THE TYPE B GRAM-CHARLIER SERIES

By Leo A. Aroian

While much attention has been devoted to the Type A Gram-Charlier series for the graduation of frequency curves, the Type B series has been somewhat neglected. However the numerical examples to be presented later will show that the Type B series is very useful for the graduation of skew frequency curves. Wicksell[1] has demonstrated that the Gram-Charlier series may be developed from the same law of probability which forms the basis of the Pearson system of frequency curves. Rietz[2] following Wicksell gives a derivation of the Gram-Charlier series based on the binomial $(q + p)^n$. Jordan[3] gives a method for fitting Type B based on certain orthogonal polynomials which he calls G. He uses factorial moments because of the resulting ease in finding the values of the constants.

We shall consider the Type B series for a distribution of equally distanced ordinates at non-negative values of x. We shall find the values of the first few terms of the series and shall also show how the values of later coefficients may easily be found. We write the Type B series in the form

$$(1)\quad F(x) = c_0 + c_1\Delta\psi(x) + c_2\Delta^2\psi(x) + c_3\Delta^3\psi(x) + c_4\Delta^4\psi(x) + c_5\Delta^5\psi(x) + c_6\Delta^6\psi(x)$$

where

$$(2)\quad \begin{aligned} \psi(x) &= \frac{e^{-m}m^x}{x!}, \qquad m = \mu_1', \text{ the mean,} \\ \Delta\psi(x) &= \psi(x) - \psi(x-1) \quad \text{for } x = 0, 1, 2, \cdots s. \end{aligned}$$

Let $f(x)$ give the ordinates of the observed distribution of relative frequencies, so that $\Sigma f(x) = 1$. To determine the coefficients $c_0, c_1, c_2, \cdots, c_6$, we have, using the method of moments,

$$(3)\quad \begin{aligned} &\Sigma[c_0\psi(x) + c_1\Delta\psi(x) + c_2\Delta^2\psi(x) + c_3\Delta^3\psi(x) + \cdots + c_6\Delta^6\psi(x)] = \Sigma f(x) = 1. \\ &\Sigma x[c_0\psi(x) + c_1\Delta\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma xf(x) = m. \\ &\Sigma x^2[c_0\psi(x) + c_1\Delta\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma x^2f(x) = \mu_2'. \\ &\Sigma x^3[c_0\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma x^3f(x) = \mu_3'. \\ &\Sigma x^4[c_0\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma x^4f(x) = \mu_4'. \\ &\Sigma x^5[c_0\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma x^5f(x) = \mu_5'. \\ &\Sigma x^6[c_0\psi(x) + \cdots\cdots + c_6\Delta^6\psi(x)] = \Sigma x^6f(x) = \mu_6'. \end{aligned}$$

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

Hence we must find the values of

$$\sum_{x=0}^{x=s} x^n \Delta^p \psi(x), \qquad \begin{matrix} n = 0, 1, 2, 3 \cdots \\ p = 0, 1, 2, 3 \cdots \end{matrix} \tag{4}$$

defining $\Delta^0\psi(x) = \psi(x)$. We assume that we are dealing with distributions in which s is large, and that the error involved in substituting $\sum_{x=0}^{\infty} x^n \Delta^p \psi(x)$ for $\sum_{x=0}^{s} x^n \Delta^p \psi(x)$ is negligible. To find these summations in a straightforward manner would involve too much labor, so we shall briefly discuss some properties of the generating function, $\psi(x) = \dfrac{e^{-m} m^x}{x!}$, the Poisson exponential, very useful in the graduation of frequency distributions of rare events. The first eight moments about the origin are:

$$\mu_0' = 1 = \Sigma\psi(x), \qquad \mu_1' = m = \Sigma x\psi(x), \qquad \mu_2' = m + m^2 = \Sigma x^2\psi(x)$$

$$\mu_3' = m + 3m^2 + m^3 = \Sigma x^3\psi(x)$$

$$\mu_4' = m + 7m^2 + 6m^3 + m^4 = \Sigma x^4\psi(x)$$

$$\mu_5' = m + 15\, m^2 + 25m^3 + 10m^3 + m^5 = \Sigma x^5\psi(x) \tag{5}$$

$$\mu_6' = m + 31m^2 + 90m^3 + 65m^4 + 15m^5 + m^6 = \Sigma x^6\psi(x)$$

$$\mu_7' = m + 63m^2 + 301m^3 + 350\, m^4 + 140m^5 + 21m^6 + m^7 = \Sigma x^7\psi(x)$$

$$\mu_8' = m + 127m^2 + 966m^3 + 1701m^4 + 1050m^5 + 256m^6 + 28m^7 + m^8 = \Sigma x^8\psi(x)$$

