ON SAMPLING FROM COMPOUND POPULATIONS*

By
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Introduction.

The decided asymmetry or the multimodality of certain frequency distributions may have prompted the idea of the possibility of the existence of frequency curves, apparently single in character, but which, on further investigation, might be shown to be actually composite. In other words, apparently homogeneous material may prove to be heterogeneous, or divisible into two or more distinct homogeneous groups.

The above ideas lead naturally to the problem of dissecting a compound frequency function into its various components. Karl Pearson[1] successfully solved such a problem, using the method of moments, on the assumption that the compound parent population was composed of two normal components. Each component curve has three parameters, the mean (or position of axis), the standard deviation, and the area (or total frequency). One requires therefore, six relations between the parameters of the given compound frequency curve, and those of its two components, in order to determine six unknowns. The ultimate solution of the problem turns on the determination of the zeros of a nonic equation, the location of whose real roots is obtained, to successive approximations, by means of the so-called Sturm's functions.

The dissection problem was taken up later, first in a paper by Charlier,[2] then in a joint paper by Charlier and Wicksell[3] who

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[1] On the dissection of frequency curves into normal curves. Karl Pearson. Phil. Trans. Roy. Soc. Lond. Vol. 185, Pt. 1, pp. 71-110. 1894A.

[2] Researches into the theory of probability. C. V. L. Charlier. Meddelanden frau Lunds Astron. Observ. Sec. 2. Bd 1. 1906.

[3] On the dissection of frequency functions. C. V. L. Charlier, and S. D Wicksell. Arkiv. fur Matematik. Astron. och Fysik. (Meddelande) Band 18. No. 6. 1923.

considerably simplified the theory, finally arriving, however, at the fundamental nonic due to Pearson, for the solution of which, they suggested the use of a graphical method. They also studied special cases of the more general problem, e.g. the means of the two components assumed known, the compound curve assumed symmetrical, or the standard deviations of the two components supposed equal. In addition, they extended the problem to the case of frequency functions of two variates.

In the present paper, I propose to investigate the sampling problem in the case of compound distribution functions, and from a consideration of the dissection problem, one is led to a division of the present investigation into two main parts, for the following reasons.

On the one hand, in sampling from a compound population, if we do not know the proportion contributed to the total frequency of the sample by each of the two components of the parent population, we are essentially sampling from a single population. That is, random samples of N are drawn from a single composite parent population made up of two components. Hence, the previously obtained results for sampling from a single parent population will be available if we derive expressions for the parameters of the compound parent in terms of the parameters of its components. This is done in Part 1.

On the other hand, however, if we know the proportion contributed to the total frequency of a sample by each of the two components, the situation differs entirely from that studied in Part 1. Here we are concerned with sampling from two distinct parent populations, and in Part 2, I develop a method for dealing with this problem. Thus, in Part 2, it is assumed that samples of r and s respectively are drawn from two distinct parent populations, and these two samples are then combined to yield a sample of $r+s=N$ from the combined populations.

Therefore in Part 1, we are essentially sampling from a single parent population, whereas in Part 2, we are sampling from multiple populations.

The developments of Part 2 yield some new sampling results for sampling from two parent populations. In Section 6, I derive expressions for the semi-invariants "of moments about a fixed point" in samples from the compound frequency function, in terms of the corresponding semi-invariants of the moments of its components. In Section 7, expressions are derived for the semi-invariants of "moments in samples from the compound population about the mean of the combined sample," in terms of r and s, and the semi-invariants of the two components themselves.

The occurrence of a certain class of well-known polynomials in the development of Section 1, is of especial interest, since these are, except perhaps for sign, the semi-invariants of the binomial distribution, and have some rather important properties, and their further study, although not pertinent to the problem in hand, should yield some very interesting results.

Section 5 is devoted to the discussion of the case in which a limiting compound frequency function exists, under certain assumptions regarding the nature of its components, where the number of the latter is allowed to increase indefinitely. This idea of a limit frequency function would appear to indicate the possibility of a new approach to the theory of frequency curves, in which the variable may now be a complete frequency distribution in itself.

This investigation was begun on the suggestion of Professor C. C. Craig, of the University of Michigan, U. S. A. to whom I am indebted for constant inspiration and guidance during its pursuit.

PART 1.

Section 1. The semi-invariants of the compound frequency function, in terms of the semi-invariants of its two normal components.

The main object of this first section, is to obtain expressions for the parameters of a compound population in terms of the parameters of its two normal components, and to this end, I shall use the following definition of the semi-invariants of Thiele.[1]

$$e^{\lambda_1 t+\lambda_2 \frac{t^2}{2!}+\lambda_3 \frac{t^3}{3!}+\cdots}=\int_{-\infty}^{\infty} f(x)\cdot e^{xt}\cdot dx.$$

I write therefore

$$f(x)=p\cdot \varphi_1(x)+q\,\varphi_2(x).$$

in which $f(x)$ is the compound frequency function, $\varphi_1(x), \varphi_2(x)$ are its two normal components, and $p+q=1$.

If L_1, L_2, etc. are the semi-invariants of $f(x)$, then

$$(1)\ e^{L_1 t+L_2\frac{t^2}{2!}+L_3\frac{t^3}{3!}+\cdots}=pe^{m_1 t+\sigma_1^2\frac{t^2}{2!}}\left\{1+\frac{q}{p}\cdot e^{(m_2-m_1)t+(\sigma_2^2-\sigma_1^2)\frac{t^2}{2!}}\right\}$$

where m_1, m_2, σ_1, σ_2, are the means and standard deviations of $\varphi_1(x), \varphi_2(x)$ respectively. For convenience, I write

$$m_2-m_1=a. \qquad \sigma_2^2-\sigma_1^2=b. \qquad \frac{q}{p}=r.$$

We wish to express the L_n in (1) in terms of the quantities m_1, m_2, σ_1, σ_2, p, or q.

Taking logarithms in (1).

[1] Numerous references relating to the theory of semi-invariants may be found at the end of "An application of Thiele's semi-invariants to the sampling problem". C. C. Craig. Metron. Vol. 7. No. 4. 1928, p. 73.

$$
\begin{aligned}
(2)\qquad & L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots \\
& = \log p + \left(m_1 t + \sigma_1^2 \frac{t^2}{2!}\right) + \log\left(1 + r e^{at + \frac{bt^2}{2!}}\right).
\end{aligned}
$$

We require now a suitable form for the expansion of the third term of the right member of (2) in successive powers of t . We have

$$
\log\left(1 + r e^{at + \frac{bt^2}{2}}\right) = \log(1+r) + \log\left\{1 + q\left(e^{at + \frac{bt^2}{2}} - 1\right)\right\}.
$$

Further

$$
(3)\qquad \log\left(1 + r e^{at + \frac{bt^2}{2}}\right) = \sum_{j=1}^{\infty} q^j \left(e^{at + \frac{bt^2}{2}} - 1\right)^j (-1)^{j+1}.
$$

The complete representation of terms of the type $e^{i(at + \frac{bt^2}{2})}$ in the right member of (3) will be

$$
\sum_{k=1}^{\infty} \sum_{i=0}^{k} \frac{(-1)^{1+i} . q^k}{k} \binom{k}{i} e^{i\left(at + \frac{bt^2}{2}\right)}.
$$

But

$$
(4)\qquad e^{i\left(at + \frac{bt^2}{2}\right)} = \sum_{j=0}^{\infty} \frac{i^j}{j!} \left(at + \frac{bt^2}{2}\right)^j.
$$

Therefore, the coefficient of $\frac{t^n}{n!}$ in the right member of (4) is

$$\sum_{j=\left[\frac{n+1}{2}\right]}^{n} \frac{n!\, a^{2j-n} \cdot b^{n-j} \cdot i^j}{2^{n-j} \cdot (2j-n)!(n-j)!}$$

where $\left[\frac{n+1}{2}\right]$ means the largest integer in $\frac{n+1}{2}$.

Then, the coefficient of $\frac{t^n}{n!}$ in the right member of (3) is

$$(5)\quad L_n = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \cdot \binom{k}{i} \sum_{j=\left[\frac{n+1}{2}\right]}^{n} \frac{n!\, a^{2j-n} \cdot b^{n-j} i^j}{2^{n-j} \cdot (2j-n)!(n-j)!}\;; n>2$$

and this is the relation sought, in which the semi-invariants of the compound frequency function are expressed in terms of the semi-invariants of its two normal components. Below, I have written out in detail the expressions for L_1 to L_9 inclusive.

$$L_1 = m_1 + aq.$$

$$L_2 = \sigma_1^2 + q\left[a^2(1-q)+b\right].$$

$$L_3 = a^3pqq_3 + 3abpq.$$

$$(6)\quad L_4 = a^4pqq_4 + 6a^2bpqq_3 + 3b^2pq.$$

$$L_5 = a^5pqq_5 + 10a^3bpqq_4 + 15ab^2pqq_3.$$

$$L_6 = a^6pqq_6 + 15a^4bpqq_5 + 45a^2b^2pqq_4 + 15b^3pqq_3.$$

$$L_7 = a^7pqq_7 + 21a^5bpqq_6 + 105a^3b^2pqq_5 + 105ab^3pqq_4.$$

$$L_8 = a^8 pqq_8 + 28a^6 bpqq_7 + 210a^4 b^2 pqq_6 + 420a^2 b^3 pqq_5 + 105 b^4 pqq_4.$$

$$L_9 = a^9 pqq_9 + 36a^7 bpqq_8 + 378a^5 b^2 pqq_7 + 1260a^3 b^3 pqq_6 + 945ab^4 pqq_5.$$

in which

$$
(7) \quad
\begin{aligned}
q_3 &= 1-2q. \\
q_4 &= 1-6q+6q^2. \\
q_5 &= 1-14q+36q^2-24q^3. \\
q_6 &= 1-30q+150q^2-240q^3+120q^4. \\
q_7 &= 1-62q+540q^2-1560q^3+1800q^4-720q^5. \\
q_8 &= 1-126q+1806q^2-8400q^3+16800q^4-15120q^5+5040q^6. \\
q_9 &= 1-254q+5796q^2-40824q^3+126000q^4-191520q^5 \\
&\quad +141120q^6-40320q^7.
\end{aligned}
$$

The expressions for the L_n in (6) have two properties, which enable one to write them down readily. In the first place, assuming that the polynomials in q (or $p = 1-q$) are suppressed, i.e. $q_2' = q_2 = pq$, $q_3' = pqq_3$, $q_4' = pqq_4$, etc., are all set equal to unity, then the resulting functions in "a" and "b" are readily obtained by means of a well-known recursion formula. Secondly, considering the polynomials q'_k as coefficients in the several terms of the original complete expressions for the L_n in (6), for $n>2$, and arranging these expressions so that their corresponding terms appear in columns, the first terms in the first column, the second terms in the second column, and so on, then every term in any diagonal array proceeding from upper left to lower right, and consisting of one and only one term from each of the expressions (6), will have the same polynomial coefficient.

I proceed now to obtain expressions for the L_n in (6), in which the individuality of the polynomials q'_2, q'_3, q'_4, etc., has been suppressed. This time I write

$$\log\left(1+re^{at+\frac{bt^2}{2}}\right) = \log\left\{1+r\sum_{k=0}^{\infty}\frac{1}{k!}\left(at+\frac{bt^2}{2}\right)^k\right\}. \tag{8}$$

The term in t^s in $\frac{t^k}{k!}\left(a+\frac{bt}{2}\right)^k$ is

$$\frac{t^s\cdot a^{2k-s}\cdot b^{s-k}}{(2k-s)!(s-k)!\,2^{s-k}}.$$

Rearranging the series in brace of right hand member of (8) in successive powers of t,

$$\log\left(1+re^{at+\frac{bt^2}{2}}\right) = \log\left\{1+r\sum_{s=0}^{\infty}\sum_{k=\left[\frac{s+1}{2}\right]}^{s}\frac{a^{2k-s}\cdot b^{s-k}\cdot t^s}{2^{s-k}(2k-s)!\,(s-k)!}\right\}$$

(9)

$$= \log(1+r)+\log\left\{1+\theta_1 t+\frac{\theta_2 t^2}{2!}+\cdots+\frac{\theta_s t^s}{s!}+\cdots\right\}.$$

where $\theta_i = q\beta_i$ $\qquad i = 1, 2, 3, \ldots\ldots$

Therefore, the coefficient of $\frac{t^s}{s!}$ in the series in the brace in the right member of (9) is θ_s, where

$$\theta_s = q \sum_{k=\left[\frac{s+1}{2}\right]}^{s} \frac{s!\, a^{2k-s} \cdot b^{s-k}}{2^{s-k} \cdot (2k-s)!\,(s-k)!}. \tag{10}$$

From equation (2) and (9) above, we have

$$\begin{aligned} & L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots\cdots \\ & = \log p + \log(1+r) + \left(m_1 t + \frac{\sigma_1^2 t^2}{2!}\right) + \log\left(1 + \theta_1 t + \frac{\theta_2 t^2}{2!} + \cdots\right). \end{aligned} \tag{11}$$

Therefore, equation (11) becomes

$$L_1 t + L_2 \frac{t^2}{2!} + \cdots = \left(m_1 t + \sigma_1^2 \frac{t^2}{2!}\right) + \log\left(1 + \theta_1 t + \theta_2 \frac{t^2}{2!} + \cdots\right). \tag{12}$$

One might note, in passing, that in (12), the θ's are playing the rôle of moments, if one recalls the definition of semi-invariants, so that it would be possible to write down a second general expression for the L's, using the well-known formula for semi-invariants in terms of moments.

