particularly distinguished by symmetry. Familiarity with this form is useful for the construction of curves of presumptive errors. But we must not expect to get it realised in all cases. For this reason I have considered it important to give, alongside of the typical curves, an example taken from real observations of a skew curve of errors, which in consequence of its marked want of symmetry deviates considerably from the typical form. Fig. 4 shows this last mentioned law of presumptive errors.

Deviation from the typical form does not indicate that the observations are not good. But it may become so glaring that we are forced by it to this conclusion. If, for instance, between the extreme values of repetitions — abscissae — there are intervals which are as free from finite ordinates as the space beyond the extremes, so that the curve of errors is divided into two or several smaller curves of errors beside one another, there can scarcely be any doubt that we have not a series of repetitions proper, but a combination of several; that is to say, different methods of observation have been used and the results mixed up together. In such cases we cannot expect that the law of large numbers will remain in force, and we had better, therefore, reject such observations, if we cannot retain them by tracing out the essential circumstances which distinguish the groups of the series, but have been overlooked.

§ 16. When a curve of presumptive errors is drawn, we can measure the magnitude of the ordinate for any given abscissa; so far then we know the law of errors perfectly, by means of the curve of errors, but certainly in the tabular form only, with all its copiousness. Whether we can advance further depends on, whether we succeed in interpolating in the table so found, and particularly on, whether we can, either from the table or direct from the curve of errors, by measurement obtain a comparatively small number of constants, by which to determine the special peculiarities of the curve.

By interpolating, by means of Newton's formula, the logarithms of the frequencies, or by drawing the curves of errors with the logarithms of the frequencies as ordinates, we often succeed, as above mentioned, in giving the curve the form of a parabola of low (and always even) degree.

Still easier is it to make use of the circumstance that fairly typical curves of errors show a single maximum ordinate, and an inflexion on each side of it, near which the curve for a short distance is almost rectilinear. By measuring the co-ordinates of the maximum point and of the points of inflexion, we shall get data sufficient to enable us to draw a curve of errors which, as a rule, will deviate very little from the original. All this, however, holds good only of the curves of presumptive errors. With the actual ones we cannot operate in this way, and the transition from the latter to the former seems in the meantime to depend on the eye's sense of beauty.

V. FUNCTIONAL LAWS OF ERRORS.

§ 17. Laws of errors may be represented in such a way that the frequency of the results of repetitions is stated as a mathematical function of the number, or numbers, expressing the results. This method only differs from that of curves of errors in the circumstance that the curve which represents the errors has been replaced by its mathematical formula; the relationship is so close that it is difficult, when we speak of these two methods, to maintain a strict distinction between them.

In former works on the theory of observations the functional law of errors is the principal instrument. Its source is mathematical speculation; we start from the properties which are considered essential in ideally good observations. From these the formula for the typical functional law of errors is deduced; and then it remains to determine how to make computations with observations in order to obtain the most favourable or most probable results.

Such investigations have been carried through with a high degree of refinement; but it must be regretted that in this way the real state of things is constantly disregarded. The study of the curves of actual errors and the functional forms of laws of actual errors have consequently been too much neglected.

The representation of functional laws of errors, whether laws of actual errors or laws of presumptive errors founded on these, must necessarily begin with a table of the results of repetitions, and be founded on interpolation of this table. We may here be content to study the cases in which the arguments (i. e. the results of the repetitions) proceed by constant differences, and the interpolated function, which gives the frequency of the argument, is considered as the functional law of errors. Here the only difficulty we encounter is that we cannot directly employ the usual Newtonian formula of interpolation, as this supposes that the function is an integral algebraic one, and gives infinite values for infinite arguments, whether positive or negative, whereas here the frequency of these infinite arguments must be $= 0$. We must therefore employ some artifice, and an obvious one is to interpolate, not the frequency itself, y, but its reciprocal, $\frac{1}{y}$. This, however, turns out to be inapplicable; for $\frac{1}{y}$ will often become infinite for finite arguments, and will, at any rate, increase much faster than any integral function of low degree.