These may be found by the formula given by Jordan,[3]

$$\mu_{s+1}' = m\left(\mu_s' + \frac{d\mu_s'}{dm}\right). \tag{6}$$

Proof:

$$\frac{d\psi(x)}{dm} = \frac{x\psi(x)}{m} - \psi(x).$$

We multiply by x^n and sum, giving (6). This result may readily be proved also by means of recursion formulas without differentiation. Now we must find the values of

$$\sum_{0}^{\infty} x^n \Delta^p \psi(x) \qquad \begin{matrix} n = 0, 1, 2, \cdots \\ p = 1, 2, 3, \cdots \end{matrix}$$

We do this by proving

$$\sum_{x=0}^{x=\infty} x^n \Delta^{s+1} \psi(x) = -\frac{d}{dm} \sum_{x=0}^{x=\infty} x^n \Delta^s \psi(x). \tag{7}$$

Now

$$\frac{d\psi(x)}{dm} = \psi(x-1) - \psi(x) = -\Delta\psi(x). \tag{8}$$

Hence

$$\frac{d}{dm}\Delta^s\psi(x) = \frac{d}{dm}\left[\psi(x) - \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s)\right],$$

since $\Delta^s\psi(x) = \psi(x) - \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s)$.

Then by (8)

$$\frac{d}{dm}\Delta^s\psi(x) = \left[\psi(x-1) - \psi(x) - \binom{s}{1}\psi(x-2) + \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-3) - \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s-1) - (-1)^s\psi(x-s)\right].$$

$$\frac{d}{dm}\Delta^s\psi(x) = -\psi(x) + \binom{s+1}{1}\psi(x-1) - \binom{s+1}{2}\psi(x-2) + \cdots - (-1)^s\psi(x-s-1). \tag{9}$$

$$= -\left[\psi(x) - \binom{s+1}{1}\psi(x-1) + \binom{s+1}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s-1)\right].$$

$$= -\Delta^{s+1}\psi(x).$$

We multiply (9) by x^n, sum with respect to x, giving (7).

Thus by use of (7) and (5) we get:

$$\Sigma\Delta^p\psi(x) = 0, \quad p = 1, 2, 3, \cdots$$

$$\Sigma x\Delta\psi(x) = -\frac{dm}{dm} = -1.$$

$$\Sigma x^2\Delta\psi(x) = -\frac{d}{dm}\Sigma x^2\psi(x) = -\frac{d}{dm}(m + m^2) = -2m - 1. \tag{10}$$

$$\Sigma x^3\Delta\psi(x) = -3m^2 - 6m - 1.$$

$$\Sigma x^4\Delta\psi(x) = -4m^3 - 18m^2 - 14m - 1.$$

$$\Sigma x^5\Delta\psi(x) = -5m^4 - 40m^3 - 75m^2 - 30m - 1.$$

$$\Sigma x^6\Delta\psi(x) = -6m^5 - 75m^4 - 260m^3 - 270m^2 - 62m - 1.$$
$$\Sigma x\Delta^2\psi(x) = 0, \quad \Sigma x^2\Delta^2\psi(x) = 2, \quad \Sigma x^3\Delta^2\psi(x) = 6m + 6.$$
$$\Sigma x^4\Delta^2\psi(x) = 12m^2 + 36m + 14.$$
$$\Sigma x^5\Delta^2\psi(x) = 20m^3 + 120m^2 + 150m + 30.$$
$$\Sigma x^6\Delta^2\psi(x) = 30m^4 + 300m^3 + 780\, m^2 + 540m + 62.$$
$$\Sigma x\Delta^3\psi(x) = 0, \quad \Sigma x^2\Delta^3\psi(x) = 0, \quad \Sigma x^3\Delta^3\psi(x) = -6.$$
$$\Sigma x^4\Delta^3\psi(x) = -24m - 36, \quad \Sigma x^5\Delta^3\psi(x) = -60m^2 - 240m - 150.$$
$$\Sigma x^6\Delta^3\psi(x) = -120m^3 - 900m^2 - 1560m - 540.$$
(10) $$\Sigma x\Delta^4\psi(x) = 0, \quad \Sigma x^2\Delta^4\psi(x) = 0, \quad \Sigma x^4\Delta^4\psi(x) = 24.$$
$$\Sigma x^5\Delta^4\psi(x) = 120m + 240, \quad \Sigma x^3\Delta^4\psi(x) = 0.$$
$$\Sigma x^6\Delta^4\psi(x) = 360m^2 + 1800m + 1560.$$
$$\Sigma x\Delta^5\psi(x) = 0, \quad \Sigma x\Delta^6\psi(x) = 0.$$
$$\Sigma x^2\Delta^5\psi(x) = 0, \quad \Sigma x^2\Delta^6\psi(x) = 0.$$
$$\Sigma x^3\Delta^5\psi(x) = 0, \quad \Sigma x^3\Delta^6\psi(x) = 0.$$
$$\Sigma x^4\Delta^5\psi(x) = 0, \quad \Sigma x^4\Delta^6\psi(x) = 0.$$
$$\Sigma x^5\Delta^5\psi(x) = -120, \quad \Sigma x^5\Delta^6\psi(x) = 0.$$
$$\Sigma x^6\Delta^5\psi(x) = -720m - 1800, \quad \Sigma x^6\Delta^6\psi(x) = 720.$$

