THE PRECISION OF THE WEIGHTED AVERAGE

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Introduction. We shall consider an infinite *universe* of elements characterized by pairs of variable quantities x_i, y_i $(i = 1, 2, 3 \cdots\cdots, \infty)$. Regarding the values of y_i as the weights to be assigned to the variates x_i the weighted average of x_i may be denoted by x_y, i.e.

$$x_y = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3 \cdots\cdots\cdots\cdots}{y_1 + y_2 + y_3 \cdots\cdots\cdots\cdots}.$$

All possible samples, each of N pairs of variates x_i, y_i $(i = 1, 2, 3 \ldots,$ that can be selected from the universe constitute the sample population.

Our problem is to obtain an expression for the probable precision of the weighted average x_y according to certain hypotheses concerning the selection of the pairs of variates in various samples. Professor Bowley discussed this problem in his paper on "Precision of Measurement Attained in Sampling"[1] presented in Rome during the Congress of Statistics 1925. In this paper Professor Bowley made no allowance for correlation between the variates x_i and y_i. In the present paper I shall attempt to eliminate this restriction.

I am greatly indebted to Professor A. L. Bowley for suggestions regarding the simplification of the proof of theorem II and for his general assistance in improving the form of this paper.

Let us suppose:

(a) the pairs of elements selected from the universe are independent of each other,

[1]Cambridge 1925.

(b) the number of pairs in each sample is so large that $\frac{1}{N}$ may be neglected,

(c) the frequency surface $f(x,y)$ is normal, i.e. the probability P_i that the particular pair x_i, y_i will be selected is,

$$P_i = \frac{1}{\sigma_x \sigma_y 2\pi\sqrt{1-r^2}}\, e^{-\frac{1}{2(1-r^2)}\left[\left(\frac{x_i-x}{\sigma_x}\right)^2+\left(\frac{y_i-y}{\sigma_y}\right)^2-\frac{2r(x_i-x)(y_i-y)}{\sigma_x\sigma_y}\right]},$$

where x, y, σ_x, σ_y and r designate the parameters characterizing the surface,

(d) the à priori chance that the parameters of (c) are equal to given values may be defined by the function $F(x_y, y, \sigma_x, \sigma_y, r)$ where this function is integrable, can be expanded in Taylor's series and converges over the whole space.

Let the calculated characteristics of the sample be,

X_y the weighted average of x_i with y_i as weights $(i=1,2,3\cdots,N)$
Y the arithmetic average of the variates $y_i, (i=1,2,3,\cdots,N)$
X the arithmetic average of the variates $x_i, ($ " " $)$
S_x the standard deviation of the variates $x_i, ($ " " $)$
S_y the standard deviation of the variates $y_i, ($ " " $)$
R the coefficient of correlation between the variates x_i and y_i $(i=1,2,3\cdots,N)$.

The expressions representing the most probable values of the weighted average and its standard deviation are independent whether the parameters of the universe are known or unknown. In Parts I, II, and III we shall consider the respective cases,

(a) when all parameters are unknown,
(b) all but y are unknown,
(c) all but y and σ_y are unknown.

In Part IV we shall consider the generalized case of Part I when there are k sets of elements, i.e. $x_i^t, y_i^t \begin{bmatrix} t=1,2\cdots\cdots k \\ i=1,2,3\cdots\cdots\infty \end{bmatrix}$

in the universe. In order to consider this case we shall, at the beginning of Part IV, slightly change the hypotheses and modify the above notation.

PART I

Case Where All Parameters Are Unknown

Theorem (1.1). If hypotheses (a) and (c) are satisfied and if $S_x S_y(1-R^2)\neq 0$ then, the most probable value of x_y is X_y.

Proof. If P_n denotes the probability of getting N particular pairs of variates, then it follows from hypotheses (a) and (c) that.

$$(1)\qquad P_n=\left(\frac{1}{\sigma_x\sigma_y 2\pi\sqrt{1-r^2}}\right)^N e^{-\frac{N}{2(1-r^2)}\left[\frac{S_x^2+(x-X)^2}{\sigma_x^2}+\frac{S_y^2+(y-Y)^2}{\sigma_y^2}-2r\frac{RS_xS_y+(x-X)(y-Y)}{\sigma_x\sigma_y}\right]}.$$

Taking the partial derivatives of P_n with respect to x, y, σ_x, σ_y and r, setting them equal to zero, and solving for x, y, σ_x, σ_y and r, yields

$$(2)\qquad \begin{cases} x=X, & \sigma_x=S_x, \\ y=Y, & \sigma_y=S_y, \end{cases} \quad r=R$$

hence $x=X$, $y=Y$, $\sigma_x=S_x$, $\sigma_y=S_y$ and $r=R$ will make P_n a maximum, and the maximum value of P_n is,

$$(3)\qquad P_{max.}=\left[\frac{1}{eS_xS_y2\pi\sqrt{1-R^2}}\right]^N.$$

The weighted average x_y and X_y can be expressed in terms of $x, y, \sigma_x, \sigma_y, r$ and X, Y, S_x, S_y, R respectively,

$$X_y = \sum_{i=1}^{N} (x_i \cdot y_i)/y_i \qquad \text{(by definition)}$$

$$= \frac{\frac{1}{N}\sum_{i=1}^{N} x_i\, y_i - XY + XY}{Y} \qquad \left(\text{since } \frac{1}{N}\sum_{i=1}^{N} y_i = Y \text{ by definition}\right)$$

(4) hence,

$$X_y = \frac{R S_x S_y}{Y} + X \qquad \left(\text{since } \frac{1}{N}\sum_{i=1}^{N} x_i y_i - XY = R S_x S_y\right)$$

similarly.

$$x_y = \frac{r \sigma_x \sigma_y}{y} + x$$

This proves theorem (1.1).

