# ON CERTAIN DISTRIBUTION FUNCTIONS WHEN THE LAW OF THE UNIVERSE IS POISSON'S FIRST LAW OF ERROR[1]

By Frank M. Weida

**Introduction.** The median, which is that value of a permuted variable which has as many observed values on one side of it as on the other, appears to be the natural competitor of the arithmetic mean when we are interested in the probable or most probable value of an unknown quantity. It is well known[2] that the law of probability, namely, Poisson's first law of error, which results from the assumption that the median is the most probable value of the unknown quantity is given by

$$f(x) = \frac{k}{\sigma} e^{-\frac{|x|}{\sigma}}. \tag{1}$$

Little is known about the form of the distribution functions of the more important statistics when the law of the "Universe" is Poisson's first law of error. It, therefore, appears to be of interest and importance to enlarge our present knowledge of distribution functions by finding certain new ones when the variable or variables are defined by (1).

In this paper we present the following results: (1) We have obtained an explicit expression for the distribution of means of samples of $n$; (2) we have obtained an explicit expression for the distribution of differences; (3) we have obtained an explicit expression for the distribution of quotients; (4) we have obtained an explicit expression for the distribution of standard deviations for samples of $n$; (5) we have obtained an explicit expression for the distribution of geometric means for samples of $n$; (6) we have obtained an explicit expression for the distribution of harmonic means for samples of $n$.

In our analysis, we have made use of the theory of characteristic functions in the sense of Levy.[3] This theory has been extended to more than one dimension by V. Romanovsky[4] and by E. K. Haviland.[5] S. Kullback,[6] in his thesis, has made further extensions and has applied them successfully to the distribution problem in statistics.

[1] Presented to the American Mathematical Society, February 23, 1935.

[2] Brunt, David: "The Combination of Observations," 1923, p. 27.

[3] Levy, P.: "Calcul des Probabilités;" pp. 153–191.

[4] Romanovsky, V.: "Sur un théorème limite du calcul des probabilités," Recueil mathématique de la Société mathématique de Moscow, Vol. 36, 1926, pp. 36–64.

[5] Haviland, E. K.: "On the inversion formula for Fourier-Stieltjes transforms in more than one dimension," American Journal of Mathematics, Vol. 57, 1935, pp. 94–101.

[6] Kullback, S.: "An application of characteristic functions to the distribution problem of statistics," Annals of Mathematical Statistics, Vol. V, No. 4, pp. 263–307.

The explicit expression for the distribution of arithmetic means of samples of $n$ is not new. This law of distribution has previously been obtained otherwise by F. Hausdorff[7] and by A. T. Craig.[8] It is inserted here to show the superiority and greater power of our method when compared with previous methods and for the completeness of our discussion. The other results offered in this paper, as far as the writer knows, are new.

1. **The distribution of arithmetic means.** Let us consider

$$f(x) = \frac{k}{\sigma} e^{-\frac{|x|}{\sigma}}, \qquad (-\alpha < x < \alpha). \tag{2}$$

If we assume that $x_1, x_2, \cdots, x_n$ are independently distributed and that each $x_i (i = 1, 2, \cdots, n)$ is distributed according to the same distribution law, namely, Poisson's first law of error, then it is fairly easy to see that the characteristic function for the law of distribution of means of samples of $n$ is given by

$$\phi(t) = \left\{\int_{-\alpha}^{\alpha} \frac{k}{\sigma} e^{it|x| - \frac{|x|}{\sigma}} dx\right\}^n. \tag{3}$$

If $u = \Sigma_i x_i$ $(i = 1, 2, \cdots, n)$, then it follows that the distribution function of $u$, namely, $P(u)$, is given by

$$P(u) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} e^{-itu} \left\{\frac{2k}{\sigma} \int_{0}^{\alpha} e^{itx - \frac{x}{\sigma}} dx\right\}^n dt, \tag{4}$$

which, upon simplification becomes

$$P(u) = \frac{2^{n-1} k^n}{\pi \sigma^n} \int_{-\alpha}^{\alpha} \frac{e^{-itu}\, dt}{(1 - \sigma it)^n}. \tag{5}$$

