ON THE POSTULATE OF THE ARITHMETIC MEAN

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Introduction

Suppose n observations have been made of an unknown quantity. It is desired to know the most probable value of the unknown. When Gauss gave his development of the so-called *Normal Law of Error*, he assumed that *the Arithmetic Mean of the n observations is the most probable value.* The question arises: Can this postulate be justified?

In the excellent book, entitled "Calculus of Observations," by Whittaker and Robinson[1] there is given a proof which purports to deduce the postulate of the Arithmetic Mean from assumptions of a more elementary nature. This proof is not correct.

Since this book has had wide circulation, it is believed that the errors in this proof should be called to the attention of the users of the book. The present paper has been prepared for this purpose. The first part of this paper points out the questionable features of the proof given in Whittaker and Robinson's book. The second part gives some critical comments on the original sources from which Whittaker and Robinson obtained their proof.

Part 1

The assumptions on which Whittaker and Robinson based their proof of the postulate of the Arithmetic Mean are:

Axiom I. The differences between the most probable value and the individual measures do not depend on the position of the null-point from which they are reckoned.

Axiom II. The ratio of the most probable value to any individual measure does not depend on the unit in terms of which the measures are reckoned.

Axiom III. The most probable value is independent of the order in which the measurements are made, and so is a symmetric function of the measures.

Axiom IV. The most probable value, regarded as a function of the individual measures, has one-valued and continuous first derivatives with respect to them.

It is fairly easy to show that if the Arithmetic Mean is the most probable value, then the above four axioms follow as conclusions. The converse, viz. if the above four axioms be assumed then the Arithmetic Mean is the most probable value, however, is not true. That is to say the above assumptions are

[1] The Calculus of Observations by E. T. Whittaker and G. Robinson, Blackie & Son, Ltd., London (1929), pp. 215–217.

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necessary conditions, but not sufficient conditions. For, consider the following function of the measures:

$$\frac{\mu_3}{\mu_2} = \frac{\frac{1}{n}\sum_{i=1}^{i=n}(x_i - \bar{x})^3}{\frac{1}{n}\sum_{i=1}^{i=n}(x_i - \bar{x})^2}$$

where $\bar{x}$ is the Arithmetic Mean of the x_i.

Clearly this function is a symmetric function of the measures (x_i) and therefore satisfies Axiom III. If the x_i are each multiplied by k then the Arithmetic Mean $(\bar{x})$ is also multiplied by k and we have

$$\frac{\frac{1}{n}\sum_{i=1}^{i=n}(kx_i - k\bar{x})^3}{\frac{1}{n}\sum_{i=1}^{i=n}(kx_i - k\bar{x})^2} = k\frac{\mu_3}{\mu_2};$$

that is to say, if we multiply the individual measures by k it is the same as multiplying the function $\frac{\mu_3}{\mu_2}$ by k and therefore the ratio of any individual measure to the most probable value (function) does not depend on the unit used. Hence the function $\frac{\mu_3}{\mu_2}$ satisfies Axiom II.

The partial derivative of $\frac{\mu_3}{\mu_2}$ with respect to x_1 is

$$\left(\left\{\sum_{i=1}^{i=n}(x_i - \bar{x})^2\right\}\left[3\left\{\sum_{i=1}^{i=n}(x_i - \bar{x})^2\right\}\left\{-\frac{\partial \bar{x}}{\partial x_1}\right\} + 3(x_1 - \bar{x})^2\frac{\partial x_1}{\partial x_1}\right]\right.$$
$$\left. - \left\{\sum_{i=1}^{i=n}(x_i - \bar{x})^3\right\}\left[2\left\{\sum_{i=1}^{i=n}(x_i - \bar{x})\right\}\left\{-\frac{\partial \bar{x}}{\partial x_1}\right\} + 2(x_1 - \bar{x})\frac{\partial x_1}{\partial x_1}\right]\right)$$
$$\div\left\{\sum_{i=1}^{i=n}(x_i - \bar{x})^2\right\}^2 = \frac{3\mu_2[(x_1 - \bar{x})^2 - \mu_2] - 2\mu_3(x_1 - \bar{x})}{n\mu_2^2},$$

since $\frac{\partial \bar{x}}{\partial x_1} = \frac{1}{n}$, and $\sum_{i=1}^{i=n}(x_i - \bar{x}) = 0$. The partial derivatives of $\frac{\mu_3}{\mu_2}$ with respect to each of the x_i are of the same literal form and clearly these partial derivatives are single valued and continuous. Therefore the function $\frac{\mu_3}{\mu_2}$ satisfies Axiom IV.