Theorem (1.2). If all four hypotheses are satisfied and if $S_x S_y (1-R^2) \neq 0$ then the *à posteriori* probability P that the sample came from the universe, the weighted average x_y of which satisfies the inequality $|x_y - X_y| \leq \epsilon$, can be expressed by,

(5)

$$P = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{\epsilon} e^{\frac{t^2}{2\sigma^2}} dt \qquad \text{where}$$

$$\sigma = \frac{S_x}{\sqrt{N}} \sqrt{1 + \left(\frac{S_y}{Y}\right)^2 \left\{1 - R^2\left[1 - \left(\frac{S_y}{Y}\right)^2\right]\right\}}$$

Proof. It follows from (4) that,

$$x - X = x_y - X_y - \frac{r\sigma_x \sigma_y}{y} + \frac{RS_x S_y}{Y}$$

Substituting the above value of $(x-X)$ in (1) we shall have,

$$(6)\left\{\begin{aligned} &P_n = \left(\frac{1}{\sigma_x \sigma_y 2\pi\sqrt{1-r^2}}\right)^N e^{-\frac{N}{2(1-r^2)}W} && \text{where} \\ &W = V^2 + \left(\frac{S_x}{\sigma_x}\right)^2 + \left(\frac{S_y}{\sigma_y}\right)^2 - 2rR\,\frac{S_x S_y}{\sigma_x \sigma_y} + (1-r^2)\left(\frac{y-Y}{\sigma_y}\right)^2 && \text{and} \\ &V = -\frac{S_x}{\sigma_x}\left(\frac{x_y - X_y}{S_x} + \frac{RS_y}{Y}\right) + r\left(\frac{y-Y}{\sigma_y} + \frac{\sigma_y}{y}\right) \end{aligned}\right.$$

Let:
$$x_y - X_y = d'S_x, \quad \sigma_x - S_x = \lambda' S_x \qquad \rho = r - R$$
$$y - Y = d'' S_y, \quad \sigma_y - S_y = \lambda'' S_y$$

then,

$$(7)\left\{\begin{aligned} &\frac{P_n}{P_{max.}} = \left[\frac{e\sqrt{1-R^2}}{(1+\lambda')(1+\lambda'')\sqrt{1-(R+\rho)^2}}\right]^N e^{-\frac{N}{2[(1-(R+\rho)^2]}W_1} && \text{in which} \\ &W_1 = V_1^2 + \frac{1}{(1+\lambda')^2} + \frac{1}{(1+\lambda'')^2} - \frac{2R(R+\rho)}{(1+\lambda')(1+\lambda'')} + [1-(R+\rho)^2]\left(\frac{d''}{1+\lambda''}\right)^2 && \text{and} \\ &V_1 = -\frac{1}{(1+\lambda')}\left(d' + \frac{RS_y}{Y}\right) + (R+\rho)\left(\frac{d''}{1+\lambda''} + \frac{1+\lambda''}{\frac{Y}{S_y} + d''}\right). \end{aligned}\right.$$

Taking the logarithm of $\frac{P}{P_{max}}$ we shall have,

$$\frac{1}{N}\log\frac{P_{max}}{P_n} = A, \quad \text{where}$$

$$(8)\begin{cases} A = const. + \log(1+\lambda_1') \\ + \log(1+\lambda'') + \frac{1}{2}\log\left[1-(R+\rho)^2\right] + \frac{1}{2\left[1-(R+\rho)^2\right]}W \end{cases}$$

Expanding A in terms of the small quantities $\lambda', \lambda'', d', d'', \rho$ to second powers inclusive and letting $K=\frac{S_y}{Y}, \lambda=\lambda'+\lambda''$ we obtain

$$(8')\begin{cases} \frac{1}{N}\log\frac{P_{max}}{P_n} = A_1 + A_2 \quad \text{where} \\ A_1 = \frac{1}{2}\left(d'\frac{\partial A}{\partial d'} + \cdots \rho\frac{\partial A}{\partial \rho}\right)^{(2)} \quad \text{or} \\ 2A_1 = const. + \frac{4(\lambda'+\frac{\lambda}{2})}{(1-R^2)} \\ + \frac{1-R^2(1-K^2)}{(1-R^2)}\left[\lambda\, R\, \frac{Kd_1' - RK(1-K^2)d'' + (1-K^2)\rho}{(1-R^2)(1-K^2)}\right]^2 \\ + \frac{1-R^2+K^2(1-K^2)}{(1-R^2)(1-K^2)}\left[\frac{\rho}{1-R^2} - K\frac{d' - K(1-K^2)d''}{1-R^2+K^2(1+R^2)}\right]^2 \\ + \frac{1+K^2\left[1-R^2(1-K^2)\right]}{1-K^2+K^2(1+R^2)}\left[d'' - \frac{R(1-K^2)d_1'}{1+K^2\left[1-R^2(1-K^2)\right]}\right]^2 + \frac{d_1^2}{1+K^2\left[1-R^2(1-K^2)\right]} \\ = A_1' + \frac{d_1^2}{1+K^2\left[1-R^2(1-K^2)\right]} \end{cases}$$

(We shall make use of the above substitution in the next paragraph.)

