ON THE POLYNOMIALS RELATED TO THE DIFFERENTIAL EQUATION

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}$$

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Introduction. In a previous issue of this Journal,[1] E. H. Hildebrandt has established the existence of a general system of polynomials $P_n(k, x)$ associated with the solutions of Pearson's Differential Equation

$$\text{(R)} \qquad \frac{1}{y}\frac{dy}{dx} = \frac{N}{D},$$

N and D being polynomials in x of degrees not exceeding one and two respectively with no factor in common.

It was shown that the polynomials $P_n(k, x) \equiv P_n$ themselves satisfy certain differential equations and a recurrence relation. The classical polynomials of Hermite, Legendre, Laguerre, and Jacobi are special types of $P_n(k, x)$. Since the classical polynomials are employed rather extensively in statistical theory, certain of their properties are of special interest.

It is the purpose of this paper to determine from Hildebrandt's general equations some new properties of $P_n(k, x)$ and to apply these properties to the classical polynomials. The paper consists of two parts. In part I some theorems are established concerning common zeros of D and P_n. In particular, a theorem is established to exhibit the conditions under which the zeros of P_n, which are not zeros of D, are simple. In part II a method is outlined for the classical polynomials by which one can determine the number and location of the real zeros in the various segments into which the zeros of D divide the x axis. The points of inflexion and the degree of the polynomials are also considered.

A new feature of the method employed is, we believe, its being based upon the use of differential equations of first order, for most part, while other investigators[2] have employed differential equations of second order. As to the results obtained, the author believes them to be partly new. They have points in common with the results of Fujiwara, Lawton and Webster.

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[1] Systems of Polynomials Connected with the Charlier Expansions, etc., Annals of Math. Stat., Vol. II, 1931, pp. 379–439.

[2] *M. Fujiwara:* On the zeros of Jacobi's Polynomials, Japanese Journal of Math., Vol. 2, 1925, pp. 1, 2.

W. Lawton: On the zeros of Certain Polynomials Related to Jacobi and Laguerre Polynomials, Bull. Am. Math. Soc., Vol. 38, 1932, pp. 442–449.

M. S. Webster: Thesis, Univ. of Penna. These results were kindly communicated to me by Dr. Webster.

I. Theorems Concerning Common Zeros of $P_n(k, x)$ and D

The following equations will be employed later:

$$(1) \qquad P_{n+1}(k, x) = [N + (k - n)D']P_n(k, x) + DP'_n(k, x).$$

$$(2) \qquad P'_{n+1}(k, x) = (n + 1)\left[N' + \frac{2k - n}{2} D''\right] P_n(k, x).$$

$$(3) \qquad \begin{aligned} P_{n+1}(k, x) = {} & [N + (k - n)D']P_n(k, x) \\ & + n\left[N' + \frac{2k - n + 1}{2} D''\right] DP_{n-1}(k, x). \end{aligned}$$

These are not explicitly given in Hildebrandt's Paper but the method of obtaining them is outlined there in detail.

We shall make use of the following lemma which we state without proof.

Lemma (1). *Let $P_n(x)$ be a polynomial of degree n. If both P_n and P'_n contain a factor $(x - \alpha)^m$, $m < n$, then P_n contains the factor $(x - \alpha)^{m+1}$.*

We also need an expression for $P^{(q)}_{n+1}(k, x)$. By repeatedly differentiating (2) and eliminating $P'_n(k, x)$ we get,

$$(4) \qquad P^{(q)}_{n+1}(k, x) = \prod_{i=0}^{q-1} (n + 1 - i)\left[N' + \frac{2k - n + i}{2} D''\right] P_{n-q+1}(k, x), \qquad q = 1, 2, \cdots (n + 1).$$

Theorem I_1. If D is a perfect square, D' is not a factor of P_{n+1} (k, x), $n = 0, 1, 2, \cdots$

Proof: Assume D' to be a factor of P_{n+1}. From (1), D' is either a factor of P_n or of $N + (k - n) D'$. But D' is not a factor of $N + (k - n) D'$ as this implies that D' is a factor of N contrary to hypothesis on (R) that D and N have no factor in common. Thus, D' is a factor of P_n, and by a repetition of the reasoning a factor finally of P_1, which as it was just pointed out, is impossible.

Theorem I_2. Set $D = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$, D not a perfect square. If $\alpha_i x + \beta_i$, $i = 1$ or 2, is a factor of P_n, then $(\alpha_i x + \beta_i)^q$ is a factor of P_{n+q-1}, $q = 1, 2, 3, \cdots$

Proof: From (1), $\alpha_i x + \beta_i$ being a factor of P_n and D, is also a factor of P_{n+1}. From (2), $\alpha_i x + \beta_i$ is a factor of P'_{n+1}. From Lemma (1) it follows that $(\alpha_i x + \beta_i)^2$ is a factor of P_{n+1}. Continued repetition of the reasoning establishes the theorem.

Corollary. If both $\alpha_1 x + \beta_1$ and $\alpha_2 x + \beta_2$ are factors of P_n, then D^q is a factor of P_{n+q-1}.

Theorem I_3. Assume D of the same form as in Theorem I_2. If $\alpha_i x + \beta_i$, $i = 1$ or 2, is a factor of P_{n+1} and no higher power of $\alpha_i x + \beta_i$ is such a factor then $\alpha_i x + \beta_i$ is a factor of $N + (k - n)D'$.

Proof: From (1), $\alpha_i x + \beta_i$ being a factor of P_{n+1} and of D is also a factor of either $N + (k - n)D'$ or of P_n. But $\alpha_i x + \beta_i$ a factor of P_n requires, from I_2, that $(\alpha_i x + \beta_i)^2$ be a factor of P_{n+1} contrary to hypothesis. Thus, $\alpha_i x + \beta_i$ is a factor of $N + (k - n)D'$.

Corollary. If $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$, $(\alpha_1, \alpha_2 \neq 0)$, *is a factor of* P_{n+1} *and no higher power of either* $\alpha_1 x + \beta_1$ *or* $\alpha_2 x + \beta_2$ *is contained in* P_{n+1} *then* $N + (k - n) D' \equiv 0$. For from I_3, $N + (k - n)D'$ contains $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$ as a factor which implies $N + (k - n)D'$, being linear, vanishes identically.

Theorem I_4. *If* $(\alpha_i x + \beta_i)^q$ *and no higher power of* $\alpha_i x + \beta_i$ *is a factor of* P_{n+q-1} *then* $\alpha_i x + \beta_i$ *and no higher power of* $\alpha_i x + \beta_i$ *is a factor of* P_n.