Now it can be shown that if h be added to each x_i, then the function $\frac{\mu_3}{\mu_2}$ is unchanged and hence this function does not satisfy Axiom I. (It should be noted that the function $\frac{\mu_3}{\mu_2}$ is invariant under the transformation specified by

Axiom I.) However, consider the function $\bar{x} + a \frac{\mu_3}{\mu_2} \equiv f$, where a is a constant independent of the x_i. Clearly, f satisfies all of the four axioms.

Thus a function, distinct from the Arithmetic Mean, has here been exhibited which satisfies the four axioms given in Whittaker and Robinson's book. Hence, these four axioms are not sufficient to establish the postulate of the Arithmetic Mean. The question arises: Where is the proof given by Whittaker and Robinson lacking in rigor? The proof given is essentially as follows. (No part of the proof given by Whittaker and Robinson is here omitted; in fact, for the sake of rigor and careful reasoning, further explanations are given and the various steps are numbered.)

(1) Suppose the most probable value is expressed in terms of the n measures $x_1, x_2, \cdots, x_n$ by the function $\phi(x_1, x_2, \cdots, x_n)$; that is to say the most probable value is some function, ϕ, of the observations, or: the most probable value $\equiv \phi(x_1, x_2, \cdots, x_n)$.

(2) By the theorem of the mean value in the differential calculus, which by Axiom IV is applicable, we have $\phi(kx_1, kx_2, \cdots, kx_n) =$

$$\phi(0, 0, \cdots, 0) + kx_1\left[\frac{\partial \phi}{\partial x_1}\right] + \cdots + kx_n\left[\frac{\partial \phi}{\partial x_n}\right],$$

where the square brackets denote that every x_i is to be replaced by $\theta k x_i$ where θ lies between 0 and 1.

(3) By Axiom II, the left hand side $= k\phi(x_1, x_2, \cdots, x_n)$.

(4) By the continuity of ϕ, postulated in Axiom IV the equation $\phi(kx_1, kx_2, \cdots, kx_n) = k\phi(x_1, x_2, \cdots, x_n)$ must hold in the limit when k is 0, that is $\phi(0, 0, \cdots, 0) = 0$.

(5) We now have

$$k\phi(x_1, x_2, \cdots, x_n) = kx_1\left[\frac{\partial \phi}{\partial x_1}\right] + \cdots + kx_n\left[\frac{\partial \phi}{\partial x_n}\right],$$

or on dividing by k,

$$\phi(x_1, x_2, \cdots, x_n) = x_1\left[\frac{\partial \phi}{\partial x_1}\right] + \cdots + x_n\left[\frac{\partial \phi}{\partial x_n}\right].$$

(6) In this last equation let $k \to 0$: then each of the quantities $\left[\frac{\partial \phi}{\partial x_i}\right]$ tends to a value which is independent of the x's and we can write $\phi(x_1, x_2, \cdots, x_n) = c_1x_1 + \cdots + c_nx_n$ where the c's are independent of the x's.

(7) By Axiom III the c's must all be equal, so

$$\phi(x_1, x_2, \cdots, x_n) = c(x_1 + x_2 + \cdots + x_n).$$

(8) From Axiom I we have

$$\phi(x_1 + h, x_2 + h, \cdots, x_n + h) = \phi(x_1, x_2, \cdots, x_n) + h.$$

(9) If in this last equation we let the x_i all approach zero then we have $cnh = h$ and therefore $c = \frac{1}{n}$ and finally

$$\phi(x_1, x_2, \cdots, x_n) = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

which states that $\phi \equiv$ the most probable value $=$ the Arithmetic Mean.