The first six θ's take the following form

$$\theta_1 = qa$$

$$\theta_2 = q(a^2 + b)$$

$$\theta_3 = q(a^3 + 3ab).$$

$$\theta_4 = q(a^4 + 6a^2b + 3b^2)$$

$$\theta_5 = q(a^5 + 10a^3b + 15ab^2)$$

$$\theta_6 = q(a^6 + 15a^4b + 45a^2b^2 + 15b^3).$$

If, in the last set of relations, we set $q = 1$, we then have $\theta_s = \beta_s$, and if in the expressions (6), q'_2, q'_3, q'_4, etc., be all set equal to unity, we shall get, for $n > 2$,

$$L_n = \beta_n .$$

I shall now show that the β's follow the recursion law

$$\beta_{s+1} = \left(a + b\frac{\partial}{\partial a}\right)\beta_s . \tag{13}$$

Now, putting $at + \frac{bt^2}{2} = \Phi(t)$, we have

$$\frac{d}{dt} \cdot e^{\Phi(t)} = (a + bt) \cdot e^{\Phi(t)}$$

$$\frac{d^2}{dt^2} \cdot e^{\Phi(t)} = \left[b + (a+bt)^2\right] \cdot e^{\Phi(t)}$$

$$\frac{d^3}{dt^3} \cdot e^{\Phi(t)} = \left[3b(a+bt) + (a+bt)^3\right] e^{\Phi(t)}.$$

and in general

$$\frac{d^s}{dt^s} \cdot e^{\Phi(t)} = P_s(b, a+bt) . e^{\Phi(t)} \tag{14}$$

where $P_s(x,y)$ is a polynomial of degree s in x and y.
It is easily shown that

$$e^{\Phi(t)} \cdot \frac{d}{dt}\left\{P_s(b, a+bt)\right\}\Bigg|_{t=0} = b \cdot \frac{\partial}{\partial a}\left\{P_s(b, a+bt) \cdot e^{\Phi(t)}\Bigg|_{t=0}\right\}. \tag{15}$$

and that

$$(16) \qquad P_s \cdot \frac{d}{dt} \cdot e^{\phi(t)} \Big|_{t=0} = a \cdot P_s \cdot e^{\phi(t)} \Big|_{t=0}$$

Now deriving the left member of (14) with respect to t, and then setting $t=0$, gives the next β, namely β_{s+1}, by definition, whilst the derivative of the right member of (14), and setting $t=0$, would equal the sum of the right members of (15) and (16), which establishes the proof.

The second property of the expressions for the L_n in (6), which requires proof, may be stated as a theorem thus—"The k-th polynomial coefficient in the expression for the semi-invariant L_{2m}, $m>1$, is identically equal to the $(k+1)$st polynomial coefficient in the expression for the semi-invariant L_{2m+1}".

For simplicity, I have considered the first and second polynomial coefficients of L_{2m} and L_{2m+1} respectively, the proof going through in exactly the same manner if perfectly general terms in these expressions were considered.

From (5), suppose that $n=2m$ (even). Then

$$(17) \quad L_{2m} = \sum_{k=1}^{2m} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \binom{k}{i} \sum_{j=m}^{2m} \frac{(2m)!\, a^{2j-2m} \cdot b^{2m-j} \cdot i^j}{2^{2m-j} \cdot (2j-2m)!\,(2m-j)!}$$

The leading term in L_{2m}, i.e. the first term in β_{2m}, multiplied by a polynomial in q, is obtained from (17), by setting $j=2m$. It is

$$\sum_{k=1}^{2m} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \binom{k}{i} \cdot a^{2m} \cdot i^{2m} .$$

Therefore the polynomial coefficient of a^{2m} i.e. the leading coefficient in L_{2m} is

$$(18) \qquad \sum_{k=1}^{2m} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \cdot \binom{k}{i} \cdot i^{2m}$$

Again, from (5), when $n = 2m+1$ (odd),

$$L_{2m+1} = \sum_{k=1}^{2m+1} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \binom{k}{i} \sum_{j=m+1}^{2m+1} \frac{(2m+1)!\, a^{2j-(2m+1)} \cdot b^{(2m+1)-j} \cdot i^j}{2^{(2m+1)-j} \cdot [2j-(2m+1)]!\,[2m+1-j]!}.$$

The second term in L_{2m+1} is obtained by setting $j = 2m$. It is

$$\sum_{k=1}^{2m+1} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \cdot \binom{k}{i} \cdot m(2m+1) \cdot a^{2m-1} \cdot b \cdot i^{2m}.$$

Therefore, the polynomial coefficient of $a^{2m-1} \cdot b$ i.e. the second coefficient in L_{2m+1} is

$$(18') \qquad \sum_{k=1}^{2m+1} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \cdot \binom{k}{i} \cdot i^{2m}.$$

Comparing (18) and (18′), which must be identically equal, if our theorem is true, it remains to show that

$$\frac{q^{2m+1}}{2m+1} \cdot \sum_{i=1}^{2m+1} (-1)^{1+i} \cdot \binom{2m+1}{i} \cdot i^{2m} = 0.$$

but it is well known that this expression is identically zero.[2]

[2] See Hall and Knight. Higher Algebra, p. 259, Ex. 2.

Section 2. A table of values of a certain class of polynomials in one variable, for different values of the argument.

In order to facilitate the actual computation of the values for the semi-invariants L_n, given in Section 1, in a particular application of the theory, when $a = m_2 - m_1$, and $b = \sigma_2^2 - \sigma_1^2$, are known, I consider the expressions for the L_n, as they appear in the form indicated in (6). Now, when $b = \sigma_2^2 - \sigma_1^2 = 0$, i.e. the two components have identical standard deviations, the set of relations (6) take the form

$$(19)\quad L'_1 = m_1 + aq \cdot L'_2 = \sigma_1^2 + a^2 pq \text{ and } L'_n = a^n pqq_n \; ; \; n > 2$$

in which q_3, q_4, etc., have the same significance as in (7). Making use of the properties of the expressions (6), which were stated at the end of Section 1, from (19) we may write the L_n, for $n > 4$, as follows

$$
(20)\quad
\begin{aligned}
L_5 &= L'_5 + 10\left(\frac{b}{a}\right)L'_4 + 15\left(\frac{b}{a}\right)^2 L'_3 .\\
L_6 &= L'_6 + 15\left(\frac{b}{a}\right)L'_5 + 45\left(\frac{b}{a}\right)^2 L'_4 + 15\left(\frac{b}{a}\right)^3 L'_3 .\\
L_7 &= L'_7 + 21\left(\frac{b}{a}\right)L'_6 + 105\left(\frac{b}{a}\right)^2 L'_5 + 105\left(\frac{b}{a}\right)^3 L'_4 .\\
L_8 &= L'_8 + 28\left(\frac{b}{a}\right)L'_7 + 210\left(\frac{b}{a}\right)^2 L'_6 + 420\left(\frac{b}{a}\right)^3 L'_5 + 105\left(\frac{b}{a}\right)^4 L'_4 .\\
L_9 &= L'_9 + 36\left(\frac{b}{a}\right)L'_8 + 378\left(\frac{b}{a}\right)^2 L'_7 + 1260\left(\frac{b}{a}\right)^3 L'_6 + 945\left(\frac{b}{a}\right)^4 L'_5 .
\end{aligned}
$$

and so on.

Therefore, for $n > 4$, the general semi-invariants L_n of (6) may be expressed in terms of the special semi-invariants L'_n, obtained from the former by setting $b=0$. From (20) in general we have

$$L_n = \sum_{k=0}^{n} L'_{n-k} \; \frac{n^{(2k)}}{2^k \cdot k!} \cdot \left(\frac{b}{a}\right)^k,$$

in which

$$L'_{n-k} = a^{n-k} pq q_{n-k} ,$$

because

$$L_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} a^{n-k} pq q_{n-k} \cdot \frac{n^{(2k)}}{2^k . k!} \left(\frac{b}{a}\right)^k$$

$$= \sum_{k=0}^{\left[\frac{n-1}{2}\right]} a^{n-2k} \cdot b^k \cdot \frac{n^{(2k)}}{2^k . k!} \; q_{n-k} ,$$

and the last expression, for $n>4$, is the equivalent of the general expression (5) for the L_n.

Further, if we consider the terms in the expressions (20) as elements in a set of diagonal arrays, as I have indicated, it is evident that, moving downwards along any particular diagonal, any term in this diagonal is obtained from the one immediately preceding it, by the use of a multiplier $M_{k\ell}\left(\frac{b}{a}\right)$. The formula for the calculation of the $M_{k\ell}$, may be derived as follows.

Consider any term of say L_n, whose numerical coefficient is $C_{k+1,n}$. Let this be the $(k+1)$st term.

Then

$$C_{k+1,n} = \frac{n^{(2k)}}{2^k . k!}$$

Similarly, take the $(k+2)$nd term of L_{n+1}, with numerical coefficient $C_{k+2,n+1}$. Then

$$C_{k+2,n+1} = \frac{(n+1)^{(2k+2)}}{2^{k+1}\cdot(k+1)!},$$

and we note that $C_{k+1,n}$ and $C_{k+2,n+1}$ are the numerical coefficients of two adjacent terms in one of the diagonal arrays mentioned above. Therefore

$$C_{k+2,n+1} = \frac{(n+1)(n-2k)}{2(k+1)}\cdot C_{k+1,n} \qquad n \geqq 2k$$

and

$$M_{k+1,n} = \frac{(n+1)(n-2k)}{2(k+1)}$$

It is of considerable interest to note, that the L'_n of (19) (for $n>2$), are, except perhaps for sign, the product of the semi-invariants λ_n $(n>1)$ of the binomial distribution and appropriate powers of "a". To show this, we need only consider the generating function for the L'_n, namely $1+q(e^{at}-1)$, and that for the λ_n, viz. $\left[1+p(e^t-1)\right]^s$, with $s=1$. Frisch[1] has obtained a recursion formula for the λ_n, which is

$$\lambda_n = pq\cdot\frac{d}{dp}\cdot\lambda_{n-1}, \qquad n>1$$

[1] Sur les semi-invariants et moments employés dans l'étude des distributions statistiques. Ragnar Frisch. Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo. 1926. No. 3, Ch. 2, p. 29.

Therefore, it is evident that the L'_n obey a corresponding recursion formula

$$L'_n = a^n \cdot pq \cdot \frac{d}{dq} \cdot L'_{n-1} \qquad n > 2$$

In fact, the polynomials $q'_2 = pq$, $q'_3 = pqq_3$, $q'_4 = pqq_4$, etc., are the same functions of q as the λ_n are functions of p, i.e.

$$\lambda_n = \lambda_n(p). \;,\; q'_n = \lambda_n(q). \qquad \text{for } n \geq 2.$$

To investigate thoroughly the properties of the polynomials[2] q'_2, q'_3, q'_4, etc., would be irrelevant to the problem in hand, but, so far as I know, such a study has not been carried out. I will, however, mention a few of these properties, which appear interesting.

1. The roots of the polynomial q'_n (for any $n \geq 2$) are all real and distinct, and these roots all lie in the interval (0, 1), zero and unity being roots of every polynomial.
2. The roots of q'_n separate the roots of q'_{n+1}
3. The polynomials q'_{2n}, of even degree in q, are symmetrical with respect to the line $x = \frac{1}{2}$, whilst those of odd degree in q, namely q'_{2n+1}, are symmetrical with respect to the point (½, 0)
4. An orthogonality property in (0, 1) holds if $m \neq n$, and $m+n$ is odd. That is

$$\int_0^1 q'_m q'_n = 0. \qquad m \neq n, \; m+n \text{(odd)}$$

but $$\int_0^1 q'_m q'_n \neq 0. \qquad m \neq n, \; m+n \text{(even)}$$

[2] These same polynomials appear as functions of p in a paper by H. C. Carver on "Fundamentals of the theory of Sampling." Amer. Statist. Assoc. Annals of Math. Statistics. Vol. I, No. 1, Feb. 1930, p. 106.

5. Further

$$\int_0^1 q'_{2n} \cdot dq = (-1)^{n-1} \cdot B_{2n}$$

in which B_{2n} is the Bernoulli number of order $2n$.

$$\int_0^1 q'_{2n+1} \cdot dq = 0.$$

In calculating the actual values of the L_n expressed in the form (20), it would obviously be very convenient to have at one's disposal a table of values of the polynomials $q'_2 = pq$, $q'_3 = pqq_3$, $q'_4 = pqq_4$, etc., for a range of values of q, since the latter, when multiplied by appropriate powers of "a", are the L'_n of (19) and (20). I have, therefore, set up such tables, for values of the variable q ranging from $.01$ to 1.0 inclusive, at intervals of $.01$.