Proof: Let us write,

(A) $P_{n+q-1} = (\alpha_i x + \beta_i)^q \phi_{n-1}$, $\phi_{n-1} \equiv$ a polynomial of degree $\leq n - 1$ which does not contain the factor $\alpha_i x + \beta_i$. Taking the $(q - 1)^{\text{st.}}$ derivative of (A) by Leibnitz Theorem, we get,

$$\text{(B)} \qquad P_{n+q-1}^{(q-1)} = \sum_{i=0}^{q-1} \binom{q-1}{i} \frac{d^i}{dx^i} (\alpha_i x + \beta_i)^q \frac{d^{q-1-i}}{dx^{q-1-i}} \phi_{n-1}.$$

On setting $q = q - 1$ in (4) there results,

$$\text{(C)} \qquad P_{n+q-1}^{(q-1)} = \prod_{i=0}^{q-2} (n + q - 1 - i) \left[N' + \frac{2k - n - q + i + 2}{2} D'' \right] P_n.$$

From (B) we see that $\alpha_i x + \beta_i$ is a factor of $P_{n+q-1}^{(q-1)}$. No higher power of $\alpha_i x + \beta_i$ is such a factor. From (C) our theorem now follows.

Corollary (1). *Under the hypotheses of Theorem* I_4, $\alpha_i x + \beta_i$ *is a factor of* $N + (k - n + 1)D'$. This follows at once from I_4 and I_3.

Corollary (2). *If* $D^q = (\alpha_1 x + \beta_1)^q (\alpha_2 x + \beta_2)^q$, $(\alpha_1, \alpha_2 \neq 0)$, *is a factor of* P_{n+q-1} *and no higher powers of either* $\alpha_1 x + \beta_1$ *or* $\alpha_2 x + \beta_2$ *are factors, then* $N + (k - n + 1)D' \equiv 0$. For the linear expression $N + (k - n + 1)D'$ contains, from Corollary (1), the quadratic factor $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$.

The following lemma can be easily established and is given without proof.

Lemma (2). *Assume D of the same form as in Theorem* I_2. *Then there is only one value of s for which* $N + sD'$ *contains* $\alpha_i x + \beta_i$ *as a factor.*

Theorem I_5. *Assume D of the same form as in Theorem* I_2. *If* $N + (k - n)D'$ *contains* $\alpha_i x + \beta_i$, $i = 1$ *or* 2, *as a factor, then* P_{n+1} *contains* $\alpha_i x + \beta_i$ *and no higher power of* $\alpha_i x + \beta_i$ *as a factor.*

Proof: From (1) we see that P_{n+1} contains $\alpha_i x + \beta_i$ at least to the first power as a factor. Again from (1), if P_{n+1} contains a higher power of $\alpha_i x + \beta_i$ as a factor, this means that both P_n and P_n' contain $\alpha_i x + \beta_i$ at least to the first power as a factor and from Lemma (1) it follows that P_n contains $\alpha_i x + \beta_i$ at least to the second power as a factor. By corollary (1) from Theorem I_4 it follows that $\alpha_i x + \beta_i$ is a factor of $N + (k - n_1)D'$ for $n_1 < n$, contrary to Lemma (2).

Theorem I_6. *If* $\alpha_1 x + \beta_1$ *and* $\alpha_2 x + \beta_2$ *are factors of* $N + (k - n_1)D'$ *and* $N + (k - n_2)D'$ *respectively,* $(\alpha_1, \alpha_2 \neq 0)$, *then* $P_\mu \equiv 0$, $\mu > n_1 + n_2$.

Proof: From Theorems I_5 and I_2 we see that $(\alpha_1 x + \beta_1)^{n_2} (\alpha_2 x + \beta_2)^{n_1}$, of degree $n_1 + n_2$, is a factor of $P_{n_2+n_1}$, of degree $n_2 + n_1$ at most. Similarly,

$(\alpha_1 x + \beta_1)^{n_2+1} (\alpha_2 x + \beta_2)^{n_1+1}$, of degree $n_2 + n_1 + 2$, is a factor of $P_{n_2+n_1+1}$, of degree $n_2 + n_1 + 1$ at most. This implies $P_{n_2+n_1+1} \equiv 0$. Hence, $P_\mu \equiv 0$, $\mu > n_1 + n_2$. In fact, (1) shows that $P_\mu \equiv 0$ implies $P_\nu \equiv 0$, $\nu > \mu$.

Theorem I_7. Assume D of the same form as in Theorem I_2. Then $P_{n+1} \equiv 0$, $P_n \not\equiv 0$, implies either $N + (k - m)D' \equiv 0$, $m \leq n$, or there exist two values of m, (m_1, m_2), such that $N + (k - m_1)D'$, $N + (k - m_2)D'$ contain as factors $\alpha_1 x + \beta_1$ and $\alpha_2 x + \beta_2$ respectively, $(m_1, m_2 \leq n)$.

Proof: Setting $P_{n+1} \equiv 0$ in (1) gives,

$$(1^0)\ [N + (k - n)D']\, P_n + DP'_n \equiv 0.$$

If $P_n \equiv$ const., 1^0 shows that $N + (k - n)D' \equiv 0$ and our theorem is verified. Suppose $P_n \not\equiv$ const. We get from (1^0),

$$P'_n = -\frac{[N + (k - n)D']P_n}{D}.$$

Thus, D is a factor of the numerator, and our theorem now follows from Corollaries (1) and (2) of Theorem I_4.

Theorem I_8. If $N + (k - m)D' \not\equiv 0$, $m = 1, 2, \cdots n$, and if $N + (k - m)D'$ contains neither $\alpha_1 x + \beta_1$, nor $\alpha_2 x + \beta_2$ as factors, then P_{n+1} and D have no factors in common. This follows at once from Theorems I_2 and I_4 which constitute a necessary and sufficient condition that P_n and D have factors in common.

Theorem I_9. If $N \equiv$ const. and if D is linear, all P_n are constants, $n = 1, 2, 3, \cdots$. This follows directly from (2).

Theorem I_{10}. If $N' + \frac{2k - m}{2} D'' \not\equiv 0$, $m = 1, 2, \cdots (n - 1)$, all zeros of P_n which are not zeros of D are simple.

Proof: Suppose P_n has a multiple zero $x = \alpha$ which is not a zero of D. Then (1) shows that α is a zero of P_{n+1}. From (2), α is a zero of P'_{n+1}. From Lemma (1), α is at least a double zero of P_{n+1}. Furthermore, (3) shows that α being a double zero of P_n and of P_{n+1} is also a double zero of P_{n-1}. By a continued application of (3), it follows that α is a double zero of P_1 which is impossible since P_1 is of degree ≤ 1.

II. Concerning the Zeros of $P_n(k, x)$

The polynomials $P_n(k, x)$ are defined by Hildebrandt[3] as follows: $P_n(k, x) = \frac{1}{y} D^{n-k} \frac{d^n}{dx^n} D^k y$ where y is a non-identically vanishing solution of the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}.$$

[3] L.c. pp. 400–401.