It should be noted that the first six steps involve only Axioms II and IV. Of these first six steps the second and sixth are questionable.

The sixth step involves the tacit assumption that the partial derivatives are functions of k. These partial derivatives are not necessarily functions of k and the example given above, viz, $f \equiv \bar{x} + a\frac{\mu_3}{\mu_2}$ is a function whose partial derivatives are independent of k; in fact no function of the form

$$F \equiv \bar{x} + \sum_{j=3}^{j=\infty} a_j \frac{\sum_{i=1}^{i=n} (x_i - \bar{x})^j}{\sum_{i=1}^{i=n} (x_i - \bar{x})^{j-1}}$$

will satisfy the tacit assumption involved in the sixth step; nor is F the most general function which will not satisfy the tacit assumption, thus take for example

$$\bar{F} \equiv \bar{x} + \frac{a\mu_3\mu_4}{b\mu_2\mu_4 + c\mu_3^2}.$$

Consider now the second step. Take the function $\phi(y_1, y_2, \cdots, y_n) = k\phi(x_1, x_2, \cdots, x_n)$. Then, by Axiom II, we have $y_i = kx_i$. Apply the Theorem of the Mean Value to $\phi(y_i)$ instead of $\phi(x_i)$. Then $\phi(y_1, y_2, \cdots, y_n) = \phi(0, 0, \cdots, 0) + y_1\left[\frac{\partial\phi}{\partial y_1}\right] + \cdots + y_n\left[\frac{\partial\phi}{\partial y_n}\right]$. Now if we replace y_i by kx_i we obtain the equation given in the second step except that the square brackets are now of the form $\left[\frac{\partial\phi(kx_1, kx_2, \cdots, kx_n)}{\partial(kx_i)}\right]$ and not $\left[\frac{\partial\phi}{\partial x_i}\right]$ as given by Whittaker and Robinson. It is difficult to decide whether by $\left[\frac{\partial\phi}{\partial x_i}\right]$ Whittaker and Robinson mean

$$\left[\frac{\partial\phi(kx_1, kx_2, \cdots, kx_n)}{\partial x_i}\right] \text{ or } \left[\frac{\partial\phi(x_1, x_2, \cdots, x_n)}{\partial x_i}\right].$$

These last two expressions are not equal. To make the second step more clear it is necessary to demonstrate that

$$\left[\frac{\partial\phi(kx_1, kx_2, \cdots, kx_n)}{\partial(kx_i)}\right] = \left[\frac{\partial\phi(x_1, x_2, \cdots, x_n)}{\partial x_i}\right],$$

and this has not been done. In order to demonstrate this equality further use must be made of Axiom II. It appears that the questionable features of the second step may be overcome by starting with the equation implied by Axiom II, thus

$$\phi(kx_1, kx_2, \cdots, kx_n) = k\phi(x_1, x_2, \cdots, x_n);$$

in other words ϕ is a homogeneous function of degree 1. Therefore use can be made of Euler's Theorem on homogeneous forms. In this way we obtain:

$$\phi = \sum_{i=1}^{i=n} x_i \frac{\partial \phi}{\partial x_i}$$

which is an abbreviation of the last equation given in the fifth step.

Now, making further use of Axiom II we have:

$$\frac{\partial \phi(kx_1, kx_2, \cdots, kx_n)}{\partial(kx_i)} = \frac{\partial}{\partial(kx_i)} k\phi(x_1, x_2, \cdots, x_n) = k \cdot \frac{1}{k} \cdot \frac{\partial}{\partial x_i} \phi(x_1, x_2, \cdots, x_n) .$$

It follows that

$$\frac{\partial \phi(x_1, x_2, \cdots, x_n)}{\partial x_i} = \frac{\partial \phi(kx_1, kx_2, \cdots, kx_n)}{\partial(kx_i)} .$$

From this development we conclude that for any function whatever which satisfies Axiom II the last equation of the fifth step cannot possibly involve k.