It is readily seen that the poles of the integrand are of the $n^{\text{th}}$ order and are those of $(1 - \sigma it)^n$. It follows by the well known Residue Theorem of Cauchy[9] that

$$P(u) = \frac{2^{n-1} k^n}{\pi \sigma^n} \cdot 2\pi i \cdot \frac{(-1)^{n-1}}{(n-1)!} \cdot \frac{1}{i^n} \cdot \frac{d^{n-1}}{dt^{n-1}} \left\{\frac{e^{-itu}}{(1 + \sigma it)^n}\right\}_{t = \frac{1}{\sigma i}}. \tag{6}$$

If now, we replace $u$ by $n|\bar{x}|$, we will obtain the desired law of the distribution of arithmetic means of samples of $n$ which is

$$P(|\bar{x}|) = \frac{2^n k^n (-1)^{n-1} n}{\sigma^n i^{n-1} (n-1)!} \cdot \frac{d^{n-1}}{dt^{n-1}} \left\{\frac{e^{-itn|\bar{x}|}}{(1 + \sigma it)^n}\right\}_{t = \frac{1}{\sigma i}} \tag{7}$$

defined for all values of $x$ on the range $(-\alpha < x < \alpha)$.

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[7] Hausdorff, F.: Beitrage zur Wahrscheinlichkeitsrechnung Königlich Sächsischen Gesellschaft der Wissenshaften zu Leipzig. Berichted über die Verhandlungen Math.-Phys. Classe, Vol. 53, 1901, pp. 152–178.

[8] Craig, A. T.: "On the distribution of certain statistics," American Journal of Mathematics, Vol. 54, 1932, pp. 353–366.

[9] Macrobert, T. M.: "Functions of a Complex Variable," 1933, pp. 57, 295.

A. T. Craig[8] has given the distribution laws of arithmetic means of samples of size 2, 3, and 4. These results as well as the results for any $n$ are readily obtained from (7).

2. **The distribution of differences.** Let us assume that the laws of distribution of $x$ and $y$ are independent and that they are given respectively by

$$f(x) = \frac{k_1}{\sigma_1} e^{-\frac{|x|}{\sigma_1}}; \qquad f(y) = \frac{k_2}{\sigma_2} e^{-\frac{|y|}{\sigma_2}}; \qquad (-\alpha < x < \alpha), \qquad (-\alpha < y < \alpha).$$

In this case, the characteristic function of the law of distribution of differences $(x - y)$ is given by

$$\phi(t) = \frac{k_1}{\sigma_1} \int_{-\alpha}^{\alpha} e^{i\,t|x| - \frac{|x|}{\sigma_1}}\, dx \cdot \frac{k_2}{\sigma_2} \int_{-\alpha}^{\alpha} e^{-it|y| - \frac{|y|}{\sigma_2}}\, dy. \tag{8}$$

Performing the operations indicated in (8) and simplifying, we find that

$$\phi(t) = \frac{4k_1k_2}{\sigma_1\sigma_2} \cdot \frac{1}{(1 - \sigma_1 it)} \cdot \frac{1}{(1 + \sigma_2 it)}. \tag{9}$$

It is fairly easy to see that the distribution law of $u$ is given by

$$P(u) = \frac{4k_1k_2}{2\pi\sigma_1\sigma_2} \int_{-\alpha}^{\alpha} \frac{e^{-itu}\, dt}{(1 - \sigma_1 it)(1 + \sigma_2 it)}. \tag{10}$$

Now, let $\{(1/\sigma_1) - it\} = v/u$, then (10) becomes

$$P(u) = \frac{2k_1k_2 e^{-\frac{u}{\sigma_1}}}{\pi i\sigma_1\sigma_2(\sigma_1 + \sigma_2)} \int_{\frac{u}{\sigma_1} - i\alpha}^{-\frac{u}{\sigma_1} + i\alpha} \frac{e^{-v}\, dv}{(-v)\left\{1 + \frac{v}{\left(\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right)}\right\}}. \tag{11}$$

The integral in (11) is convergent because

$$\lim_{v \to \alpha} \left| v^m \frac{e^{-v}\, dv}{(-v)\left\{1 + \frac{v}{\left(\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right)}\right\}} \right| = 0.$$