The Jacobi Polynomials are defined as follows:

$$J_n(x, \alpha, \beta) = x^{1-\alpha}(1-x)^{1-\beta} \frac{d^n}{dx^n}[x^{n+\alpha-1}(1-x)^{n+\beta-1}], \ \alpha, \beta$$

real. It follows that $J_n(x, \alpha, \beta)$ is a special type of $P_n(k, x)$ with $N \equiv (-\beta-\alpha) x + \alpha$, $D \equiv x(1-x)$, $n = k+1$, whence,

$$N' = -\beta-\alpha, \qquad D' = 1-2x, \qquad D'' = -2; \qquad D(0) = D(1) = 0,$$

$$P_1(k, x) \equiv N + kD' = 0 \text{ for}$$

$$x = \frac{\alpha + k}{\alpha + \beta + 2k}, \qquad P_1'(k, x) = -\beta - \alpha - 2k.$$

In determining the number and location of the real zeros of the Jacobi Polynomials we employ the following notations:

$P_i(k, x) = 0$ for $x = \alpha_{i,k,j}$, $\ i = 1, 2, \cdots k+1; k = 0, 1, 2, \cdots; j = 1, 2, \cdots i.$

$\alpha_{i,k,j} \leq \alpha_{i+1,k,j}$

$$\theta = N' + \frac{2k-n}{2} D'' = -\beta - \alpha - 2k + n, \quad n = 1, 2, \cdots k,$$

$$\mu = [N + (k-n)D']_{x=0} = \alpha + (k-n),$$

$$\nu = [N + (k-n)\, D']_{x=1} = -\beta - (k-n).$$

We proceed to determine the number of real zeros of the Jacobi Polynomials on the intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$ into which the zeros of D divide the x axis.[4] The proofs proceed by mathematical induction. We first determine the location of the real zeros of $P_n(k, x)$, $n = 1, 2, \cdots k+1$, by successive applications of (1) and (2). We then use the relation $P_{k+1}(k, x) \equiv J_{k+1}(x, \alpha, \beta)$.

Several cases concerning possible values of α and β should be considered. In order to bring out the method of procedure only two such cases will be fully discussed here. The results for other possible cases will be merely listed.

A_1 : $\alpha < 0, \beta < 0, |\alpha| < |\beta|$, $\alpha, \beta, \alpha+\beta$ *not integers.*

Let k_1 be the greatest integer contained in α,

" k_2 " " " " " " β,

" k_3 be the greatest integral value of k for which $\alpha + \beta + 2k < 0$. Then

$$0 \leq k_1 \leq k_3 \leq k_2 .$$

[4] In the case $\alpha, \beta > 0$ these zeros all lie, as is known, on $(0, 1)$.

$A_{11}: 0 \leq k \leq k_1$. *We then have* $\theta > 0, \mu < 0, \nu > 0, 0 < \alpha_{1k1} < 1, P_1' > 0$. *Then* $J_{k+1}(x, \alpha, \beta)$ *has* $\dfrac{(1)^k + (-1)^k}{2}$ *zeros in* 0, 1. *These are the only real zeros.*

Proof: Consider first $P_1(k, x)$. Its only zero is at $\alpha_{1,k,1}$, where $0 < \alpha_{1,k,1} < 1$. Furthermore, $P_1' > 0$. Also $P_1 > 0$ for $x > \alpha_{1,k,1}$ and < 0 for $x < \alpha_{1,k,1}$. From (1) we see that $P_2(k, \alpha_{1,k,1}) > 0$, (since $P_1(k, \alpha_{1,k,1}) = 0, D(\alpha_{1,k,1}) > 0$ and $P_1' > 0$). From (2) it follows that $P_2'(k, x) < 0$ for $x < \alpha_{1,k,1}$, $P_2'(k, \alpha_{1,k,1}) = 0$, $P_2'(k, x) > 0$ for $x > \alpha_{1,k,1}$. These conclusions follow from remarks concerning the sign of θ, the fact that $P_1(k, \alpha_{1,k,1}) = 0$, and from remarks concerning the sign of P_1 to the left and to the right of $x = \alpha_{1,k,1}$. Thus, $P_2(k, x) > 0$ for all real x and hence has no real zeros. By employing (2), it is now evident that $P_3'(k, x) > 0$. From (1) and remarks concerning μ and ν we see that $P_3(k, 0) < 0$ and $P_3(k, 1) > 0$. Thus $P_3(k, x)$ has a single real zero $\alpha_{3,k,1}$, $0 < \alpha_{3,k,1} < 1$. The reasoning from P_3 to P_4 is analogous to that from P_1 to P_2. By continuing this procedure we finally conclude that $P_{k+1}(k, x)$, $(\equiv J_{k+1}(x, \alpha, \beta))$, has but one real zero, (in 0, 1), if k is even and no real zeros if k is odd.

A_{12}: $k_1 < k \leq k_3$. *Set* $k = k_1 + q, q = 1, 2, \cdots, k_3 - k_1$. *Here* $\theta > 0$, $\mu > 0, n = 1, 2, \cdots q - 1, \mu < 0, n = q, q + 1, \cdots, q + k_1$. $\nu > 0, \alpha_{1,k,1} < 0$, $P_1'(k, x) > 0$. $J_{k_1} + q + 1$ (x, α, β) *has* q *distinct zeros in* $(-\infty, 0)$ *and* $\dfrac{(1)^{k_1} + (-1)^{k_1}}{2}$ *zeros in* 0, 1. *These are the only real zeros.*

Proof: First consider the sequence $P_n(k, x)$ $n = 1, 2, \cdots q$, since the conditions on θ, μ, and ν do not change over this range of n. Now $P_1(k, \alpha_{1,k,1}) = 0$, $\alpha_{1,k,1} < 0$. Furthermore since $P_1' > 0$ we have $P_1 > 0$ for $x > \alpha_{1,k,1}$ and < 0 for $x < \alpha_{1,k,1}$. Pass now to $P_2(k, x)$. Since $D(\alpha_{1,k,1}) < 0$ and $P_1'(k, \alpha_{1,k,1}) > 0$, we see from (1) that $P_2(k, \alpha_{1,k,1}) < 0$. Moreover (2) shows $P_2'(k, \alpha_{1,k,1}) = 0$, $P_2'(k, x) < 0$ for $x < \alpha_{1,k,1}$ and > 0 for $x > \alpha_{1,k,1}$. Thus $P_2(k, x) < 0$ and a relative minimum at $x = \alpha_{1,k,1}$. Since $|P_2(k, \pm\infty)| = \infty$, we see that $P_2(k, x)$ has two real zeros of which the left most, $\alpha_{2,k,1}$, is in $(-\infty, 0)$. Again $\mu > 0$ together with (1) assures $P_2(k, 0) > 0$. Thus $\alpha_{2,k,2}$ is in $(\alpha_{1,k,1}, 0)$, hence in $(-\infty, 0)$. By continuing this reasoning on the successive $P_n(k, x)$, $n = 1, 2, \cdots q$, we conclude that $P_q(k, x)$ has q zeros in $-\infty, 0$ and $P_q'(k, \alpha_{q,k,1}) < 0$.