It is to be observed that only functional values are recorded here for $.01 \leqq q \leqq .50$, since we would merely repeat these values when $.50 \leqq q \leqq 1.0$, in the case of the polynomials q'_{2n}, of even degree, whilst in the case of those of odd degree, namely q'_{2n+1}, there would merely be a change of sign. For it is easily seen that, writing, $q'_n = \lambda_n(q)$,

$$\lambda_n(q) = (-1)^n \lambda_n(p), \qquad (p+q) = 1.$$

Hence

$$\lambda_{2n}(q) = \lambda_{2n}(p).$$

and

$$\lambda_{2n+1}(q) = -\lambda_{2n+1}(p).$$

I have calculated the exact functional values of all the polynomials q'_n, for $2 \leqq n \leqq q$ and these values appear in the tables for those cases in which $n \leqq 4$, but for $n > 4$ the functional values are written down correct only to eight decimal places. Each polynomial is set out in detail below.

$$q'_2 = q - q^2:$$

$$q'_3 = q - 3q^2 + 2q^3.$$

$$q'_4 = q - 7q^2 + 12q^3 - 6q^4.$$

$$q'_5 = q - 15q^2 + 50q^3 - 60q^4 + 24q^5.$$

$$q'_6 = q - 31q^2 + 180q^3 - 390q^4 + 360q^5 - 120q^6.$$

$$q'_7 = q - 63q^2 + 602q^3 - 2100q^4 + 3360q^5 - 2520q^6 + 720q^7.$$

$$q'_8 = q - 127q^2 + 1932q^3 - 10206q^4 + 25200q^5 - 31920q^6$$

$$+ 20160q^7 - 5040q^8.$$

$$q'_9 = q - 255q^2 + 6050q^3 - 46620q^4 + 166824q^5$$

$$- 317520q^6 + 332640q^7 - 181440q^8 + 40320q^9$$

q	q'_2	q'_3	q'_4	q'_5
.01	.0099	.0097 02	.0093 1194	.0085 4940
.02	.0196	.0188 16	.0172 9504	.0143 9048
.03	.0291	.0273 54	.0240 1914	.0178 0198
.04	.0384	.0353 28	.0295 5264	.0190 4886
.05	.0475	.0427 50	.0339 6250	.0183 8250
.06	.0564	.0496 32	.0373 1424	.0160 4106
.07	.0651	.0559 86	.0396 7194	.0122 4974
.08	.0736	.0618 24	.0410 9824	.0072 2104
.09	.0819	0671 58	.0416 5434	.0011 5512
.10	.0900	.0720 00	.0414 0000	−.0057 6000
.11	.0979	.0763 62	.0403 9354	−.0133 4808
.12	.1056	.0802 56	.0386 9184	−.0214 4440
.13	.1131	.0836 94	.0363 5034	−.0298 9550
.14	.1204	.0866 88	.0334 2304	−.0385 5882
.15	.1275	.0892 50	.0299 6250	−.0473 0250
.16	.1344	.0913 92	.0260 1984	−.0560 0502
.17	.1411	.0931 26	.0216 4474	−.0645 5494
.18	.1476	.0944 64	.0168 8544	−.0728 5064
.19	.1539	.0954 18	.0117 8874	−.0807 9996
.20	.1600	.0960 00	.0064 0000	−.0883 2000
.21	.1659	.0962 22	.0007 6314	−.0953 3676
.22	.1716	.0960 96	−.0050 7936	−.1017 8488
.23	.1771	.0956 34	−.0110 8646	−.1076 0738
.24	.1824	.0948 48	−.0172 1856	−.1127 5530
.25	.1875	.0937 50	−.0234 3750	−.1171 8750
.26	.1924	.0923 52	−.0297 0656	−.1208 7030
.27	.1971	.0906 66	−.0359 9046	−.1237 7722
.28	.2016	.0887 04	−.0422 5536	−.1258 8872
.29	.2059	.0864 78	−.0484 6886	−.1271 9184
.30	.2100	.0840 00	−.0546 0000	−.1276 8000
.31	.2139	.0812 82	−.0606 1926	−.1273 5264
.32	.2176	.0783 36	−.0664 9856	−.1262 1496
.33	.2211	.0751 74	−.0722 1126	−.1242 7766
.34	.2244	.0718 08	−.0777 3216	−.1215 5658
.35	.2275	.0682 50	−.0830 3750	−.1180 7250
.36	.2304	.0645 12	−.0881 0496	−.1138 5078
.37	.2331	.0606 06	−.0929 1366	−.1089 2110
.38	.2356	.0565 44	−.0974 4416	−.1033 1720
.39	.2379	.0523 38	−.1016 7846	−.0970 7652
.40	.2400	.0480 00	−.1056 0000	−.0902 4000
.41	.2419	.0435 42	−.1091 9366	−.0828 5172
.42	.2436	.0389 76	−.1124 4576	−.0749 5864
.43	.2451	.0343 14	−.1153 4406	−.0666 1034
.44	.2464	.0295 68	−.1178 7776	−.0578 5866
.45	.2475	.0247 50	−.1200 3750	−.0487 5750
.46	.2484	.0198 72	−.1218 1536	−.0393 6246
.47	.2491	.0149 46	−.1232 0486	−.0297 3058
.48	.2496	.0099 84	−.1242 0096	−.0199 2008
.49	.2499	.0049 98	−.1248 0006	−.0099 9000
.50	.2500	.0000 00	−.1250 0000	.0000 0000

q	q_6	q_7	q_8	q'_9
.01	.0070 7614	.0042 8133	-.0008 6757	-.0098 9983
.02	.0089 7874	-.0007 0941	-.0168 9834	-.0405 4526
.03	.0066 5276	-.0120 6717	-.0398 1333	-.0700 8269
.04	.0009 5797	-.0273 1414	-.0632 2635	-.0843 1166
.05	-.0073 2687	-.0443 6381	-.0823 9570	-.0752 0088
.06	-.0175 0006	-.0614 8682	-.0939 9591	-.0395 8596
.07	-.0289 3297	-.0772 7839	-.0959 0866	.0219 6545
.08	-.0410 6621	-.0906 2746	-.0870 3171	.1061 3469
.09	-.0534 0591	-.1006 8733	-.0671 0523	.2076 9234
.10	-.0655 2000	-.1068 4800	-.0365 5440	.3202 0992
.11	-.0770 3465	-.1087 0989	.0036 5241	.4366 5482
.12	-.0876 3077	-.1060 5914	.0521 3085	.5498 7950
.13	-.0970 4057	-.0988 4432	.1071 9393	.6530 1537
.14	-.1050 4428	-.0871 5459	.1669 5230	.7397 8144
.15	-.1114 6687	-.0711 9919	.2294 0230	.8047 1682
.16	-.1161 7493	-.0512 8831	.2925 0255	.8433 4602
.17	-.1190 7356	-.0278 1528	.3542 3967	.8522 8519
.18	-.1201 0342	-.0012 3997	.4126 8411	.8292 9704
.19	-.1192 3784	.0279 2652	.4660 3653	.7733 0158
.20	-.1164 8000	.0591 3600	.5126 6560	.6843 4944
.21	-.1118 6020	.0918 1615	.5511 3781	.5635 6392
.22	-.1054 3324	.1253 8237	.5802 3985	.4130 5748
.23	-.0972 7586	.1592 4855	.5989 9440	.2358 2811
.24	-.0874 8429	.1928 3676	.6066 6961	.0356 4020
.25	-.0761 7187	.2255 8594	.6027 8320	-.1831 0547
.26	-.0634 6676	.2569 5961	.5871 0148	-.4155 0871
.27	-.0495 0971	.2864 5266	.5596 3389	-.6563 1025
.28	-.0344 5199	.3135 9717	.5206 2368	-.9000 3211
.29	-.0184 5333	.3379 6740	.4505 3509	-1.1411 1524
.30	-.0016 8000	.3591 8400	.4100 3760	-1.3740 5184
.31	.0156 9709	.3769 1728	.3399 8768	-1.5935 1019
.32	.0335 0413	.3908 8987	.2614 0845	-1.7944 5014
.33	.0515 6609	.4008 7856	.1754 6769	-1.9722 2773
.34	.0697 0833	.4067 1546	.0834 5462	-2.1226 8757
.35	.0877 5812	.4082 8856	-.0132 4420	-2.2422 4216
.36	.1055 4606	.4055 4163	-.1131 6952	-2.3279 3728
.37	.1229 0738	.3984 7352	-.2148 1421	-2.3775 0310
.38	.1396 8328	.3871 3695	-.3166 4796	-2.3893 9063
.39	.1557 2201	.3716 3683	-.4171 4155	-2.3627 9368
.40	.1708 8000	.3521 2800	-.5147 9040	-2.2976 5632
.41	.1850 2283	.3288 1262	-.6081 3721	-2.1946 6636
.42	.1980 2614	.3019 3716	-.6957 9341	-2.0552 3535
.43	.2097 7648	.2717 8896	-.7764 5916	-1.8814 6575
.44	.2201 7200	.2386 9258	-.8489 4184	-1.6761 0632
.45	.2291 2312	.2030 0569	-.9121 7270	-1.4424 9645
.46	.2365 5311	.1651 1489	-.9652 2163	-1.1845 0081
.47	.2423 9851	.1254 3119	-1.0073 0991	-1.1474 9129
.48	.2466 0959	.0843 8534	-1.0378 2075	-.6129 8577
.49	.2491 5060	.0424 2304	-1.0563 0756	-.3091 2081
.50	.2500 0000	.0000 0000	-1.0625 0000	.0000 0000

Section 3. Approximate expressions for the semi-invariants of α_3, α_4, σ_x in samples from the compound frequency function.

In the paper of C. C. Craig, already cited, the author obtained the following results for sampling from a single parent population.

(1) Expressions[1] for the sampling characteristics of the correlation functions for ν_2, ν_3, and ν_2, ν_4, in terms of N, the size of the sample, and the characteristics of the population itself.

(2) Expressions[2] for the sampling characteristics of the distribution functions for α_3, α_4, and σ_x, in terms of certain "g" functions, the latter being defined by

$$g_{k\ell}(\nu_m, \nu_n) = \frac{S_{k\ell}(\nu_m, \nu_n)}{\nu_2^{\frac{km+\ell n}{2}}} \tag{21}$$

in which $S_{k\ell}(\nu_m, \nu_n)$ are the characteristics of the correlation function for ν_m, ν_n.

I can now make use of the results indicated above, in conjunction with the relations (6) of the present paper, in order to determine approximate expressions for the semi-invariants of α_3, α_4, σ_x, in samples from the compound frequency function, retaining only terms of order -2 and higher in N in using expressions (1), and only those "g" functions in using the expressions (2) which are of order -2 and higher in N, where $g_{k\ell}$ is of order $\frac{1}{N^{k+\ell-1}}$.

A. The semi-invariants of α_3, viz. b_1, b_2, b_3, etc.

From definition (21) and the relations (1) for $m=2$, $n=3$, and making use of the following notation

[1] Loc. Cit. p. 57 et seq.
[2] Loc. Cit. p. 50 et seq.

(22) $\phi_i = \frac{L_i}{L_2^{\frac{i}{2}}}$, $\phi_{ikj}^{rst} = \phi_i^r \cdot \phi_k^s \cdot \phi_j^t$, etc.

in which the L_n are the same as in (6) of Section 1, I obtain the following set of "g" functions.

$$g_{10} = \frac{1}{N}\left\{(N-1)\right\}. \qquad \text{using for brevity } g_{K\ell} \equiv g_{K\ell}(V_2, V_3).$$

$$g_{01} = \frac{1}{N^2}\left\{(N-1)(N-2)\phi_3\right\}.$$

$$g_{20} = \frac{1}{N^3}\left\{(N-1)^2\phi_4 + 2N(N-1)\right\}.$$

(23)

$$g_{11} = \frac{1}{N^4}\left\{(N-1)^2(N-2)\phi_5 + 6N(N-1)(N-2)\phi_3\right\}.$$

$$g_{02} = \frac{1}{N^5}\left\{(N-1)^2(N-2)^2\phi_6 + 9N(N-1)(N-2)^2\phi_4 + 9N(N-1)(N-2)^2\phi_3^2 + 6N^2(N-1)(N-2)\right\}.$$

$$g_{30} = \frac{1}{N^5}\left\{(N-1)^3\phi_6 + 12N(N-1)^2\phi_4 + 4N(N-1)(N-2)\phi_3^2 + 8N^2(N-1)\right\}$$

$$g_{21} = \frac{1}{N^6}\left\{(N-1)^3(N-2)\phi_7 + 16N(N-1)^2(N-2)\phi_5 + 12N(N-1)(N-2)(2N-3)\phi_{43} + 48N^2(N-1)(N-2)\phi_3\right\}.$$

$$g_{12} = \frac{1}{N^7}\left\{(N-1)^3(N-2)^2\phi_8 + 21N(N-1)^2(N-2)^2\phi_6 + 6N(N-1)(N-2)^2(8N-11)\phi_{53} + 9N(N-1)(N-2)^2(3N-5)\phi_4^2 + 18N^2(N-1)(N-2)(6N-11)\phi_4 + 18N^2(N-1)(N-2)(9N-20)\phi_3^2 + 36N^3(N-1)(N-2)\right\}$$

$$g_{03}=\frac{1}{N^8}\Big\{(N-1)^3(N-2)^3\phi_9+27N(N-1)^2(N-2)^3\phi_7+27N(N-1)(N-2)^3(3N-4)\phi_{63}$$
$$+27N(N-1)(N-2)^3(4N-7)\phi_{54}+54N^2(N-1)(N-2)^2(4N-7)\phi_5$$
$$+162N^2(N-1)(N-2)^2(5N-12)\phi_{43}+36N^2(N-1)(N-2)(7N^2-30N+34)\phi_3^3$$
$$+108N^8(N-1)(N-2)(5N-12)\phi_3\Big\}.$$

and on substituting these values for the "g"s in the expressions for the semi-invariants of α_3 from the relations (2), I get:—

$$b_1=\alpha_3\Big\{\frac{13}{16}+\frac{1}{N}\Big(\frac{783}{32}+\frac{645}{64}\phi_4\Big)+\frac{1}{N^2}\Big(-\frac{9155}{64}-\frac{5385}{64}\phi_4-\frac{8505}{256}\phi_4^2+\frac{245}{32}\phi_6$$
$$+\frac{245}{8}\phi_3^2\Big)\Big\}+\Big\{\frac{11}{8}\phi_3+\frac{1}{N}\Big(-\frac{333}{8}\phi_3-\frac{69}{16}\phi_5-\frac{75}{16}\phi_{34}\Big)+\frac{1}{N^2}\Big(\frac{8273}{32}\phi_3$$
$$-\frac{39}{16}\phi_5-\frac{75}{16}\phi_7+\frac{1875}{32}\phi_{34}-\frac{35}{16}\phi_{36}-\frac{35}{4}\phi_3^2+\frac{735}{32}\phi_{45}+\frac{945}{128}\phi_{34}^{12}\Big)\Big\}.$$

$$b_2=\alpha_3^2\Big\{\frac{1}{N}\Big(\frac{99}{2}+\frac{99}{4}\phi_4\Big)+\frac{1}{N^2}\Big(-\frac{2313}{4}-\frac{819}{2}\phi_4+\frac{225}{8}\phi_6+\frac{225}{2}\phi_3^2-\frac{2565}{16}\phi_4^2\Big)\Big\}$$
$$+2\alpha_3\Big\{\frac{1}{N}\Big(-81\phi_3-\frac{21}{2}\phi_5-9\phi_{34}\Big)+\frac{1}{N^2}\Big(\frac{7479}{8}\phi_3+\frac{33}{4}\phi_5-\frac{33}{2}\phi_7+\frac{2547}{8}\phi_{34}$$
$$-\frac{45}{8}\phi_{36}-\frac{45}{2}\phi_3^3+\frac{825}{8}\phi_{45}+\frac{855}{32}\phi_{34}^{12}\Big)\Big\}+\Big\{\frac{1}{N}\Big(24+36\phi_4+4\phi_6+9\phi_{35}$$
$$+\frac{189}{2}\phi_3^2+\frac{9}{4}\phi_{34}^{21}\Big)+\frac{1}{N^2}\Big(-90+135\phi_4+108\phi_6+9\phi_8+27\phi_4^2-\frac{2331}{2}\phi_3^2$$
$$-39\phi_5^2-\frac{567}{2}\phi_{34}^{21}-\frac{123}{2}\phi_{35}+\frac{33}{4}\phi_{37}-\frac{165}{4}\phi_{345}-24\phi_{46}\Big)\Big\}.$$

