COMBINING TWO PROBABILITY FUNCTIONS

By

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The object of this paper is to show results which arise from combining two probability functions in finding the probability function for the sum of two independent variables. The first part presents the sum function when the probability law for each individual variable is "one-half" of the Pearson Type X law. From this law arise certain ideas concerning the Beta function which are not presented by texts treating this subject.

The second part presents some peculiar probability functions when special laws for the individual variables are considered. Here certain laws with infinite discontinuities are combined.

I. The probability function for the sum of n variables when each is subject to the function e^{-x}.

Let the probability that the chance variable x_1 lies in the interval (x_1, x_1+dx_1) be to within infinitesimals of higher order $f(x_1)\,dx_1$ and the probability that the chance variable x_2 lies in the interval (x_2, x_2+dx_2) be to within infinitesimals of higher order $g(x_2)\,dx_2$, where x_1 and x_2 may have respectively any real value.

By a well known theorem, the probability that the sum, $x_1+x_2=z$, lies in the interval $(z, z+dz)$ is, to within infinitesimals of higher order,

$$F(z)\,dz = \int_{-\infty}^{\infty} f(x_1)\cdot g(z-x_1)\,dx_1 \cdot dz.$$

Let

$$\begin{aligned} f(x_1) &= e^{-x_1} && \text{for} \quad (0,\infty) \\ &= 0 && \text{elsewhere,} \end{aligned}$$

and

$$\begin{aligned} g(x_2) &= e^{-x_2} && \text{for} \quad (0,\infty) \\ &= 0 && \text{elsewhere.} \end{aligned}$$

According to the above theorem, the probability function for the sum, $x_1 + x_2 = z$, is

$$
\begin{aligned}
F_2(z) &= \int_0^z e^{-x_1} e^{-(z-x_1)} dx_1 \\
&= z e^{-z} \quad \text{for } (0,\infty) \\
&= 0 \quad \text{elsewhere,}
\end{aligned}
$$

which is a Pearson Type III function. The probability functions or laws for x_1 and x_2 are discontinuous at the origin, while the law for the sum is continuous from minus infinity to plus infinity.

By using $F_2(y)$ and $g(x_3)$, the frequency function for the sum, $x_1 + x_2 + x_3 = z$, is

$$
\begin{aligned}
F_3(z) &= \int_0^z x e^{-x} e^{-(z-x)} dx \\
&= z^2 e^{-z}/2 \quad \text{for } (0,\infty) \\
&= 0 \quad \text{, elsewhere; where } x_1 + x_2 = y.
\end{aligned}
$$

In general, if the probability function for the individual variable x_i is

$$
\begin{aligned}
f_i(x_i) &= e^{-x_i}, \quad \text{for } (0,\infty) \\
&= 0 \quad \text{, elsewhere,}
\end{aligned}
$$

then the probability function for the sum, $\sum_{i=1}^{n} x_i = z$ is

$$
\begin{aligned}
F_n(z) &= (z^{n-1} e^{-z})/(n-1)! \quad \text{for } (0,\infty) \\
&= 0 \quad \text{elsewhere.}
\end{aligned}
$$

This is also a Type III law. Others have studied this law and have obtained functions for the sum and the average.[1]

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[1] Mayr—Wahrscheinlichkeitsfunktionen und ihre Anwendungen—Monatshefte für Math. und Phys., Vol. 30, 1920. p. 20.

Church—On the mean and squared standard deviation of small samples from any population—Biometrika, Vol. 18, 1926. pp. 321-394.

Irwin—On the frequency distribution of the means of samples from a population having any law of frequency with finite moments, with special reference to Pearson Type II—Biometrika, Vol. 19, 1927. pp. 225-239.