Next, consider the sequence $P_n(k, x)$, $n = q + 1, q + 2, \cdots q + k_1 + 1$. Over this range of n we have $\theta > 0, \mu < 0, \nu > 0$. From what has just been shown, $P_q(k, \alpha_{q,k,i}) = 0$, $-\infty < \alpha_{q,k,i} < 0$, $i = 1, 2, \cdots q$. Also $P_q'(k, \alpha_{q,k,i})$, $i = 1, 2, \cdots q$, is alternately negative and positive. Suppose q odd, (similar reasoning holds for q even). Thus, we suppose $P_q'(k, \alpha_{q,k,1}) < 0$, $P_q'(k, \alpha_{q,k,q}) < 0$, $P_q(k, x) > 0$ for $x < \alpha_{q,k,1}$ and < 0 for $x > \alpha_{q,k,q}$. (1) shows $P_{q+1}(k, \alpha_{q,k,i})$, $i = 1, 2, \cdots q$, to be alternately positive and negative. Thus, the zeros $\alpha_{q,k,i}$ are separated by $q - 1$ zeros of $P_{q+1}(k, x)$. Since from (1), $P_{q+1}(k, \alpha_{q,k,1}) > 0$ and from (2) $P_{q+1}'(k, x) > 0$ for $x < \alpha_{q,k,1}$, there exists a zero $\alpha_{q+1,k,1}$ in $(-\infty, \alpha_{q,k,1})$. Thus far, we have established the existence of q zeros of $P_{q+1}(k, x)$ in $(-\infty, 0)$. q being odd, we have from (1), $P_{q+1}(k, \alpha_{q,k,q}) > 0$. Also from (2),

$P'_{q+1}(k, x) < 0$ for $x > \alpha_{q,k,q}$. Again from (1) and assumptions regarding μ and ν it follows that $P_{q+1}(k, 0) > 0$, $P_{q+1}(k, 1) < 0$. Thus, $P_{q+1}(k, x)$ has a zero $\alpha_{q+1,k,q+1}$ in (0, 1). There being no extrema for $P_{q+1}(k, x)$ other than the $\alpha_{q,k,i}$, $i = 1, 2, \cdots q$, (as (2) shows), we have thus proved that $P_{q+1}(k, x)$ has q distinct zeros in $(-\infty, 0)$ and a single zero in (0, 1). Reasoning similarly from $P_{q+1}(k, x)$ to $P_{q+2}(k, x)$ we establish the existence of q distinct zeros $\alpha_{q+2,k,i}$, $i = 1, 2, \cdots q$, in $(-\infty, 0)$ with $\alpha_{q+2,k,1}$ in $(-\infty, \alpha_{q+1,k,1})$ and $\alpha_{q+2,k,i}$, $i = 2, 3, \cdots q$, separating $\alpha_{q+1,k,i}$, $i = 1, 2, \cdots q$. From (1) we see that $P_{q+2}(k, \alpha_{q+1,k,q}) < 0$ and $P_{q+2}(k, \alpha_{q+1,k,q+1}) < 0$. The only extrema of $P_{q+2}(k, x)$, (as (2) shows), are located at $\alpha_{q+1,k,i}$, $i = 1, 2, \cdots q + 1$. Again, by (2), $P'_{q+2}(k, x) < 0$ for $x > \alpha_{q+1,k,q+1}$; hence there can be no real zeros of P_{q+2} except the q zeros in $(-\infty, 0)$ already found. The reasoning from P_{q+2} to P_{q+3} is similar to that from P_q to P_{q+1}. Thus, $P_{q+k_1+1} \equiv J_{k_1+q+1}$ has q distinct zeros in $(-\infty, 0)$ together with one zero in (0, 1) for k_1 even. For k_1 odd, there are q distinct zeros in $(-\infty, 0)$ only. The results are the same whether q is odd or even.

The results for the remaining sub-cases under case A_1 are given in the table which follows. For completeness, the results for cases A_{11} and A_{12} are included in the tabulation. A few words of explanation are necessary to clarify the conditions under which the various sub-cases in the table occur. Let $|\alpha| = k_1 + q$, $|\beta| = k_2 + h$, $h, q < 1$. If $q + h < 1$, then $|\alpha + \beta| = k_1 + k_2$ and we have either,

A_{131} : $k_1 + k_2$ *even*, $2k_3 = k_1 + k_2 \equiv k_3 - k_1 = k_2 - k_3$.

A_{132} : $k_1 + k_2$ *odd*, $2k_3 = k_1 + k_2 - 1 \equiv k_3 - k_1 = k_2 - k_3 - 1$.

Again if $1 < q + h < 2$, then $|\alpha + \beta| = k_1 + k_2 + 1$ and we have either,

A_{133} : $k_1 + k_2 + 1$ *even*, $2k_3 = k_1 + k_2 + 1 \equiv k_3 - k_1 = k_2 - k_3 + 1$.

A_{134} : $k_1 + k_2 + 1$ *odd*, $2k_3 = k_1 + k_2 \equiv k_3 - k_1 = k_2 - k_3$

In cases A_{141} and A_{151} we assume $|\alpha + \beta| = k_1 + k_2 + p$, $p < 1$, while in cases A_{142} and A_{152}, $|\alpha + \beta| = k_1 + k_2 + p$, $1 < p < 2$. The complete results for case A_1 follow. (See page 213.)

A_2 : $\alpha < 0, \beta < 0, |\alpha| < |\beta|$, α, β *not integers*, $\alpha + \beta =$ *integer*. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. In Case A_{21}, $\beta + \alpha$ is odd while in Case A_{22}, $\beta + \alpha$ is even. (See page 214.)

A_3 : $\alpha < 0$, $\beta < 0$, $\alpha = -k_1$, *integer*, β *not an integer*, $|\alpha| < |\beta|$. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. There are two sub-cases, A_{31} : the greatest integral value of $\alpha + \beta$ is odd, A_{32} : this integral value is even. (See page 215.)

A_4 : $\alpha < 0$, $\beta < 0$, α *not an integer*, $\beta = -k_1$, *integer*, $|\alpha| < |\beta|$. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. There are two sub-cases, A_{41} : the integral part of $\alpha + \beta$ is odd, A_{42} : this integral value is even. (See page 216).