Hence, we find that

$$P(u) = -\frac{2k_1k_2 e^{-\frac{u}{\sigma_1}}}{\pi i\sigma_1\sigma_2(\sigma_1 + \sigma_2)} \int_{\alpha}^{(0+)} \frac{e^{-v}\, dv}{(-v)\left\{1 + \frac{v}{\left(\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right)}\right\}} \tag{12}$$

which upon simplification becomes

$$P(u) = \frac{4k_1k_2}{\sigma_1\sigma_2(\sigma_1 + \sigma_2)} W_{0,\frac{1}{2}}\left\{\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right\}, \tag{13}$$

where $W_{0,\frac{1}{2}}\left\{\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right\}$ is the *confluent hypergeometric function.*[10]

It is well known that

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\alpha t^{-k-\frac{1}{2}+m}\left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\, dt$$

for all values of $k$ and $m$ and for all values of $z$ except negative real values. Clearly,

$$W_{0,\frac{1}{2}}\left\{\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right\} = \frac{e^{-\frac{\sigma_1+\sigma_2}{\sigma_1\sigma_2\cdot 2} u}}{\Gamma(1)} \int_0^\alpha e^{-t}\, dt$$

which, upon simplification becomes

$$W_{0,\frac{1}{2}}\left\{\frac{\sigma_1 + \sigma_2}{\sigma_1\sigma_2} u\right\} = e^{-\frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2} u}. \tag{14}$$

Hence, we now find that

$$P(u) = \frac{4k_1k_2}{\sigma_1\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2} u}. \tag{15}$$

If now, we replace $u$ by $|x| - |y|$, we will obtain the desired law of distribution of differences which is

$$P(|x| - |y|) = \frac{4k_1k_2}{\sigma_1\sigma_2(\sigma_1 + \sigma_2)} e^{-\frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2}\{|x| - |y|\}}. \tag{16}$$

3. **The distribution of ratios.** We assume that the laws of distribution of $x$ and $y$ are independent and that they are given respectively by

$$f(x) = \frac{k_1}{\sigma_1} e^{-\frac{|x|}{\sigma_1}}; \quad f(y) = \frac{k_2}{\sigma_2} e^{-\frac{|y|}{\sigma_2}}; \quad (-\alpha < x < \alpha), \quad (-\alpha < y < \alpha).$$

[10] Whittaker, E. T. and Watson, G. N.: "A course in modern Analysis," 1915, pp. 333–334.

Let $u = \log|x| - \log|y|$. The characteristic function of the law of distribution of quotients is then given by

$$\phi(t) = \frac{k_1}{\sigma_1}\int_{-\alpha}^{\alpha} e^{-\frac{|x|}{\sigma_1}}(|x|)^{it}\,dx \cdot \frac{k_2}{\sigma_2}\int_{-\alpha}^{\alpha} e^{-\frac{|y|}{\sigma_2}}(|y|)^{-it}\,dy$$
$$= \frac{4k_1k_2}{\sigma_1\sigma_2}\int_0^{\alpha} e^{-\frac{x}{\sigma_1}}x^{it}\,dx \int_0^{\alpha} e^{-\frac{y}{\sigma_2}}y^{-it}\,dy. \qquad (17)$$

Now, let $s = x/\sigma_1$ and $w = y/\sigma_2$, then clearly

$$\phi(t) = 4k_1k_2\sigma_1^{it}\sigma_2^{-it}\int_0^{\alpha} e^{-s}s^{it}\,ds \int_0^{\alpha} e^{-w}w^{-it}\,dw,$$

whence

$$\phi(t) = 4k_1k_2\sigma_1^{it}\sigma_2^{-it}\,\Gamma(it+1)\Gamma(1-it). \qquad (18)$$

It follows that the distribution law of $u$ is given by

$$P(u) = \frac{4k_1k_2}{2\pi}\int_{-\alpha}^{\alpha} e^{-itu+i\log\sigma_1 t - i\log\sigma_2 t}\,\Gamma(it+1)\,\Gamma(1-it)\,dt$$

which upon simplification, becomes

$$P(u) = \frac{2k_1k_2}{\pi}\int_{-\alpha}^{\alpha} e^{-i(u-\log\sigma_1+\log\sigma_2)t}\,\Gamma(it+1)\,\Gamma(1-it)\,dt. \qquad (19)$$