$$b_3=\alpha_3^3\Big\{\frac{1}{N^2}\Big(\frac{4671}{8}-\frac{3969}{8}\phi_4+\frac{351}{16}\phi_6+\frac{351}{4}\phi_3^2-\frac{6075}{32}\phi_4^2\Big)\Big\}$$
$$+3\alpha_3^2\Big\{\frac{1}{N^2}\Big(\frac{3537}{4}\phi_3+36\phi_5-\frac{99}{8}\phi_7+\frac{1863}{4}\phi_{34}-\frac{27}{8}\phi_{36}-\frac{27}{2}\phi_3^3+117\phi_{45}\Big)\Big\}$$
$$+3\alpha_3\Big\{\frac{1}{N^2}\Big(-54+135\phi_4+\frac{369}{4}\phi_6-\frac{2403}{2}\phi_3^2+\frac{27}{4}\phi_8-\frac{441}{2}\phi_{35}-\frac{81}{2}\phi_4^2$$

$$+\frac{9}{2}\phi_{37}-\frac{1323}{4}\phi_{34}^{21}-36\phi_{345}-\frac{99}{4}\phi_{46}-\frac{363}{8}\phi_5^2\Big)\Big\}+\Big\{\frac{1}{N^2}\Big(-432\phi_3$$

$$-513\phi_5-\frac{189}{2}\phi_7-810\phi_{34}-\frac{7}{2}\phi_9-108\phi_{36}-\frac{27}{2}\phi_{45}+1710\phi_3^3$$

$$-\frac{9}{2}\phi_{38}+\frac{891}{2}\phi_{35}^{21}+\frac{81}{2}\phi_{56}+\frac{27}{2}\phi_{346}+\frac{243}{2}\phi_{34}^{31}+\frac{99}{4}\phi_{35}^{12}\Big)\Big\}.$$

$b_4=0$

In a similar manner I obtain

B. The semi-invariants of α_4, viz. c_1, c_2, c_3, etc.

In this case *$m=2$, $n=4$, and the "g" functions this time are

$$g_{10}=\frac{1}{N}\Big\{(N-1)\Big\}.$$

* for brevity $g_{k\ell}\equiv g_{k\ell}(v_2,v_4)$

$$g_{01}=\frac{1}{N^3}\Big\{N(N^2-4N+6)\phi_4+3N(N^2-2N+1)\Big\}.$$

$$g_{20}=\frac{1}{N^3}\Big\{N(N-2)\phi_4+2N(N-1)\Big\}.$$

$$g_{11}=\frac{1}{N^2}\Big\{(N-5)\phi_6+2(7N-25)\phi_4+6(N-5)\phi_3^2+12(N-2)\Big\}.$$

$$(24)\quad g_{02}=\frac{1}{N^2}\Big\{(N-8)\phi_8+4(7N-46)\phi_6+48(N-8)\phi_{35}+2(17N-128)\phi_4^2$$
$$+12(17N-88)\phi_4+72(3N-20)\phi_3^2+24(4N-13)\Big\}.$$

$$g_{30}=\frac{1}{N^2}\Big\{\phi_6+12\phi_4+4\phi_3^2+8\Big\}.$$

$$g_{21}=\frac{1}{N^2}\Big\{\phi_8+26\phi_6+40\phi_{35}+34\phi_4^2+176\phi_4+144\phi_3^2+72\Big\}.$$

$$g_{12}=\frac{1}{N^2}\Big\{\phi_X+44\phi_{28}+56\phi_{37}+208\phi_{46}+116\phi_5^2+596\phi_{26}^{21}$$

$$+2016\phi_{235}+1512\phi_{24}^{12}+1584\phi_{34}^{21}+2832\phi_{24}^{31}+4176\phi_{23}^{22}+768\phi_{2}^{5}\Big\}$$

$$g_{03}=\frac{1}{N^2}\Big\{\phi^{\cdot}_{\underline{\mathrm{XII}}}+66\phi_{2\cdot\underline{\mathrm{X}}}+208\phi_{39}+492\phi_{48}+768\phi_{57}+462\phi_{6}^{2}+1476\phi_{28}^{21}$$

$$+7344\phi_{237}+13776\phi_{246}+7848\phi_{25}^{12}+8280\phi_{36}^{21}+25344\phi_{345}$$

$$+5672\phi_{4}^{3}+11448\phi_{26}^{31}+73440\phi_{235}^{211}+51048\phi_{24}^{22}+119232\phi_{234}^{121}$$

$$+12456\phi_{3}^{4}+49248\phi_{24}^{41}+110592\phi_{23}^{32}+9504\phi_{2}^{6}\Big\}$$

Therefore, substituting from (24) into the expressions for the semi-invariants of α_4 given by the relations (2) we have—

$$c_1=\alpha_4\Big\{-2+\frac{1}{N}\Big(50+21\phi_4\Big)+\frac{1}{N^2}\Big(401+16\phi_6-75\phi_4^2+522\phi_4+64\phi_3^2\Big)\Big\}$$

$$+\Big\{\Big(6+2\phi_4\Big)+\frac{1}{N}\Big(-174-8\phi_6-9\phi_4^2-169\phi_4-48\phi_3^2\Big)+\frac{1}{N^2}\Big(1269-9\phi_8$$

$$-92\phi_6+489\phi_4^2+15\phi_4^3-420\phi_3^2-360\phi_{35}+272\phi_{34}^{21}+1341\phi_4+44\phi_{46}\Big)\Big\}.$$

$$c_2=\alpha_4^2\Big\{\frac{1}{N}\Big(128+64\phi_4\Big)+\frac{1}{N^2}\Big(-1688+72\phi_6-462\phi_4^2-1256\phi_4+288\phi_3^2\Big)\Big\}$$

$$+2\alpha_4\Big\{\frac{1}{N}\Big(-384-22\phi_6-20\phi_4^2-408\phi_4-132\phi_3^2\Big)+\frac{1}{N^2}\Big(5664-35\phi_8$$

$$-282\phi_6+2800\phi_4^2+6808\phi_4-1200\phi_3^2-1400\phi_{35}+1464\phi_{34}^{21}$$

$$+240\phi_{46}+66\phi_4^3\Big)\Big\}+\Big\{\frac{1}{N}\Big(1320+7\phi_8+244\phi_6+494\phi_4^2+4\phi_4^3+2376\phi_4$$

$$+1800\phi_3^2+336\phi_{35}+16\phi_{46}+96\phi_{34}^{21}\Big)+\frac{1}{N^2}\Big(-32208-140\phi_8-6544\phi_6$$

$$-2440\phi_4^3-37192\phi_4^2-75000\phi_4-48168\phi_3^2-7056\phi_{35}+704\phi_{28}$$

$$+896\phi_{37}-532\phi_{46}+1856\phi_5^2+9536\phi_{26}^{21}+32256\phi_{235}+24192\phi_{24}^{12}$$

$$-384\phi_{34}^{21}+16\phi_{\underline{\mathrm{X}}}+45312\phi_{24}^{31}+66816\phi_{23}^{22}+12288\phi_2^5-36\phi_{48}-1840\phi_{345}$$

$$-80\phi_6^2-2880\phi_3^4+11920\phi_{26}^{21}-960\phi_{36}^{21}-84\phi_{46}^{21}-504\phi_{34}^{22}\Big)\Big\}.$$

$$
\begin{aligned}
C_3=\frac{1}{N^2}\Big\{&\alpha_4^3\big(-2080+64\phi_6-648\phi_4^2-1824\phi_4+256\phi_3^2\big)\\
&+3\alpha_4^2\big(7488-28\phi_8-16\phi_6+4608\phi_4^2+10592\phi_4+288\phi_3^2\\
&-1120\phi_{35}+360\phi_{46}+2176\phi_{34}^{21}\big)+3\alpha_4\big(-28896+12\phi_{\underline{X}}-88\phi_8\\
&-5344\phi_6-2752\phi_4^3-38776\phi_4^2-67872\phi_4-37728\phi_3^2-3528\phi_3^4\\
&+528\phi_{28}+672\phi_{37}-2008\phi_{46}+1392\phi_5^2+9216\phi_2^5+7152\phi_{26}^{21}\\
&+24192\phi_{235}+18144\phi_{24}^{12}-10800\phi_{34}^{21}+33984\phi_{24}^{31}+50112\phi_{23}^{22}\\
&-4416\phi_{35}+7152\phi_{26}^{21}-48\phi_{48}-2368\phi_{345}-98\phi_6^2-1176\phi_{36}^{21}\\
&-80\phi_{46}^{21}-480\phi_{34}^{22}\big)+\big(114912-5\phi_{\underline{\underline{XII}}}-18\phi_{X}-330\phi_{2X}-6\phi_{4X}-1040\phi_{39}\\
&-1332\phi_{48}-3840\phi_{57}-168\phi_6^2-7380\phi_{28}^{21}-36720\phi_{237}-68880\phi_{246}\\
&-39240\phi_{25}^{12}-12240\phi_{36}^{21}-72576\phi_{345}+23120\phi_4^3-57240\phi_{26}^{31}\\
&-367200\phi_{235}^{211}-255240\phi_{24}^{22}-596160\phi_{234}^{121}+35568\phi_3^4-246240\phi_{24}^{41}\\
&-552960\phi_{23}^{32}-47520\phi_2^6-792\phi_{28}-1008\phi_{37}+47064\phi_{46}-2088\phi_5^2\\
&-10728\phi_{26}^{21}-36288\phi_{235}-27216\phi_{24}^{12}+330480\phi_{34}^{21}-50976\phi_{24}^{31}\\
&-75168\phi_{23}^{22}-13824\phi_2^5-264\phi_{248}-336\phi_{347}+3048\phi_{46}^{21}-696\phi_{45}^{12}\\
&-3576\phi_{246}^{211}-12096\phi_{2345}-9072\phi_{24}^{13}+17424\phi_{34}^{22}-16992\phi_{24}^{32}\\
&-25056\phi_{234}^{221}-4608\phi_{24}^{51}+72\phi_{68}+432\phi_{38}^{21}+1008\phi_8+38160\phi_6\\
&+3456\phi_{356}+20736\phi_{35}^{31}+48384\phi_{35}+305496\phi_4^2+362304\phi_4\\
&+277344\phi_3^2+24\phi_{48}^{21}+1152\phi_{345}^{121}+42\phi_{46}^{12}+1512\phi_{34}^{41}+504\phi_{346}^{211}+816\phi_4^4\big)\Big\}.
\end{aligned}
\tag{25}
$$

$$C_4=0$$

C. The semi-invariants of $\sigma_x \, (=\sqrt{V_2})$, viz. d_1, d_2, d_3, etc.

Now $g_r \equiv g_{r0}$ and $g_r(V_2) = \dfrac{S_r(V_2)}{L_2^r}$ from the relation (21).

Therefore, the "g" functions here are—

$$
\begin{aligned}
& g_1 = \frac{1}{N}\left\{(N-1)\right\}, \qquad \text{for brevity } g_r = g_r(V_2) \\
(26) \qquad & g_2 = \frac{1}{N^2}\left\{(N-2)\Phi_4 + 2(N-1)\right\}, \\
& g_3 = \frac{1}{N^2}\left\{\Phi_6 + 12\Phi_4 + 4\Phi_3^2 + 8\right\}.
\end{aligned}
$$

On substituting from (26) into the expressions for the semi-invariants of σ_x from the relations (2) gives—

$$
d_1 = \sigma_x\left\{\frac{23}{16} + \frac{1}{N}\left(-\frac{25}{32} - \frac{11}{64}\Phi_4\right) + \frac{1}{N^2}\left(\frac{89}{64} - \frac{3}{32}\Phi_6 + \frac{43}{64}\Phi_4 - \frac{3}{8}\Phi_3^2 + \frac{75}{256}\Phi_4^2\right)\right\},
$$

$$
d_2 = \frac{L_2}{4}\left\{\frac{1}{N}\left(2+\Phi_4\right) + \frac{1}{N^2}\left(-7 + \frac{1}{2}\Phi_6 - 4\Phi_4 + 2\Phi_3^2 - \frac{7}{4}\Phi_4^2\right)\right\},
$$

$$
d_3 = \frac{\sigma_x^3}{8}\left\{\frac{1}{N^2}\left(5 - \frac{1}{2}\Phi_6 + 3\Phi_4 - 2\Phi_3^2 + \frac{9}{4}\Phi_4^2\right)\right\},
$$

$$
d_4 = 0.
$$

Section 4. The case in which the compound frequency function may possess non-zero semi-invariants of all orders.

Instead of the components of the compound frequency function being considered normal, I now assume that they may possess non-zero semi-invariants as far as the third order, and I again derive expressions for the parameters of the compound in terms of the parameters of the components. The method of derivation is entirely analogous to that in Section 1, where the components were normal. The expressions for the L_n, the semi-invariants of the compound are seen to be more complicated in the present case.

but this complexity is more apparent than real. In fact, I have succeeded in deducing a rather simple general law, by means of which, these expressions may be written out, and this law is still applicable, even if the two components should possess non-zero semi-invariants of higher orders than the third.