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $(1, \infty)$
A_{11}	J_{k+1};	$0 \leq k \leq k_1$;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{12}	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{131}, A_{133};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{132}, A_{134};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	1
A_{141}	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$;	$\frac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$;	$\frac{(1)^{k_2-k_1+q+1} + (-1)^{k_2-k_1+q+1}}{2}$
A_{142}	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	$\frac{(1)^{q} + (-1)^{q}}{2}$;	$\frac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$;	$\frac{(1)^{k_2-k_1+q} + (-1)^{k_2-k_1+q}}{2}$
A_{151}	$J_{k_1+k_2+q+1}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$\frac{(1)^{k_2} + (-1)^{k_2}}{2}$
A_{152}	$J_{k_1+k_2+2}$;		$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$;	0;	$\frac{(1)^{k_2+1} + (-1)^{k_2+1}}{2}$
	$J_{k_1+k_2+q+1}$;	$q = 2, 3, \cdots$;	Same zeros as in A_{151} for corresponding values of q.		

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $(1, \infty)$
A_{211}, A_{221}	J_{k+1};	$0 \leq k \leq k_1$	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{212}, A_{222}	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{213}	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{223}	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$;	0
A_{214}, A_{224}	J_{k_2+q+1};	$q = 1, 2, \cdots k_1$;	0;	$\frac{(1)^{k_1-q} + (-1)^{k_1-q}}{2}$;	0
A_{215}	$J_{k_1+k_2+2}$;	$J =$ const, > 0, k_1 odd. $J =$ const, < 0, k_1 even.			
	$J_{k_1+k_2+q+2}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$
A_{225}	$J_{k_1+k_2+2}$;	$J =$ const, > 0, k_1 odd. $=$ const, < 0, k_1 even.			
	$J_{k_1+k_2+q+2}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$

Cases	Polynomial	Range of Sub-Script	Zeros in			
			$(-\infty, 0)$	$x = 0$	$(0, 1)$	$(1, \infty)$
A_{311}, A_{321};	J_{k+1};	$0 \leq k < k_1$;	0;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{312}, A_{322};	J_{k_1+q+1};	$q = 0, 1, \cdots, k_3 - k_1$;	q;	$k_1 + 1$;	0;	0
A_{313};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$k_1 + 1$;	0;	1
A_{323};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$k_1 + 1$;	0;	0
A_{314};	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	0;	$k_1 + 1$;	0;	$\frac{(1)^q + (-1)^q}{2}$
A_{324};	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	0;	$k_1 + 1$;	0;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$
A_{315};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	0;	$k_1 + 1$;	$q - 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$
A_{325};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	0;	$k_1 + 1$;	$q - 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $x = 1$	Zeros in $(1, \infty)$
A_{411}, A_{421};	J_{k+1};	$0 \leq k \leq k_1$;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0;	0
A_{412}, A_{422};	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	0
A_{413};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	1
A_{423};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	0
A_{414}, A_{424};	J_{k_2+q+1};	$q = 0, 1, 2, \cdots, k_1$;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$;	0;	$k_2 + 1$;	0
A_{415}, A_{425};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$k_2 + 1$;	0

$A_5 : \alpha < 0, \beta < 0, |\alpha| < |\beta|, \alpha = -k_1$ *integer*, $\beta = -k_2$ *integer*. Define k_1, k_2, k_3 as in A_1. In cases A_{51} and A_{52}, $\alpha + \beta$ is odd and even respectively.

Cases	Polynomial	Range of Sub-Script	Zeros in		
			$(-\infty, 0)$	$x = 0$	$(0, 1)$
A_{511}, A_{521};	J_{k+1};	$0 \leq k < k_1$;	0;	0;	$\frac{(1)^k + (-1)^k}{2}$;
A_{512}, A_{522};	J_{k_1+q+1};	$q = 0, 1, 2, \cdots, k_3 - k_1$;	q;	$k_1 + 1$;	0
A_{513};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$k_1 + 1$;	0
A_{523};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q + 1$;	$k_1 + 1$;	0
A_{514}, A_{524};	$J_{k_2+q+1} \equiv 0$;	$q = 0, 1, 2, \cdots k_1$.			
A_{515}, A_{525};	$J_{k_1+k_2+q+1} \equiv 0$;	$q = 1, 2, 3, \cdots$			

If assumptions are identical with those of A_5 except $|\alpha| = |\beta|$, then for $0 \leq k < k_1$, the results agree with A_{511} and $J_{k_1+q+1} \equiv 0, q = 0, 1, 2, \cdots$.

$A_6 : \alpha > 0, \beta < 0, |\alpha| > |\beta|, \beta$ *not an integer*. Let k_1 be the largest integer in β.

Case	Polynomial	Range of Sub-Script	Zeros in	
			$(0, 1)$	$(1, \infty)$
A_{61}	J_{k+1}	$0 \leq k < k_1$	0	$\frac{(1)^k + (-1)^k}{2}$
A_{62}	J_{k_1+q+1}	$q = 1, 2, 3, \cdots$	q	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

A_7 : *Same assumptions as in* A_6 *except* $\beta = -k_1$, *integer*.

Case	Polynomial	Range of Sub-Script	Zeros in		
			$(0, 1)$	$x = 1$	$(1, \infty)$
A_{71}	J_{k+1}	$0 \leq k \leq k_1 - 1$	0	0	$\frac{(1)^k + (-1)^k}{2}$
A_{72}	J_{k_1+q+1}	$q = 0, 1, 2, \cdots$	q	$k_1 + 1$	0

$A_8 : \alpha > 0, \beta < 0, |\alpha| = |\beta|$. $J_1 = \alpha$ and results for $J_n, n > 1$ are identical with those in A_7 and A_6 respectively according as β is or is not an integer.

$A_9 : \alpha > 0, \beta < 0, |\alpha| < |\beta|$; $\beta, \alpha + \beta$, *not integers*.

Let k_1 be the greatest integer in $\alpha + \beta$.
" k_2 " " " " " β.
" k_3 " " " " for which $\alpha + \beta + 2k < 0$.

Then $0 \leq k_3 \leq k_1 \leq k_2$.

Case	Polynomial	Range of Sub-Script	Zeros in		
			$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
A_{91};	J_{k+1};	$0 \leq k \leq k_3$;	$k + 1$;	0;	0
A_{921};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3$; k_1 even;	$k_3 - q + 1$;	0;	0
A_{922};	J_{k_3+q+1};	$q = 1, 2, \cdots, (k_3 + 1)$; k_1 odd;	$k_3 - q + 2$;	0;	1
A_{93};	J_{k_1+q+1};	$q = 1, 2, \cdots, (k_2 - k_1)$;	0;	0;	$\frac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$
A_{94};	J_{k_2+q+1};	$q = 1, 2, 3, \cdots$;	0;	q;	$\frac{(1)^{k_2} + (-1)^{k_2}}{2}$

A_{10} : *Same assumptions as in* A_9 *but now* $|\alpha| = |\beta|$. Then $k_1 = k_3 = 0$, $J_1 = \alpha$, and results for J_n, $n > 1$ are the same as in A_{93} and A_{94}.