Therefore the probability of getting a particular set of N pairs of variates can be expressed approximately by,

$$P_n = a \text{ const. times } e^{-NA_1} \tag{9}$$

Then it follows from hypothesis (d) and (8′) that the *à posteriori* probability P that the sample came from the universe—the weighted average x_y of which satisfies the inequality $[x_y - X_y] \leq \epsilon$, whatever the parameters x, y, σ_x, σ_y and r may be—is expressed by,

$$P = \frac{\int_{X_y-\epsilon}^{X_y+\epsilon} \int_{-\infty}^{\infty} \cdots \int_{-1}^{1} F(x_y, \ldots r) e^{-N(A_1+A_2)} \, dx_y \cdots dr}{\int_{-\infty}^{\infty} \cdots \int_{-1}^{1} F(x_y, \ldots r) e^{-N(A_1+A_2)} \, dx_y \cdots dr}. \tag{10}$$

We may write,

$$\left\{ \begin{aligned} & F(x_y, y, \sigma_x, \sigma_y, r) e^{-N(A_1+A_2)} \\ & = F(x_y, y, \sigma_x, \sigma_y, r) e^{-N(A_1'/2+A_2) - \frac{\left(-\frac{Nd_1^2}{1+K^2[1-R^2(1-K^2)]}\right)}{2}} \\ & = F_1(x_y, y, \sigma_x, \sigma_y, r) e^{-\frac{-N(x_y - X_y)^2}{\sigma_0^2}} \end{aligned} \right. \tag{11}$$

where

$$\sigma_0 = \frac{S_x}{\sqrt{2}} \sqrt{1+K^2\left[1-R^2(1-K^2)\right]}, \quad \text{since } \left(\frac{x_y - X_y}{S_x}\right) = d_1$$

and

$$F_1(x_y, y, \sigma_x, \sigma_y, r) = F(x_y, \sigma_x, \sigma_y, r) e^{-N(A_1'/2+A_2)}$$

$$
(12)\begin{cases}
\text{Let, } & E=\sqrt{N}(xy-X_y) \\
\text{and} & \varphi\left(\dfrac{E}{\sqrt{N}}\right)=\displaystyle\int_{\infty}^{\infty}\cdots\int_{-1}^{1} F_1\left(\frac{E}{\sqrt{N}}+X_y, y\ldots r\right)dy\ldots dr \\
\text{then, } & P=\displaystyle\int_{-\epsilon\sqrt{N}}^{\epsilon\sqrt{N}} \varphi\left(\frac{E}{\sqrt{N}}\right)e^{-\left(\frac{E}{\sigma_0}\right)^2}dE\Big/\int_{\infty}^{\infty}\varphi\left(\frac{E}{\sqrt{N}}\right)e^{-\left(\frac{E}{\sigma_0}\right)}dE.
\end{cases}
$$

It follows from (8′), (11), and (12) and hypothesis (d) that $\varphi\left(\frac{E}{\sqrt{N}}\right)$ can be developed in Taylor's series for all values of N, hence,

$$
(13)\begin{cases}
\varphi\left(\dfrac{E}{\sqrt{N}}\right)=\varphi(0)+\dfrac{E}{\sqrt{N}}\varphi'(0)+\dfrac{1}{2!}\left(\dfrac{E}{\sqrt{N}}\right)^2\varphi''(0)+\dfrac{1}{3!}\left(\dfrac{E}{\sqrt{N}}\right)^3\varphi'''(0)+\cdots \\
\quad=\varphi(0)+\left(\dfrac{E}{\sqrt{N}}\right)\varphi'(0)+\dfrac{1}{N}\left[\dfrac{1}{2!}E^2\varphi''(0)+\dfrac{E^3}{3!\sqrt{N}}\varphi'''(0)\right. \\
\quad\left.+\dfrac{E^4}{4!(\sqrt{N})^2}\varphi^{IV}(0)+\cdots\right]=\varphi(0)+\left(\dfrac{E}{\sqrt{N}}\right)\varphi'(0)+O\left(\dfrac{1}{N}\right).
\end{cases}
$$

Neglecting terms of order of $\left(\frac{1}{N}\right)$ we shall have,

$$
(14)\quad P=\int_{-\epsilon\sqrt{N}}^{\epsilon\sqrt{N}}\left[\varphi(0)+\frac{E}{\sqrt{N}}\varphi'(0)\right]e^{-\left(\frac{E}{\sigma_0}\right)^2}dE\Big/\int_{-\infty}^{\infty}\left[\varphi(0)+\left(\frac{E}{\sqrt{N}}\right)\varphi'(0)\right]e^{-\left(\frac{E}{\sigma_0}\right)^2}dE
$$

but, $\displaystyle\int_{\epsilon\sqrt{N}}^{\epsilon\sqrt{N}} Ee^{-\left(\frac{E}{\sigma_0}\right)^2}dE=\int_{-\infty}^{\infty} Ee^{-\left(\frac{E}{\sigma_0}\right)^2}dE=0$ (odd function)

and $\displaystyle\int_{-\infty}^{\infty} e^{-\left(\frac{E}{\sigma_0}\right)^2}dE=\sqrt{\pi}\sigma_0$

hence, $\displaystyle P=\frac{2}{\sqrt{\pi}\sigma_0}\int_0^{\epsilon\sqrt{N}} e^{-\left(\frac{E}{\sigma_0}\right)^2}dE$

Let $\sigma=\dfrac{\sigma_0}{\sqrt{2N}}$ and $E=\sqrt{N}t$

then $\displaystyle P=\frac{1}{\sqrt{2\pi}\sigma}\int_0^{\epsilon} e^{2\frac{t^2}{\sigma^2}}dt$ *This proves theorem (1.2)*