I now write

$$(27)\quad e^{L_1 t + L_2\frac{t^2}{2} + L_3\frac{t^3}{3!}+\cdots} = pe^{m_1 t + \sigma_1^2\frac{t^2}{2} + {}_1\lambda_3\frac{t^3}{3!}}\left\{1+re^{at+\frac{bt^2}{2!}+\frac{ct^3}{3!}}\right\},$$

where

$$a = m_2 - m_1, \qquad b = \sigma_2^2 - \sigma_1^2, \qquad c = {}_2\lambda_3 - {}_1\lambda_3, \qquad r = \frac{q}{p},$$

and m_1, m_2, σ_1, σ_2, have the same significance as in Section 1, and ${}_1\lambda_3$, ${}_2\lambda_3$, are the third semi-invariants of $\phi_1(x)$, $\phi_2(x)$ respectively, and as before $p+q=1$. Taking logarithms in (27), we have

$$(28)\quad L_1 t + L_2\frac{t^2}{2} + L_3\frac{t^3}{3!}+\cdots = \log p + \Bigl(m_1 t + \frac{\sigma_1^2 t^2}{2} + \frac{{}_1\lambda_3 t^3}{3!} + \log\left\{1+re^{ab+\frac{bt^2}{2!}+\frac{ct^3}{3!}}\right\};$$

Further

$$(29)\quad \log\left\{1+re^{at+\frac{bt^2}{2!}+\frac{ct^3}{3!}}\right\} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cdot q^k}{k}\left(e^{B}-1\right)^k + \log(1+r),$$

in which

$$B = at + \frac{bt^2}{2!} + \frac{ct^3}{3!} .$$

The right member of (29), may be put into the form

$$\left\{ \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \binom{k}{i} \sum_{j=0}^{\infty} \frac{i^j B^j}{j!} \right\} + \log(1+r)$$

Therefore equating corresponding coefficients of $\frac{t^n}{n!}$ in the right and left members of (28), for $n > 3$, gives

(30)

$$L_n = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^{1+i} \cdot q^k}{k} \binom{k}{i} \sum \frac{i^j \cdot n!}{\alpha! \beta! \gamma! \cdots (2!)^\beta (3!)^\gamma} \cdot (a)^\alpha (b)^\beta (c)^\gamma ,$$

where the last summation is taken over all values of j, such that the following diophantine equations are satisfied.

$$\begin{aligned} &\alpha + \beta + \gamma = j , \\ &\alpha + 2\beta + 3\gamma = n . \end{aligned} \tag{31}$$

Using (30), I obtain, for $n = 1$ to 8 inclusive, the first eight semi-invariants of the compound as follows,

$$L_1 = m_1 + aq .$$

$$L_2 = \sigma_1^2 + q\left\{a^2(1-q) + b\right\} .$$

$$L_3 = {}_1\lambda_3 + a^3 q'_3 + 3abq'_2 + cq .$$

$$L_4 = a^4 q'_4 + 6a^2 b q'_3 + 3b^2 q'_2 + 4acq'_2 .$$

$$L_5 = a^5q'_5 + 10a^3bq'_4 + 15ab^2q'_3 + 10a^2cq'_3 + 10bcq'_2$$

$$(32)\quad L_6 = a^6q'_6 + 15a^4bq'_5 + 45a^2b^2q'_4 + 15b^3q'_3 + 20a^3cq'_4 + 60abcq'_3 + 10c^2q'_2.$$

$$L_7 = a^7q'_7 + 21a^5bq'_6 + 105a^3b^2q'_5 + 105ab^3q'_4 + 35a^4cq'_5 + 210a^2bcq'_4 + 70ac^2q'_3 + 105b^2cq'_3$$

$$L_8 = a^8q'_8 + 28a^6bq'_7 + 210a^4b^2q'_6 + 420a^2b^3q'_5 + 105b^4q'_4 + 56a^5cq'_6 + 560a^3bcq'_5 + 280a^2c^2q'_4 + 840ab^2cq'_4 + 280bc^2q'_3.$$

in which $q'_2 \equiv q_2 = pq$, $q'_3 = pqq_3$, etc., and are in fact the same polynomials that occurred in the discussion of the case for normal components in *Section 1.*

The expression for the L_n in (5) may be put into a form similar that that of (30), and then, if we compare these two forms, it is obvious that no new polynomials q'_n will occur, in addition to those which appeared in (5), and this would be true however many non-zero semi-invariants the two components may have.

Using the results established for the L_n in Section 1, it is evident that, if we consider a particular semi-invariant, say L_n, of (5), the terms in the right member can be readily written down, if we determine all the j part partitions of n, where j and n are fixed, using the integers 1 and 2 as part magnitudes. Suppose we have α parts, each equal to 1, and β parts, each equal to 2, where $\alpha + \beta = j$ and $\alpha + 2\beta = n$, then such a partition corresponds to a term of the type $a^\alpha b^\beta q'_j$ (omitting the numerical coefficient), the factor $a^\alpha b^\beta$ arising from the last summation of (5), which is clearly seen, if the latter be put into the same form as (30). In addition, it is to be noted that any j part partition of n will be unique for this case.

Now if we consider the case of the present *Section,* for $n > 3$ the L_n of (30) will be seen to contain, as well as the same terms of the corresponding L_n of (5), some additional terms, the latter appearing on account of the fact that, since the integer 3 is now admitted along with 1 and 2 as a part magnitude, the j part partitions of n will no longer be unique for every possible value of j. These j part partitions of n will, moreover, give rise to terms of the type $a^{\alpha} b^{\beta} c^{\gamma} q'_j$ where the relations (31) are satisfied. Further, the total number of terms in a given L_n, for n not too large, can be readily obtained by making use of the so-called "enumerating function," discussed in works on combinatorial analysis, which enables one to determine the number of partitions of a given integer n, when the number of parts j, and the part magnitudes 1, 2, 3, etc. are fixed.

It would appear, from the above discussion that the partition method of obtaining the terms of L_n could be carried over to the most general case, in which the components may possess non-zero semi-invariants of all orders.

I shall now indicate, without going into detail, that, if in the expressions (30) for $n > 3$, I set every q'_j equal to unity, then the L_n become functions of a, b, and c only, which I call β_n, and the latter obey a recursion law analogous to the one established for the β_s of (10), namely

$$\beta_{n+1} = \left(a + b\frac{\partial}{\partial a} + c\frac{\partial}{\partial b}\right)\beta_n, \tag{33}$$

where now

$$\beta_n = \frac{d^n}{dt^n}\left\{e^{at + \frac{bt^2}{2!} + \frac{ct^3}{3!}}\right\}\Bigg|_{t=0} \text{ by definition.}$$

Putting $\phi(t) = at + \frac{bt^2}{2!} + \frac{ct^3}{3!}$

$$(34) \quad \frac{d^n}{dt^n}\left\{e^{\phi(t)}\right\} = P_n\left(c,\, b+ct,\, a+bt+\frac{ct^2}{2}\right).\, e^{\phi(t)},$$

where $P_n(x, y, z)$ is an n-th degree polynomial in x, y, and z. Again, it is readily seen that

$$(35) \quad e^{\phi(t)}.\frac{d}{dt}\left\{P_n\left(c,\, b+ct,\, a+bt+\frac{ct^2}{2}\right)\right\}\Bigg|_{t=0} = \left(b\cdot\frac{\partial}{\partial a} + c\frac{\partial}{\partial b}\right)\left\{P_n\left(c,\, b+ct,\, a+bt+\frac{ct^2}{2}\right).e^{\phi(t)}\Bigg|_{t=0}\right\},$$

and that

$$(36) \quad P_n\cdot\frac{d}{dt}\left\{e^{\phi(t)}\right\}\Bigg|_{t=0} = a\cdot P_n\cdot e^{\phi(t)}\Bigg|_{t=0}.$$

Now deriving the left member of (34), with respect to t, and then setting $t=0$ gives the next β, viz. β_{n+1}, whilst the derivative of the right member of (34), then setting $t=0$, would equal the sum of the right members of (35) and (36), and thus the law in this case is established.

It is at once apparent that the recursion formulae of (13) and (33) may be generalized, so that, if the two components should possess non-zero semi-invariants of all orders, the law for the β_n would then be

$$\beta_{n+1} = \left(a + b\frac{\partial}{\partial a} + c\frac{\partial}{\partial b} + d\frac{\partial}{\partial c} + \text{etc.}, \ldots\right)\beta_n.$$

a, b, c, d, etc. being the differences between the 1st, 2nd, etc. semi-invariants of the two components, respectively. Thus it appears that the actual writing down of the expressions for the parameters of a compound frequency function in terms of the parameters of its two components may be reduced to a partititon process, and a taking of derivatives.

Section 5. The limiting compound frequency function, when the number of components is allowed to become indefinitely large.

It is to be noted that if the compound is assumed to be composed of a greater number of components than two, then the mathematical development beoomes heavy, but a rather interesting case arises, when we consider the form of the limiting compound frequency function, when its components, infinite in number, and identical in form, each contribute the same proportion to the total frequency of the compound, and have their means distributed according to the known frequency law $f(x)$.

First of all I consider the compound to be composed of a finite number of components, say $M+1$, of the type indicated, and later pass to the limit, allowing M to become indefinitely large.

I write now

$$(37)\quad e^{L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots} = \sum_{i=1}^{M+1} p_i \cdot e^{\lambda_{i1} t + \lambda_{i2} \frac{t^2}{2!} + \lambda_{i3} \frac{t^3}{3!} + \cdots}$$

in which $p_i = \frac{1}{M+1}$ for all $i = 1, 2, 3, \ldots$ etc. and λ_{i1}, λ_{i2}, etc., are the 1st, 2nd, etc. semi-invariants respectively of the i-th component. The right member of (37) may be written

$$e^{\lambda_{11} t + \lambda_{12} \frac{t^2}{2!} + \lambda_{13} \frac{t^3}{3!} + \cdots} \cdot \frac{1}{M+1} \left\{ 1 + \sum_{i=1}^{M} e^{(m_{i+1} - m_1)t} \right\},$$

in which m_1 is the mean of some component and m_{i+1} is the mean of the $(i+1)$st component.

If we now assume that $m_1 = \lambda_{11} = 0$, then

$$e^{L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots}$$

(38)
$$= e^{\lambda_{12}\frac{t^2}{2} + \lambda_{13}\frac{t^3}{3!} + \cdots} \cdot \frac{1}{M+1}\left\{1 + \sum_{i=2}^{M+1} e^{m_i t}\right\},$$

and the right member of the last relation is the generating function for the moments of the compound frequency function. Allowing M to become indefinitely large, we have

$$\lim_{M\to\infty} \cdot \frac{1}{M+1}\left\{1 + e^{m_2 t} + e^{m_3 t} + \cdots\cdots\cdot e^{m_{M+1} t} + \cdots\cdots\right\}$$

$$= \int_{-\infty}^{\infty} \cdot e^{xt} \cdot f(x) \cdot dx = G_x(t).$$

Therefore the limit of the generating function (the right member of (38) is given by

$$e^{\lambda_{12}\frac{t^2}{2} + \lambda_{13}\frac{t^3}{3!} + \cdots\cdots} \cdot G_x(t),$$

so that the semi-invariants L_n of the limiting compound frequency function are given by

$$e^{L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots} = e^{\lambda_{12}\frac{t^2}{2!} + \lambda_{13}\frac{t^3}{3!} + \cdots} \cdot G_x(t).$$

From this last relation, it will be seen that the mean of the limiting compound is equal to the mean of the means of the components. Further, if the means of the components are normally distributed, and the components themselves are normal, then the limiting compound frequency function is also normal. More generally, if the components are normal, and their means follow any frequency law $F(x)$, then the limiting compound function also follows this same law. If now, considering the most general case of all, where the components may have non-zero semi-invariants of all orders, and the means of the components are distributed according to the frequency law $F(x)$, then the semi-invariants of the limiting compound frequency function may always be calculated, and will be given by

$$L_k = \lambda_{1k} + \ell_k ,$$

in which L_k, λ_{1k}, ℓ_k, are the k-th semi-invariants of the limiting compound function, of one of the components, and of $F(x)$ respectively.

This shows that the variate z of the limiting compound frequency function, is distributed as if it were the sum of two independent variates, one of which is distributed according to the law of the means, and the other according to one of the components. To write down the actual distribution function for the limiting compound is quite another matter, but since we may write, in the limit, when $M \to \infty$,

$$e^{L_1 t + L_2 \frac{t^2}{2!} + L_3 \frac{t^3}{3!} + \cdots} = e^{\lambda_{12} \frac{t^2}{2!} + \lambda_{13} \frac{t^3}{3!} + \cdots} \cdot e^{\ell_1 t + \ell_2 \frac{t^2}{2!} + \cdots\cdots}$$

then, the distribution function sought, provided it fulfills the necessary conditions, may be given formally by means of the Fourier Integral Theorem.

PART 2.

As indicated previously, in the introduction to this paper, we are concerned in this second part with an entirely new problem, in which we are now sampling from two distinct parent populations instead of from only one, as in Part 1. Hence, in order to obtain the desired sampling results, we must have recourse to an entirely different method of treatment from any we have made use of heretofore. I shall suppose that $\phi_1(x)$, $\phi_2(x)$, the two parent populations may possess non-zero semi-invariants of all orders, and that a random sample of r is taken from the first population, and a random sample of s from the second population, these two samples being then combined to give the composite sample from the combined populations.

Section 6. The semi-invariants of "moments about a fixed point," in samples from the compound frequency function.
With the above hypotheses, I shall derive in this section, expressions for the semi-invariants of "moments about a fixed point" in samples from the compound population.

