ON DIFFERENTIAL OPERATORS DEVELOPED BY O'TOOLE

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1. O'Toole in his paper 'Symmetric Functions and Symmetric Functions of Symmetric Functions' [Ann. Statist. 2. (1931)102–49], has expressed Monomial Symmetric Functions $\sum a^{p_1} b^{p_2} c^{p_3} \cdots$, in terms of power-sums, s_r.

The Monomial Symmetric Functions can be written in partition notation as $\binom{k_1\, k_2\, k_3}{p_1 p_2 p_3} \cdots)$ where k_1, k_2, $\cdots$ denote the repetitions of parts.

To express $\binom{k_1\, k_2\, k_3}{p_1 p_2 p_3} \cdots)$ as a function of s_r', O'Toole has developed operators d_r and D_r, connected by the formulae,[1]

$$d_r = \frac{d}{ds_r},$$

$$\text{(A)} \qquad rd_r = \frac{(-1)^{r+1} \sum (-1)^{r+k}(k-1)!\, r \cdot D_A^{k_1} D_B^{k_2} \cdots}{k_1!\, k_2! \cdots;}$$

$$\text{(B)} \qquad r!\, D_r = \frac{\sum r!\, d_A^{k_1} d_B^{k_2} \cdots}{k_1!\, k_2! \cdots},$$

$$\text{where } k_1 A + k_2 B + \cdots = r$$
$$k_1 + k_2 + \cdots = k.$$

In this paper it will be shown that these operational relations are easily deduced from the operators d_r and D_r of Hammond, used for expressing Monomial Symmetric Functions as functions of Elementary Symmetric Functions, a_r.

For the sake of distinction I shall use q_r and Q_r for the operators employed by O'Toole and keep d_r and D_r for Hammond's Operators.

Macmahon has dealt with Hammond's operators in his Combinatory Analysis Vol. I. Cambridge University Press (1915), where they are defined[2] as

$$D_r = \frac{1}{r!}(d_1^r) \qquad \text{and} \qquad d_r = \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + a_2 \frac{d}{da_{r+2}} + \cdots, \qquad (1).$$

2. It is known[3] that

$$\log(1 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots) = -\left(s_1 x + \frac{1}{2} s_2 x^2 + \cdots + \frac{1}{r} s_r x^r + \cdots\right).$$

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[1] O'Toole, Loc. cit., p. 120.

[2] Macmahon. Comb. Analysis. I. 27–28.

[3] Ibid., p. 6.

Now operate on the right hand side with d_r, and with its equivalent in (1) on the left hand side. Equating coefficients of x^r on both sides, we obtain,

$$d_r s_r = (-1)^{r-1} r; \qquad d_r s_k = 0 \text{ when } r \neq k,$$

which yields

$$d_r = (-1)^{r-1} r \frac{d}{ds_r} = (-1)^{r-1} r q_r. \tag{2}$$

The operator Q_r exactly behaves like D_r. From the formula[4]

$$d_r - D_1 d_{r-1} + D_2 d_{r-2} - D_3 d_{r-3} + \cdots + (-1)^r r d_r = 0,$$

which is in complete correspondence with Newton's recurrence relation, we derive

$$\begin{aligned} d_1 &= D_1 \\ d_2 &= D_1^2 - 2D_2 \\ d_3 &= D_1^3 - 3D_1D_2 + 3D_3. \\ &\cdots\cdots \end{aligned} \tag{C}$$

By multinomial theorem

$$d_r = \frac{\sum (-1)^{r+k}(k-1)!\, r D_A^{k_1} D_B^{k_2} \cdots}{k_1! k_2! \cdots}$$

using (2) we at once get

$$r q_r = (-1)^{r+1} \frac{\sum (-1)^{r+k}(k-1)!\, r Q_A^{k_1} Q_B^{k_2} \cdots}{k_1! k_2! \cdots}$$

which is the result (A) obtained by O'Toole.

From (C), D_r follows in terms of d_r and thence with (2) Q_r can be expressed in terms of q_r. Using multinomial theorem we arrive at

$$r! Q_r = \frac{\sum r!\, d_A^{k_1} d_B^{k_2} \cdots}{k_1! k_2! \cdots}$$

which is (B). Hence both the results of O'Toole have been deduced.

3. In his second paper[5] O'Toole defines symmetric functions for more than one system of Variates. I call such symmetric functions *Hyper Symmetric Functions.*

The Hyper operators are developed to express Hyper symmetric functions in terms of hyper power-sums. They are defined by O'Toole by the following relations, taking into consideration two systems of variates only,

$$d_{pq}^r = \frac{d^r}{ds_{pq}};$$

[4] p. 29.

[5] Ann. Stat. 3. (1932), 56-63.

(A)′ $$d_{pq} = \frac{\sum (-1)^{k+1}(k-1)!\,D^{k_1}_{p_1q_1}D^{k_2}_{p_2q_2}\cdots}{k_1!\,k_2!\cdots}$$

(B′) $$D_{pq} = \frac{\sum d^{k_1}_{p_1q_1}d^{k_2}_{p_2q_2}\cdots}{k_1!\,k_2!\cdots}$$

$$\text{where } k_1p_1 + k_2p_2 + \cdots = p$$
$$k_1q_1 + k_2q_2 + \cdots = q$$

These relations readily follow from Macmahon's[6] hyper operators g_{pq} and G_{pq}. These operators came into existence with the problem of expressing hyper symmetric functions in terms of hyper elementary symmetric functions and they are connected by the following relations.

I. $$(-1)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq} = \frac{\sum (-1)^{k-1}(k-1)!}{k_1!\,k_2!\cdots}\,G^{k_1}_{p_1q_1}G^{k_2}_{p_2q_2}\cdots$$

II. $$(-1)^{p+q-1}G_{pq} = \sum \frac{[(p_1+q_1-1)!]^{k_1}}{p_1!\,q_1!}\,\frac{[(p_2+q_2-1)!]^{k_2}}{p_2!\,q_2!}\cdots\frac{(-1)^{k-1}}{k_1!\,k_2!\cdots}(g_{p_1q_1})^{k_1}(g_{p_2q_2})^{k_2}\cdots.$$

Macmahon[7] has also shown that

$$g_{pq}s_{pq} = (-1)^{p+q-1}\frac{p!\,q!}{(p+q-1)!};$$

from which we get

$$g_{pq} = (-1)^{p+q-1}\frac{p!\,q!}{(p+q-1)!}\,d_{pq} \tag{3}$$

The operator G behaves like D of O'Toole. Now using (3) we derive from (I) the result (A') arrived at by O'Toole without reference to Macmahon. Similarly from II. using (3) (B') is deduced.

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[6] Macmahon. Comb. Analysis. Vol. II. Cambridge University Press (1916), p. 302.
[7] Macmahon Op. Cit., p. 304.