Now, let $(1 - it) = -v$, then (19) becomes

$$P(u) = \frac{4k_1k_2}{2\pi i}\int_{-1-i\alpha}^{-1+i\alpha} e^{-v\{u-\log\sigma_1+\log\sigma_2\}-\{u-\log\sigma_1+\log\sigma_2\}}\,\Gamma(2+v)\,\Gamma(-v)\,dv. \qquad (20)$$

Since it can be shown that[11]

$$(1/2\pi i)\int_{-1-i\alpha}^{-1+i\alpha} e^{-vu}\Gamma(2+v)\,\Gamma(-v)\,dv = \Gamma(2)\{1+(1/e^u)\}^{-2},$$

we find that (20) becomes

$$P(u) = \frac{4k_1k_2e^{-u}\sigma_1}{\sigma_2}\,\Gamma(2)\left\{1+\frac{\sigma_1}{\sigma_2e^u}\right\}^{-2}. \qquad (21)$$

Now, put $e^u = |x|/|y| = R$, whence from (21) we will obtain the desired law of distribution of quotients which is

$$P(R) = \frac{4k_1k_2\sigma_1\Gamma(2)}{\sigma_2R}\left\{1+\frac{\sigma_1}{\sigma_2R}\right\}^{-2}. \qquad (22)$$

[11] Macrobert, T. M., "Functions of a Complex Variable," 1933, pp. 114, 139, 151. Whittaker, E. T. and Watson, G. N., "A course in modern Analysis," 1915, pp. 283.

4. **The distribution of variances and standard deviations.** If we assume that the variance and standard deviation are calculated about a sample mean and if we let $u = \sum_{i=1}^{n-1} x_i^2$, and if the $x_i$ are independently distributed and each $x_i$ is distributed according to the same distribution law, namely, Poisson's first law of error, then it is clear that the characteristic function for the law of distribution of variances of samples of $n$ is

$$\phi(t) = \left\{\frac{k}{\sigma}\int_{-\alpha}^{\alpha} e^{-\frac{|x|}{\sigma}+itx^2}\,dx\right\}^{n-1} = \left\{\frac{2k}{\sigma}\int_0^{\alpha} e^{itx^2-\frac{x}{\sigma}}\,dx\right\}^{n-1}. \tag{23}$$

Let $I$ represent the integral in the right-hand member of (23). We obtain that $(dI/d\sigma) = I/\sigma^2$; whence $I = Ce^{-\frac{1}{\sigma}}$. Making use of the conditions:

$$\sigma \to \alpha, \qquad I \to \int_0^{\alpha} e^{itx^2}\,dx = e^{\frac{1}{4}\pi i}\frac{\sqrt{\pi}}{\sqrt{t}},$$

$\sigma \to \alpha$, $Ce^{-\frac{1}{\sigma}} \to C$, whence we find that

$$\int_0^{\alpha} e^{itx^2-\frac{x}{\sigma}}\,dx = e^{\frac{1}{4}\pi i}\frac{\sqrt{\pi}}{\sqrt{t}}\,e^{-\frac{1}{\sigma}}.$$

Clearly, it follows that

$$\phi(t) = \frac{2^{n-1}k^{n-1}e^{\frac{(n-1)\pi i}{4}}\pi^{\frac{n-1}{2}}}{\sigma^{n-1}t^{\frac{n-1}{2}}}\,e^{-\frac{n-1}{\sigma}}. \tag{24}$$

We now find that the distribution law of $u$ is given by

$$P(u) = \frac{2^{n-1}k^{n-1}e^{\frac{(n-1)\pi i}{4}}\pi^{\frac{n-1}{2}}e^{-\frac{n-1}{\sigma}}}{2\pi\sigma^{n-1}}\int_{-\alpha}^{\alpha}\frac{e^{-itu}}{t^{\frac{n-1}{2}}}\,dt. \tag{25}$$

Evaluating the integral in (25) with a suitably chosen contour,[12] we find that

$$P(u) = \frac{2^{n-1}k^{n-1}\pi^{\frac{n-1}{2}}e^{-\frac{n-1}{\sigma}}2\pi}{2\pi\sigma^{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}\,u^{\frac{n-3}{2}}e^{-u}. \tag{26}$$

Now, let $u = \sum_{i=1}^{n} x_i^2 = ns^2$, whence from (26) we will obtain the desired law of distribution of variances which is

$$P(s^2) = \frac{2^{n-1}k^{n-1}\pi^{\frac{n-1}{2}}e^{-\frac{n-1}{\sigma}}}{\sigma^{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}\,n^{\frac{n-3}{2}}(s^2)^{\frac{n-3}{2}}e^{-s^2}. \tag{27}$$

[12] Macrobert, T. M., "Functions of a Complex Variable," 1933, p. 67.