A_{11} : *Same assumptions as in* A_9 *except* $\beta = -k_2$, *integer.*

Case	Polynomial	Range of Sub-Script	Zeros in			
			$(-\infty, 0)$	$(0, 1)$	$x = 1$	$(1, \infty)$
$A_{11,1}$	Same as A_{91}					
$A_{11,2}$	Same as A_{92}					
$A_{11,3}$	Same as A_{93}					
$A_{11,4}$	J_{k_2+q+1};	$q = 1, 2, 3, \cdots$;	0;	q;	$k_2 + 1$;	0

A_{12} : $\alpha > 0$, $\beta < 0$, $|\alpha| < |\beta|$, β *not an integer.* $\alpha + \beta =$ *odd integer.* Define k_1, k_2, k_3 as in A_9.

A_{13} : *Same assumptions as in* A_{12} *except* $\alpha + \beta =$ *even integer.*

Cases	Polynomial	Range of Sub-Script	Zeros in
			$(-\infty, 0)$
$A_{12,1}$, $A_{13,1}$;	Same as A_{91}		
$A_{12,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3$;	$k_3 - q + 1$
	$J_{2k_3+3} =$ const. > 0;		
$A_{13,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3 + 1$;	$k_3 - q + 2$
	$J_{2k_3+3} =$ const. > 0;		
$A_{12,3}$, $A_{13,3}$;	Same as A_{93}		
$A_{12,4}$, $A_{13,4}$;	Same as A_{94}		

A_{14} : *Same assumptions as in* A_{12}, *except* $\beta = -k_2$ *integer.* Cases $A_{14,1}$, $A_{14,2}$ and $A_{14,3}$ have the same results as $A_{12,1}$, $A_{12,2}$, and $A_{12,3}$ respectively. $A_{14,4}$ has the same results as $A_{11,4}$.

A_{15} : *Same assumptions as* A_{13} *except* $\beta = -k_2$, *integer.* Cases $A_{15,1}$, $A_{15,2}$, and $A_{15,3}$ have the same results as $A_{13,1}$, $A_{13,2}$, and $A_{13,3}$ respectively. $A_{15,4}$ has the same results as $A_{11,4}$.

A_{16} : $\alpha = 0, \beta < 0, \beta$ — *not an integer.*

Let k_1 be the largest integer contained in β.

" k_3 be the largest integer for which $\beta + 2k < 0$.

Case	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$x = 0$	$(0, 1)$	$(1, \infty)$
$A_{16,1}$;	J_{k+1};	$0 \le k \le k_3$;	k;	1;	0;	0
$A_{16,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_1 - k_3$;	$k_3 - q$;	1;	0;	0; k_1 even
			$k_3 - q + 1$;	1;	0;	1; k_1 odd
$A_{16,3}$;	J_{k_1+q+1};	$q = 1, 2, 3, \cdots$;	0;	1;	$q - 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

A_{17} : $\alpha = 0, \beta = -k_1$ — *odd integer.* Define k_3 as in A_{16}.

A_{18} : $\alpha = 0, \beta = -k_1$ — *even integer.* Define k_3 as in A_{16}.

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$x = 0$	$(0, 1)$	$x = 1$
$A_{17,1}$, $A_{18,1}$;	Same as $A_{16,1}$					
$A_{17,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_1 - k_3 - 1$;	$k_3 - q$;	1;	0;	0
$A_{18,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3 + 1$;	$k_3 - q + 1$;	1;	0;	0
$A_{17,3}$, $A_{18,3}$;	$J_{k_1+1} \equiv 0$					
	J_{k_1+q+1};	$q = 1, 2, 3, \cdots$;	0;	1;	$q - 1$;	$k_1 + 1$

A_{19} : $\alpha = 0, \beta = 0.$ $J_1 \equiv 0.$

J_{k+1} has $k - 1$ zeros in (0, 1), 1 zero at $x = 0$, 1 zero at $x = 1$, $k = 1, 2, 3, \cdots$.

From the definition of $J_n(x, \alpha, \beta)$ it is readily seen that $J_n(x, \alpha, \beta) \equiv (-1)^n J_n(1 - x, \beta, \alpha)$. Thus, a transformation of x to $1 - x$ interchanges α and β. The interval $(-\infty, 0)$ is transformed into $(1, \infty)$ and vice-versa. The points $x = 0$ and $x = 1$ are interchanged. Consequently, in all previous results we may interchange properly α and β.

In the foregoing results, the only real multiple zeros that can occur are at either $x = 0$ or $x = 1$. In the process of determining the degree of multiplicity of such zeros use was made of Theorem I_2.

Points of Inflexion. By taking (4), setting $k = n$, and replacing N' and D''

by their values for Jacobi polynomials, we get: $P''_{n+1}(n, x) = (n + 1)(n)[\beta + \alpha + n][\beta + \alpha + n + 1] P_{n-1}(n, x)$. From definitions of $P_n(k, x)$ and $J_n(x, \alpha, \beta)$ we easily verify that,

$$P_n(n \pm q, x) \equiv J_n(x, \alpha \pm q + 1, \beta \pm q + 1), \text{ whence,}$$

$$J''_n(x, \alpha, \beta) = (n + 1)(n)[\beta + \alpha + n][\beta + \alpha + n + 1] J_{n-1}(x, \alpha + 2, \beta + 2).$$

We conclude that if neither $\alpha + \beta + n$ nor $\alpha + \beta + n + 1$ vanishes, the points of inflexion of $J_{n+1}(x, \alpha, \beta)$ are at the zeros of odd order of $J_{n-1}(x, \alpha + 2, \beta + 2)$.

The Degree of $J_n(x, \alpha, \beta)$. In analyzing the results of cases A_1 to A_{19} inclusive, it is noted that in some cases the number of real zeros of J_n is less than n. The question naturally arises whether the degree of J_n is n or less, for then we can determine the number of its imaginary zeros. The explicit expression of $J_n(x, \alpha, \beta)$ is known from which the degree of J_n can be found for various α and β. However, the degree of J_n can be found from (4).

Since $J_{n+1}(x, \alpha, \beta) \equiv P_{n+1}(n, x)$, let us replace k by n in (4) and at the same time replace N' and D'' by their values for Jacobi Polynomials. Thus, we get:

$$J^{(q)}_{n+1}(x, \alpha, \beta) = \prod_{i=0}^{q-1} (n + 1 - i)[-\beta - \alpha - n - i] P_{n-q+1}(n, x), \tag{5}$$
$$n = 0, 1, 2, \cdots; q = 0, 1, \cdots, (n + 1).$$

We may establish the following results.