In order to overcome the defect in the sixth step it is necessary to make a more restrictive assumption. If in place of Axiom IV, we assume that "*The most probable value, regarded as a function of the individual measures, has first partial derivatives with respect to them which are constant,*" then the equation given in the sixth step can be rigorously established.

After the equation of the sixth step is rigorously established there remains an objection in the seventh step. The axioms do not explicitly state that the n observations must be functionally independent. Therefore suppose the x_i are functionally dependent according to the relation $x_i = y_i z$ where the y_i are all constant. Then the function $f \equiv \bar{x} + \frac{\mu_3}{\mu_2}$ will have partial derivatives with respect to the x_i which are unequal and constant; yet at the same time the function f is a symmetrical expression of the n variables.

Hence in order to establish the postulate of the Arithmetic Mean along the lines followed by Whittaker and Robinson it is necessary to make another restrictive assumption slightly different from that proposed in the last paragraph but one, and assume (in addition to Axioms I and II) that *the function has partial derivatives with respect to the x_i which are equal.*

Part 2

The first original paper consulted was one by Schiaparelli.[2] In this paper nine propositions are presented four of which are also called lemmas. From a strict mathematical point of view the four propositions which Schiaparelli calls lemmas are really postulates. Schiaparelli discusses these four lemmas at length; three of these lemmas are the first three axioms given in Whittaker and Robinson's book. The fourth one is: "When, in the function ϕ, all the variables (x_i) take the same value a, the function itself becomes equal to a," (This, as a matter of fact, is the definition of an average).

In his discussion of these lemmas, which are based partly on practical and partly on philosophical grounds, Schiaparelli points out that they are justified from the practical or statistical nature of the problem involved in arriving at the most probable value (Schiaparelli uses the term "true value") of a set of observations. In the present writer's opinion, these discussions are the most excellent part of Schiaparelli's paper. These discussions are even more significant in view of the fact that the later writers on this subject make no attempt whatsoever to justify the use of their postulates.

Schiaparelli remarks that we should have no reason for not expecting that a small change in a single observation should produce a small change in the function ϕ; but he does not make this remark in the form of an explicit postulate. This could have been done and, moreover, such a postulate of continuity could be justified from the practical nature of the problem. It seems that a more elegant procedure would have been to deduce the continuity of the function and its derivatives from Axioms I and II. It will be shown later that this is possible. From his remark on the continuity of the function, Schiaparelli concludes that the partial derivatives of ϕ with respect to the x_i exist and are continuous. His method of arriving at this conclusion is not valid, for it is well known that an arbitrarily assumed function may be everywhere continuous and yet possess a derivative at no point.

Schiaparelli's Proposition III states: "When in the function ϕ all the x_i take the same value, then the $\dfrac{\partial\phi}{\partial x_i}$ become equal to each other." This Proposition is false. To show this, consider the function

$$f \equiv \bar{x} + \frac{\mu_3}{\mu_2},$$

where the

$$\frac{\partial f}{\partial x_i} = \frac{1}{n} + \frac{3\mu_2[(x_i - \bar{x})^2 - \mu_2] - 2\mu_3(x_1 - \bar{x})}{n\mu_2^2}.$$

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[2] Giovanni Schiaparelli—Come si possa giustificare l'uso della media aritmetica nel calcoło die risultati d'osservazione, Rendiconti Reale Instituto Lombardo di Scienze e lettere, Vol. XL (1907), pp 752–764.