Finally we substitute from (5) and (10) into (3), and for μ_n' we substitute $\mu_n' = \sum_{r=0}^{n} \binom{n}{r} \mu_{n-r} m^r$. Hence

$$c_0 = 1$$
$$c_1 = 0$$
$$c_2 = \tfrac{1}{2}(\mu_2 - m).$$
(11) $$c_3 = -\frac{1}{3!}(\mu_3 - 3\mu_2 + 2m).$$
$$c_4 = \frac{1}{4!}[\mu_4 - 6\mu_3 + \mu_2(11 - 6m) + 3m(m - 2)].$$
$$c_5 = -\frac{1}{5!}[\mu_5 - 10\mu_4 - \mu_3(10m - 25) + 50\mu_2(m - 1) - 4m(5m - 6)].$$
$$c_6 = \frac{1}{6!}[\mu_6 - 15\mu_5 + \mu_4(85 - 15m) + \mu_3(130m - 225) + \mu_2(45m^2 - 375m + 274) - 15m^3 + 130m^2 - 120m].$$

It may be asked whether criteria may be given as guides for the use of Type B. In general Type B may be tried if either the skewness of the distribution to be fitted is considerable, $\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} > .6$, or if $m = \mu_2 = \mu_3$ approximately. The latter condition strictly would mean that $\psi(x)$ alone is sufficient for a good graduation, if the fourth moment, μ_4, is not used. The examples which follow are arranged to facilitate comparison with the Pearson system of frequency curves. We have an example each of Type I, III, IV, V, VI, and an example of the normal curve.

Type I. Table 1. Here $\alpha_3 > .6$ although $m \neq \mu_2 \neq \mu_3$. The first four moments, unadjusted, give an excellent fit by Type B, which is not quite as good as Type I. The degrees of freedom, according to Fisher,[4] have been taken into consideration here in applying the x^2 test. The two classes 13, 14, were grouped together for the x^2 test. The actual numerical work is easily done on a calculating machine, although logarithms are necessary to find the value of e^{-m}. This example and the remaining are all taken from Elderton[5] with the exception of Type IV which is from A. Fisher.[6]

Type III. Table 2. The unadjusted moments are used. Here $\alpha_3 = 2.0833 > .6$, and $m = \mu_2$ approximately. The fit by Type B is slightly better than that by Type III. We have for Type III $P(x^2 \geq 12.8) = .007$, $n = 3$, while for Type B, $P(x^2) \geq 9.4 = .025$ $n = 3$. Moreover the standard error of prediction for Type III is 11.2 and for Type B is 7.7.

Type IV. Table 3. The rough moments were used. Although $\alpha_3 = .48 < .6$, Type B gives a fine fit since $m = \mu_2 = \mu_3$ approximately. Here the results are given for Type B using 2, 3, and 4 terms of the series. This was done to show how the distribution changes with the addition of more terms. The superiority of Type B over Type IV is evident. The results for Type IV are taken from the class notes of Professor C. C. Craig.

Type V. Table 4. Using the adjusted moments we have a comparison among Types V, A, and B. While the graduations may seem satisfactory, the x^2 test shows that the fit is poor in each case. The order of merit is Type V, Type B, and then Type A. The negative frequencies which appear in Type B may be due to the use of the adjusted moments. If we use the rough moments, the negative frequencies disappear. On the whole the fit by means of the adjusted moments is superior.