Calling the required semi-invariants S_1, S_2, S_3. arc, we have, by definition

$$e^{S_1 t + S_2 \frac{t^2}{2!} + S_3 \frac{t^3}{3!} + \cdots}$$

(39)

$$= \int_{-\infty}^{\infty} \int [\phi_1(x)]^r [\phi_2(x)]^s \cdot e^{\left\{\sum_{i=1}^{r} {}_1x_i^n + \sum_{j=r+1}^{r+s} {}_2x_j^n\right\}\frac{t}{N}} (dx)^{r+s}$$

in which $\phi_1(x)$, $\phi_2(x)$ are the initial parent frequency functions, and ${}_1x_i$, ${}_2x_j$, indicates that the variate was taken from the first and second parent respectively. By a suitable transformation of the parameter in the power of the exponential which appears in the right member of (39), this same member may be put into the required form

$$\int_{-\infty}^{\infty}\!\!\int [\phi_1(x)]^r \cdot e^{\frac{rt}{N}\left\{\frac{1}{r}\sum_{i=1}^{r} {}_1x_i^n\right\}} \cdot (dx)^r \cdot \int_{-\infty}^{\infty}\!\!\int [\phi_2(x)]^s \cdot e^{\frac{st}{N}\left\{\frac{1}{s}\sum_{j=r+1}^{r+s} {}_2x_j^n\right\}} (dx)^s.$$

On equating corresponding coefficients of $\frac{t^k}{k!}$ in the last expression and the left member of (39), I get

$$S_k(V_n') = \frac{r^k \cdot S_k'(V_n') + s^k \cdot S_k''(V_n')}{(r+s)^k},$$

in which $S_1'(V_n')$, $S_2'(V_n')$, $S_3'(V_n')$, etc., and $S_1''(V_n')$, $S_2''(V_n')$, $S_3''(V_n')$, etc., are the 1st, 2nd, 3rd, etc., semi-invariants for V_n', in samples from the two component populations $\phi_1(x)$, and $\phi_2(x)$, respectively, the values of which are well known.[1]

Section 7. The semi-invariants of "moments about the mean" in samples from the compound frequency function.

Employing the same sampling procedure as in the last section, I wish now to consider the semi-invariants of "moments in samples from the combined population, about the mean of the combined sample" and to express them in terms of α_1, α_2, and β_1, β_2, (the semi-invariants if the component distributions $\phi_1(x)$, $\phi_2(x)$ respectively), and r and s.

In order to obtain the desired results, I have made use of a

[1] Loc. Cit. pp. 12-13.

modification and extension of a method originally employed by C. C. Craig[1] for the case of sampling from one normal parent population. I shall first develop the theory for my case, on the basis that the two parent populations may possess non-zero semi-invariants of all orders, imposing the condition of normality only when actually computing the desired results. The mean of the combined sample is

$$\nu_1' = \sum_{i=1}^{r} {}_1x_i + \sum_{j=r+1}^{r+s} {}_2x_j$$

We wish to find the semi-invariants δ_k of $\nu_n \equiv \sum_{i=1}^{N} \frac{\delta_i^n}{N}$ in which $\delta_i = x_i - \nu_1'$ and $r+s=N$, for particular values of n, and for infinitely many sets of $r+s$ variates, assuming that each member of each set is independent of all the rest. The N δ's in each set satisfy $\sum_{i=1}^{N} \delta_i = 0$.

Now, let $F(\delta_1, \delta_2 \cdots \delta_{N-1})$ be the correlation function of the first $N-1$ δ's. Then $F(\delta_1, \delta_2 \cdots \delta_{N-1}) d\delta_1 d\delta_2 \cdots d\delta_{N-1}$ gives the probability that the first $N-1$ δ's fall simultaneously within a cell

$$\left(\delta_1 \pm \tfrac{1}{2} d\delta_1,\ \delta_2 \pm \tfrac{1}{2} d\delta_2, \cdots \cdots, \delta_{N-1} \pm \tfrac{1}{2} d\delta_{N-1}\right)$$

The semi-invariants of $F(\delta_1, \delta_2 \cdots \delta_{N-1})$ are defined by

$$(40)\quad e^{\left(\sum_{i=1}^{N-1} \lambda_i t_i\right) + \frac{1}{2}\left(\sum_{i=1}^{N-1} \lambda_i t_i\right)^{(2)} + \frac{1}{3!}\left(\sum_{i=1}^{N-1} \lambda_i t_i\right)^{(3)} + \cdots}$$

$$= \int d\delta_1 \int d\delta_2 \int_{-\infty}^{\infty} \cdots \int d\delta_{N-1} \cdot F(\delta_1, \delta_2 \cdots \delta_{N-1}) \cdot e^{\sum_{i=1}^{N-1} \delta_i t_i}$$

$$= 1 + \left(\sum_{i=1}^{N-1} \nu_i t_i\right) + \frac{1}{2!}\left(\sum_{i=1}^{N-1} \nu_i t_i\right)^{(2)} + \frac{1}{3!}\left(\sum_{i=1}^{N-1} \nu_i t_i\right)^{(3)} + \cdots\cdots$$

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[1] Loc. Cit. pp. 1 to 35.

where e.g.

$$(41) \quad \left(\sum_{i=1}^{2} \lambda_i t_i\right)^{(2)} = \lambda_{20} t_1^2 + 2\lambda_{11} t_1 t_2 + \lambda_{02} t_2^2 ;$$

Setting

$$\delta_i = \sum_{j=1}^{N} a_{ij} x_j \qquad a_{ij} = -\frac{1}{N},\ i \neq j \cdot$$
$$a_{ii} = \frac{N-1}{N}$$

we have

$$e^{\left(\sum_{i=1}^{N-1} \lambda_i t_i\right) + \frac{1}{2!}\left(\sum_{i=1}^{N-1} \lambda_i t_i\right)^{(2)} + \cdots\cdots}$$

$$= \int dx_1 \int_{-\infty}^{\infty} dx_2 \int \cdots \int dx_N \left[\phi_1(x)\right]^r \left[\phi_2(x)\right]^s \cdot e^{\sum_{i=1}^{N-1} \sum_{j=1}^{N} a_{ij} x_j t_i}$$

$$= e^{\alpha_1\left(\sum_{i=1}^{N-1} a_{i1} t_i\right) + \frac{\alpha_2}{2!}\left(\sum_{i=1}^{N-1} a_{i1} t_i\right)^2 + \cdots\cdots\cdots}$$

$$e^{\alpha_1\left(\sum_{i=1}^{N-1} a_{i2} t_i\right) + \frac{\alpha_2}{2!}\left(\sum_{i=1}^{N-1} a_{i2} t_i\right)^2 + \cdots\cdots\cdots}$$

$$\vdots$$

$$e^{\alpha_1\left(\sum_{i=1}^{N-1} a_{ir} t_i\right) + \frac{\alpha_2}{2!}\left(\sum_{i=1}^{N-1} a_{ir} t_i\right)^2 + \cdots\cdots\cdots}$$

$$e^{\beta_1\left(\sum_{i=1}^{N-1} a_{i,r+1} t_i\right) + \frac{\beta_2}{2!}\left(\sum_{i=1}^{N-1} a_{i,r+1} t_i\right)^2 + \cdots\cdots}$$

$$e^{\beta_1\left(\sum_{i=1}^{N-1} a_{i,r+2} t_i\right) + \frac{\beta_2}{2!}\left(\sum_{i=1}^{N-1} a_{i,r+2} t_i\right)^2 + \cdots\cdots}$$

$$\vdots$$

$$e^{\beta_1\left(\sum_{i=1}^{N-1} a_{i,r+s} t_i\right) + \frac{\beta_2}{2!}\left(\sum_{i=1}^{N-1} a_{i,r+s} t_i\right)^2 + \cdots\cdots}$$

in which α_1, α_2, etc., are the 1st, 2nd, etc. semi-invariants of the first component $\phi_1(x)$, and β_1, β_2, etc. are the corresponding semi-invariants for the second component $\phi_2(x)$. It follows then, that

$$\left(\sum_{i=1}^{N-1} \lambda_i t_i\right)^{(k)} = \alpha_k \left\{\left(\sum_{i=1}^{N-1} a_{i1} t_i\right)^k + \left(\sum_{i=1}^{N-1} a_{i2} t_i\right)^k + \cdots + \left(\sum_{i=1}^{N-1} a_{ir} t_i\right)^k\right\}$$

$$+\beta_k \left\{\left(\sum_{i=1}^{N-1} a_{i,r+1} t_i\right)^k + \left(\sum_{i=1}^{N-1} a_{i,r+2} t_i\right)^k + \cdots + \left(\sum_{i=1}^{N-1} a_{i,r+s} t_i\right)^k\right\}$$

or

$$\lambda_{k_1 k_2 \cdots k_{N-1}} = \alpha_k \left\{ a_{11}^{k_1} a_{21}^{k_2} \cdots a_{N-1,1}^{k_{N-1}} + a_{12}^{k_1} a_{22}^{k_2} \cdots a_{N-1,2}^{k_{N-1}} + \cdots + a_{1r}^{k_1} a_{2r}^{k_2} \cdots a_{N-1,r}^{k_{N-1}} \right\}$$

$$+ \beta_k \left\{ a_{1,r+1}^{k_1} a_{2,r+1}^{k_2} \cdots a_{N-1,r+1}^{k_{N-1}} + a_{1,r+2}^{k_1} a_{2,r+2}^{k_2} \cdots a_{N-1,r+2}^{k_{N-1}} + \cdots \right.$$

$$\left. + a_{1,r+s}^{k_1} a_{2,r+s}^{k_2} \cdots a_{N-1,r+s}^{k_{N-1}} \right\},$$

where $k_1 + k_2 + \cdots\cdots + k_{N-1} = k$.

so that

$$\lambda_{k_1 k_2 \cdots k_{N-1}} = \alpha_k \left\{ \sum_{j=1}^{r} a_{1j}^{k_1} a_{2j}^{k_2} \cdots\cdots a_{N-1,j}^{k_{N-1}} \right\}$$

$$+\beta_k \left\{ \sum_{j=r+1}^{r+s} a_{1j}^{k_1} a_{2j}^{k_2} \cdots\cdots a_{N-1,j}^{k_{N-1}} \right\},$$

By substituting for the a's from the relation (41) in the last equation, the latter may be reduced to the following convenient form, where, now, $\lambda_{k_1k_2\cdots k_{N-1}} = \lambda_{k_1k_2\cdots\cdots k_N}$, provided that, at least one of the k_i in $\lambda_{k_1k_2'\cdots k_N}$ is equal to zero.

$$
\begin{aligned}
\lambda_{k_1k_2\cdots k_N} = \frac{1}{N^k}\Bigg\{ & \alpha_k\left(\sum_{i=1}^{r}(-1)^{k-k_i}(N-1)^{k_i}\right) \\
(42) \qquad & +\beta_k\left(\sum_{i=r+1}^{r+s}(-1)^{k-k_i}(N-1)^{k_i}\right)\Bigg\}.
\end{aligned}
$$

In the case of one parent population, all $\lambda_{k_1k_2\cdots k_N}$ of the same type, i.e. whose subscripts $k_1k_2\cdots\cdots k_N$ are merely different permutations of the same set of integers $k_1, k_2, \ldots, k_N$, were equal. This is no longer true for two parent populations, for we must now distinguish between the δ's which arise from observations from the population $\phi_1(x)$, and those from $\phi_2(x)$. I therefore introduce, at this point, what I shall call the "bar notation". For example, from the relation (42), all $\lambda_{k_1k_2\cdots k_r|k_{r+1}\cdots k_{r+s}}$ will be of the same type, and therefore equal, if the first r subscripts are merely different permutations of the same set of integers $k_1, k_2, \cdots k_r$ whilst, quite apart from the first r subscripts, the last s subscripts are also different permutations of the same set of integers k_{r+1}, $k_{r+2}, \ldots k_{r+s}$. In writing down the semi-invariants of the correlation function F, using the "bar notation", for convenience, I shall suppress zero subscripts. Further, on account of the relation (42), all $\lambda_{k_1k_2\cdots k_r|k_{r+1}\cdots k_{r+s}}$ for which $k_1+k_2+\cdots k_r+k_{r+1}+\cdots+k_{r+s}\geq 3$ will vanish, since we are assuming now, that our two parent populations are normal. As a matter of fact, the only $\lambda_{k_1k_2\cdots k_r|k_{r+1}\cdots k_{r+s}}$ that I shall require here are the following,

$$\lambda_{1|0} = \frac{1}{N}\left\{N\alpha_1\right\}.$$

$$\lambda_{0|1} = \frac{1}{N}\left\{N\beta_1\right\}.$$

$$\lambda_{2|0} = \frac{1}{N^2}\left\{r\alpha_2 + s\beta_2 + N(N-2)\alpha_2\right\}.$$

$$\lambda_{0|2} = \frac{1}{N^2}\left\{r\alpha_2 + s\beta_2 + N(N-2)\beta_2\right\}.$$

$$\lambda_{11|0} = \frac{1}{N^2}\left\{r\alpha_2 + s\beta_2 - 2N\alpha_2\right\}.$$

$$\lambda_{0|11} = \frac{1}{N^2}\left\{r\alpha_2 + s\beta_2 - 2N\beta_2\right\}.$$

$$\lambda_{1|1} = \frac{1}{N^2}\left\{r\alpha_2 + s\beta_2 - N(\alpha_2 + \beta_2)\right\}.$$

The above expressions were obtained from (42), after assuming, (without loss of generality,) that $r\alpha_1 + s\beta_1 = 0$. If, instead of this last assumption, I had assumed that $\lambda_{1|0}$ or $\lambda_{0|1}$ were equal to zero, not only would the symmetry of the final results have been destroyed, but the amount of labour necessary to obtain them would also have been doubled. The symmetric substitution actually made, required that only half the final number of terms be obtained, the remaining half in any particular result being readily written down by interchanging the α's and β's as well as r and s.