Now, when the x_i all approach a then both f and $\frac{\partial f}{\partial x_i}$ become indeterminate forms. However, in this case f takes an indeterminate form which can be evaluated and it can be shown that $\frac{\mu_3}{\mu_2}$ will always have the value zero, i.e., f will have the value a when all the $x_i = a$; while the $\frac{\partial f}{\partial x_i}$ can take any value whatever and in general the $\frac{\partial f}{\partial x_i}$ will not be equal when the $x_i \rightarrow a$. To illustrate: Consider the observations $y_1 = 1$, $y_2 = 3$, $y_3 = 4$ then $\bar{y} = 8/3$ and $\mu_2 = 14/9$ and $\mu_3 = -20/27$ whence $f = 8/3\text{–}10/21$. Now assume that these three observations all approach 2 in a certain way, i.e., let $x_i = 2 + (y_i - 2)z$. Then $\bar{x} = 2 + (\bar{y} - 2)z = 2 + (2/3)z$.

$$\mu_2(x_i) = z^2 \frac{1}{n} \Sigma (y_i - \bar{y})^2 = (14/9)z^2$$

and

$$\mu_3(x_i) = z^3 \frac{1}{n} \Sigma (y_i - \bar{y})^3 = (-20/27)z^3$$

whence $f = 2 + (2/3)z - (10/21)z$. Clearly as $z \rightarrow 0$ the $x_i \rightarrow 2$ and $f \rightarrow 2$. However,

$$\left.\frac{\partial f}{\partial x_1}\right]_{x_1=2+(y_1-2)z} = \frac{1}{3} + \frac{131}{294},$$

$$\left.\frac{\partial f}{\partial x_2}\right]_{x_2=2+(y_2-2)z} = \frac{1}{3} - \frac{253}{294},$$

$$\left.\frac{\partial f}{\partial x_3}\right]_{x_3=2+(y_3-2)z} = \frac{1}{3} + \frac{122}{294}.$$

Thus the $\frac{\partial f}{\partial x_i}$ are not functions of z and as the $x_i \rightarrow 2$ the $\frac{\partial f}{\partial x_i}$ remain constant and unequal.

From his conclusion that the derivatives of ϕ exist and from Axiom I, Schiaparelli obtains the equation, $\sum_{i=1}^{i=n} \frac{\partial \phi}{\partial x_i} = 1$, (this equation being his Proposition V) in the following way: Since the derivatives of ϕ exist, then by the Theorem of the mean value,

$$\phi(x_1 + h, x_2 + h, x_3 + h, \cdots, x_n + h)$$
$$= \phi(x_1, x_2, \cdots, x_n) + h\left(\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \cdots + \frac{\partial \phi}{\partial x_n}\right). \quad \text{(A)}$$

By Axiom I:

$$\phi(x_1 + h, x_2 + h, \cdots, x_n + h) = \phi(x_1, x_2, \cdots, x_n) + h.$$

Whence $\sum_{i=1}^{i=n} \frac{\partial\phi}{\partial x_i} = 1$. Now this equation is correct but the above proof of it is not convincing. Clearly, according to the Theorem of the mean value, in equation (A) it is necessary to replace each x_i in the $\frac{\partial\phi}{\partial x_i}$ by θx_i where θ is between 0 and 1.

Schiaparelli's Proposition VII states in effect that the $\frac{\partial\phi}{\partial x_i}$ are invariant under the transformation $x_i' = x_i + h$ where h is constant, and his Proposition IX states that the $\frac{\partial\phi}{\partial x_i}$ are invariant under the transformation $x_i' = kx_i$ where k is a constant. These two propositions are correct and are correctly established. Making use of his Propositions III (which is false), V, VII and IX, Schiaparelli proceeds to the establishment of the postulate of the Arithmetic Mean, as follows:

Let $a = \phi(x_i)$. As the x_i vary, then a varies but for a particular set of x_i then a is a constant. Now by Axiom I we have

$$a + (m-1)\,a = \phi(x_1 + (m-1)a, x_2 + (m-1)a, \cdots, x_n + (m-1)a) = m\,a$$

for all values of $m > 1$. Then by Axiom II:

$$a = \phi\left(\frac{x_1 + (m-1)\,a}{m}, \frac{x_2 + (m-1)\,a}{m}, \cdots, \frac{x_n + (m-1)\,a}{m}\right)$$

$$= \phi\left(\frac{x_1 - a}{m} + a, \frac{x_2 - a}{m} + a, \cdots, \frac{x_n - a}{m} + a\right).$$

And by Propositions VII and IX, the $\frac{\partial\phi}{\partial x_i}$ are unchanged during the above transformations. Hence the last equation is true when $m \to \infty$ and by Proposition III (false) the $\frac{\partial\phi}{\partial x_i} = \frac{1}{n}$ as when $m \to \infty$, $\phi(x_i) = a$. In this final proof Schiaparelli gives a geometric illustration of each step.