C_1) If $\alpha + \beta$ is not an integer, the degree of $J_{n+1}(x, \alpha, \beta)$ is $n + 1$, $n = 0, 1, 2, \cdots$.

In fact, in order for $J^{(q)}_{n+1}$ to vanish, we see from (5) that either some factor $-\beta - \alpha - n - i$ vanishes or $P_{n-q+1}(n, x)$ vanishes identically. We first show that the latter is not possible. Now $P_1(n, x) \equiv N + nD' \equiv (-\beta - \alpha - 2n)x + \alpha + n \not\equiv 0$ since $\beta + \alpha$ is not an integer. Consequently, if $P_\mu(n, x) \equiv 0$, $\mu > 0$, $\mu \leq n + 1$ there will be a first value of μ, $(\mu = \nu)$, for which $P_\nu(n, x) \equiv 0$ but $P_{\nu-1}(n, x) \not\equiv 0$. By virtue of Theorem I_7 this means that either $N + (n - p)D' \equiv [-\beta - \alpha - 2(n - p)]x + \alpha + n - p \equiv 0$, $p \leq \nu$, or else there exist two values of p, (p_1, p_2), such that $[-\beta - \alpha - 2(n - p_1)]x + \alpha + n - p_1$ and $[-\beta - \alpha - 2(n - p_2)]x + \alpha + n - p_2$ are divisible by x and $1 - x$ respectively, $p_1, p_2 \leq \nu - 1$, $p_1 \neq p_2$. Since, however, $\alpha + \beta$ is not an integer we see that, $[-\beta - \alpha - 2(n - p)]x + \alpha + n - p \not\equiv 0$, n and p being integers. This eliminates the first possibility that $P_\mu(n, x) \equiv 0$, $\mu \leq n + 1$. Again, if, $[-\beta - \alpha - 2(n - p_1)]x + \alpha + n - p_1$ is divisible by x, we have $\alpha + n - p_1 = 0$ or α an integer. For $(\alpha + n - p_2) - [\beta + \alpha + 2(n - p_2)]x \equiv (\alpha + n - p_2)\left[1 - \frac{(\alpha + n - p_2) + (\beta + n - p_2)}{(\alpha + n - p_2)} x\right]$ to be divisible by $1 - x$ requires $\beta + n - p_2 = 0$ or β, an integer. α and β are therefore both integers contrary to hypothesis. Thus, in (5), no polynomial $P_{n-q+1}(k, x) \equiv 0$ and $J_{n+1}(x, \alpha, \beta) \not\equiv 0$. Replacing q by $n + 1$ in (5) leads to,

(6) $$J_{n+1}^{(n+1)}(x, \alpha, \beta) = \prod_{i=0}^{n} (n + 1 - i) [-\beta - \alpha - n - i] P_0(n, x),$$

$$n = 0, 1, 2, \cdots.$$

Thus $J_{n+1}^{(n+1)} \not\equiv 0$, (since $P_0(n, x) = 1$ and no factor $- \beta - \alpha - n - i$ can vanish) and the degree of J_{n+1} is precisely $n + 1$. From similar reasoning we prove:

C_2) If $\alpha + \beta > 0$ the degree of J_{n+1} is $n + 1$, $n = 0, 1, 2, \cdots$.

C_3) If $\alpha + \beta = 0$, then (I) $J_1 = \alpha$ and (II) J_{n+1} is of degree $n + 1$, $n = 1, 2, 3, \cdots$

C_4) If $\alpha + \beta = -M$ – integer, $M > 0$, β, α not integers, then,

(I) For $n < M$, the degree of J_{n+1} is min. $(n + 1, M - n)$.

(II) $n = M$, $J_{n+1} \equiv$ const.

(III) $n > M$, the degree of J_{n+1} is $n + 1$.

C_5) If $\alpha + \beta = - M$ – integer, $M > 0$, α, β integers, $\alpha > 0$, $\beta < 0$, then,

(I) For $n < M$, the degree of J_{n+1} is min. $(n + 1, M - n)$.

(II) $n = M$, $J_{n+1} \equiv$ const.

(III) $n > M$, the degree of J_{n+1} is $n + 1$.

C_6) If $\alpha + \beta = - M$ – integer, $M > 0$, $\alpha = -k_1$-integer, $\beta = -k_2$-integer, $k_1 < k_2$ then,

(I) For $n < k_2$, J_{n+1} is of degree $n + 1$.

(II) $n \geq k_2$, $J_{n+1} \equiv 0$.

C_7) If $\alpha + \beta = - M$ – integer, $M > 0$, $\alpha = \beta = -k_1$-integer, then,

(I) For $n < k_1$, J_{n+1} is of degree $n + 1$,

(II) $n \geq k_1$, $J_{n+1} \equiv 0$.

The Laguerre Polynomials. These are defined as follows:

$$L_n \equiv L_n(x, \alpha) = x^{1-\alpha} e^x \frac{d^n}{dx^n} [e^{-x} x^{n+\alpha-1}], \qquad n = 0, 1, 2, \cdots;$$

α – real. We see that L_n is a special case of $P_n(k, x)$ with $N \equiv - x + \alpha$, $D \equiv x$, $n = k + 1$. It follows that $\theta = -1$, $\mu = \alpha + k - n$, $\alpha_{1k1} = \alpha + k$, and $P_1'(k, x) = 1$. These can be used in determining the location of the real zeros of L_n, as was done for J_n. The discussion here is somewhat simplified since L_n has but one parameter, α, and the x-axis is divided by the zeros of $D(x)$ into two segments only, namely, $(- \infty, 0)$ and $(0, \infty)$.

The following results are easily obtained.

B_1 : $\alpha > 0$, $L_n(x, \alpha)$ has n distinct zeros in $(0, \infty)$, $n = 1, 2, 3, \cdots$. This result is well known.

B_2 : $\alpha = 0$. $L_{n+1}(x, \alpha)$ has n distinct zeros in $(0, \infty)$ and a simple zero at $x = 0$, $n = 0, 1, 2, \cdots$.

B_3 : $\alpha < 0$, α, *not an integer*. Let k_1 be the largest integer contained in α.

(I) $L_{k+1}(x, \alpha)$ has $\dfrac{(1)^k + (-1)^k}{2}$ zeros in $(- \infty, 0)$, $0 \leq k \leq k_1$,

(II) $L_{k_1+q+1}(x, \alpha)$ has q distinct zeros in $(0, \infty)$ and $\frac{(1)^{k_1} + (-1)^{k_1}}{2}$ zeros in $(-\infty, 0), q = 0, 1, 2, \cdots$.