Now, let $P(V_n)$ be the probability function for $V_n \equiv \sum_{i=1}^{N} \frac{\delta_i^n}{N}$

The semi-invariants of $P(V_n)$ are then defined by

$$(43)\quad \begin{aligned} e^{S_1 t + S_2 \frac{t^2}{2!} + S_3 \frac{t^3}{3!} + \cdots} &= \int_{-\infty}^{\infty} P(V_n)\cdot e^{V_n \cdot t}\cdot dV_n \\ &= \int d\delta_1 \int_{-\infty}^{+\infty} d\delta_2 \cdots \int d\delta_N \cdot F(\delta_1, \delta_2 \cdots \delta_N)\cdot e^{\sum_{i=1}^{N} \delta_i^n \frac{t}{N}} \end{aligned}$$

Regarding the use of $F(\delta_1, \delta_2, \cdots \delta_N)$ instead of $F(\delta_1, \delta_2, \cdots \delta_{N-1})$ in the above relation, see paper by C. C. Craig.[1]

We wish now to express the semi-invariants S_K in terms of the semi-invariants $\lambda_{K_1 K_2 \cdots\cdots K_N}$ of the correlation function $F(\delta_1, \delta_2 \cdots\cdots \delta_N)$ for the δ's. The semi-invariants $L_{rst\ldots\ldots}$ of the correlation function for δ_i^n are defined by

$$(44)\quad \begin{aligned} e^{\left(\sum_{i=1}^{N} L_i t_i\right) + \frac{1}{2!}\left(\sum_{i=1}^{N} L_i t_i\right)^{(2)} + \frac{1}{3!}\left(\sum_{i=1}^{N} L_i t_i\right)^{(3)} + \cdots\cdots} \\ = \int d\delta_1 \int d\delta_2 \int_{-\infty}^{+\infty} \cdots\cdots \int d\delta_N F(\delta_1, \delta_2 \cdots \delta_N)\cdot e^{\sum_{i=1}^{N} \delta_i^n t_i} \\ = 1 + \left(\sum_{i=1}^{N} V_{ni} t_i\right) + \left(\sum_{i=1}^{N} V_{ni} t_i\right)^{(2)} + \cdots\cdots \end{aligned}$$

by expansion of the exponential function. Then, comparing the

[1] Loc. Cit. pp. 18 to 19.

relations in (43) and (44). it is readily seen that

$$(45) \qquad S_k = \frac{1}{N^k} \sum \frac{k!}{k_1!\, k_2! \cdots\cdots k_r!} \cdot L_{k_1 k_2 \cdots k_r}$$

in which the summation is taken over all values of $k_1, k_2, \ldots\, k_r$, such that

$$k_1 + k_2 + \cdots + k_r = k.$$

Making use of the explicit relations for semi-invariants in terms of moments and vice versa, we have from (44) and (40)

$$(46) \qquad \left(\sum_{i=1}^{N} L_i t_i\right)^{(k)} = \sum\sum\cdots\sum \frac{(-1)^{(r+s+t+\cdots)-1}\left[(r+s+t+\cdots)-1\right]!\,k!}{(a!)^r (b!)^s (c!)^t \cdots\cdots} \times \frac{\left[\left(\sum_{i=1}^{N} \nu_{ni} t_i\right)^{(a)}\right]^r \left[\left(\sum_{i=1}^{N} \nu_{ni} t_i\right)^{(b)}\right]^s \cdots\cdots}{r!\, s!\, t!}$$

$$(47) \qquad \left(\sum_{i=1}^{N} \nu_i t_i\right)^{(n)} = \sum\sum\cdots\sum \frac{n!\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(a)}\right]^r \cdot \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(b)}\right]^s \cdots\cdots}{(a!)^r (b!)^s (c!)^t \cdots \quad r!\, s!\, t! \cdots\cdots}$$

where, in both these relations

$$a > b > c > \cdots\cdots\cdots$$

and

$$ar + bs + ct + \cdots = n$$

From the relations (46) and (47), the L_n can be found in terms of the moments of F, and these, in turn, can be found in terms of the semi-invariants of F, by equating the coefficients of like powers of the t's on both sides of the two equations. Examples of the kind of relations obtained from (46) and (47) in particular cases would be as follows,

$$\left(\sum_{i=1}^{N} L_i t_i\right)^{(3)} = \left(\sum_{i=1}^{N} \nu_{2i} t_i\right)^{(3)} - 3\left(\sum_{i=1}^{N} \nu_{2i} t_i\right)^{(2)} \left(\sum_{i=1}^{N} \nu_{2i} t_i\right) + 2\left(\sum_{i=1}^{N} \nu_{2i} t_i\right).$$

Therefore

$$L_{210\ldots0} = \nu_{420\ldots0} - \nu_{40\ldots0} \cdot \nu_{020\ldots0}$$

$$- 2\nu_{220\ldots0}\, \nu_{20\ldots0} + 2\nu^2_{20\ldots0}\, \nu_{020\ldots0}.$$

$$\left(\sum_{i=1}^{N} \nu_i t_i\right)^{(6)} = \left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(6)} + 15\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(4)} \left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}$$

$$+ 10\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(3)}\right]^2 + 15\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^3.$$

Therefore

$$\nu_{420\ldots0} = \lambda_{420\ldots0} + \lambda_{40\ldots0} \cdot \lambda_{020\ldots0} + 8\lambda_{310\ldots0}\lambda_{110\ldots0}$$

$$+ 6\lambda_{220\ldots0}\,\lambda_{20\ldots0} + 6\lambda^2_{210\ldots0} + 4\lambda_{30\ldots0}\,\lambda_{120\ldots0}$$

$$+ 3\lambda^2_{20\ldots0} \cdot \lambda_{020\ldots0} + 12\lambda_{20\ldots0} \cdot \lambda^2_{110\ldots0}.$$

In my work, I actually make use of the following relations obtained from (47), with certain terms omitted, which vanish when each of the parent distributions is normal.

(48)

(i) $\left(\sum_{i=1}^{N} V_i t_i\right) = \left(\sum_{i=1}^{N} \lambda_i t_i\right).$

(ii) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(2)} = \left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)} + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(1)}\right]^2.$

(iii) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(3)} = 3\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\left(\sum_{i=1}^{N} \lambda_i t_i\right) + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^3.$

(iv) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(4)} = 3\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^2 + 6\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^2 + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^4.$

(v) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(5)} = 15\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^2\left(\sum_{i=1}^{N} \lambda_i t_i\right)$

$+10\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^3\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)} + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^5$

(vi) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(6)} = 45\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^2\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^2$

$+15\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^4\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)} + 15\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^3 + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^6.$

(vii) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(7)} = 105\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^2\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^3$

$+21\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^5 + 105\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^3\left(\sum_{i=1}^{N} \lambda_i t_i\right) + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^7.$

(viii) $\left(\sum_{i=1}^{N} V_i t_i\right)^{(8)} = 210\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^2\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^4 + 28\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)} \times$

$\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^6 + 420\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^3\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^2 + 105\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}\right]^4 + \left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^8.$

By substituting the expressions for the L's given in (46), into the right member of (45), a direct expression for the S_K's in terms of the moments of F is obtained, viz.

$$S_k = \frac{1}{N^k}\Sigma\ldots\Sigma L_{k_1k_2\ldots k_r}\frac{k!}{k_1!\,k_2!\ldots k_r!} = \frac{1}{N^k}\left(\sum_{i=1}^{N} L_i\right)^{(k)}.$$

omitting the parameters t_i, and in which $\left(\sum_{i=1}^{N} L_i\right)^{(k)}$ is given by (46). From relation (43) to this point, the theory follows exactly that given in the paper already cited.

I shall next quote the final expressions A for some of the $S_k(v_n)$ obtained by C. C. Craig,[1] in terms of the moments of F, for the case of one parent population, and then I shall write down the modified expressions B, when two parent populations are involved

A. 1. $S_1(v_n) = \frac{1}{N}\left\{N v_{n,0}\right\}$.

2. $S_2(v_n) = \frac{1}{N^2}\left\{N v_{2n,0} + N(N-1)v_{n,n,0} - N^2 v^2_{n,0}\right\}$.

B. 1. $S_1(v_n) = \frac{1}{N}\left\{r v_{n|0} + s v_{0|n}\right\}$

2. $$S_2(v_n) = \frac{1}{N^2}\Big\{r v_{2n|0} + s v_{0|2n} + v(r-1)v_{n,n|0} + s(s-1)v_{0|n,n}$$

$$+ 2rs v_{n|n}\Big\} - \left[r^2 v^2_{n|0} + s^2 v^2_{0|n} + 2rs v_{n|0}\cdot v_{0|n}\right].$$

In the paper mentioned above, expressions were also derived, using a method similar to the one already indicated, for the semi-invariants of the correlation function of two moments about the mean. I have made use of a modification of only one of these expressions as follows,

[1] Loc. Cit. p. 22.

$$\text{A. } S_{11}(\nu_m, \nu_n) = \frac{1}{N^2}\left\{N\nu_{m+n,0} + N(N-1)\nu_{m,n,0} - N^2\nu_{m,0}\nu_{n,0}\right\}$$

$$\text{B. } S_{11}(\nu_m, \nu_n) = \frac{1}{N^2}\Big\{r\nu_{m+n|0} + s\nu_{0|m+n} + r(r-1)\nu_{m,n|0}$$

$$+ s(s-1)\nu_{0|m,n} + rs\,\nu_{m|n} + rs\,\nu_{n|m}$$

$$-\left[r^2\nu_{m|0}\,\nu_{n|0} + s^2\nu_{0|m}\,\nu_{0|n} + rs\,\nu_{m|0}\nu_{0|n} + rs\,\nu_{0|m}\,\nu_{n|0}\right]\Big\}.$$

Here again, for the moments of the correlation function, I employ the "bar notation", its meaning being exactly the same as in the case of the $\lambda_{k_1k_2\cdots k_r|k_{r+1}\cdots k_{r+s}}$, the discussion regarding identical types of moments and their equality, corresponding also in every detail. Once more, zero subscripts are suppressed.

It now becomes necessary to express the modified moments in the expressions B, for particular values of m and n, in terms of the $\lambda_{k_1k_2\cdots k_r|k_{r+1}\cdots\cdots k_{r+s}}$ of the correlation function F. To this end, I make use of the relations (48), in conjunction with the so-called "D_s operator of Hammond"[1] which splits off a total integral part s, made up by addition from any or all of a permutation of integers.

At this point also, it is necessary to modify somewhat the use of the D_s operator, because of the "bar notation" used to designate the moments and the semi-invariants of the correlation function F, when two parent populations are being considered. In making up the total integral part s, split off from the permutation of integers

[1] MacMahon—Combinatorial Analysis, Vol. I, p. 27.

$k_1 k_2 \ldots\ldots k_r \mid k_{r+1} \ldots\ldots k_{r+s}$, the parts split off from the set of integers $k_1 k_2 \ldots\ldots k_r$ must be kept distinct from those parts which are split off from the set of integers $k_{r+1}, k_{r+2}, \ldots k_{r+s}$, and this same rule applies also to the residual permutations from each of these two sets, after all the parts, with sum s have been finally split off. Hence, the use of the "bar notation" to effect this distinction. To illustrate exactly what is meant here, suppose that I wish to express $\nu_{3|2}$ in terms of $\lambda_{k_1 k_2 \cdots k_r \mid k_{r+1} \cdots k_{r+s}}$. In this case, I shall use the relation (v) of the set of equations (48), and I shall merely consider the contribution made by the second term in the right member of (v) to the final expression for $\nu_{3|2}$, the other terms of (v) being treated in a similar manner. Now $\nu_{3|2}$ (omitting a numerical factor) is the coefficient of $t_1^3 t_2^2$ in the left member of (v). I therefore seek the corresponding coefficient of $t_1^3 t_2^2$ in the second term of the right member of (v) this term being $10\left[\left(\sum_{i=1}^{N} \lambda_i t_i\right)\right]^3 \left(\sum_{i=1}^{N} \lambda_i t_i\right)^{(2)}$. Using the modified form of the D_s operator, we have

$$D_2 D_1^3 (3|2) = (2|0) D_1^2 (1|2) + (0|2) D_1^2 (3|0) + (1|1) D_1^2 (2|1)$$

$$= 3(2|0)(1|0)(0|1)^2 + (0|2)(1|0)^3 + 3(1|1)(1|0)^2(0|1)$$

Now, we are able to write down immediately the terms in $\lambda_{k_1 k_2 \cdots k_r \mid k_{r+1} \cdots\cdots\cdots k_{r+s}}$ which arise. They are

$$(49) \qquad 3\lambda_{2|0}\lambda_{1|0}\lambda_{0|1}^2 + \lambda_{0|2}\lambda_{1|0}^3 + 6\lambda_{1|1}\lambda_{1|0}^2\lambda_{0|1}$$

Ordinarily, the numerical coefficients in (49), will need to be multiplied by an integral factor, obtained as follows. A term $t_1^3 t_2^2$ may be chosen from the expansion of $\left(\sum_{i=1}^{N} v_i t_i\right)^{(5)}$ in $\frac{5!}{3!\,2!}$ or 10 (in general, in C_1) ways. The numerical coefficient of the second term in the right member of (V) is also 10 (and in general, is, say C_2). The required factor for the above example is unity (or, in general, the quotient $\frac{C_2}{C_1}$. It should be noticed in addition, that the sum of the coefficients in the final expression (49) should equal the numerical coefficient of the second term in the right member of (V), with which we started, and this is seen to be the case.

As a check, one may observe, that if in the results which I obtain for $S_k(v_n), S_{k\ell}(v_m, v_n)$, the two normal parent populations are identified, then the results for a single normal parent population are obtained. Note that, to get the results as usually given for a single normal parent population, one would further have to set $\alpha_1 = \beta_1 = 0$.