It is both interesting and strange to know that in closing his paper Schiaparelli does not claim that the Arithmetic Mean is the only function which will satisfy all of his postulates. In fact he himself points out that the function ϕ, implicitly defined by the equation $\sum_{i=1}^{i=n} (\phi - x_i)^m = 0$ where m is an odd integer > 1 will satisfy all of his postulates. Furthermore he points out that this function will not satisfy his Proposition III. Schiaparelli's object was to establish the postulate of the Arithmetic Mean without any appeal to the concept of probability. To accomplish this he made four assumptions each of which he justified by *a priori* reasoning. Then he proceeded with the above proof. Why he should have been satisfied with his own proof after perceiving the function defined by $\sum_{i=1}^{i=n} (\phi - x_i)^m = 0$ is hard to understand.

The second paper[3] consulted was also by Schiaparelli. It is merely an abridged form of the one just discussed. Schiaparelli wrote two earlier papers on this same subject (altogether Schiaparelli wrote four papers on it) but it was inferred from the footnotes in his paper, which has just been discussed at length, that it contained all of the material of the two earlier papers with which he himself was satisfied. Therefore Schiaparelli's two earlier papers were not consulted.

The third paper consulted was that by Broggi.[4] Broggi states that the purpose of his paper is to establish the postulate of the Arithmetic Mean by purely analytic methods which are more brief than Schiaparelli's method. Broggi words the assumptions upon which he bases his proof as follows:

1. ϕ is a symmetric function of its n variables;
2. The partial derivatives are single-valued and finite;
3. We have $\phi(kx_1, kx_2, \cdots, kx_n) = k\phi(x_1, x_2, \cdots, x_n)$;
4. We have $\phi(x_1 + h, x_2 + h, \cdots, x_n + h) = \phi(x_1, x_2, \cdots, x_n) + h$, that is to say for 2:

$$\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + \cdots + \frac{\partial\phi}{\partial x_n} = 1\,. \tag{a}$$

Broggi does not explain why he used the postulate 2 but presumably it was in order to exclude the function defined by $\sum_{i=1}^{i=n} (\phi - x_i)^m = 0$. Consider the special case where $m = 3$. Then $n\phi^3 - 3\phi^2\,\Sigma x_i + 3\phi\,\Sigma x_i^2 - \Sigma x_i^3 = 0$. Let $p = 3\left(\frac{1}{n}\,\Sigma x_i^2 - \bar{x}^2\right)$ and $q = \frac{3\bar{x}}{n}\,\Sigma x_i^2 - 2\bar{x}^3 - \frac{1}{n}\,\Sigma x_i^3$. Also put $R = (p/3)^3 + (q/2)^2$ and let A be the *real* cube root of $-q/2 + \sqrt{R}$ and B be the *real* cube root of $-q/2 - \sqrt{R}$. Then the three branches of ϕ can be explicitly written

$$\phi_1 = A + B + \bar{x}$$

$$\phi_2 = \omega A + \omega^2 B + \bar{x}$$

$$\phi_3 = \omega^2 A + \omega B + \bar{x}$$

where ω and ω^2 are the two complex cube roots of unity. Now while ϕ does not satisfy the postulate that the function be single valued, ϕ_1 satisfies this postulate as well as all the others and so does ϕ_2 and also ϕ_3. Hence, Broggi's failure to comment at length on the function $\sum_{i=1}^{i=n} (\phi - x_i)^m = 0$ is unsatisfying. As a matter of fact Broggi fails to point out any of the defects of Schiaparelli's

[3] Giovanni Schiaparelli—Come si possa giustificare l'uso della media aritmetica nel calcolo delle misure, senza fare alcuna ipotesi sulla legge di probabilità degli errori accidentali, Astronomische Nachrichten, Band 176 (1907) pp. 206–212.