Type VI. Table 5. Type VI using the adjusted moments gives an excellent fit. Even though α_3 is considerable, and $\mu_2 = \mu_3$ approximately, four moments with Type B give a poor fit, and five moments, adjusted, achieve a very small gain. Five moments using the unadjusted moments give some improvement, but the -2 frequency in the first class is objectionable.

Normal Curve. Table 6. The normal curve provides a fine fit. $P(x^2 \geq .9) = .96$, $n = 6$. The first two and the last two classes were grouped together for the test. The fit by Type B is less probable, $P(x^2 \geq 8) = .15$, $n = 5$. Type B has two discrepancies, the negative frequencies, and the fact that the total frequencies (neglecting the -1) is 352. That Type B does so well is in itself quite amazing!

TABLE 1

x	Actual frequency	Frequency computed by Pearson Type I	Frequency given by Type B
0	34	44	42.4
1	145	137	121.3
2	156	149	168.7
3	145	142	156.8
4	123	127	120.5
5	103	108	94.9
6	86	88	82.9
7	71	69	72.2
8	55	51	56.7
9	37	36	38.0
10	21	24	23.1
11	13	14	12.0
12	7	7	5.7
13	3	3	2.4
14	1	1	.9

$m = 4.175$ $\alpha_3 = .712247$ Type I $P(x^2 \geq 4.36) = .88$

$\mu_2 = 7.66237$ $\alpha_4 = 2.95214$ n (number of degrees of freedom) $= 9$

$\mu_3 = 15.1069$ $c_2 = 1.74368$

$\mu_4 = 173.326$ $c_3 = -.078298$ Type B $P(x^2 \geq 9.67) = .37$

$c_4 = +.094592$ $n = 9$

$$F(x) = \psi(x) + 1.74368\,\Delta^2\psi(x) - .078298\,\Delta^3\psi(x) + .094592\,\Delta^4\psi(x).$$

TABLE 2

x	Actual frequency	Frequency computed by Type III	Fequency by Type B
0	44	59	48.1
1	135	111	121.6
2	45	45	58.5
3	12	20	10.4
4	8	9	3.5
5	3	4	4.3
6	1	2	2.9
7	3	1	1.2

$m = 1.33466$ $\alpha_3 = \dfrac{\mu_3}{\mu_2^{3/2}} = 2.0833$ $c_2 = .05356$

$\mu_2 = 1.44179$ $c_3 = -.32510$

$\mu_3 = 3.60662$

$$F(x) = \psi(x) + .05356\Delta^2\psi(x) - .32510\Delta^3\psi(x)$$

TABLE 3

Number of alpha particles from a bar of polonium in intervals of $\frac{1}{8}$ of one minute

x	Frequency	Type IV	Type B 2 terms	Type B 3 terms	Type B 4 terms
0	57	50	49.5	49.0	58.2
1	203	183	201.3	201.0	199.8
2	383	392	403.4	404.3	386.1
3	525	544	532.3	533.8	523.9
4	532	539	520.6	521.5	532.1
5	408	417	402.6	402.5	418.2
6	273	250	254.8	254.4	260.2
7	139	131	137.1	136.7	134.0
8	45	61	64.0	63.9	56.7
9	27	26	26.1	26.2	22.9
10	10	12	9.4	9.6	8.6
11	4	4	3.0	3.1	3.6
12	0	1	.9	.9	1.6
13	1	0	.2	.2	.8
14	1	0	.0	.0	.3

$m = 3.87155$ $\alpha_3 = .47844$

$\mu_2 = 3.69477$ $\alpha_4 = 3.506536$

$\mu_3 = 3.39791$

$\mu_4 = 47.86888$

$$F(x) = \psi(x) - .08839\Delta^2\psi(x) - .00930\Delta^3\psi(x) + .16810\Delta^4\psi(x).$$

Type B, 4 terms $P(x^2 \geq 4.50) = .72, n = 7$

Type IV $P(x^2 \geq 10.8) = .15, n = 7$

TABLE 4

Mortality Among Female Nominees

x	Deaths	Elderton Type V	Type A	Type B 2 terms	Type B 3 terms	Type B 5 terms	Type B 5 terms
0	4	4	2	1.4	−6.9	−.4	4.1
1	18	10	15	26.3	7.1	9.4	13.1
2	53	80	78	109.7	100.1	84.6	77.4
3	265	261	235	248.3	268.4	252.3	242.5
4	438	441	426	379.5	418.8	425.9	427.4
5	525	480	521	432.7	461.0	484.0	494.1
6	342	381	411	388.8	388.4	402.6	408.1
7	253	247	225	285.4	263.5	259.0	253.9
8	128	137	107	170.8	145.5	132.2	124.9
9	82	68	66	84.3	68.3	58.6	54.1
10	28	32	44	32.9	28.2	26.2	26.4
11	12	14	22	8.6	11.0	13.9	16.4
12	8	6	8	−.01	4.7	8.2	10.7
13	5	3	2	−2.1	2.1	4.3	5.9
14	1	1	0	−1.5	1.3	2.0	2.5