The law of distributions of standard deviations can be obtained at once from (27) since $d(s^2) = 2s\,ds$.

We shall now give the specific laws of distribution of variances for samples of size 1, 2, 3, 4, and 5 when the law of the "Universe" is Poisson's first law of error. From (27),

For $n = 1$,

$$P(s^2) = 0, \qquad (0 < s^2 < \infty). \tag{28}$$

For $n = 2$,

$$P(s^2) = \frac{2^{\frac{1}{2}} k e^{-\frac{1}{\sigma}} e^{-s^2}}{\sigma s}, \qquad (0 < s^2 < \infty). \tag{29}$$

For $n = 3$,

$$P(s^2) = \frac{4k^2 \pi e^{-\frac{2}{\sigma}} e^{-s^2}}{\sigma^2}, \qquad (0 < s^2 < \infty). \tag{30}$$

For $n = 4$,

$$P(s^2) = \frac{32k^3 \pi e^{-\frac{3}{\sigma}} s e^{-s^2}}{\sigma^3}, \qquad (0 < s^2 < \infty). \tag{31}$$

For $n = 5$,

$$P(s^2) = \frac{80k^4 \pi^2 e^{-\frac{4}{\sigma}} s^2 e^{-s^2}}{\sigma^4}, \qquad (0 < s^2 < \infty). \tag{32}$$

5. **The distribution of geometric means.** As before, we assume that the $x_i$ are independently distributed and each $x_i$ is distributed according to the same distribution law, namely, Poisson's first law of error. Then, clearly, the characteristic function for the law of distribution of geometric means of samples of $n$ is

$$\phi(t) = \left\{\int_{-\alpha}^{\alpha} \frac{k}{\sigma} e^{-\frac{|x|}{\sigma}} |x|^{it}\, dx\right\}^n = \left\{\frac{2k}{\sigma}\int_0^{\alpha} e^{-\frac{x}{\sigma}} x^{it}\, dx\right\}^n. \tag{33}$$

Now, put $s = x/\sigma$, then (33) becomes

$$\phi(t) = \left\{2k\sigma^{it}\int_0^{\alpha} e^{-s} s^{it}\, ds\right\}^n = 2^n k^n \sigma^{nit}\{\Gamma(it + 1)\}^n. \tag{34}$$

It follows at once that the distribution law of $u$ is

$$P(u) = \frac{2^n k^n}{2\pi}\int_{-\alpha}^{\alpha} e^{-i(u+n\log\sigma)t}\{\Gamma(it + 1)\}^n\, dt. \tag{35}$$

Now, let $it + 1 = -v$, then (35) becomes

$$P(u) = \frac{-2^n k^n}{2\pi i} e^{u+n\log\sigma} \int_{-1-i\alpha}^{-1+i\alpha} e^{v(u+n\log\sigma)} \{\Gamma(-v)\}^n \, dv. \tag{36}$$

It is well known that (10)

$$\{\Gamma(-v)\}^n = \frac{(-1)^n \pi^n}{\sin^n \pi v \{\Gamma(v+1)\}^n}. \tag{37}$$

Using (37) in (36), we readily find that

$$P(u) = \frac{-2^n k^n}{2\pi i} e^{u+n\log\sigma} \int_{-i-i\alpha}^{-1+i\alpha} \frac{e^{v(u+n\log\sigma)}(-1)^n \pi^n}{\{\Gamma(v+1)\}^n \sin^n \pi v} \, dv. \tag{38}$$

It is fairly easy to see that the poles of the integrand in (38) are the poles of $\{\Gamma(-v)\}^n$ and that these poles are of the $n^{\text{th}}$ order. Applying the well known Residue Theorem of Cauchy (8), we find that

$$P(u) = 2^n k^n e^{u+n\log\sigma} \sum_{a=0}^{\alpha} \frac{(-1)^{n+na+1}}{(n-1)!} \left\{ \frac{d^{n-1}}{dv^{n-1}} \left[ \frac{e^{v(u+n\log\sigma)}}{\{\Gamma(v+1)\}^n} \right] \right\}_{v=a}. \tag{39}$$