[4] Ugo Broggi—Sur Le Principe De La Moyenne Arithmetique, L'Enseignement Mathematique, XI (1909) pp. 14–17.

paper, with the possible exception that he shows Schiaparelli's postulate which states $\phi = a$ when each of the $x_i = a$ to be a consequence of Axioms I and II. This is done so casually that it makes one wonder whether Broggi really was aware of the fact that Schiaparelli's postulates are not independent.

Broggi proves the Lemma: "A homogeneous function of the first degree which is a solution of the equation of partial derivatives (a) is an integral function." This Lemma is correct and is correctly proved but its wording is apt to be misleading; in fact it appears that its true meaning was not clear to Broggi himself. For, while the function ϕ cannot be of the form $\frac{\psi}{\chi}$ where ψ is a homogeneous function of the p^{th} degree which satisfies Axiom I and χ a homogeneous function of the $(p - 1)^{th}$ degree which also satisfies Axiom I, the Lemma does not mean and Broggi has not proved that ϕ cannot be of the form $\phi = \Omega + \frac{\psi}{\chi}$ where Ω is an integral function satisfying Axioms I and II and ψ and χ are homogeneous functions of the p^{th} and $(p - 1)^{th}$ degrees respectively which are *invariant* under the transformation specified in Axiom I. By reason of this oversight, Broggi concludes that any function satisfying Axioms I and II must be linear in its n variables, a conclusion which is erroneous.

The fourth paper consulted was that by Schimmack.[5] Schimmack's paper is in three sections. The first section contains the proof which is essentially that which Whittaker and Robinson give. In the second section Schimmack gives a different proof, from a set of new postulates. The new set of postulates is:

Axiom I′ = Axiom I.

Axiom II′—The most probable value is independent of the sense of direction of the scale upon which the observed values (and the most probable value) are reckoned, that is to say,

$$\phi(-x_1, -x_2, \cdots, -x_n) = -\phi(x_1, x_2, \cdots, x_n).$$

Axiom III′ = Axiom III.

Axiom IV′—If from n observed values, the most probable value be computed and if one obtains an additional observed value then the most probable value of the $n + 1$ observed values is the same as the most probable value of $n + 1$ quantities consisting of the initial most probable value counted n times and the $(n + 1)^{th}$ observed value, namely:

$$\phi_{n+1}(x_1, \cdots, x_{n+1}) = \phi_{n+1}(\phi_n, \cdots, \phi_n, x_{n+1}).$$

In explaining the object of this second section, Schimmack says that postulating the existence of the derivatives (Axiom IV) seems unjustified and ought to be avoided and only such axioms made which the intrinsic character of the problem justifies. In connection with this statement of Schimmack's it appears that the intrinsic character of the problem certainly does not justify Axiom IV′. In

[5] Rudolf Schimmack—Der Satz vom arithmetischen Mittel in axiomatischer Begründung, Mathematische Annalen, Band 68 (1909) pp. 125–132, 304.

fact, Axiom IV′ appears to be quite artificial. Moreover, Schimmack does not attempt to justify Axiom IV′ by *a priori* reasoning as Schiaparelli does for Axioms I, II, and III. While, if the Arithmetic Mean is the most probable value, Axiom IV′ follows, since it is a property of the Arithmetic Mean, it does not seem to be in keeping with the intrinsic character of the problem to use this property as a starting point for later deductions.