PART II

CASE WHERE y IS CONSTANT[1]

Theorem (2.1). If hypotheses (a) and (c) are satisfied and if $S_x, S_y(1-R)^2 \neq 0$ then,

$$
(1)\begin{cases}
(1.a) & \sigma_1 = S_x\sqrt{1+\bar{R}K_1^2} \\
(1.b) & \sigma_2 = S_y\sqrt{1+K_1^2} \\
(1.c) & r_0 = R\sqrt{(1+K_1^2)/(1+\bar{R}K_1^2)} \\
(1.d) & x_0 = X_y + \sigma_1 r_0\left[k + \frac{y}{Y}(k_1 - k)\right]
\end{cases}
$$

where σ_1, σ_2, r_0 and x_0 are the most probable values of σ_x, σ_y, r and x_y respectively, and,

$$
(2)\qquad K = \frac{S_y}{Y}, \quad K_1 = \frac{y-Y}{S_y}
$$
$$
k = \frac{\sigma_2}{y}, \quad k_1 = \frac{y-Y}{\sigma_2}
$$

Proof. The probability of getting N particular pairs of variates is given by (6) of Part I. Taking the partial derivatives of P_n with respect to σ_x, σ_y, r and x_y, and setting them equal to zero, we obtain,

$$
(3)\begin{cases}
(3.a) & \frac{\partial P_n}{\partial \sigma_x} = 2(1-r^2) + \sigma_x W'_{\sigma_x} = 0 \\
(3.b) & \frac{\partial P_n}{\partial \sigma_y} = 2(1-r^2) + \sigma_y W'_{\sigma_y} = 0 \\
(3.c) & \frac{\partial P_n}{\partial r} = 2(1-r^2) - 2rW - (1-r^2)W'_r = 0 \\
(3.d) & \frac{\partial P_n}{\partial x_y} = V = 0
\end{cases}
$$

[1] Case where all the parameters but y are unknown.

where W'_{σ_x}, W'_{σ_y} and W'_r mean the partial derivatives of W with respect to σ_x, σ_y and r respectively,

But,

$$\sigma_x W'_{\sigma_x} = 2V\sigma_x V'_{\sigma_x} - 2\left(\frac{S_x}{\sigma_x}\right)^2 + 2rR\frac{S_x S_y}{\sigma_x \sigma_y}$$

$$\sigma_y W'_{\sigma_y} = 2V\sigma_y V'_{\sigma_y} - 2\left(\frac{S_x}{\sigma_x}\right)^2 + 2rR\frac{S_x S_y}{\sigma_x \sigma_y} - 2(1-r^2)\left(\frac{y-Y}{\sigma_y}\right)^2$$

$$W'_r = 2VV'_r - 2R\frac{S_x S_y}{\sigma_x \sigma_y} - 2r\left(\frac{y-Y}{\sigma_y}\right)^2$$

since $V=0$, we obtain,

$$(3')\begin{cases} (3'.a) \quad (1-r^2) - \left(\frac{S_x}{\sigma_x}\right)^2 + rR\frac{S_x S_y}{\sigma_x \sigma_y} = 0 \\ (3'.b) \quad (1-r^2) - \left(\frac{S_y}{\sigma_y}\right)^2 + rR\frac{S_x S_y}{\sigma_x \sigma_y} - (1-r^2)\left(\frac{y-Y}{\sigma_y}\right)^2 = 0 \\ (3'.c) \quad r(1-r^2) - r\left[\left(\frac{S_x}{\sigma_x}\right)^2 + \left(\frac{S_y}{\sigma_y}\right)^2\right] + (1+r^2)R\left(\frac{S_x S_y}{\sigma_x \sigma_y}\right)^2 = 0 \end{cases}$$

Solving for σ_x, σ_y and r from $(3')$ and making use of the substitutions from (2) we get the most probable value of σ_x, σ_y and r,

$$(4)\begin{cases} (4.a) \quad \sigma_x = \sigma_1 = S_x\sqrt{1+\overline{R_1 K_1}^2} \\ (4.b) \quad \sigma_y = \sigma_2 = S_y\sqrt{1+K_1^2} \\ (4.c) \quad r = r_0 = R\sqrt{(1+K_1^2)/(1+\overline{RK_1}^2)} \\ \text{and from (3.d) we obtain,} \\ (4.d) \quad x_y = x_0 = X_y - \frac{RS_x S_y}{Y} + r_0\sigma_1(k+k_1) \end{cases}$$

Since $k_1 = \frac{k_1\sigma_2}{S_y}$ we get from (4.a), (4.b) and (4.c),

$$(4'.a) \qquad S_x = \sigma_1\sqrt{1-\overline{r_0 k_1}^2}$$

(4′.b) $$S_y = \sigma_2 \sqrt{1-k_1^2}$$

(4′.c) $$R = r_o \sqrt{(1-k_1^2)/(1-\bar{r}_o k^2)}$$

hence,

$$R S_x S_y = \sigma_1 \sigma_2 r_o (1-k_1^2)$$

Substituting the above value of $R\, S_x\, S_y$ in (4.d) we obtain,

(4′.d) $$x_y = x_o = X_y + \sigma_1 r_o \left[k + \frac{y}{Y}(k_1 - k)\right]$$

This proves theorem (2.1)