Adjusted moments:

$m = 5.30435$ $\alpha_3 = .703564$

$\mu_2 = 3.573345$ $\alpha_4 = 3.996196$

$\mu_3 = +4.752437$

$\mu_4 = 51.02659$

$\mu_5 = 193.439125$

Rough moments:

$m = 5.30435$

$v_2 = 3.65668$

$v_3 = 4.752437$

$v_4 = 52.85276$

$v_5 = 197.39949$

Type A: $f(t) = \varphi(t) + .117261\,\varphi^3(t) + .041508\varphi^4(t)$

Type B: $F(x) = \psi(x) - .86550\Delta^2\psi(x) - .77352\Delta^3\psi(x) + .02814\Delta^4\psi(x) + .57459\Delta^5\psi(x)$

Using uncorrected moments

Type B: $F(x) = \psi(x) - .82384\Delta^2\psi(x) - .73185\Delta^3\psi(x) + .03192\Delta^4\psi(x) + .94033\Delta^5\psi(x)$

(last column above)

TABLE 5

x	Frequency	Type VI	Type B 4 terms	Type B 5 terms
0	1	1	−9.5	−2.0
1	56	50	83.2	69.9
2	167	168	141.6	143.1
3	98	100	102.3	110.7
4	34	36	41.5	40.2
5	9	10	8.7	4.6
6	2	2	.05	2.0
7	1	.5	−.4	1.0

Corrected moments:	Rough moments:
$m = 2.402174$	$m = 2.402174$
$\mu_2 = .928835$	$\mu_2 = 1.012169$
$\mu_3 = .893096$	$\mu_3 = .893096$
$\mu_4 = 4.088800$	$\mu_4 = 4.313176$
	$\mu_5 = 11.28304$
	$\alpha_3 = .87704$
	$\alpha_4 = 4.2101$

Type B, adjusted moments:

$$F(x) = \psi(x) - .73667\Delta^2\psi(x) - .48516\Delta^3\psi(x) - .06424\Delta^4\psi(x) + .10365\Delta^5\psi(x)$$

*Type B, rough moments:

$$F(x) = \psi(x) - .69805\Delta^2\psi(x) - .44654\Delta^3\psi(x) - .06587\Delta^4\psi(x) + .15165\Delta^5\psi(x)$$

* This is used in last column of above. There is a slight error here, which however will not affect the results materially. The third decimal place may be slightly wrong.

TABLE 6

Normal curve

x	Frequency	Normal curve	Type F
0	.6	.6	2.3
1	2.8	2.7	4.7
2	11.5	10.9	8.7
3	27.7	30.1	25.2
4	59.1	58.4	55.2
5	84.7	80.1	79.5
6	74.1	76.9	80.1
7	50.5	52.2	58.1
8	23.2	25.0	29.7
9	12.2	8.4	8.6
10	1.3	2.4	−.9

Moments corrected:

$m_1 = 5.393443$

$\mu_2 = 2.769635$

$\mu_3 = .029805, \mu_4 = 22.40663$

$\alpha_3 = .0064$

$\alpha_4 = 2.920997$

Type B: $F(x) = \psi(x) - 1.3119\Delta^2\psi(x) - .4179\Delta^3\psi(x) + 2.1625\Delta^4\psi(x)$

Colorado State College

REFERENCES

(1) Wicksell, S. D.: Contributions to the analytical theory of sampling, Archiv för Matematik, Astronomi och Fysik, Vol. 17 no. 19 (1923) pp. (1–46)

(2) Rietz, H. L.: Mathematical Statistics. pp. (156–172).

(3) Jordan, C.: Statistique Mathématique pp. 36–40, 90–92, 98–102, 240, 282–283.

(4) Fisher, R. A.: Statistical Methods for Research Workers. (5th edition.)

(5) Elderton, W. P.: Frequency Curves and Correlation.

(6) Fisher, A.: The Mathematical Theory of Probabilities.