C. C. Craig—Sampling when the parent population is a Pearson Type III—Biometrika, Vol. 21, 1929. pp. 287-293.

A. T. Craig—On the distribution of certain Statistics—Am. Jour. of Math., Vol. 54, No. 2, 1932. pp. 353-366.

Baten—Frequency laws for the sum of n variables, which are subject each to given frequency laws—Metron, Vol. X, No. 3, 1932. pp. 75-91.

The purpose of developing this law for the sum of n independent variables is to show how certain finite summations are evaluated. An interesting summation arises when f and g are interchanged in certain cases. For example the law for the sum,

$$z = \sum_{i=1}^{n} x_i$$

is $F_n(z)$, and the law for the sum,

$$z = \sum_{i=1}^{n+1} x_i \text{ is } \int_0^z F_n(x) f_1(z-x)\,dx = \int_0^z f_1(x)\, F_n(z-x)\,dx =$$

$$\text{(a)} \qquad F_{n+1}(z) = \frac{1}{(n-1)!}\int_0^z e^{-x} e^{-z+x}\left[z^{n-1} + (-1)^1\, {}_{n-1}C_1\, z^{n-2} x + \cdots + (-1)^{n-1} x^{n-1}\right]dx$$

$$= e^{-z} z^n \left[\sum_{r=0}^{n-1} \frac{(-1)^r\, {}_{n-1}C_r}{r+1}\right] \Big/ (n-1)!$$

$$= 0, \quad \text{elsewhere.}$$

Since the probability function for the sum of the first $n+1$ variables, when each is subject to f_i , is

$$\text{(b)} \qquad z^n e^{-z}/n! \qquad \text{, for the positive axis}$$

then (a) and (b) are equal and the summation in the above expression for (a) is equal to $1/n$; hence

$$\sum_{r=0}^{n-1} \frac{(-1)^r\, {}_{n-1}C_r}{r+1} = 1/n \ .$$

If the probability function for the sum of the first $2n$ variables is obtained by "combining" the probability function for the sum of the first n variables with the probability function for the sum of the following n variables, another interesting summation arises. This summation is a Beta function in disguise. For example the probability function for the sum, $x_1 + x_2 + x_3 + x_4 = z$, is $z^3 e^{-z}/3!$ for positive z and zero elsewhere, and the probability function for the sum, $x_5 + x_6 + x_7 + x_8 = v$, is $v^3 e^{-v}/3!$ for positive v and zero elsewhere. The probability function for the sum, $z + v = \sum_{i=1}^{8} x_i = w$,

is $$F(w) = \frac{1}{3!\,3!}\int_0^w e^{-z} e^{-w+z} z^3 (w^3 - 3w^2 z + 3w z^2 - z^3)\,dz$$

$$= \frac{1}{3!\,3!} e^{-w}\left(1/4 - 3/5 + 3/6 - 1/7\right) w^7 \text{ ; for the positive axis}$$

and zero elsewhere. The quantity in parentheses has for numerators the coefficients of the binomial $(a-b)^3$, while the denominators begin with a number greater by one than the exponent of the binomial and increase by unity from term to term. The above form suggests the following integral

$$\sum_{r=0}^{3} \frac{(-1)^r {}_3C_r}{r+4} = \int_0^1 x^3(1-x)^3 dx = B(4,4).$$

In general the probability function for the sum of the first $2n$ variables, by using the probability function for the sum of the first n and the probability function for the sum of the following n, is

$$\frac{w^{2n-1} e^{-w}}{(n-1)!\,(n-1)!} \sum_{r=0}^{n-1} \frac{(-1)^r {}_{n-1}C_r}{r+n}$$, for $(0,\infty)$ and zero elsewhere.

The summation can be written as a definite integral

$$\sum_{r=0}^{n-1} \frac{(-1)^r {}_{n-1}C_r}{r+n} = \int_0^1 x^{n-1}(1-x)^{n-1} dx = B(n,n) = \frac{\Gamma(n)\cdot\Gamma(n)}{\Gamma(2n)}.$$

If the probability law for the sum of n independent variables is obtained by combining the probability law for the sum of the first s variables with the law for the sum of the following $n-s$ variables the following summation arises which is also equal to a Beta function. This summation is

$$\sum_{r=0}^{n-s-1} \frac{(-1)^r {}_{n-s-1}C_r}{s+r} = \int_0^1 x^{s-1}(1-x)^{n-s-1} dx = B(s, n-s).$$

This idea concerning the Beta function appears to be new.