If we denote the maximum probability by P_{max} then,

(5) $$P_{max} = \left[\frac{1}{2\pi e \sigma_1 \sigma_2 \sqrt{1-r_o^2}}\right]^N$$

Theorem (2.2). If all four hypotheses are satisfied and if $S_x\, S_y(1-R^2) \neq 0$ then the *à posteriori* probability P that the sample came from the universe, the weighted average x_y of which satisfies the inequality $|\, x_y - X_y| \leq \epsilon$, can be expressed by,

(6)
$$P = \frac{2}{\sqrt{2\pi}\,\sigma_Y} \int_o^{\epsilon} e^{-\frac{t^2}{2\sigma_Y^2}}\, dt \qquad \text{where}$$

$$\sigma_Y = \frac{\sigma_1}{\sqrt{N}} \sqrt{k^2(1+r_o^2) + (1-r_o^2)\frac{(1+kk_1)^2}{(1-k_1)^2}}$$

Proof. Let $n_1 = 1-\bar{r}_o k_1^2,\ n_2 = 1-k_1^2 > 0$ then by substituting the values of S_x, S_y and R from (4′.a), (4′.b) and (4′.c) we get,

[2]In this case the function $F(x_y, y, \sigma_x, \sigma_y, r)$ in (d) is $F(x_y, \sigma_x, \sigma_y, r)$

$$(7)\begin{cases} P_n=\left(\dfrac{1}{\sigma_x\sigma_y 2\pi\sqrt{(1-r^2)}}\right)^{N} e^{-\frac{N}{2(1-r^2)}W} & \text{where} \\ W=V^2+n_1\left(\dfrac{\sigma_1}{\sigma_x}\right)^2+n_2\left(\dfrac{\sigma_2}{\sigma_y}\right)^2-2rr_0n_2\dfrac{\sigma_1\sigma_2}{\sigma_x\sigma_y}+(1-r^2)\left(\dfrac{k_1\sigma_2}{\sigma_y}\right)^2 \\ V=-\dfrac{x_y-x_0}{\sigma_x}-\dfrac{r_0\sigma_1}{\sigma_x}(k+k_1)+r\left(\dfrac{y-Y}{\sigma_y}+\dfrac{\sigma_y}{y}\right) & \text{and} \end{cases}$$

hence,

$$(8)\quad \frac{P_n}{P_{max}}=\left(e\frac{\sigma_1}{\sigma_x}\frac{\sigma_2}{\sigma_y}\sqrt{\frac{1-r_0^2}{1-r^2}}\right)^{N} e^{-\frac{N}{2(1-r^2)}W}$$

Taking the logarithm of $\frac{P}{P_{max}}$ and letting

$$\sigma_x=\sigma_1(1+\lambda'),\qquad r=(r_0+\rho)$$
$$\sigma_y=\sigma_2(1+\lambda''),\qquad x_y=x_0+\sigma_1 d'$$

we shall have,

$$(9)\quad \frac{1}{N}\log\frac{P_{max}}{P_n}=A \qquad \text{where}$$

$$A=Const.+\log(1+\lambda')+\log(1+\lambda'')+\tfrac{1}{2}\log\left[1-(r_0+\rho)^2\right]+\frac{1}{2\left[1-(r_0+\rho)^2\right]}W.$$

Expanding A in terms of the small quantities λ', λ'', d' and ρ we obtain,

$$(10)\begin{cases} \dfrac{1}{N}\log\dfrac{P_{max}}{P_n}=const.+A_1+A_2 \quad \text{where} \\ A_1=\dfrac{1}{2(1-r_0^2)}\Big\{d'^2+\left[2-r_0^2+r_0^2k(k+2k_1)\right]\lambda'^2+\left[2-r_0^2+r_0^2k(k-2k_1)\right]\lambda''^2 \\ \quad+\left[\dfrac{1+r_0^2}{1-r_0^2}+k(k+2k_1)\right]\rho^2-2r_0(k+k_1)d'd_r'-2r_0(k+k_1)d'\lambda'' \\ \quad-2(k+k_1)d'\rho-2r_0^2(1-k_1^2)\lambda'\lambda'' \\ \quad-2r_0\left[1-k(k+2k_1)\right]\lambda'\rho-2r_0(1-k_1^2)\lambda''\rho\Big\} \quad \text{and} \\ A_2=\sum\limits_{n=3}^{\infty}\dfrac{1}{n!}\left(\lambda'\dfrac{\partial A}{\partial\lambda'}+\lambda''\dfrac{\partial A}{\partial\lambda''}+\dfrac{d'\partial A}{\partial d'}+\dfrac{\rho\partial A}{\partial\rho}\right)^{(n)} \end{cases}$$

The expression representing the value of A_1 is quadratic in form in terms of the variables λ', λ'', d' and ρ where all the coefficients are positive,

$$
\begin{aligned}
A_1 = {} & \frac{2-r_0^2+r_0^2k(k+2k_1)}{2(1-r_0^2)}\left\{\lambda' - r_0\frac{(k_1+k)d'+r_0(1-k^2)d''+\left[1-k(k+2k_1)\right]\rho}{2-r_0^2+r_0^2k(k+2k_1)}\right\}^2 \\
& + \frac{4\left[1-r_0^2+r_0^2k(k+2k_1)\right]}{2(1-r_0^2)\left[2-r_0^2+r_0^2k(k+2k_1)\right]}\left\{\lambda'' - r_0\frac{\left[k(1-r_0^2k_1^2)-k_1(1-r_0^2)\right]d'+(1-k^2)\rho}{2\left[1-r_0^2+r_0^2k^2(1-r_0^2k_1^2)\right]}\right\}^2 \\
(11) \quad & + \frac{(1-r_0^2)(1+kk_1)^2+(1+r_0^2)k^2(1-k_1^2)}{2\left[1-r_0^2+r_0^2k(k+2k_1)\right]}\left\{\frac{\rho}{1-r_0^2} - \frac{\left[k+k_1-r_0^2k_1(1+kk_1)\right]d'}{(1-r_0^2)(1+kk_1)^2+(1+r_0^2)k^2(1-k_1^2)}\right\}^2 \\
& + \frac{d'^2(1-k_1^2)}{2\left[(1-r_0^2)(1+kk_1)^2+(1+r_0^2)k^2(1-k_1^2)\right]} \, .
\end{aligned}
$$