I derived the following results, which have been checked by calculating the corresponding results for a single normal parent population, without assuming that the first order semi-invariants of the type $\lambda_{100\cdots 00}$ are equal to zero.

$$s_1(v_2) = \frac{1}{N^2}\left\{(N-1)(r\alpha_2 + s\beta_2) + N(r\alpha_1^2 + s\beta_1^2)\right\}.$$

$$s_1(v_3) = \frac{1}{N^2}\left\{3(N-2)(r\alpha_1\alpha_2 + s\beta_1\beta_2) + N(r\alpha_1^3 + s\beta_1^3)\right\}.$$

$$s_1(v_4) = \frac{1}{N^3}\left\{\left[3Nr\left\{N(N-2)^2 + r(2N-3)\right\}\right]\alpha_2^2 + \left[3Ns\left\{N(N-2)^2 + s(2N-3)\right\}\right]\beta_2^2\right.$$

$$+N^4(r\alpha_1^4+s\beta_1^4)+6N^2\left[r(N^2-2N+r)\alpha_1^2\alpha_2+s(N^2-2N+s)\beta_1^2\beta_2\right]$$

$$+\left[6Nrs(2N-3)\right]\alpha_2\beta_2+6N^2rs\left[\alpha_1^2\beta_2+\alpha_2\beta_1^2\right]\Big\}$$

$$s_2(v_2)=\frac{2}{N^4}\Big\{r(N^2-2N+r)\alpha_2^2+s(N^2-2N+s)\beta_2^2+2rs\Big[(r\alpha_1^2\beta_2+s\alpha_2\beta_1^2)$$

$$+\alpha_2\beta_2+(s\alpha_1^2\alpha_2+r\beta_1^2\beta_2)-2(s\alpha_1\alpha_2\beta_1+r\alpha_1\beta_1\beta_2)\Big]\Big\}.$$

$$s_2(v_3)=\frac{3}{N^6}\Big\{6r\left[N^4-N^3+N^2s^2-s(r^2+5Ns)\right]\alpha_1^2\alpha_2^2+6s\Big[N^4-N^3+N^2r^2$$

$$-r(s^2+5Nr)\Big]\beta_1^2\beta_2^2+3N^2rs\Big[(s\alpha_1^4\alpha_2+r\beta_1^4\beta_2)+(r\alpha_1^4\beta_2+s\alpha_2\beta_1^4)$$

$$-2(s\alpha_1^2\alpha_2\beta_1^2+r\alpha_1^2\beta_1^2\beta_2)\Big]+3rs\Big[4(s^2\alpha_1\alpha_2^2\beta_1+r^2\alpha_1\beta_1\beta_2^2)$$

$$+8rs(\alpha_1\alpha_2\beta_1\beta_2)-2\left\{s(N^2-2N-r)\alpha_2^2\beta_1^2+r(N^2-2N-s)\alpha_1^2\beta_2^2\right\}$$

$$+2\Big\{\left[N(6-N)(r-s)+2r^2\right]\alpha_2\beta_1^2\beta_2+\left[N(6-N)(s-r)+2s^2\right]\alpha_1^2\alpha_2\beta_2\Big\}$$

$$+\Big\{\left[r(20-12N+3N^2)-N(2N^2-10N+16)\right]\alpha_2^2\beta_2+\Big[s(20-12N+3N^2)$$

$$-N(2N^2-10N+16)\Big]\alpha_2\beta_2^2\Big\}\Big]+r\Big[r^2(3N^2-12N+20)-Nr(6N^2-30N+48)$$

$$+N^2(5N^2-24N+32)\Big]\alpha_2^3+s\Big[s^2(3N^2-12N+20)-Ns(6N^2-30N+48)$$

$$+N^2(5N^2-24N+32\Big]\beta_2^3\Big\}.$$

$$s_2(v_4)=\frac{8}{N^8}\Big\{3N^2r(7N^4-20N^3-8N^3r+38N^2r+4N^2r^2-28Nr^2+7r^3)\alpha_1^4\alpha_2^2$$

$$+3N^2s(7N^4-20N^3-8N^3s+38N^2s+4N^2s^2-28Ns^2+7s^3)\beta_1^4\beta_2^2$$

$$+2N^4rs\Big[(s\alpha_1^6\alpha_2+r\beta_1^6\beta_2)+(r\alpha_1^6\beta_2+s\alpha_2\beta_1^6)-2(s\alpha_1^3\alpha_2\beta_1^3$$

$$+r\alpha_1^3\beta_1^3\beta_2)\Big]+3N^2r^2s^2\Big[7(\alpha_1^4\beta_2^2+\alpha_2^2\beta_1^4)-4(\alpha_1\beta_1^3\beta_2^2$$

$$+\alpha_1^3\alpha_2^2\beta_1)+12(\alpha_1^2\alpha_2\beta_1^2\beta_2)\Big]+18N^2rs(s^2\alpha_1^2\alpha_2^2\beta_1^2+r^2\alpha_1^2\beta_1^2\beta_2^2)$$

$$+12N^2rs\Big[(N^2-N^2r-2s^2)\alpha_1\alpha_2\beta_1^3\beta_2+(N^2-N^2s-2r^2)\alpha_1^3\alpha\beta_1\beta_2\Big]$$

$$+6N^2rs\Big[(5N^2+2N^2r-14Nr+7r^2)\alpha_1^4\alpha_2\beta_2+(5N^2+2N^2s$$

$$-14Ns+7s^2)\alpha_2\beta_1^4\beta_2\Big]+12rs\Big[(-2N^3-3N^2rs+12N^2r$$

$$-15Nr^2+5r^3)\alpha_1\alpha_2^3\beta_1+(-2N^3-3N^2rs+12N^2s-15Ns^2$$

$$+5s^3)\alpha_1\beta_1\beta_2^3\Big]+12N^2rs\Big[s(-N^2+2N-r)\alpha_1\alpha_2^2\beta_1^3+r(-N^2$$

$$+2N-s)\alpha_1^3\beta_1\beta_2^2\Big]+6rs\Big[(3N^2r^2-6N^3+15N^2s-12Ns^2+3s^3$$

$$+8rs^2)\alpha_1^2\beta_2^3+(3N^2s^2-6N^3+15N^2r-12Nr^2+3r^3$$

$$+8r^2s)\alpha_2^3\beta_1^2\Big]+36rs\Big[(-5rs^2-N^4s-N^2r^2+3N^3s-2N^3$$

$$+3N^2r)\alpha_1\alpha_2^2\beta_1\beta_2+(-5r^2s-N^4r-N^2s^2+3N^3r-2N^3$$

$$+3N^2s)\alpha_1\alpha_2\beta_1\beta_2^2\Big]+9rs(9N^3-22N^2-36Nrs+72rs$$

$$+6N^2rs)\alpha_2^2\beta_2^2+18rs\Big[(5r^2+7N^2r^2-14N^3r+N^4r+8N^4$$

$$-29Nr^2+43N^2r-21N^3)\alpha_1^2\alpha_2^2\beta_2+(5s^3+7N^2s^2-14N^3s$$

$$+N^4s+8N^4-29Ns^2+43N^2s-21N^3)\alpha_2\beta_1^2\beta_2^2\Big]+18rs\Big[(N^3$$

$$+4N^2s+5N^2rs+5r^2s-19Nrs)\alpha_1^2\alpha_2\beta_2^2+(N^3+4N^2r$$

$$+5N^2rs+5rs^2-19Nrs)\alpha_2^2\beta_1^2\beta_2\Big]+3r(36r^3-18Nr^3$$

$$+75N^2r^2+3N^2r^3-12N^3r^2-126N^3r+28N^4r+84N^4$$

$$-32N^5+4N^6-126Nr^2+162N^2r-78N^3)\alpha_2^4+3s(36s^3$$

$$-16Ns^3+75N^2s^2+3N^2s^3-12N^3s^2-126N^3s+28N^4s+84N^4$$

$$-32N^5+4N^6-126Ns^2+162N^2s-78N^3)\beta_2^4+6r(5r^4-39Nr^3$$

$$+102N^2r^2+3N^4r^2+9N^2r^3-36N^3r^2-121N^3r+57N^4r$$

$$-6N^3r+60N^4-42N^5+8N^6)\alpha_1^2\alpha_2^3+6s(5s^4-39Ns^3$$

$$+102N^2s^2+3N^4s^2+9N^2s^3-36N^3s^2-121N^3s+57N^4s$$

$$-6N^5s+60N^4-42N^5+8N^6)\beta_1^2\beta_2^3+6rs(72r^2+66N^2-51N^3$$

$$+75N^2r-36Nr^2-126Nr+6N^2r^2+11N^4-12N^3r)\alpha_2^3\beta_2$$

$$+6rs(72s^2+66N^2-51N^3+75N^2s-36Ns^2-126Ns+6N^2s^2$$

$$+11N^4-12N^3s)\alpha_2\beta_2^3\Big\}.$$

$$s_{11}(v_2,v_3)=\frac{6}{N^5}\Big\{r\big[s^2(N-3)+N(N^2-N-s)\big]\alpha_1\alpha_2^2+s\big[r^2(N-3)$$

$$+N(N^2-N-r)\big]\beta_1\beta_2^2+rs\big[s(3-N)\alpha_2^2\beta_1+r(3-N)\alpha_1\beta_2^2\big]$$

$$+rs\Big[\{(N-2)(r-s)+2s\}\alpha_1\alpha_2\beta_2+\{(N-2)(s-r)+2r\}\alpha_2\beta_1\beta_2\Big]$$

$$+Nrs\Big[(s\alpha_1^3\alpha_2+r\beta_1^3\beta_2)-(s\alpha_1^2\alpha_2\beta_1+r\alpha_1\beta_1^2\beta_2)$$

$$-(r\alpha_1^2\beta_1\beta_2+s\alpha_1\alpha_2\beta_1^2)+(r\alpha_1^3\beta_2+s\alpha_2\beta_1^3)\Big]\Big\}.$$

$$s_{11}(v_2,v_4)=\frac{4}{N^6}\Big\{3r(13N^2r-7N^3+2N^2r^2+3N^4-4N^3r+2r^3$$

$$-9Nr^2)\alpha_1^2\alpha_2^2+3s(13N^2s-7N^3+2N^2s^2+3N^4-4N^3s$$

$$+2s^3-9Ns^2)\beta_1^2\beta_2^2+3r(6N^2+N^4+3r^2-5N^3-Nr^2+4N^2r$$

$$-8Nr)\alpha_2^3+3s(6N^2+N^4+3s^2-5N^3-Ns^2+4N^2s-8Ns)\beta_2^3$$

$$+6rs(2r^2-4Nr+N^2r+2N^2-N^3)\alpha_1\alpha_2^2\beta_1+6rs(2s^2$$

$$-4Ns+N^2s+2N^2-N^3)\alpha_1\beta_1\beta_2^2+2N^2rs\big[(s\alpha_1^4\alpha_2+r\beta_1^4\beta_2)$$

$$-(s\alpha_1^3\alpha_2\beta_1+r\alpha_1\beta_1^3\beta_2)-(s\alpha_1\alpha_2\beta_1^3+r\alpha_1^3\beta_1\beta_2)$$

$$
\begin{aligned}
&+(r\alpha_1^4\beta_2+s\alpha_2\beta_1^4)\Big]+3rs\Big[2(4rs-N^3)\alpha_1\alpha_2\beta_1\beta_2+2(2r^2-5Nr\\
&+2N^2+N^2r)\alpha_1^2\alpha_2\beta_2+2(2s^2-5Ns+2N^2+N^2s)\alpha_2\beta_1^2\beta_2\\
&+(N^2-2Ns+3rs+s^2)\alpha_1^2\beta_2^2+(N^2-2Nr+3rs+r^2)\alpha_2^2\beta_1^2\\
&+(9r-8N+4N^2-3Nr)\alpha_2^2\beta_2+(9s-8N+4N^2-3Ns)\alpha_2\beta_2^2\Big]\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
s_{11}(v_3,v_4)=\frac{12}{N^7}\Big\{&Nr(6r^3-26Nr+37N^2r+4N^2r^2-8N^3r-20N^3\\
&+7N^4)\alpha_1^3\alpha_2^2+Ns(6s^3-26Ns^2+37N^2s+4N^2s^2-8N^3s\\
&-20N^3+7N^4)\beta_1^3\beta_2^2+r(75Nr^2-107N^2r+58N^3-19r^3\\
&+6Nr^3-30N^2r^2+54N^3r+3N^3r^2-6N^4r-42N^4\\
&+8N^5)\alpha_1\alpha_2^3+s(75Ns^2-107N^2s+58N^3-19s^3+6Ns^3\\
&-30N^2s^2+54N^3s+3N^3s^2-6N^4s-42N^4+8N^5)\beta_1\beta_2^3\\
&+Nrs\Big[(-N^2r+2N^2-2s^2+4rs)\alpha_1^3\beta_2^2+(-N^2s+2N^2-2r^2\\
&+4rs)\alpha_2^2\beta_1^3\Big]+N^3rs\Big[(s\alpha_1^5\alpha_2+r\beta_1^5\beta_2)-(s\alpha_1^3\alpha_2\beta_1^2\\
&+r\alpha_1^2\beta_1^3\beta_2)-(r\alpha_1^3\beta_1^2\beta_2+s\alpha_1^2\alpha_2\beta_1^3)+(r\alpha_1^5\beta_2+s\alpha_2\beta_1^5)\Big]\\
&+Nrs\Big[(-N^3+11N^2+5N^2r-16r^2-28rs)\alpha_1^3\alpha_2\beta_2+(-N^3\\
&+11N^2+5N^2s-16s^2-28rs)\alpha_2\beta_1^3\beta_2\Big]+3Nrs\Big[(-N^2s\\
&+3N^2-5Nr+2r^2)\alpha_1\alpha_2^2\beta_1^2+(-N^2r+3N^2-5Ns\\
&+2s^2)\alpha_1^2\beta_1\beta_2^2\Big]+3Nrs\Big[(-N^2r+N^2-2Ns+4rs)\alpha_1\alpha_2\beta_1^2\beta_2\\
&+(N^2s+N^2-2Nr+4rs)\alpha_1^2\alpha_2\beta_1\beta_2\Big]+3Nrs\Big[s(N\\
&-2r)\alpha_1^2\alpha_2^2\beta_1+r(N-2s)\alpha_1\beta_1^2\beta_2^2\Big]+3rs(19r^2+41Nr\\
&-23N^2+6Nr^2-17N^2r+11N^3+2N^3r-N^4)\alpha_1\alpha_2^2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
&+3rs(-19s^2+41Ns-23N^2+6Ns^2-17N^2s+11N^3\\
&+2N^3s-N^4)\alpha_2\beta_1\beta_2^2+3rs(-19rs+6Nrs+7Ns\\
&-N^2-N^3r-N^2s+3N^2r)\alpha_1\alpha_2\beta_2^2+3rs(-19rs+6Nrs\\
&+7Nr-N^2-N^3s-N^2r+3N^2s)\alpha_2^2\beta_1\beta_2+rs\Big[(3N^3-9N^2s\\
&+19rs-8Nr+6N^2s)\alpha_1\beta_2^3+(3N^3-9N^2+19rs-8Ns\\
&+6Nr^2)\alpha_2^3\beta_1\Big]\Big\}.
\end{aligned}
$$

G W Brown