Now, since $u = \log|x_1| + \log|x_2| + \cdots + \log|x_n|$, then clearly, the distribution law of the geometric mean, $G$, is obtained from the law of distribution for $u$ by means of the transformation

$$u = \log (G)^n.$$

Hence, from (39), we find the desired law of distribution of geometric means of samples of $n$ which is

$$P\{G\} = \frac{2^n k^n G^n \sigma^n}{\Gamma(n)} \sum_{a=0}^{\alpha} (-1)^{n+na+1} \left\{ \frac{d^{n-1}}{dv^{n-1}} \left[ \frac{G^{nv} \sigma^{nv}}{\{\Gamma(v+1)\}^n} \right] \right\}_{v=a}. \tag{40}$$

6. **The distribution of harmonic means.** Let us assume that $f(x)$ is the law of distribution for $x$. It is well known[13] that the law of distribution of $x' = 1/x$ is given by

$$F(x') = (1/x'^2) f(1/x')$$

if $1/x$ is continuous on the range of definition of $f(x)$. Now, in case $f(x)$ is Poisson's first law of error, we find that

$$F(x') = F(1/x) = \frac{k}{\sigma} x^2 e^{-\frac{|x|}{\sigma}}; \qquad (-\alpha \leqq x < 0), \quad (0 < x \leqq \alpha). \tag{41}$$

[13] Dodd, E. L., "The frequency law of a function of one variable," Bulletin of the American Mathematical Society, Vol. 31, 1925, p. 28; "The frequency law of a function of variables with given frequency laws," Annals of Mathematics, Second Series, Vol. 27, 1925–26, p. 18.

We assume that the $x'_i$ are independently distributed and each $x'_i$ is dis-distributed according to the same law of distribution, whence we find that the characteristic function for the law of distribution of harmonic means of samples of $n$ is

$$\phi(t) = \left\{ \int_0^\alpha \frac{k}{\sigma} e^{it|x| - \frac{|x|}{\sigma}} x^2 \, dx \right\}^n, \tag{42}$$

from which, after simplification, we find that

$$\phi(t) = \frac{k^n 2^n \sigma^{2n}}{(1 - \sigma it)^{3n}}. \tag{43}$$

We now find that the law of distribution for $u$ is

$$P(u) = \frac{2^n k^n \sigma^{2n}}{2\pi} \int_{-\alpha}^{\alpha} \frac{e^{-itu}}{(1 - \sigma it)^{3n}} \, dt,$$

which, after evaluation and simplification, becomes

$$P(u) = \frac{2^n k^n}{\sigma^n \Gamma(3n)} u^{3n-1} e^{-\frac{u}{\sigma}}. \tag{44}$$

Recalling that in this case, $u = 1/|x_1| + 1/|x_2| + \cdots + 1/|x_n|$, we make the transformation $u = n/H$, where $H$ is the harmonic mean; whence, from (44), we find that the desired law of distribution of harmonic means of samples of $n$ is given by

$$P(H) = \frac{2^n k^n n^{3n-1}}{\sigma^n \Gamma(3n)} \cdot H^{1-3n} e^{-\frac{n}{\sigma H}}. \tag{45}$$

7. **Conclusions.** We have shown that the same analysis is applicable to find the explicit expression for all the distribution laws we have discussed in this paper.

THE GEORGE WASHINGTON UNIVERSITY,
WASHINGTON, D. C.

## ERRATA

In my paper* there appear two blunders which were called to my attention by A. T. Craig.

In section 4, pages 107–108, headed "The distribution of variances and standard deviations," I have obtained the distribution function of the sum of the squares of $n - 1$ independent values of $x$ and not the distribution function of the sum of the squares of the deviations from the sample mean of the $n$ independent values of $x$.

In section 2, pages 104–105, headed "The distribution of differences," I have obtained the distribution function of the differences of *absolute values* and not the distribution function of the actual differences.

* Weida, F. M., "On Certain Distribution Functions when the Law of the Universe is Poisson's First Law of Error," Annals of Mathematical Statistics, Vol. VI, No. 2, June, 1935, pp. 102-110.