For the rest of the proof of this theorem we proceed as in Part I and can obtain,

$$
(12) \quad \sigma_Y = \frac{\sigma_1}{\sqrt{N}}\sqrt{k(1+r_0^2)+(1-r_0^2)\frac{(1+kk_1)^2}{(1-k_1)^2}} \, .
$$

Notice that if $y = Y$ *then,*

$$
K_1 = k_1 = 0 \, , \quad \sigma_1 = S_x \, , \quad \sigma_2 = S_x \, , \quad r_0 = R \quad \textit{and} \quad x_0 = X_y
$$

$$
(13) \qquad \sigma_Y = \sigma_{Y'} = \frac{S_x}{\sqrt{N}}\sqrt{1-R^2+k^2(1+R^2)}
$$

hence $\sigma_{Y'} < \sigma$ *if* $R \neq 0$ *where* σ *is given by (5) Part I.*

PART III

Case Where y and σ_y are Constants[3]

Theorem (3.1). If hypotheses (a) and (c) are satisfied and if $S_x S_y (1-R)^2 \neq 0$ then,

[3]Case where all the parameters but y, σ_y are unknown.

$$(1)\begin{cases} (1.a) & \sigma_1 = \dfrac{S_x}{S_y}\sqrt{R^2\sigma_y^2 + (1-R^2)S_y^2} \\ (1.b) & r_0 = R\sigma_y \Big/ \sqrt{R^2\sigma_y^2 + (1-R^2)S_y^2} \\ (1.c) & x_0 = X_y - \dfrac{RS_xS_y}{Y} + \dfrac{RS_xS_y}{S_y}\left(\dfrac{y-Y}{\sigma_y} + \dfrac{\sigma_y}{y}\right) = X + \dfrac{RS_xC}{a} \end{cases}$$

where σ_1, r_0 and x_0 are the most probable values of σ_x, r and x_y respectively and

$$\frac{S_y}{Y} = K; \quad \frac{S_y}{\sigma_y} = a; \quad \frac{y-Y}{\sigma_y} + \frac{\sigma_y}{y} = c$$

Theorem (3.2). If all four hypotheses[4] are satisfied and if $S_x S_y (1-R^2) \neq 0$ then the *à posteriori* probability P that the sample came from the universe, the weighted average x_y of which satisfies the inequality $|x_y - X_y| \leq \epsilon$, can be expressed by.

$$(2) \qquad P = \frac{2}{2\pi\sigma_{Y_1}}\int_0^{\epsilon} e^{-\frac{t^2}{2\sigma_{Y_1}^2}}\,dt \quad \text{where}$$

$$\sigma_{Y_1} = \frac{\sigma_1}{\sqrt{N}}\sqrt{(1-r_0^2)\left[1+\left(\frac{c}{a}\right)\right]}$$

Notice that if $y - Y$ *and* $\sigma_y = S_x$ *then,*

$$(3)\begin{cases} \sigma_1 = S_x, \quad r_0 = R; \quad x_0 = X_y & \text{and} \\ \sigma_{Y_1} = \sigma_{Y_1'} = \dfrac{S_x}{\sqrt{N}}\sqrt{(1-R^2)(1+k^2)} \end{cases}$$

hence $\sigma_{Y_1} < \sigma_{Y'}$ *if* $R \neq 0$
where $\sigma_{Y'}$ *is given by (12) Part II.*

As the proofs of theorems (3.1) and (3.2) do not differ from the proofs of theorems (2.1) and (2.2.), we shall omit them.*

[4]In this case the function $F(x_y, y, \sigma_x, \sigma_y, r)$ in (d) is $F(x_y, \sigma_x, r)$.

*Part I and II were presented in Wilno during the II Assembly of Polish Mathematicians.

PART IV

In this Part we shall consider the generalized case of Part I where there are k sets of elements characterized by pairs of variable quantities, $x_i^\ell, y_i^\ell \begin{Bmatrix} \ell = 1, 2, 3, \cdots\cdots, k \\ i = 1, 2, 3, \cdots\cdots, \infty \end{Bmatrix}$

$$\text{Let,} \qquad x = \frac{\sum_1^k x_y{}^\ell A_\ell}{\sum_1^k A_\ell} = \sum_1^k x_y{}^\ell \frac{A_\ell}{A}$$

where x_y^ℓ is the weighted average of the variates x_i^ℓ, with y_i^ℓ as weights, and A_ℓ the sum of these weights. Our problem is to obtain an expression for the probable precision of the quantity x according to certain hypotheses.