II. Combining two probability functions.

Combining here shall mean finding the probability function for the sum of the variables from the probability functions of the individual variables. Many peculiar functions arise when various laws are used for the probability functions of the individual variables. This section presents a few of them.

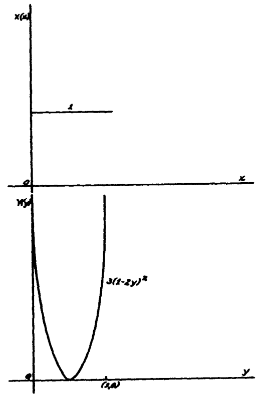
Let

$$f(x) = 1, \text{ for } (0,1) \text{ and zero elsewhere,}$$

and

$$g(y) = 3(1-2y)^2, \text{ for } (0,1) \text{ and zero elsewhere.}$$

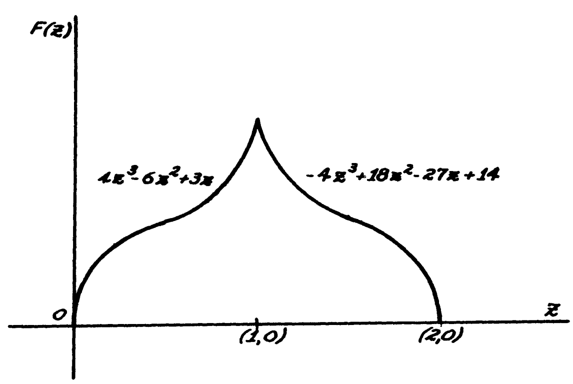
These laws are drawn below. Both have two points of discontinuity.



The probability function for the sum, $x + y = z$ is

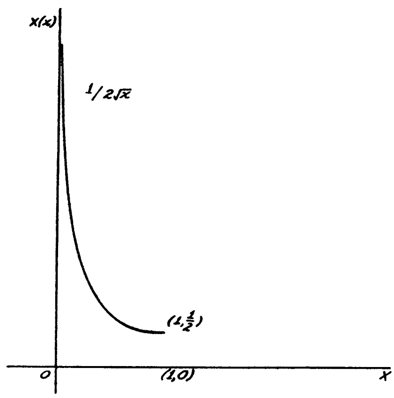
$$F(z) = \begin{cases} (4z^3 - 6z^2 + 3z) & \text{, for the interval } (0,1) \\ (-4z^3 + 18z^2 - 27z + 14) & \text{, for } (1,2) \\ 0, & \text{elsewhere.} \end{cases}$$

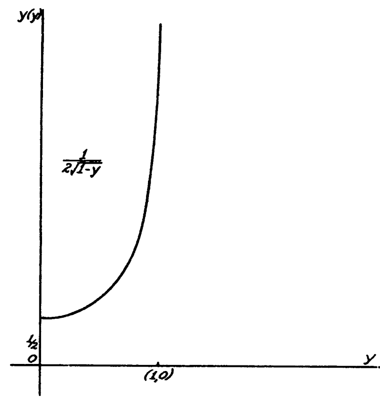
$F(z)$ is continuous, symmetrical about the line $z = 1$ with large slope at the points (0,0), (1,1) and (2,0). There is a cusp at (1,1). $F(z)$ is drawn below.



2. Let $f(x) = \frac{1}{2\sqrt{x}}$, for (0,1) and zero elsewhere, and $g(y) = 1/2\sqrt{1-y}$, for (0,1) and zero elsewhere. The function $f(x)$ is the probability function for the square of the variable if the probability law for the variable is unity in (0,1) and zero elsewhere. The function $f(x)$ approaches infinity at the origin,

while $g(y)$ is a similar curve turned in the opposite direction and has an infinite slope at (1,0).