As regards Schimmack's objections to Axiom IV, all of the conditions specified by it can be deduced from the first two Axioms except that the derivatives must be single-valued. To show that this is true, consider an arbitrary function which satisfies Axioms I and II. Let this function be $\phi(x_1, x_2, \cdots, x_n)$. We do not know that ϕ is continuous or that ϕ has any derivatives. All we assume is that ϕ satisfies the first three Axioms and it is here proven that ϕ must be continuous and have continuous partial derivatives. By Axiom I we can give increments to the x_i; hence we give each x_i the same increment, Δx, and then subtract ϕ and we have: $\phi(x_1 + \Delta x, x_2 + \Delta x, \cdots, x_n + \Delta x) - \phi(x_1, x_2, \cdots, x_n) = \Delta\phi$ but by Axiom I, $\Delta\phi = \Delta x$. Therefore $\frac{\Delta\phi}{\Delta x} = 1 = \frac{d\phi}{dx}$. In other words, the total derivative of ϕ exists and is constant. Therefore the total derivative of ϕ is continuous. But since the total derivative exists, all of the partial derivatives exist. By Axiom II, ϕ is a homogeneous function of the first degree. Applying Euler's Theorem for homogeneous forms, we have $\phi = x_1 \frac{\partial\phi}{\partial x_1} + x_2 \frac{\partial\phi}{\partial x_2} + \cdots + x_n \frac{\partial\phi}{\partial x_n}$. Since the total derivative of ϕ is everywhere continuous, ϕ is also everywhere continuous. Thus, the right hand side of the above equation is everywhere continuous and each partial derivative is therefore everywhere continuous.

As regards that part of Axiom IV which requires the $\frac{\partial\phi}{\partial x_1}$ to be single valued, it would seem more satisfactory to postulate that the function ϕ is single-valued, for the single-valuedness of a derivative does not insure the single-valuedness of the integral while the single-valuedness of a function does insure the single-valuedness of the derivative where the derivative exists.

In the third section of his paper, Schimmack shows Axioms I, II, III, and IV to be independent, and likewise Axioms I, II′, III and IV′.

Schimmack does not mention any of the questionable features of Schiaparelli's and Broggi's papers.

The fifth paper consulted was that by Suto.[6] Suto's assumptions are:

1°. $\phi(x, x, \cdots, x) = x$ (This is Schiaparelli's).

2°. $\phi(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) - \phi(x_1, x_2, \cdots, x_n)$ depends on the values of $y_1, y_2, \cdots, y_n$ only.

3°. = Axiom III = Axiom III′.

[6] Onosaburo Suto—Law of the Arithmetical Mean, Tohoku Mathematical Journal, Vol. 6 (1914) pp. 79–81.

Suto says he believes these assumptions to be more simple and natural than Schimmack's Axioms I′–IV′. However, assumption 2° appears to be quite artificial and very restrictive. Suto does not even attempt to justify it by *a priori* reasoning.

Suto shows his three Axioms to be independent. It is interesting to know that Suto has established the postulate of the Arithmetic Mean rigorously using only three postulates while Schiaparelli, Broggi and Schimmack failed using four postulates. In this connection it should be observed that when Axiom IV as given by Whittaker and Robinson is replaced by "The most probable value, regarded as a function of the individual measures, has first partial derivatives with respect to them which are equal" as suggested at the end of Part 1, then Axiom III can be deduced as a consequence of Axioms I, II and the reworded Axiom IV, so that three Axioms only are sufficient to deduce the postulate of the Arithmetic Mean. However, it would be difficult to justify the reworded Axiom IV from the nature of this problem of the Arithmetic Mean.

Suto does not point out any of the defects of the preceding papers.

The last paper consulted was that by Beetle.[7] It deals with the third section of Schimmack's paper. Beetle also fails to point out any of the defects of the preceding papers.

Conclusion

The postulate of the Arithmetic Mean can be rigorously established, without the use of the concept of probability, if sufficiently restrictive assumptions are made. The writers making sufficiently restrictive assumptions have failed to justify the use of them. Several proofs of the postulate of the Arithmetic Mean are clearly erroneous. The existing attempts to establish the postulate of the Arithmetic Mean without any appeal to the concept of probability are, therefore, unsatisfactory.

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[7] R. D. Beetle—On the complete independence of Schimmack's postulates for the Arithmetic Mean, Mathematische Annalen, Band 76 (1915) pp. 444–446.