We shall replace hypothesis (b) of the introduction by hypotheses (b′) and (b″) where,

(b′) the number ($N = N_1 + N_2 + N_3 + \cdots\cdots + N_k$) of pairs in each sample is so large that $\frac{1}{N}$ may be neglected,

(b″) each of the numbers $N_\ell (\ell = 1, 2, 3, \cdots\cdots, k)$ of pairs from separate sets is so large that $\frac{N_\ell}{N}$ has a significant value, i.e.,

$$\frac{N_\ell}{N} \geq \omega_0 > 0$$

Let us replace in hypothesis (c) P_i by P_i^ℓ ($\ell = 1, 2, 3, \cdots, k$) and $x, y, \sigma_x, \sigma_y, r$ by $x^\ell, y^\ell, \sigma_x^\ell, \sigma_y^\ell, r_\ell$ and refer to the corresponding general hypothesis by (c′). Likewise if in hypothesis (d) we replace $F(x_y, y, \sigma_x, \sigma_y, r$ by $F(x_y^\ell, y^\ell, \sigma_x^\ell, \sigma_y^\ell, r_\ell)$ we obtain the generalized hypothesis (d′).

We shall denote the calculated characteristics of the sample by

$$X_y^\ell, X^\ell, Y^\ell, S_x^\ell, S_y^\ell, R_\ell \; (i = 1, 2, 3, \cdots; N_\ell)$$

corresponding to the values X_y, X, Y, S_x, S_y, R as defined in the introduction page 197.

Theorem (4.1). If hypotheses (a) and (c$'$) are satisfied and if $(1-R)S_x^\ell S_y^\ell \neq 0$ then the most probable value of x is X where,

$$X = \sum_1^k X_y^\ell \frac{A_\ell}{A}$$

*Proof**. Let P_n be the probability of getting a given set of N pairs of variates x_i^ℓ, y_i^ℓ, then it follows from hypotheses (a) and (c$'$) that,

$$P_n = \prod_1^k \frac{e^{-\frac{N_\ell}{2(1-r_\ell^2)} W_\ell}}{\left(2\pi\sigma_x^2\sigma_y^2\sqrt{1-r_\ell^2}\right)^{N_\ell}} \qquad \text{, where}$$

(1)

$$W_\ell = V_\ell^2 + \left(\frac{S_x^\ell}{\sigma_x^\ell}\right)^2 + \left(\frac{S_y^\ell}{\sigma_y^\ell}\right)^2 - 2r_\ell R_\ell \frac{S_x^\ell S_y^\ell}{\sigma_x^\ell \sigma_y^\ell} + (1-r_\ell^2)\left(\frac{y^\ell - Y^\ell}{\sigma_y^2}\right)^2 \qquad \text{, and}$$

$$V_\ell = -\frac{S_x^\ell}{\sigma_x^\ell}\left(\frac{x_y^\ell - x_y}{S_x^\ell} + \frac{R_\ell S_y^\ell}{Y^\ell}\right) + r_\ell\left(\frac{y^\ell - Y^\ell}{\sigma_y^\ell} + \frac{\sigma_y{}^\ell}{y^\ell}\right)$$

*The proofs of the theorems (4.1) and (4.2) shall be given in very abbreviated form as the method of proofs of these theorems does not differ from the proofs of theorem (2.1) and (2.2) of Part I.

Let,

$$
(2)\qquad \left.\begin{array}{l}
x - X = D\,;\quad x_y^{\ell} - X_y^{\ell} = S_x^{\ell} d_{\ell}\,; \\[2ex]
y^{\ell} - Y^{\ell} = S_y^{\ell} \delta_{\ell}\,;\quad \sigma_x^{\ell} = S_x^{\ell}(1+\lambda_{\ell}');\\[2ex]
\sigma_y^{\ell} = S_y^{\ell}(1+\lambda_{\ell}'');\quad r_{\ell} = R_{\ell}(1+\rho_{\ell})\\[2ex]
S_x^{\ell}\dfrac{A_{\ell}}{A} = \alpha_{\ell}\,;\quad \dfrac{S_y}{Y_{\ell}} = K_{\ell}
\end{array}\right\}\ \ell = 1, 2, \cdots k.
$$

then,

$$
(3)\qquad D = \sum_{1}^{k} \alpha_{\ell}\, d_{\ell}
$$

and we can also express the unknown quantity $d_{\ell} = (\ell = 1, 2, \cdots, k)$

in terms of D and the independent variable $\eta_{\ell}\ (\ell = 1, 2, 3, \cdots\cdots, k-1)$ as follows,

$$
(4)\qquad \begin{array}{l}
d_{\ell} = \dfrac{1}{\alpha_{\ell}}\left(\dfrac{D}{k} - \eta_{\ell}\right);\quad \ell = 1, 2, \cdots, k-1\,;\\[2ex]
d_k = \dfrac{1}{\alpha_k}\left(\dfrac{D}{k} + \sum_{1}^{k-1} \eta_{\ell}\right)
\end{array}
$$

Hence it follows from (I) that,

$$P_n = \prod_{\ell=1}^{k} \frac{e^{-\frac{N_\ell W_\ell}{2[1-R_1^2(1+\rho_\ell)^2]}}}{\left[(2\pi S_x^\ell S_y^\ell (1+\lambda'_\ell)(1+\lambda''_\ell)\sqrt{1-R_\ell^2(1+\rho_\ell)^2})\right]^{N_\ell}} \quad \text{where}$$