$B_4 : \alpha < 0, \alpha = -k_1 -$ *integer.*

(I) $L_{k+1}(x, \alpha)$ has $\frac{(1)^k + (-1)^k}{2}$ zeros in $(-\infty, 0), 0 \leq k \leq k_1$.

(II) $L_{k_1+q+1}(x, \alpha)$ has q distinct zeros in $(0, \infty)$ and a zero of order $k_1 + 1$ at $x = 0, q = 0, 1, 2, \cdots$.

The Degree of $L_n(x, \alpha)$. We show first that here $P\mu(n, x) \not\equiv 0, \mu = 1, 2, \cdots n + 1$. By definition, $P_1(n, x) \equiv N + nD' \equiv -x + \alpha + n \not\equiv 0$. Let us rewrite (2) for our present situation thus:

(2°) $P'_\mu(n, x) = -\mu P_{\mu-1}(n, x)$. If, now, $P_\mu(n, x) \equiv 0$, then from (2°) it follows that $P_{\mu-1}(n, x) \equiv 0$. Continuing this reasoning, we finally arrive at a contradiction, namely, $P_1(n, x) \equiv 0$. If in (4) we set $q = n + 1$ and replace N' and D'' by their values we get:

$$L_{n+1}^{(n+1)}(x, \alpha) = (-1)^{n+1}(n + 1)! \qquad P_0(n, x) = (-1)^{n+1}(n + 1)!$$

Hence, L_{n+1} is of degree $n + 1$. Note that this holds regardless of the value of α contrary to what was found for Jacobi Polynomials.

Points of Inflexion. By a procedure analogous to that used for Jacobi Polynomials we can show that the points of inflexion of $L_{n+1}(x, \alpha)$ are located at the zeros of odd order of $L_{n-1}(x, \alpha + 2)$.

The Polynomials $P_n(0, x)$. If we set $k = 0$ in (1), (2), and (3) we obtain the following relationships for $P_n(0, x)$[5] $\equiv P_n(x) \equiv P_n$.

(7) $P_{n+1}(x) = [N - nD'] P_n(x) + DP'_n(x)$.

(8) $P'_{n+1}(x) = (n + 1)\left[N' - \frac{n}{2} D''\right] P_n(x)$.

(9) $P_{n+1}(x) = [N - nD'] P_n(x) + n\left(N' - \frac{n-1}{2} D''\right) DP_{n-1}(x)$.

Theorems I_1 to I_{10} inclusive, with $k = 0$, hold for $P_n(x)$. In addition, the following theorems hold for P_n.

Theorem H_1. *Suppose N linear and* $D(x) > 0$ *for all x. Furthermore, let* $N' - \frac{m}{2} D'' < 0, m = 1, 2, 3, \cdots$. *Then* P_n *has n real, distinct zeros which separate the zeros of* P_{n+1}.

Proof: Denote the zeros of P_n by $\alpha_{n,i}, i = 1, 2, \cdots n, \alpha_{n,i} < \alpha_{n,i+1}$. Suppose $N' > 0$. N being linear has a single zero α_{11}. Furthermore, since $P_1 \equiv N_1$, then $P_1 < 0$ for $x < \alpha_{11}$ and > 0 for $x > \alpha_{11}$. We pass now to P_2. From (7), we see that $P_2(\alpha_{11}) > 0$, (since $D > 0$ and $P'_1 > 0$). Also (8) shows $P'_2(x) > 0$

[5] E. H. Hildebrandt, loc. cit. pp. 399.

for $x < \alpha_{11}$ and < 0 for $x > \alpha_{11}$. This follows from what was noted concerning the sign of P_1 for $x > \alpha_{11}$ and $x < \alpha_{11}$, together with the hypothesis that $N' - \frac{m}{2} D'' < 0$. Thus, there exists a zero of P_2 in $(-\infty, \alpha_{11})$ and a zero in (α_{11}, ∞) and our theorem holds for $n = 1$. Assume that the theorem is true for $n = h$. The sequence $P_h'(\alpha_{h,i})$, $i = 1, 2, \cdots h$, is alternately positive and negative. Since, from (8), the only extrema of P_{h+1} are at $\alpha_{h,i}$, $i = 1, 2, \cdots h$, we conclude that there are $h - 1$ zeros of P_{h+1} separating the $\alpha_{h,i}$, $i = 1, 2, \cdots h$. Since $P_h'(\alpha_{h,1}) > 0$ we conclude that $P_h < 0$ for $x < \alpha_{h,1}$. This fact, combined with (8), shows $P_{h+1}'(x) > 0$ for $x < \alpha_{h,1}$. $P_{h+1}(\alpha_{h,1})$ being positive, it follows that there exists a zero of P_{h+1} in $(-\infty, \alpha_{h,1})$. Similar reasoning establishes the existence of a zero of P_{h+1} in $(\alpha_{h,h}, \infty)$. Our theorem is thus established for $N' > 0$. The case $N' < 0$ can be similarly treated.

Theorem H_2: If $D(x) > 0$ for all x, $D'' < 0$, $N' - \frac{m}{2} D'' < 0$, $N' = 0$, $N \not\equiv 0$, then P_n, $n = 2, 3, \cdots$, has $n - 1$ real, distinct zeros which are separated by the zeros of P_{n-1}.

Proof: Since $P_1 \equiv N =$ const., we see from (7) that P_2 is linear. The reasoning of Theorem H_1 applies where we now start with P_2.

Theorem H_3: If $D(x) > 0$ for all x, except $x = \beta$, where D has a double zero and if $N' \not\equiv 0$, $N' - \frac{n}{2} D'' < 0$, $n = 1, 2, 3, \cdots$, then P_n has n real, distinct zeros which separate those of P_{n+1}.

Proof: Theorem I_1 with $k = 0$ assures us that P_n and D have no zeros in common. The proof now follows the line of reasoning of Theorem H_1.

Theorem H_4: If $D(x) > 0$ for all x except $x = \beta$ where D has a double zero and if $N' = 0$, $N \not\equiv 0$, $N' - \frac{m}{2} D'' < 0$, $m = 1, 2, 3, \cdots$, then P_n has $n - 1$ real, distinct zeros which separate those of P_{n+1}, $n = 1, 2, 3, \cdots$. This theorem follows from H_3 as did H_2 from H_1.

Points of Inflexion. Setting $k = 0$ in (4) leads to,

$$P_{n+1}'' = (n + 1)(n)\left[N' - \frac{n}{2} D''\right]\left[N' - \frac{n-1}{2} D''\right] P_{n-1}.$$

This shows, under the assumptions of Theorems H_1 to H_4 inclusive, that the points of inflexion of P_{n+1} are at the zeros of P_{n-1}.

Hermite Polynomials. Theorem H_1 and statement immediately above concerning points of inflexion apply directly to Hermite Polynomials where $N \equiv -x$ and $D \equiv \sigma^2$.

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