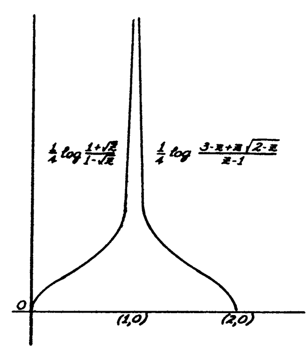




The law for the sum, $x+y=z$ is

$$F(z)=\begin{cases} 1/4\cdot\log\left[(1+\sqrt{z})/(1-\sqrt{z})\right], & \text{for } (0,1) \\ 1/4\cdot\log\left[(3-z+z\sqrt{2-z})/(z-1)\right], & \text{for } (1,2) \\ 0, & \text{elsewhere.} \end{cases}$$

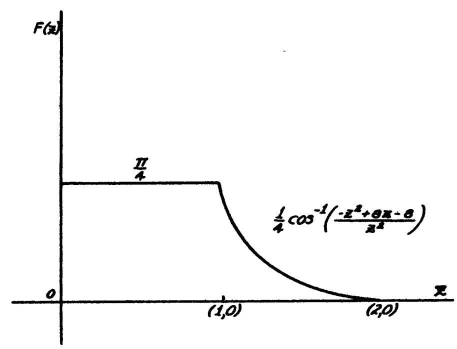
$F(z)$ is somewhat of a surprise for it is equal to zero at the origin and the point (2,0) and approaches infinity from the right and from the left at the point (1,0). The slope of the law for the sum is infinite at the origin and at the points (2,0) and (1,0). $F(z)$ appears below.



3. Let $f(x)=1$, for (0,1) and zero elsewhere and $g(y)=1$ for the interval (0,1) and zero elsewhere; then the probability law of $w=x^2$ is $h(w)=\frac{1}{2\sqrt{w}}$, for the interval (0,1) and zero elsewhere. Let $u=y^2$, then the probability function for u is $k(u)=\frac{1}{2\sqrt{u}}$, for the interval (0,1) and zero elsewhere. According to the theorem used in part I the probability law for the sum, $x^2+y^2=z$, is

$$F(z) = \begin{cases} \pi/4, & \text{for the interval } (0,1) \\ 1/4 \cdot \text{arc cos } \dfrac{-z^2+8z-8}{z^2}, & \text{for } (1,2) \\ 0, & \text{elsewhere.} \end{cases}$$

The plot of $F(z)$ is below.



The functions $h(w)$ and $k(u)$ are J-shaped functions with infinite slope at the origin and are equal to $f(x)$ in example 2. The law for the sum of the squares in this case has one point of discontinuity which is at the origin. The function for the sum is constant throughout the interval (0,1) and is equal to an inverse cosine function throughout the interval (1,2).

4. If $f(x)=3(1-2x)^2$ for the interval (0,1) and zero elsewhere and $g(y)=3(1-2y)^2$ for the interval (0,1) and zero elsewhere, then the law for the sum, $x+y=z$, is

$$F(z) = \begin{cases} .6(8z^5-40z^4+80z^3-60z^2+15z), & \text{for the interval } (0,1) \\ .6(-8z^5+40z^4-80z^3+100z^2-95z+46) & \text{for } (1,2) \\ 0, & \text{elsewhere.} \end{cases}$$

The function $F(z)$ has three modes and has its highest point where one would least expect it, and has large slopes at the origin, and at the points (1,0), (2,0). To appreciate the nature of $F(z)$ here tne graphs of the functions for x and y should be examined. They are U-shaped curves which are tangent to the horizontal axis at the middle of the interval (0,1). See the second figure in 1. $F(z)$ is shown in the following figure.