(5)
$$W_\ell = V_\ell^2 + \frac{1}{(1+\lambda'_\ell)^2} + \frac{1}{(1+\lambda''_\ell)^2} - 2R_\ell^2 \frac{1+\rho_\ell}{(1+\lambda'_\ell)(1+\lambda''_\ell)} + \left[1-R_1^2(1+\rho_\ell)^2\right]\left(\frac{\delta_\ell}{1+\lambda''_\ell}\right)^2$$

and

$$V_\ell = -\frac{1}{1+\lambda'_\ell}(d_\ell + R_\ell K_\ell) + R_\ell(1+\rho_\ell)\left(\frac{\delta_\ell}{1+\lambda''_\ell} + \frac{1+\lambda''_\ell}{\delta_\ell + K_\ell}\right)$$

where d_ℓ are to be found from the equations (3) and $(\ell = 1, 2 \cdots k)$

Taking the partial derivitives of P_n with respect to D and η_ℓ we obtain,

(6)
$$\frac{\partial P_n}{\partial D} = \frac{1}{k}\sum_{1}^{k} \frac{1}{\alpha_k} \cdot \frac{\partial P_n}{\partial d_\ell}$$

$$\frac{\partial P_n}{\partial \eta_\ell} = \frac{1}{\alpha_k}\frac{\partial P_n}{\partial d_k} - \frac{1}{\alpha_\ell} \cdot \frac{\partial P_n}{\partial d_\ell} \cdot \quad \ell = 1, 2, \cdots k-1$$

It can be easily vertified that $\frac{\partial P_n}{\partial D} = \frac{\partial P_n}{\partial \eta_\ell} = 0$ if and only if

$$\frac{\partial P_n}{\partial d_\ell} = 0$$

The probability P_n treated as the function of variables

$D, \eta_1, \cdots \eta_{k-1}, \delta_1, \cdots \delta_k, \lambda'_1, \cdots \lambda'_k, \lambda''_2 \cdots \lambda''_k, \rho_1, \cdots \rho_k$, is a maximum when,

$$D = \eta_1 = \cdots \eta_{k-1} = \delta_\ell = \lambda'_\ell = \lambda^2_\ell = \rho_\ell = 0, (\ell = 1, 2, \cdots k)$$

This proves theorem (4.1).

Theorem (4.2). If all hypotheses are satisfied and if then the *a' posteriori* probability that the sample came from the universe, the quantity x of which satisfies the inequality $|x-X| \leq \epsilon$ may be expressed by,

$$P = \frac{2}{2\pi\sigma} \int_0^{\epsilon} e^{-\frac{t^2}{2\sigma^2}} dt, \qquad \text{where}$$

(7)
$$\sigma = \sqrt{\sum_1^k \left(s_x^\ell \frac{A_\ell}{A}\right)^2 \frac{\Phi_\ell}{N_\ell}}, \qquad \text{and}$$

$$\Phi_\ell = 1 + \left(\frac{s_y^\ell}{Y^\ell}\right)^2 \left\{1 - R_\ell^2\left[1 - \left(\frac{s_y^\ell}{Y^\ell}\right)^2\right]\right\}, (\ell = 1, 2, \cdots k)$$

Proof. Let P_{max} denote the maximum probability, then it follows from (6) that,

(8)
$$P_{max} = e^{-N} \prod_1^k \frac{1}{\left(2\pi s_x^\ell s_y^\ell \sqrt{1-R_\ell^2}\right)^{N_\ell}} \qquad \text{and}$$

$$\frac{P_n}{P_{max}} = e^{N} \prod_1^k \left[\frac{\sqrt{1-R_\ell^2}}{(1+\lambda_\ell')(1+\lambda_\ell'')\sqrt{1-R_\ell^2(1+\rho_\ell)^2}}\right]^{N_\ell} e^{-\frac{n_\ell W_\ell}{2[1-R_\ell^2(1+\rho_\ell)^2]}}$$

$$= e^{N} \prod_1^k \left[\frac{\sqrt{1-R_\ell^2}}{(1+\lambda_\ell')(1+\lambda_\ell'')\sqrt{1-R_\ell^2(1+\rho_i)^2}}\right]^{Nw_\ell} e^{-\frac{Nw_\ell W_\ell}{2[1-R_\ell^2(1+\rho_\ell)^2]}}$$

where the value of W_ℓ given by (5) and $w_\ell = \frac{N_\ell}{n}$

As in Part I or Part II if we expand the $\log \frac{P_n}{P_{max}}$ in terms of $D, \eta_1, \cdots \eta_{k-1}, \delta_1 \cdots \delta_k; \lambda_1' \cdots \lambda_1^k, \lambda_2' \cdots \lambda_2^k, \rho_1 \cdots \rho_k$, the first term that does not vanish is quadratic in form in terms of the variables,

$D, \eta_1, \cdots \eta_{k-1}; \delta_1, \cdots \delta_k; \lambda_1', \cdots \lambda_1^k; \lambda_2', \cdots \lambda_2^k; \rho_1, \cdots \rho_k;$

and this in turn by linear transformation can be expressed as,

$$
(9)\left\{
\begin{array}{ll}
N(C_1 D^2 + C_1 \bar{\eta}_1^2 + \cdots + C_{5k} \bar{\rho}_k^2); & \rho_\ell > 0 \; (\ell = 1, 2, 3, 4, 5k) \\[2ex]
C_1 = \dfrac{1 \cdot}{\sum_1^k \dfrac{\alpha_\ell^2 \varphi_\ell}{\omega_\ell}} & \text{when} \\[2ex]
\varphi_\ell = 1 + K_\ell^2 \left[1 - R_\ell^2 (1 - k_\ell^2 \right] & \text{and}
\end{array}
\right.
$$

To complete the proof we proceed as in Part I.

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